

Numerical simulations of stochastic electrical circuits using C#

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Modelling of physical systems by ordinary differential equations ignores stochastic effects. By incorporating random elements into the differential equations, a system of stochastic differential equations (SDEs) arises.

A general N-dimensional SDE can be written in vector form as

$$d\mathbf{X}(t) = \mathbf{A}(t, \mathbf{X}(t)) dt + \sum_{j=1}^M \mathbf{B}^j(t, \mathbf{X}(t)) dW^j(t),$$

where $\mathbf{A} : \langle 0, T \rangle \times \mathbb{R}^N \rightarrow \mathbb{R}^N$, $\mathbf{B} : \langle 0, T \rangle \times \mathbb{R}^N \rightarrow \mathbb{R}^{N \times M}$ are functions and $W^1(t), \dots, W^M(t)$ are independent Wiener processes representing the noise. (A process $W(t)$ is called the Wiener process if it has independent increments, $W(0) = 0$ and $W(t) - W(s)$ distributed $N(0, t - s)$.) The solution is a stochastic vector process $\mathbf{X}(t) = (X^1(t), \dots, X^N(t))$.

By an SDE we understand in fact an integral equation

$$\mathbf{X}(t) = \mathbf{X}_0 + \int_{t_0}^t \mathbf{A}(s, \mathbf{X}(s)) ds + \sum_{j=1}^M \int_{t_0}^t \mathbf{B}^j(s, \mathbf{X}(s)) dW^j(s),$$

where the integral with respect to ds is the Lebesgue integral and the integrals with respect to $dW^j(s)$ are stochastic integrals, called the Itô integrals (see [1]). Although the Itô integral has some very convenient properties, the usual chain rule of classical calculus doesn't hold. Instead, the appropriate stochastic chain rule, known as Itô formula, contains an additional term, which, roughly speaking, is due to the fact that the stochastic differential $(dW(t))^2$ is equal to dt in the mean square sense, i.e. $E[(dW(t))^2] = dt$, so the second order term in $dW(t)$ should really appear as a first order term in dt .

We present an application of the Itô stochastic calculus to the problem of modelling inductor-resistor electrical circuits. The electrical current $i(t)$ at time t in a simple RL electrical circuit satisfies the differential equation

$$L \frac{di(t)}{dt} + R i(t) = u(t), \quad i(0) = i_0,$$

where the resistance R and the inductance L are constants and $u(t)$ denotes the potential source at time t (see [2]). Now we allow some randomness in the electrical source as well as in the resistance. The SDE describing this situation is

$$dI(t) = \frac{1}{L}(u(t) - RI(t)) dt - \frac{\alpha}{L} I(t) dW_1(t) + \frac{\beta}{L} dW_2(t), \quad I(0) = I_0,$$

where α and β are non negative constants. Their magnitudes determine the deviation of the stochastic case from the deterministic one. We consider both the initial condition and the current at time t as random variables and denote them by capital letters. We present the analytical solution of this SDE and also show the expectation and the second moment equation of $I(t)$.

In order to simulate $I(t)$ numerical techniques have to be used (see [3]). To generate numerical solutions and their graphical representations we use the programming language C# (see [4]), which is a part of the new MS .NET platform. To make matrix manipulation and visualization simpler we use the component library LinAlg described in [5]. LinAlg

is a set of classes that enables vectorial programming and incorporates a wide range of numerical, statistical and graphical methods. We also compute the confidence intervals for the trajectories of the solution. The results were verified in an experiment by measurements on inductor-resistor electrical circuits.

References

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