Verified localization of trajectories with prescribed behaviour in the forced damped pendulum

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In mathematics, it is quite difficult to define exactly what chaos really means. In particular, it is easier to prepare a list of properties which describe a so called chaotic system than give a precise definition. A dynamic system is generally classified as chaotic if it is sensitive to its initial conditions. Chaos can be also characterized by dense periodic orbits and topological transitivity.

While studying computational approximations of solutions of differential equations, it is an important question is whether the given equation has chaotic solutions. The nature of chaos implies that the numerical simulation must be carried out carefully, considering fitting measures against possible distraction due to accumulated rounding errors. Unfortunately except a few cases, the recognition of chaos has remained a hard task that is usually handled by theoretical means [3].

In our present studies, we investigate a simple mechanical system, Hubbard's sinusoidally forced damped pendulum [3]. Applying rigorous computations, his 1999 conjecture on the existence of chaos was proved in Bánhelyi *et al.* [1] in 2008 but the problem of finding chaotic trajectories remained entirely open. This time, we present a fitting verified numerical technique capable to locate finite trajectory segments theoretically with arbitrary prescribed qualitative behaviour and thus shadowing different types of chaotic trajectories with large precision. For example, we can achieve that our pendulum goes through any specified finite sequence of gyrations by choosing the initial conditions correctly.

To be able to provide solutions with mathematical precision, the computation of trajectories has to be executed rigorously. Keeping in mind this intention, we calculated the inclusion of a solution of the differential equation with the VNODE algorithm [5] and based on the PROFIL/BIAS interval environment [4]. The search for a solution point is a global optimization problem to which we applied the C version of the GLOBAL algorithm, a clustering stochastic global optimization technique [2]. This method is capable to find the global optimizer points of moderate dimensional global optimization problems, when the relative size of the region of attraction of the global minimizer points are not very small.

References

[1] BÁNHELYI, B., T. CSENDES, B.M. GARAY, AND L. HATVANI, *A computer–assisted proof for Σ_3–chaos in the forced damped pendulum equation*, SIAM J. Appl. Dyn. Syst., 7, 843–867 (2008).

[2] CSENDES, T., *Nonlinear parameter estimation by global optimization – efficiency and reliability*, Acta Cybernetica, 8, 361-370 (1988).

[3] HUBBARD, J.H., *The forced damped pendulum: chaos, complication and control*, Amer. Math. Monthly, 8, 741–758 (1999).

[4] KNÜPPEL, O., *PROFIL – Programmer's Runtime Optimized Fast Interval Library*, Bericht 93.4., Technische Universität Hamburg-Harburg (1993).

[5] NEDIALKOV, N.S., *VNODE – A validated solver for initial value problems for ordinary differential equations*, Available at `www.cas.mcmaster.ca/~nedialk/Software/VNODE/VNODE.shtml` (2001).