Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **28**, 1–22; https://doi.org/10.14232/ejqtde.2020.1.28 www.math.u-szeged.hu/ejqtde/

On the existence and multiplicity of eigenvalues for a class of double-phase non-autonomous problems with variable exponent growth

Vasile Florin Uță[✉]

Department of Mathematics, University of Craiova, Str. A. I. Cuza, nr. 13, 200585 Craiova, Romania

Received 2 February 2020, appeared 2 May 2020
Communicated by Patrizia Pucci

Abstract. We study the following class of double-phase nonlinear eigenvalue problems

$$-\operatorname{div}\left[\phi(x,|\nabla u|)\nabla u+\psi(x,|\nabla u|)\nabla u\right]=\lambda f(x,u)$$

in Ω, $u=0$ on $\partial\Omega$, where Ω is a bounded domain from $\mathbb{R}^N$ with smooth boundary and the potential functions ϕ and ψ have $(p_1(x);p_2(x))$ variable growth. The main results of this paper are to prove the existence of a continuous spectrum consisting in a bounded interval in the near proximity of the origin, the fact that the multiplicity of every eigenvalue located in this interval is at least two and to establish the existence of infinitely many solutions for our problem. The proofs rely on variational arguments based on the Ekeland's variational principle, the mountain pass theorem, the fountain theorem and energy estimates.

Keywords: double-phase differential operator, continuous bounded spectrum, variable exponent, multiplicity of eigenvalues, multiple types of solutions.

2020 Mathematics Subject Classification: 35P30, 49R05, 58C40.

1 Introduction

The recent study of various mathematical models described by variational problems with non-standard variable growth conditions is motivated by many phenomena that arise in applied sciences. For instance, in some cases, to describe the behavior of some materials which are not homogeneous the classical theory of $L^p(\Omega)$ and $W^{1,p}(\Omega)$ Lebesgue and Sobolev spaces has proven its limitation.

An example of such type of materials are the thermorheological and electrorheological fluids. For a good description from the partial differential equations point of view of these types of materials we refer to V. Rădulescu [23] and V. Rădulescu, D. Repovš [24]. We remark also that the variable exponent analysis for some nonlinear problems plays a crucial role in the development of robotics, aircraft and airspace and the image restoration.

[✉] Email: uta.vasi@yahoo.com

In this paper we are interested in the study of a class of non-autonomous eigenvalue problems with a variable $(p_1(x); p_2(x))$-grow rate condition, which are described by the fact that the associated energy density changes its ellipticity and growth properties according to the point.

Our study is based on some new type of non-homogeneous differential operators developed by I. H. Kim and Y. H. Kim [12], which allow us to analyze some problems that imply the possibility of lack of uniform convexity. In this paper we extend the results of I. H. Kim and Y. H. Kim by studying a double-phase problem. Moreover, for the best of our knowledge for this type of operators it is not established yet the possibility of existence and multiplicity for some eigenvalues in the near proximity of the origin, even in the simpler case when the differential operator is driven by only one potential function. This paper also aim to extend the spectral analysis for this kind of problems made by S. Baraket, S. Chebbi, N. Chorfi, V. Rădulescu in [2].

Therefore we consider the following double-phase nonlinear eigenvalue problem:

$$\begin{cases} -\operatorname{div}\left[\phi(x,|\nabla u|)\nabla u + \psi(x,|\nabla u|)\nabla u\right] = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $\lambda \in \mathbb{R}$ is a real parameter.

The study of these types of problems was motivated by the fact that we may need to model a composite that changes its hardening exponent according to the point. For more details about integral functionals with nonstandard (p,q)-growth conditions, we refer to P. Marcellini [13, 14]. These types of problems was also studied by G. Mingione *et al.* [3, 6, 7], where the associated energies are of type

$$u \mapsto \int_\Omega \left(|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)}\right) dx \tag{1.1}$$

and

$$u \mapsto \int_\Omega \left[|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)} \log(e+|x|)\right] dx, \tag{1.2}$$

where $p_1(x) \le p_2(x)$, $p_1 \ne p_2$, for all $x \in \Omega$ and $a(x) \ge 0$.

These problems describe the behavior of two materials with variable power hardening exponents $p_1(x)$ and $p_2(x)$ and the coefficient $a(x)$ dictates the geometry of a composite of the two materials.

