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The damped Fermi–Pasta–Ulam oscillator

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Abstract. The system

$$\ddot{q}_k + \gamma \dot{q}_k = V'(q_{k+1} - q_k) - V'(q_k - q_{k-1}) \qquad (k = 1, \dots, N-2)$$

is considered, where $0 < \gamma = \text{const.}$, $2 < N \in \mathbb{N}$, $V : (A, B) \to \mathbb{R}$ $(-\infty \leq A < B \leq \infty)$ is a strictly convex, two times continuously differentiable function. We connect to the system three kinds of boundary conditions: $q_0(t) = 0$, $q_{N-1}(t) = L = \text{const.} > 0$ (fixed endpoints – this is the original Fermi–Pasta–Ulam oscillator provided that the damping coefficient γ equals zero); $q_1(t) - q_0(t) = L/(N-1)$, $q_{N-1}(t) - q_{N-2}(t) = L/(N-1)$ (free endpoints); $q_0(t) = -(K - q_{N-2}(t))$, $q_{N-1}(t) = q_1(t) + K$, $K = \text{const.}$ (cycle). We prove that the unique equilibrium state of the system with fixed endpoints is asymptotically stable. We also prove that the system with free endpoints and the cycle asymptotically stop at an equilibrium state along their arbitrary motion, i.e., for every motion there is $q_1^\infty \in \mathbb{R}$ such that $\lim_{t\to\infty} q_k(t) = q_1^\infty + (k-1)\bar{r}$, $\lim_{t\to\infty} \dot{q}_k(t) = 0$ $(k = 1, \dots, N-2)$, where the constant $\bar{r}$ is defined by the equation $V'(\bar{r}) = 0$.

Keywords: asymptotic stability, asymptotic stop, invariance principle, total mechanical energy.

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1 Introduction

In early 1950's physicist Enrico Fermi, computer scientist John R. Pasta and mathematician Stanisław Ulam took the initiative in investigating nonlinear dynamical problems "experimentally" by the use of computers. The first model they chose was a series of masses placed along a line and coupled to their nearest neighbors by springs [2]. They obtained this system as the discretization of a partial differential equation model of a string.

If one linearizes the system, or in other words, if the connecting springs are linear, i.e., the restoring forces depend on the displacements linearly (Hooke's law), then the system of ordinary differential equations describing motions of the coupled system is linear. It is known that the general solution is the sum of the "normal modes" of the oscillation corresponding to the eigenvalues and the eigenvectors of the matrix of the system. The mechanical energy of the oscillator is distributed between the normal modes and it wanders between the modes.

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For the original nonlinear system, Fermi, Pasta, and Ulam expected "thermalization". This would be a process, during which the oscillator would tend to equalize and the process would lead to the "equipartition of energy." However, they were surprised: the thermalization did not occur, instead the energy wandered between the modes for a while, then eventually almost all the energy returned to the initial mode. This exciting experience led to interesting new theories and concepts in mathematics and mathematical physics [3–6,9].

In this paper we investigate what happens if the damping is taken into account. One expects that the mechanical energy will be dissipated and the system "asymptotically stops." We prove that this conjecture is true. We consider three versions of the model, which differ from each others only in boundary conditions. In the first version the endpoints of the masses-springs chain are fixed – this is the original Fermi–Pasta–Ulam model. In the second one the endpoints of the chain are free. In the third variant the masses are placed along a circle and the first and the last one are also connected by a spring ("cycle"). It will be pointed out that the system with fixed endpoints has a unique equilibrium position, but the other two models have infinitely many ones. We prove that in the case of fixed endpoints the unique equilibrium state is globally asymptotically stable, and the other two systems asymptotically stop along their arbitrary motion. The letter property means that along every motion velocities tend to zero and displacements tend to an equilibrium position as the time tends to infinity.

