Quasi-Orders on Automata

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A reflexive and transitive binary relation is called a *quasi-order*, or a *preorder*, in some sources. As known, quasi-orders play an important role in many mathematical theories, in Mathematical Logic, Algebra, Topology, Computer Science etc. The main aim of the present paper is to give some applications of quasi-orders in Automata Theory.

A nonempty set Q equipped with a quasi-order is said to be a *quasi-ordered set*. If Q can be represented as a disjoint union of a family $\{Q_\alpha\}_{\alpha\in Y}$ of its subsets such that the elements from different Q_α's are incomparable, then Q is said to be a *direct sum* of quasi-ordered sets Q_α, $\alpha \in Y$. Direct sum decompositions of quasi-ordered sets were investigated by Ćirić, Bogdanović and Kovačević in [3], 1998, through ideals, filters (or dual ideals), and *double ideals*, which are defined as subsets which are both ideals and filters. It was proved that double ideals of a quasi-ordered set Q form a complete atomic Boolean algebra which is the center both of the lattice $\boldsymbol{I}(Q)$ of ideals and the lattice $\boldsymbol{F}(Q)$ of filters of Q, and that the following holds:

Theorem 2 *Every quasi-ordered set Q can be uniquely represented as a direct sum of direct sum indecomposable automata.*

This is the greatest direct sum decomposition of Q and its summands are exactly the atoms of the Boolean algebra of double ideals of Q.

Using the same concept we established a correspondence between direct sum decompositions of Q and direct product decompositions of the lattices $\boldsymbol{I}(Q)$ and $\boldsymbol{F}(Q)$.

The mentioned results can be applied to automata if they are considered as quasi-ordered sets with respect to quasi-orders defined in a natural way. Namely, if A is an automaton (deterministic or nondeterministic), then we define a quasi-order $\preceq$ on A saying that $a \preceq b$ if and only if there exists an input word which takes a into b. Here we show that the lattice of subautomata of the automaton A is isomorphic to the lattice of filters of the quasi-ordered set $(A, \preceq)$, and that the lattice of direct sum decompositions of the automaton A is isomorphic to the lattice of direct sum decomposition of the quasi-ordered set $(A, \preceq)$. As consequences we obtain the results given by Ćirić and Bogdanović in [5], 1999, which say that every automaton A can be represented as a direct sum of direct sum indecomposable automata A_α, $\alpha \in Y$, and that the lattice $\mathrm{Sub}(A)$ of subautomata of A can be represented as a direct product of direct product indecomposable lattices L_α, $\alpha \in Y$, where $L_\alpha \cong \mathrm{Sub}(A_\alpha)$, for every $\alpha \in Y$.

We also give other applications of quasi-orders to study of direct sum and subdirect product decompositions of deterministic (not necessarily finite) automata. Namely, we define a quasi-order π on an automaton A to be *positive* if $a\,\pi\, au$, for every state a and input word u. If π is a positive quasi-order on A, then a state a is called *π-reversible* if $au\,\pi\, a$, for every input word u, and A is said to be *π-connected* if for any two states a, b there exists an input word v such that $a\,\pi\, bv$. States which are $\preceq$-reversible are called *reversible*, whereas $\preceq$-connected automata are exactly the connected ones. An automaton is *reversible* if any its state is reversible. Using these concepts we prove the following:

Theorem 3 *Let A be an automaton with a countable input alphabet. Then the following conditions are equivalent:*

(i) *There exists a positive quasi-order π on A such that A is π-connected and every π-reversible state of A is an ordinary reversible state.*

(ii) *A satisfies one of the following two conditions:*

  (1) *A is an extension of a reversible automaton by a trap-connected automaton.*

  (2) *A does not have a trap and it is a subdirect product of countably many trap-connected automata.*

On the other hand, we define a positive quasi-order π on A to satisfy the *quadrangle property* if for any two states a, b, $a\,\pi\, b$ implies that for every input word u there exists an input word v such that $au\,\pi\, bv$. The next three theorems demonstrate the role of quasi-orders with the quadrangle property in direct sum decompositions of automata.

Theorem 4 *If π is a positive quasi-order on an automaton A having the quadrangle property, then A can be represented as a direct sum of automata A_α, $\alpha \in Y$, such that any A_α is direct sum indecomposable and π_α-connected, where π_α is the restriction of π onto A_α.*

Conversely, if A is a direct sum of automata A_α, and for any $\alpha \in Y$, A_α is π_α-connected, then $\pi = \bigcup_{\alpha \in Y} \pi_\alpha$ is a positive quasi-order on A with the quadrangle property.

Theorem 5 *If A is an automaton with a countable input alphabet, then the following conditions are equivalent:*

(i) *There exists a positive quasi-order π on A having the quadrangle property such that every π-reversible state of A is an ordinary reversible state.*
(ii) *A is a direct sum of direct sum indecomposable automata A_α, $\alpha \in Y$, such that any A_α is either an extension of a reversible automaton by a trap-connected automaton or it does not have a trap and it is a subdirect product of countably many trap-connected automata.*

Theorem 6 *The following conditions on an automaton A are equivalent:*

(i) *The quasi-order $\preceq$ on A has the quadrangle property;*
(ii) *A is locally connected, i.e. every monogenic subautomaton of A is connected;*
(iv) *A is a direct sum of connected automata;*
(v) *$D(H) = \{a \in A \mid au \in H,$ for some input word $u\}$ is a subautomaton of A, for every subautomaton H of A.*

References

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Verification solutions of packing circle problems

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The discussed packing circle problem can be formalized in the following way: place a given number of n equal circles without overlapping into a unit square maximizing the size of the circles. In the last decade some computer-aided proofs came to light presenting globally optimal packings based on traditional real arithmetic. Due to the well-known representing and rounding problems of real numbers, methods with guaranteed accuracy are required to verify these results. In our present study two different kinds of interval branch-and-bound algorithms are introduced, providing reliable optimal solutions for both local and global cases.