The analytical approximation of the nilpotent operators and its applications

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The Lukasiewicz or nilpotent operator family plays an important role in fuzzy logic. The operators have the following common form:

$$c(x, y) = f^{-1}([f(x) + f(y)])$$

where $f(x)$ is the generator function of the operator, and

$$[x] = \begin{cases} 1, & \text{if } 1 \leq x \\ x, & \text{if } 0 < x < 1 \\ 0, & \text{if } 0 \geq x \end{cases}$$

These operators are widely used in fuzzy logic due to their good theoretical properties, i.e. using the nilpotent class of operators the residual and material implications coincide, the law of excluded middle and the law of non-contradiction both hold, etc.

Despite these good theoretical properties this operator family is rarely used in practice. The reason is the non analytical property of the $[x]$ function.

On one hand a logical expression using nilpotent conjunction and disjunction must be properly discussed to be evaluated, and this is a rather difficult process. On the other hand robotics and fuzzy control generally use the derivatives of the fuzzy operators and the $[x]$ function is not differentiable. As a consequence, using standard, gradient based optimization techniques is impossible.

In this work we propose an approximation of the $[x]$ function by means of sigmoid function:

$$B(x, \lambda) = 1 + \ln \left(\frac{\sigma(x-1, \lambda)}{\sigma(x, \lambda)} \right)^{1/\lambda}$$

The approximation $B(x, \lambda)$ is differentiable and gives back the $[x]$ function if $\lambda \to \infty$. Another advantage is that $B'(x, \lambda)$ is simple and can be expressed by sigmoid functions. We also investigate the possible uses of this approximately nilpotent operator.