2 The models

Let $N > 2$ be a natural number. Suppose that $N-2$ mass points of mass 1 can move along a line, and the neighboring mass points are connected by springs of the same kind. Let $q_k(t)$ ($k \in \overline{1, N-2}$) denote the coordinate of the kth mass point on the line at time $t \geq 0$; $p_k(t) := \dot{q}_k(t)$ is the derivative of $q_k(t)$ (velocity). Let $-V(q)$ be the force function of the springs, where $V : \mathbb{R} \supset (A, B) \to \mathbb{R}$ ($-\infty \leq A < B \leq \infty$) is a strictly convex, two times continuously differentiable function. Consider the representation of the Fermi–Pasta–Ulam oscillator given in [1, Figure 3.4]. If $\gamma > 0$ denotes the damping coefficient, then the equations of motions are

$$\ddot{q}_k + \gamma \dot{q}_k = V'(q_{k+1} - q_k) - V'(q_k - q_{k-1}) \qquad (k \in \overline{1, N-2}); \tag{2.1}$$

q_0 is the coordinate of the left-hand end of the first spring, q_{N-1} is the coordinate of the right-hand end of the last spring; there are no mass points at these ends. The endpoints of the chain are connected to unmovable walls:

$$q_0 = 0, \quad q_{N-1} = L \tag{2.2}$$

where L denotes the distance of the walls along the line. Introducing the notations

$$r_i := q_{i+1} - q_i \qquad (i \in \overline{0, N-2}) \tag{2.3}$$

we can rewrite system (2.1) into the equivalent system of first order equations

$$\dot{q}_k = p_k, \quad \dot{p}_k = V'(r_k) - V'(r_{k-1}) - \gamma p_k \qquad (k \in \overline{1, N-2}). \tag{2.4}$$

The *state variables* of system (2.4) determining a state of the system are $q_1, q_2, \dots, q_{N-2}$; $p_1, p_2, \dots, p_{N-2}$, which are independent of each other. However, (2.1) and (2.4) contains also variables q_0, q_{N-1}, they are not independent, they are determined by boundary conditions in terms of $q_1, q_2, \dots, q_{N-2}$. We consider three cases:

(A) *fixed endpoints*:

$$q_0(t) \equiv 0, \quad q_{N-1}(t) \equiv L; \tag{2.5}$$

(B) *free endpoints*: If there is an $L_0 > 0$ with $V'(L_0) = 0$, then we can require the boundary conditions

$$r_0(t) = q_1(t) - q_0(t) \equiv L_0, \quad r_{N-2}(t) = q_{N-1}(t) - q_{N-2}(t) \equiv L_0. \tag{2.6}$$

The third model is the *cycle*. Suppose that $N-2$ mass points are placed along a circle of arc length K and the neighboring mass points are connected by springs, so the number of springs is $N-2$. Let us fix a point O of the circle and denote by q_k the length of the arc between O and the kth mass point in the anticlockwise direction. If we use notations (2.3), then the boundary conditions are

(C) *cycle*:

$$r_0(t) \equiv r_{N-2}(t) = K - (r_1 + \cdots + r_{N-3}), \tag{2.7}$$

or, equivalently,

$$q_0 := -(K - q_{N-2}), \quad q_{N-1} := q_1 + K. \tag{2.8}$$

3 Equilibria

We are looking for equilibria

$$q_1 = \bar{q}_1 = \text{const.}, \ldots, q_{N-2} = \bar{q}_{N-2} = \text{const.}; \qquad p_1 = 0, \ldots, p_{N-2} = 0$$

of (2.4)&(2.5), (2.4)&(2.6), and (2.4)&(2.8). Since V is strictly convex, from (2.4) for $\bar{r}_i := \bar{q}_{i+1} - \bar{q}_i$ we obtain

$$\bar{r}_0 = \bar{r}_1 = \cdots = \bar{r}_{N-2},$$

i.e.,

$$\bar{q}_1 - \bar{q}_0 = \bar{q}_2 - \bar{q}_1 = \cdots = \bar{q}_{N-1} - \bar{q}_{N-2}.$$

If the endpoints are fixed, then

$$\text{(A) fixed endpoints:} \qquad \bar{q}_k = kL_0 \quad (k \in \overline{1, N-2}) \tag{3.1}$$

where $L_0 := L/(N-1)$. There is one and only one equilibrium position. Without loss of the generality we can suppose that

$$V(L_0) = 0, \quad V'(L_0) = 0. \tag{3.2}$$

In fact, define the function

$$\tilde{V}(r) := V(r) - V(L_0) - V'(L_0)(r - L_0) \qquad (r \in \mathbb{R}). \tag{3.3}$$

Then $\tilde{V}(L_0) = 0$, $\tilde{V}'(L_0) = 0$, and $\tilde{V}'(r) - \tilde{V}'(s) = V'(r) - V'(s)$ for all $r, s \in \mathbb{R}$. Obviously, if in (2.4) we change force function V to $\tilde{V}$, then the new equation is equivalent to the old one.

