Investigation of a Delayed Differential Equation with Verified Computing Technique

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Consider the following delayed differential equation:

$$y' = -\alpha\left(e^{y(t-1)} - 1\right),$$

where $\alpha \in \mathbb{R}^+$ is a parameter.

When $\alpha \leq 1.5$, it is known, that the trajectory converges to zero, and when $\alpha \geq \pi/2$, the trajectory converges to different periodic solutions. Thus, the question is the $[1.5, \pi/2]$ interval. We want to prove that there do not exist periodic solutions of this delayed differential equation with any $\alpha \in [1.5, \pi/2]$ parameter as conjectured in [6].

The analysis of this problem is very hard with numerical methods, hence in the first part we consider an easier problem. We are interested in checking whether for all $\alpha \in \left[\frac{3}{2}, \frac{\pi}{2}\right]$, there exists a unit length time segment where the absolute value of the solution is less than 0.075. Let the initial function be

$$\phi(s) \equiv -11,$$

where $s \in [-1, 0]$.

Most verified techniques for solving ordinary differential equations apply a Taylor series. Our technique is based on the same idea too. The general form of the Taylor-series is:

$$y(x) = \sum_{k=0}^{n-1} \frac{(x-x_0)^k y^{(k)}(x_0)}{k!} + r_n. \tag{1}$$

Using the mean-value theorem, r_n can be bounded by

$$r_n = \frac{(x-x_0)^n}{n!} y^{(n)}(x^*), \tag{2}$$

for some $x^* \in [x_0, x]$ ($x_0 \leq x$).

If we want a better approximation of the solution, we have to use higher derivatives. We can characterize the higher derivatives with this formula:

$$y^{(k)}(t) = -\alpha y^{(k-1)}(t-1) + \sum_{i=1}^{k-1} \left(\begin{array}{c} k-2 \\ i-1 \end{array} \right) y^{(i)}(t-1) y^{(k-i)}(t).$$

In this case the verification means verification in the mathematical sense, hence rounding and other errors were considered and bounded. Instead of real numbers, we can also calculate with intervals. In case the bounds of the result interval are not representable, then they are rounded outward. In this problem we used the multiple precision interval arithmetic libraries (C-XSC, PROFIL/BIAS) [5, 3].

To provide a mathematical proof, it is not enough to use interval arithmetic, we have to use the formula in a correct, suitable form. We can use the Taylor-series to bound the results:

$$Y(t_1) = \sum_{i=0}^{n-1} Y^i(t_0) \frac{(t_1 - t_0)^i}{i!} + Y([t_0, t_1])^n \frac{(t_1 - t_0)^n}{n!},$$

$$y([t_0, t_1]) = \sum_{i=0}^{n-1} Y^i(t_0) \frac{([0, t_1 - t_0])^i}{i!} + Y([t_0, t_1])^n \frac{([0, t_1 - t_0])^n}{n!}.$$

We use two fix length lists to store the solution bounds. The first list contains the solution and the derivatives on time intervals, which cover the unit length time segment. The other list stores the solution and the derivatives in concrete time points. We calculate the new elements of the lists with the earlier discussed formula. The oldest elements are deleted from the lists, and the new ones are inserted. This technique has three parameters: step length, maximum derivate rank, and a precision of the interval arithmetic. We combine our method and an optimization technique to determine the optimal values for these parameters.

We proved the above original statement with this technique for some tiny intervals around certain computer representable numbers. But we were not able to prove it for all points of the α parameter interval, due to the large amount of necessary CPU time. We show the details of the newest program which is based on the earlier technique and the idea some theoretical results. This compound method is able to prove the original conjecture for $\alpha \in [1.50, 1.568]$.

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