As we mentioned before our nonhomogeneous differential operator corresponds to the type of double-phase operators, fact that is induced by the presence of the potential functions ϕ and ψ. In order to make a better connection with the work of Mingione *et al.*, we remark that our potential functions ϕ and ψ may behave as it follows

- $\phi(x,t) = t^{p(x)-2}$, case in which we can also embed the description given by (1.1) for the fact that our operators extends the case when

  $$-\operatorname{div}\left[\phi(x,|\nabla u|)\nabla u + \psi(x,\nabla u)\nabla u\right] = -\operatorname{div}\left[a(x)|\nabla u|^{p_1(x)-2}\nabla u + b(x)|\nabla u|^{p_2(x)-2}\nabla u\right],$$

  for some functions $a(x), b(x) \in L^\infty(\Omega)_+$;

- $\phi(x,t) = (1+|t|^2)^{\frac{p(x)-2}{2}}$, case in which we obtain the generalized mean curvature operator;

- $\phi(x,t) = \left(1 + \frac{t^{p(x)}}{\sqrt{1+t^{2p(x)}}}\right) t^{p(x)-2}$, case in which we obtain the corresponding differential operator that describe the capillary phenomenon.

For this cases, in order to obtain the description given by (1.2) we have to analyze the following differential operator:

$$-\operatorname{div}\left[\phi(x,|\nabla u|)\nabla u + a(x)\psi(x,|\nabla u|)\log(e+|x|)\nabla u\right].$$

As we mentioned before the main results of this paper is to establish the fact that for every $\lambda > 0$ small enough we have two different solutions and the fact that our problem (P_λ) admits a sequence of solutions with higher and higher energies provided only by the restriction $\lambda > 0$. The first solution is obtained as a local minimum near the origin. To this end we refer to [9,17] and [24, Chapter 2] for more details about the method used to point out this type of solutions. Our second solution is obtained as a mountain pass critical point. For a comprehensive study of this type of solutions we refer to the following works of P. Pucci, J. Serrin [21,22], P. Pucci, V. Rădulescu [19]. The third type of solutions is obtained as high energy solutions by employing the fountain theorem. For more details about this critical point technique we refer to the following works: [10,12,25,28].

Also more details about existence and nonexistence results related to variable exponent equations can be found in the following works [4,11], while more critical point techniques and qualitative analysis for double-phase operators can be found in [1,5,20].

Moreover, we make a parallel between the techniques used to point out our results and between our methods and some other techniques used so far to describe some spectral properties of these types of operators. For more details we mention the following works [2,12,26,27].

Also in the final part of this paper are given some examples and remarks in order to illustrate the validity of the general results obtained throughout this work.

2 The functional framework

Throughout this section we will introduce the necessary information about the functional framework that we will need in the study of problem (P_λ). To this end we will give a brief description of variable exponent Lebesgue and Sobolev spaces. Most of the following properties and results can be found in the following books by J. Musielak [18], L. Diening, P. Hästö, P. Harjulehto, M. Růžička [8], V. Rădulescu, D. Repovš [24].

First we assume that $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary. Let

$$C_+(\Omega) = \left\{ p \in C(\overline{\Omega}) : \min_{x\in\Omega} p(x) > 1 \right\},$$

and for any continuous function $p : \overline{\Omega} \to (1,+\infty)$, we have

$$p^- = \inf_{x\in\Omega} p(x) \qquad \text{and} \qquad p^+ = \sup_{x\in\Omega} p(x).$$

For any $p \in C_+(\overline{\Omega})$, with $p < +\infty$ we define the variable exponent Lebesgue space as if follows

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ a measurable function} : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

which endowed with the following Luxemburg norm

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}$$

becomes a Banach space. For any $1 < p(x) < +\infty$ as defined before, $L^{p(x)}(\Omega)$ is reflexive, uniformly convex Banach space, and moreover for any measurable bounded exponent p, $L^{p(x)}(\Omega)$ is separable.

Remark 2.1. This space is a special case of an Orlicz–Musielak space and its dual space is defined as $L^{p'(x)}(\Omega)$, where $p'(x)$ is the conjugate exponent of $p(x)$, in the sense that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

If p and q are two variable exponents and $p(x) \leq q(x)$ for almost all $x \in \Omega$, with $|\Omega| < \infty$, then there exists the following continuous embedding

$$L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega),$$

where by $|\Omega|$ we denote the Lebesgue measure of Ω.