If the endpoints are free, then $\bar{r}_i = L_0$ $(i \in \overline{0, N-2})$ where L_0 is defined by the properties $V'(L_0) = 0$, $V(L_0) = 0$, and, consequently

$$\text{(B) free endpoints:} \qquad \bar{q}_1 \in \mathbb{R},\ \bar{q}_k = \bar{q}_1 + (k-1)L_0 \quad (k \in \overline{1, N-2}),$$

i.e., equilibrium positions form a line in $\mathbb{R}^{N-2}$.

In the case of cycle we have $\bar{r}_i = K_0 := K/(N-2)$ $(i \in \overline{0, N-2})$ and, therefore

$$\text{(C) cycle:} \qquad \bar{q}_1 \in \mathbb{R},\ \bar{q}_k = \bar{q}_1 + (k-1)K_0 \quad (k \in \overline{1, N-2}),$$

where we can also suppose that $V'(K_0) = 0$, $V(K_0) = 0$. Equilibrium positions also form a line in $\mathbb{R}^{N-2}$.

4 Total mechanical energy

Without loss of the generality we can suppose in cases (A) and (B), that $V(L_0) = 0$, while in case (C), that $V(K_0) = 0$. The total mechanical energy $H = H(q_1, \dots, q_{N-2}; p_1, \dots, p_{N-2})$ is equal to the sum of the kinetic and potential energy:

(A) fixed endpoints: $H_A = (1/2)\sum_{k=1}^{N-2}(p_k)^2 + \sum_{j=1}^{N-3} V(r_j) + V(q_1) + V(L - q_{N-2})$;

(B) free endpoints: $H_B = (1/2)\sum_{k=1}^{N-2}(p_k)^2 + \sum_{j=1}^{N-3} V(r_j)$;

(C) cycle: $H_C = (1/2)\sum_{k=1}^{N-2}(p_k)^2 + \sum_{j=1}^{N-3} V(r_j) + V(r_0)$.

With the notation $H_1 := (1/2)\sum_{k=1}^{N-2}(p_k)^2 + \sum_{j=1}^{N-3} V(r_j)$, for the derivative of H_1 with respect to (2.4) we have

$$\begin{aligned} \dot{H}_1 = & \sum_{k=1}^{N-2} \big(p_k V'(r_k) - p_k V'(r_{k-1})\big) + \sum_{j=1}^{N-3} \big(p_{j+1} V'(r_j) - p_j V'(r_j)\big) \\ & - \gamma \sum_{k=1}^{N-2} (p_k)^2 = -\gamma \sum_{k=1}^{N-2} (p_k)^2 - p_1 V'(r_0) + p_{N-2} V'(r_{N-2}). \end{aligned}$$

In case (A) from $(V(q_1) + V(L - q_{N-2}))^{\cdot} = p_1 V'(r_0) - p_{N-2} V'(r_{N-2})$ we get

$$\dot{H}_A = \dot{H}_A(q_1, \dots, q_{N-2}; p_1, \dots, p_{N-2}) = -\gamma \sum_{k=1}^{N-2} (p_k)^2. \tag{4.1}$$

In case (B) we have $V'(r_0) = V'(L_0) = 0$, $V'(r_{N-2}) = V'(L_0) = 0$, therefore we obtain

$$\dot{H}_B = \dot{H}_B(q_1, \dots, q_{N-2}; p_1, \dots, p_{N-2}) = -\gamma \sum_{k=1}^{N-2} (p_k)^2. \tag{4.2}$$

In case (C) we know that $(V(r_0))^{\cdot} = (V(q_1 + K - q_{N-2}))^{\cdot} = p_1 V'(r_0) - p_{N-2} V'(r_{N-2})$, so we have

$$\dot{H}_C = \dot{H}_C(q_1, \dots, q_{N-2}; p_1, \dots, p_{N-2}) = -\gamma \sum_{k=1}^{N-2} (p_k)^2. \tag{4.3}$$

Formulae (4.1), (4.2), and (4.3) describes how the total mechanical energy varies along motions.

5 Asymptotic stability for the oscillator with fixed endpoints

In proofs of the main theorems we will use the invariance principle [8]. Consider the system of differential equations $\dot{x} = f(x)$, where $f : \Omega \to \mathbb{R}^n$ ($\Omega \subset \mathbb{R}^n$ is open) is continuously differentiable. A set $M \subset \Omega$ is called *invariant* if for every point $x_* \in M$ the trajectory starting from x_* remains in M.

