Operations on Signed Distance Functions

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Abstract: We present a theoretical overview of signed distance functions and analyze how sphere tracing algorithms slow down in the presence of set operations on said implicit representations.

Introduction and Previous Work

Surface representations for real-time graphics rely on linear approximations. With the advent of hardware accelerated tessellation units, parametric surfaces gained momentum in real-time computer graphics; however, implicit mappings are still considered infeasible for high-performance applications.

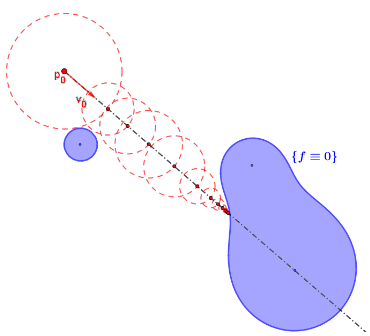
Nevertheless, implicit functions simplify some otherwise difficult operations. For example, the result of set operations on surfaces can be trivially represented by minimum, maximum, and negation of their implicit functions.

Our paper focuses on a particular class of implicit representations, signed distance functions, and their lower bounds. It has been noted by Hart in [1] that they can be rendered efficiently using a technique called sphere tracing.

We discuss this class of functions and highlight their theoretical aspects that have practical consequences in rendering.

Signed Distance Function

From any sample point, a distance function $f : \mathbb{R}^3 \to \mathbb{R}$ evaluates to the distance to the surface $\{f \equiv 0\} := \{\boldsymbol{x} \ : \ f(\boldsymbol{x}) = 0\}$. That is, for any $\boldsymbol{p} \in \mathbb{R}^3$ point in space, the sphere of radius $f(\boldsymbol{p})$ around it is disjoint from the $\{f \equiv 0\}$ surface. This property demonstrates that the sphere tracing algorithm from Figure 1 can be applied to find the first ray-surface intersection.



(a) *Unbounding spheres*

In : $\boldsymbol{p}, \boldsymbol{v} \in \mathbb{R}^3, |\boldsymbol{v}| = 1$ ray; SDFE $f : \mathbb{R}^3 \to \mathbb{R}$
Out: $t \in [0, +\infty)$ distance traveled along the ray
$t := 0; \quad i := 0;$
for $i < i_{max}$ ***and*** $f(\boldsymbol{p} + t \cdot \boldsymbol{v}) > \epsilon$ **do**
  $t := t + f(\boldsymbol{p} + t \cdot \boldsymbol{v});$
  $i := i + 1;$
end

(b) *Sphere tracing adapted from [1]*

Figure 1: *An overview of the sphere tracing root-finding algorithm*

Singed distance functions (SDFs) can represent an entire volume by classifying the points of $\mathbb{R}^3$ belonging to its "interior" ($\{f < 0\}$), "exterior" ($\{f > 0\}$), or to the surface ($\{f \equiv 0\}$). Formally, the continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ is an SDF if $|f(\boldsymbol{p})| = d(\boldsymbol{p}, \{f \equiv 0\})$ for any point $\boldsymbol{p} \in \mathbb{R}^3$. Note that a distance function is also a signed distance function. In summary, we have the following equivalent definition to describe SDFs:

Theorem 1 (SDF equivalence)**.** *The function $f : \mathbb{R}^3 \to \mathbb{R}$ is a singed distance function if, and only if there exists a $\emptyset \neq D \subseteq \mathbb{R}^3$ set for which*

$$f(\boldsymbol{p}) = \begin{cases} d(\boldsymbol{p}, \partial D) & ha \ \boldsymbol{p} \notin D \\ -d(\boldsymbol{p}, \partial D) & ha \ \boldsymbol{p} \in D \end{cases}, \tag{1}$$

where $\partial D := \{f \equiv 0\}$ is the boundary of volume D.