Let $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ then the following Hölder type inequality occurs:

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)} |v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.1}$$

A crucial role in manipulating the variable exponent Lebesgue spaces is played by the modular function associated to these types of spaces. We define the modular of $L^{p(x)}(\Omega)$ by the function $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ such that

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

If $p(x) \not\equiv$ constant in Ω and $u, (u_n)_n \in L^{p(x)}(\Omega)$, then the following relations hold true:

$$|u|_{p(x)} < 1 \Rightarrow \; |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}, \tag{2.2}$$

$$|u|_{p(x)} > 1 \Rightarrow \; |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}, \tag{2.3}$$

$$|u|_{p(x)} = 1 \Rightarrow \; \rho_{p(x)}(u) = 1, \tag{2.4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \; \rho_{p(x)}(u_n - u) \to 0. \tag{2.5}$$

We define in what follows the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}.$$

On $W^{1,p(x)}(\Omega)$ we can define the following equivalent norms:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

and

$$\|u\| = \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right) dx \leq 1\right\}.$$

Since our problem necessitates that the function $u = 0$ on $\partial\Omega$, we define the associated space $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p(x)}$ as it follows

$$W_0^{1,p(x)}(\Omega) = \left\{ u; u|_{\partial\Omega} = 0, u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Taking account of [12] for $p \in C_+(\overline{\Omega})$ it holds true the following Poincaré type inequality

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \tag{2.6}$$

for $C > 0$ a constant which depends on p and Ω.

Remark 2.2. If $\Omega \subset \mathbb{R}^N$ is a bounded domain, and the function p which dictates the variable exponent is global log-Hölder continuous the norm $|\nabla u|_{p(x)}$ is equivalent with $\|u\|_{p(x)}$ on $W_0^{1,p(x)}(\Omega)$.

Remark 2.3. If $p^- > 1$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are reflexive, uniformly convex Banach spaces. Furthermore if p is measurable and bounded then our spaces are separable.

Remark 2.4 ([24])**.** If $p, q, r \in C_+(\Omega)$ with $p^+ < N$, and $p(x) < r(x) < q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$, for any $x \in \Omega$, then the following embeddings hold true

$$\begin{aligned} &W_0^{1,r(x)}(\Omega) \hookrightarrow W_0^{1,p(x)}(\Omega) \quad &&\text{(continuous embedding)}, \\ &W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \quad &&\text{(continuous and compact embedding)}. \end{aligned}$$

3 Basic hypotheses and auxiliary results

In this section we will establish the main conditions imposed on the potential functions ϕ and ψ which drive us to our double-phase differential operator from the problem (P_λ) and some auxiliary results that will help us pointing out our solutions.

We assume that:

(S_1) $\phi, \psi : \Omega \times [0, \infty) \to [0, \infty)$ and

- $\phi(\cdot, t)$, $\psi(\cdot, t)$ are measurable on Ω for all $t \geq 0$;
- $\phi(x, \cdot)$, $\psi(x, \cdot)$ are locally absolutely continuous on $[0, \infty)$ for almost all $x \in \Omega$.

(S_2) There exist some functions v_1 and v_2 such that $v_1 \in L^{p_1'(x)}(\Omega)$ and $v_2 \in L^{p_2'(x)}(\Omega)$ and a constant $\xi > 0$ such that

- $|\phi(x, |t|)t| \leq v_1(x) + \xi|t|^{p_1(x)-1}$,
- $|\psi(x, |t|)t| \leq v_2(x) + \xi|t|^{p_2(x)-1}$

for almost all $x \in \Omega$, and all $t \in \mathbb{R}^N$.

(S_3) There is a strictly positive constant c such that the following statements are verified for almost all $x \in \Omega$ and all $t > 0$:

- $\phi(x, t) \geq ct^{p_1(x)-2}$ and $t\frac{\partial\phi}{\partial t} + \phi(x, t) \geq ct^{p_1(x)-2}$,
- $\psi(x, t) \geq ct^{p_2(x)-2}$ and $t\frac{\partial\psi}{\partial t} + \psi(x, t) \geq ct^{p_2(x)-2}$.