Invariance Principle. *Suppose that there exists a set $E \subset \Omega$, closed in Ω such that for every solution $t \mapsto x(t)$ one has $x(t) \to E$ as $t \to \infty$. If the positive half trajectory $\cup_{t\geq 0} x(t)$ is bounded, then $x(t) \to M$ as $t \to \infty$ where M is the largest invariant subset of E.*

Theorem 5.1. *The unique equilibrium (3.1) of the system (2.4)&(2.5) with fixed endpoints is asymptotically stable, i.e., it is stable in Lyapunov sense, and for every solution of (2.4) starting from a neighborhood of (3.1) with sufficiently small velocities we have*

$$\lim_{t\to\infty} q_k(t) = kL_0, \quad \lim_{t\to\infty} p_k(t) = 0 \qquad (k \in \overline{1, N-2}). \tag{5.1}$$

Proof. Let us introduce the new variables

$$x_j := q_j - jL_0, \quad y_k := p_k \qquad (j \in \overline{0, N-1}, k \in \overline{1, N-2}). \tag{5.2}$$

The model (2.4)&(2.5) and the mechanical energy H_A have the following forms in the new variables:

$$\dot{x}_k = y_k, \quad \dot{y}_k = V'(x_{k+1} - x_k + L_0) - V'(x_k - x_{k-1} + L_0) - \gamma y_k \qquad (k \in \overline{1, N-2}), \tag{5.3}$$

$$H_A = \frac{1}{2}\sum_{k=1}^{N-2}(y_k)^2 + \sum_{j=0}^{N-2} V(s_j + L_0); \quad s_j := x_{j+1} - x_j. \tag{5.4}$$

We have to prove that the zero solution of (5.3) is globally asymptotically stable. Define the function

$$a(u) := \min\{V(L_0 - u); V(L_0 + u)\} \qquad (u \geq 0), \tag{5.5}$$

which is strictly increasing and continuous on $[0, \infty)$, and $\lim_{u\to\infty} a(u) = \infty$. With the notations

$$\mathbf{x} := (x_1, x_2, \ldots, x_{N-2}) \in \mathbb{R}^{N-2}, \quad \mathbf{y} := (y_1, y_2, \ldots, y_{N-2}) \in \mathbb{R}^{N-2},$$
$$(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2(N-2)}, \quad \|\mathbf{x}\| := \max\{|x_1|, \ldots, |x_{N-2}|\}$$

obviously,

$$H_A(\mathbf{x}, \mathbf{y}) \geq a(|x_1|) + a(|s_1|) + \cdots + a(|s_{N-3}|) + a(|x_{N-2}|) + \frac{1}{2}\sum_{k=1}^{N-2} y_k^2, \tag{5.6}$$

and the state space, where the right-hand side of (5.3) is determined, is

$$\Omega := \left\{(x, y) \in \mathbb{R}^{2(N-2)} : x_{j+1} - x_j \in (A - L_0, B - L_0), y \in \mathbb{R}\right\}. \tag{5.7}$$

For $\varepsilon > 0$ given let initial values $\|\mathbf{x}(0)\|$, $\|\mathbf{y}(0)\|$ be so small that

$$H_A(\mathbf{x}(0), \mathbf{y}(0)) < \min\left\{\frac{\varepsilon^2}{2}; a\left(\frac{\varepsilon}{N-1}\right)\right\}. \tag{5.8}$$

Then for all t we have $H_A(\mathbf{x}(t), \mathbf{y}(t)) \le H_A(\mathbf{x}(0), \mathbf{y}(0))$, from which there follows

$$|\mathbf{y}(t)| < \varepsilon \qquad (t \ge 0). \tag{5.9}$$

On the other hand, from (5.6) and (5.8) we obtain

$$a(|x_1(t)|) \le V(x_1(t) + L_0) \le a\left(\frac{\varepsilon}{N-1}\right),$$

whence we have $|x_1(t)| < \varepsilon/(N-1)$. In the same way we have

$$|x_2(t) - x_1(t)| < \frac{\varepsilon}{N-1}, \dots, |x_{N-2}(t) - x_{N-3}(t)| < \frac{\varepsilon}{N-1},$$
$$|x_{N-2}(t)| < \frac{\varepsilon}{N-1}.$$

Therefore,

$$|x_2(t)| < |x_2(t) - x_1(t)| + |x_1(t)| \le 2\frac{\varepsilon}{N-1} \le \varepsilon,$$

and so on,

$$|x_{k+1}(t)| < |x_{k+1}(t) - x_k(t)| + |x_k(t)| \le \frac{\varepsilon}{N-1} + k\frac{\varepsilon}{N-1} \le \varepsilon$$

for $k \in \overline{1, N-3}$. This together with (5.9) means that the zero solution of (5.3) is stable. It has remained to prove that the zero solution is attractive.

Stability implies that every solution starting from a neighborhood of $x = y = 0$ is bounded. At first we prove that for these solutions velocities y_k tend to zero as $t \to \infty$. In fact, if this is not true, then from (4.1) there follows the existence of $k_* \in \overline{1, N-2}$, $0 < \varepsilon_1 < \varepsilon_2$, and sequence $(\alpha_n, \beta_n)_{n=1}^{\infty}$ such that

$$\alpha_n < \beta_n < \alpha_{n+1}, \quad \lim_{n\to\infty} \alpha_n = \infty,$$
$$y_{k_*}^2(\alpha_n) = \varepsilon_1,\ y_{k_*}^2(\beta_n) = \varepsilon_2, \qquad \varepsilon_1 \le y_{k_*}^2(t) \le \varepsilon_2 \quad (\alpha_n \le t \le \beta_n)$$

for all $n \in \mathbb{N}$. From (4.1) we obtain

$$-H_A(\mathbf{x}(\alpha_n), \mathbf{y}(\alpha_n)) + H_A(\mathbf{x}(\beta_n), \mathbf{y}(\beta_n)) \ge \gamma \int_{\alpha_n}^{\beta_n} y_{k_*}^2(t)\,\mathrm{d}t \ge \gamma\varepsilon_1(\beta_n - \alpha_n),$$

whence, taking into account also $H_A \ge 0$, we have $\lim_{n\to\infty}(\beta_n - \alpha_n) = 0$. However, this is impossible, because, according to (5.3), the derivative of $y_{k_*}^2$ is bounded.

Now we apply the invariance principle. We have proved that the trajectory of every solution tends to the set

$$E := \{(\mathbf{x}, \mathbf{y}) \in \Omega : \mathbf{y} = \mathbf{0}\}$$

as $t \to \infty$, and every positive half trajectory is bounded. E consists of equilibrium states of (5.3). By (3.1) the maximal invariant subset M of E is equal to the singleton $\{(\mathbf{0}, \mathbf{0})\}$. Application of the invariance principle yields the attractivity of $(\mathbf{0}, \mathbf{0})$. □

Remark 5.2. It is easy to see that if $A = -\infty$ and $B = \infty$, then the equilibrium state is *globally* asymptotically stable in Theorem 5.1, i.e., (5.1) is satisfied for every motion.

Remark 5.3. Fermi, Pasta and Ulam investigated only the case of fixed endpoints (A). They directly considered the form (5.3) of the model. They wrote: "If x_i denotes the displacement of the i-th point from its original position...", and the "original position" should be understood as the equilibrium position $q_i = i(L/(N-1)) = iL_0$ (see (5.2)). A comparison between (5.2) and the equations of Fermi, Pasta and Ulam [2, p. 979, (1) and (2)] shows that their force function satisfies either

$$V'(r + L_0) = r + \alpha r^2 \qquad (\alpha \geq 0),$$

or

$$V'(r + L_0) = r + \beta r^3 \qquad (\beta \geq 0),$$

where "α and β were chosen so that at the maximum displacement the nonlinear term was small, e.g., the order of one-tenth of the linear term." Since $V(L_0) = 0$, this means that either

$$V(s) = \frac{1}{2}(s - L_0)^2 + \frac{\alpha}{3}(s - L_0)^3,$$

or

$$V(s) = \frac{1}{2}(s - L_0)^2 + \frac{\beta}{4}(s - L_0)^4,$$

and $A = -1/(2\alpha)$, $B = \infty$ or $A = -\infty$, $B = \infty$, respectively.

6 Asymptotic stop for the oscillators with free endpoints and the cycle

We return to the original common system (2.4). Exchange the state variables $q_1, q_2, \dots, q_{N-2}$; $p_1, p_2, \dots, p_{N-2}$ for $q_1, r_1, r_2, \dots, r_{N-3}; p_1, p_2, \dots, p_{N-2}$. The universal model (2.4) in the new state variables has the form

$$\begin{aligned} \dot{r}_m &= p_{m+1} - p_m \qquad (m \in \overline{1, N-3}), \\ \dot{p}_k &= V'(r_k) - V'(r_{k-1}) - \gamma p_k \qquad (k \in \overline{1, N-2}) \end{aligned} \tag{6.1}$$

with the boundary conditions (2.6) (free endpoints) and (2.7) (cycle), respectively. We omitted the equation $\dot{q}_1 = p_1$ because q_1 can be separated from the other state variables: at first we solve (6.1), then we compute $q_1(t)$. Let us consider (6.1) as a system in the state space $\mathbb{R}^{2N-5}$ and find the equilibrium states

$$\begin{aligned} r_1 = \overline{r}_1 = \text{const.}, \dots, r_{N-3} = \overline{r}_{N-3} = \text{const.}; \\ p_1 = \overline{p}_1 = \text{const.}, \dots, p_{N-2} = \overline{p}_{N-2} = \text{const.} \end{aligned}$$

in this space. From the first block of equations (6.1) we obtain that there is a $\overline{p}$ =const. such that $\overline{p}_1 = \cdots = \overline{p}_{N-2} = \overline{p}$. Summing the equations of the second block of (6.1) we get

$$0 = V'(\overline{r}_{N-2}) - V'(\overline{r}_0) - \gamma(N-2)\overline{p}.$$

By boundary conditions (2.6) and (2.7) $\overline{r}_{N-2} = \overline{r}_0$, so $\overline{p} = 0$, therefore from the equations of the second block of (6.1) it follows that

$$\overline{r}_0 = \overline{r}_1 = \ldots = \overline{r}_{N-2} = \overline{r}.$$

According to (2.6) and (2.7) constant $\bar{r}$ is determined by the equation $V'(\bar{r}) = 0$, i.e., $\bar{r} := L/(N-1) = L_0$, respectively, $\bar{r} := K/(N-2) = K_0$ (see (2.2)). This means that (6.1) has one and only one equilibrium state

$$(\bar{r}, \dots, \bar{r}; 0, \dots, 0) \in \mathbb{R}^{2N-5}.$$

Introduce the notations

$$\mathbf{r} := (r_1, \dots, r_{N-3}) \in \mathbb{R}^{N-3}, \quad \mathbf{p} := (p_1, \dots, p_{N-2}) \in \mathbb{R}^{N-2};$$
$$\bar{\mathbf{r}} := (\bar{r}, \dots, \bar{r}) \in \mathbb{R}^{N-3}; \quad \|\cdot\| : \text{ the maximum norm in } \mathbb{R}^l.$$

Lemma 6.1. *The unique equilibrium state $\mathbf{r} = \bar{\mathbf{r}}, \mathbf{p} = \mathbf{0}$ of (6.1) is stable and attractive in $\mathbb{R}^{2N-5}$ for both systems (6.1)&(2.6) and (6.1)&(2.7).*

Proof. Obviously, $V(r) \geq a(|r - \bar{r}|)$ $(r \in \mathbb{R})$ is satisfied with function a defined in (5.5). Therefore H_B and H_C are positive definite. By (4.2) and (4.3), Lyapunov's theorem on the stability guarantees stability.

On the other hand, if $\|(\mathbf{r}, \mathbf{p})\| \to \infty$, then $H_B(\mathbf{r}, \mathbf{p}) \to \infty$ and $H_C(\mathbf{r}, \mathbf{p}) \to \infty$. The maximal invariant subset of the set

$$E := \{(\mathbf{r}, \mathbf{p}) \in \mathbb{R}^{2N-5} : \dot{H}_B(\mathbf{r}, \mathbf{p}) \equiv \dot{H}_C(\mathbf{r}, \mathbf{p}) = 0\} = \{(\mathbf{r}, \mathbf{p}) : \mathbf{p} = 0\}$$

is the unique equilibrium state $(\bar{\mathbf{r}}, \mathbf{0})$. Applying the invariance principle we get attractivity. □

Lemma 6.2. *Every solution $t \mapsto (\mathbf{q}(t), \mathbf{p}(t))$ of (2.4) is bounded on $[0, \infty)$.*

Proof. Introduce the notations

$$Q(t) := \sum_{k=1}^{N-2} q_k(t), \quad P(t) := \sum_{k=1}^{N-2} p_k(t).$$

If we sum the equations for $\dot{p}_k$'s in (2.4) then by (2.6) and (2.7) we get

$$\dot{P}(t) = -\gamma P(t) \Longrightarrow P(t) = P(0)e^{-\gamma t},$$

from which by integration we obtain

$$Q(t) - Q(0) = P(0)\frac{1}{\gamma}\left(1 - e^{-\gamma t}\right). \tag{6.2}$$

In consequence of Lemma 6.1 this means that every $q_k(t)$ is bounded and the assertion of Lemma 6.2 is true. □

Theorem 6.3. *The system with free endpoints (2.4)&(2.6) and the cycle (2.4)&(2.8) asymptotically stop along every motion. This means that for every motion $t \mapsto (\mathbf{q}(t), \mathbf{p}(t))$ there exists a $q_1^\infty \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} q_k(t) = q_1^\infty + (k-1)\bar{r}, \quad \lim_{t\to\infty} p_k(t) = 0 \qquad (k \in \overline{1, N-2}),$$

where $\bar{r}$ is determined by the equation $V'(\bar{r}) = 0$ (i.e., $\bar{r} = L_0$ and $\bar{r} = K_0$, respectively).

Proof. By Lemma 6.1 for every $\varepsilon > 0$ there exists a $\bar{t}(\varepsilon)$ such that

$$|r_m(t) - \bar{r}| < \varepsilon \qquad (t > \bar{t}(\varepsilon); m \in \overline{1, N-3}).$$

Let $\varepsilon > 0$ be fixed sufficiently small, it will be restricted exactly later (see (6.3)).

Thanks to Lemma 6.1 it is enough to prove that $t \mapsto q_1(t)$ has a finite limit as $t \to \infty$. Using the method of contradiction, let us suppose that this is not true. Then, in consequence of Lemma 6.2, the upper and the lower limit of q_1 are finite and different, i.e., there are S, T $(S < T)$ and a sequence $(s_n, t_n)_{n=1}^{\infty}$ such that

$$s_1 > \bar{t}(\varepsilon),\; s_n < t_n < s_{n+1}, \quad \lim_{n\to\infty} s_n = \infty,$$
$$q_1(s_n) = S,\; q_1(t_n) = T;\; s_n \le t \le t_n \Longrightarrow S \le q_1(t) \le T \qquad (n \in \overline{1, \infty}).$$

Let us fix ε so that

$$0 < \varepsilon < \frac{T-S}{4N}. \tag{6.3}$$

Then

$$q_1(t_n) - q_1(s_n) = T - S;$$
$$q_2(t_n) - q_2(s_n) \ge q_1(t_n) + (\bar{r} - \varepsilon) - (q_1(s_n) + (\bar{r} + \varepsilon)) \ge T - S - 2\varepsilon.$$

By induction we get

$$q_k(t_n) - q_k(s_n) \ge T - S - 2(k-1)\varepsilon \ge (T-S) - \frac{T-S}{2} = \frac{T-S}{2} \qquad (k \in \overline{3, N-2}).$$

Therefore

$$Q(t_n) - Q(s_n) \ge (N-2)\frac{T-S}{2} \qquad (n \in \overline{1, \infty}).$$

On the other hand, in view of (6.2) we have

$$Q(t_n) - Q(s_n) = P(0)\frac{1}{\gamma}\left(e^{-\gamma s_n} - e^{-\gamma t_n}\right) \to 0 \quad (n \to \infty),$$

which is a contradiction, i.e., $t \mapsto q_1(t)$ has a finite limit as $t \to \infty$. □

7 An outlook

The cycle is important from the point of view of a further development of the Fermi–Pasta–Ulam problem. Suppose that we have infinitely many mass points in the lattice. Then, instead of (2.1), one has to consider the system

$$\ddot{q}_m + \gamma \dot{q}_m = V'(q_{m+1} - q_m) - V'(q_m - q_{m-1}) \qquad (m \in \mathbb{Z}). \tag{7.1}$$

With

$$r_m := q_{m+1} - q_m \qquad (m \in \mathbb{Z})$$

the equivalent system of first order differential equations is

$$\dot{q}_m = p_m, \quad \dot{p}_m = V'(r_m) - V'(r_{m-1}) - \gamma p_m \qquad (m \in \mathbb{Z}). \tag{7.2}$$

If the total mechanical energy

$$H = \sum_{m\in\mathbb{Z}} \left(\frac{1}{2}(p_m)^2 + V(r_m) \right)$$

is divergent, then the problem is extremely difficult. However, if initial values are periodic in r_m's and p_m's, then the system can be reduced to a cycle and one can apply Theorem 6.3.

Let $M \geq 1$ be an arbitrary natural number, and consider system (7.2) with the initial values

$$q_m(0) = q_m^0, \ p_m(0) = p_m^0 \quad (r_m^0 = r_{m+M}^0, \ p_m^0 = p_{m+M}^0) \qquad (m \in \mathbb{Z}). \tag{7.3}$$

Obviously, if $t \mapsto (\mathbf{q}(t), \mathbf{p}(t)) = (\mathbf{q}(t; 0, \mathbf{q}^0, \mathbf{p}^0), \mathbf{p}(t; 0, \mathbf{q}^0, \mathbf{p}^0))$ is the solution of the initial value problem (7.2)&(7.3), then

$$r_m(t) \equiv r_{m+M}(t), \ p_m(t) \equiv p_{m+M}(t) \qquad (t \in \mathbb{R}; \ m \in \mathbb{Z}),$$

so (7.2)&(7.3) is equivalent to the cycle (2.4)&(2.8) with $K := MK_0$, $N := M + 2$.

Corollary 7.1. *The infinite system (7.2) asymptotically stops along every motion with periodic initial values (7.3). This means that for every such motion $t \mapsto (\mathbf{q}(t), \mathbf{p}(t))$ there exists a $q_1^\infty \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} q_m(t) = q_1^\infty + (m-1)\bar{r}, \quad \lim_{t\to\infty} p_m(t) = 0 \qquad (m \in \mathbb{Z}), \tag{7.4}$$

where $\bar{r}$ is determined by the equation $V'(\bar{r}) = 0$.

Possessing this corollary we conjecture that system (7.2) asymptotically stops along *arbitrary* motion:

Conjecture 7.2. *The infinite system (7.2) asymptotically stops along its every motion, i.e., (7.4) holds for every solution of (7.2) with some q_1^∞.*

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References

[1] C. Chicone, *Ordinary differential equations with applications*, Texts in Applied Mathematics, Vol. 34, Springer-Verlag, New York, 1999. https://doi.org/10.1007/b97645; MR1707333

[2] E. Fermi, J. Pasta, S. Ulam, Studies of nonlinear problems I (1955), I. Los Alamos report LA-1940, published later in *Collected Papers of Enrico Fermi*, E. Segré (Ed.), University of Chicago Press, 1965.

[3] S. Flach, A. Gorbach, Discrete breathers in Fermi–Pasta–Ulam lattices, *Chaos* **15**(2005), 015112, 10 pp. https://doi.org/10.1063/1.1839151; MR2133463

[4] R. Khomeriki, S. Lepri, S. Ruffo, Pattern formation and localization in the forced-damped FPU lattice, published online on arXiv, 2 Jul 2001. https://arxiv.org/abs/nlin/0107002v1

[5] R. Khomeriki, Interaction of a kink soliton with a breather in a Fermi–Pasta–Ulam chain, *Phys. Rev. E* **65**(2002), 025505. https://doi.org/10.1103/PhysRevE.65.026605

[6] R. S. Palais, The symmetries of solitons, *Bull. Amer. Math. Soc. (N.S.)* **34**(1997), 339–403. https://doi.org/10.1090/S0273-0979-97-00732-5; MR1462745

[7] N. Rouche, J. Mawhin, *Équations différentielles ordinaires. Tome II: Stabilité et solutions périodiques,* (in French) Masson et Cie, Éditeurs, Paris, 1973. MR0481182

[8] S. H. Saperstone, *Semidynamical systems in infinite-dimensional spaces*, Applied Mathematical Sciences, Vol. 37, Springer-Verlag, New York–Berlin, 1981. https://doi.org/10.1007/978-1-4612-5977-0; MR638477

[9] T. P. Weissert, *The genesis of simulation in dynamics. Pursuing the Fermi–Pasta–Ulam problem*, Springer-Verlag, New York, 1997. https://doi.org/10.1007/978-1-4612-1956-9; MR1477158