Let us now impose some conditions on the reaction term (right-hand side) of the problem (P_λ). We define $f : \Omega \times \mathbb{R} \to \mathbb{R}$ as a Carathéodory function (i.e. $f(\cdot, z)$ is measurable for all $z \in \mathbb{R}$ and $f(x, \cdot)$ is continuous for almost all $x \in \Omega$) satisfying the following hypotheses:

(R_1) $zf(x,z) \geq 0$ for almost all $(x,z) \in \Omega \times \mathbb{R}$, and there exists a function $m \in L^\infty(\Omega) \setminus \{0\}$, $m(x) \geq m^- > 0$, where m^- is a constant, for all $x \in \Omega$ such that

$$|f(x,z)| \leq m(x)|z|^{q(x)-1} \quad \text{for almost all } x \in \Omega, \text{ all } z \in \mathbb{R}.$$

(R_2) There exist some strictly positive constants A and η such that

$$0 < \eta F(x,z) \leq zf(x,z) \quad \text{for almost all } x \in \Omega, z \in \mathbb{R} \setminus \{0\},$$

where $F(x,z) = \int_0^z f(x,t)dt$, $\eta > p_2^+$ and $|z| > A$.

By hypothesis (R_1) we obtain that

(R_3) $F(x,z) \leq \dfrac{m(x)}{q(x)}|z|^{q(x)}$ for all $(x,z) \in \Omega \times \mathbb{R}$.

(R_4) There exists a constant $C_F > 0$ such that

$$|z|^{q(x)} \leq C_F F(x,z), \text{ for all } (x,z) \in \Omega \times \mathbb{R}.$$

Now we assume that $p_1, p_2, q \in C_+(\Omega)$. Our variable exponents exhibits the following behavior

$$\begin{cases} 1 < q^- < p_1^- \leq p_1(x) \leq p_1^+ < p_2^- \leq p_2(x) \leq p_2^+, \\ p_2^+ < p_1^*(x) \text{ and } q^+ < p_1^*(x), \end{cases} \tag{3.1}$$

where $p_1^*(x) = \frac{Np_1(x)}{N-p_1(x)}$ is the critical Sobolev exponent, for all $x \in \overline{\Omega}$.

Remark 3.1. At this point we do not have any information on the behavior of the quantity $\sup_{x\in\Omega} q(x)$, beside the fact that it is a subcritical exponent.

Remark 3.2. Taking account on the relation (3.1) and the embedding theorems for variable exponent Lebesgue and Sobolev spaces we will choose $W = W_0^{1,p_2(x)}(\Omega)$ as functional space for the solutions of problem (P_λ), and for the simplicity of the writing by $\|\cdot\|$ we will denote the norm associated to $W_0^{1,p_2(x)}(\Omega)$ ($\|\cdot\|_{p_2(x)}$).

Definition 3.3. We say that $u \in W \setminus \{0\}$ is a weak solution of the problem (P_λ) if

$$\int_\Omega [\phi(x, |\nabla u|)\nabla u \nabla \varphi + \psi(x, |\nabla u|)\nabla u \nabla \varphi]\, dx = \lambda \int_\Omega f(x,u)\varphi dx$$

for all $\varphi \in W$.

In order to establish the desired spectral properties for our problem we define the energy functional associated to the problem (P_λ) as it follows

$$T_\lambda : W \to \mathbb{R},$$
$$T_\lambda(u) = S(u) - \lambda R(u),$$

where

$$S(u) = \int_\Omega S_0(x, |\nabla u|)dx, \quad \text{with} \quad S_0(x,t) = \int_0^t \phi(x,s)sds + \int_0^t \psi(x,s)sds$$

and

$$R(u) = \int_\Omega F(x,u)dx.$$

An important role in the analysis made by using the energy functional T_λ is played by the fact that the part of the functional driven by our double-phase operator (left-hand side of the problem) satisfy the following hypothesis

(S_4) For all $x \in \overline{\Omega}$, all $t \in \mathbb{R}^N$, the following estimate holds true:

$$0 \leq [\phi(x,|t|) + \psi(x,|t|)]\, |t|^2 \leq \omega S_0(x,|t|),$$

for a constant $\omega > 1$.

Remark 3.4. We can observe that the functional T_λ is of class $C^1(W, \mathbb{R})$ (for more details we refer to [12, Lemmas 3.2, 3.4] and [2, Section 4]).

In order to reveal the eigenvalues associated to our differential operator we will point out that the critical points of the energy functional T_λ are weak solutions for the problem (P_λ), so they are eigenfunctions to their corresponding eigenvalues denoted by λ.

Firstly we need to prove some useful properties related by the geometry of the energy functional T_λ.

Proposition 3.5. *There exists $\lambda_{\phi,\psi} > 0$ such that for any $0 < \lambda < \lambda_{\phi,\psi}$ there exist two strictly positive constants r and δ such that $T_\lambda(u) \geq \delta > 0$ for any $u \in W$ with $\|u\| = r$.*

Proof. We will compute first the part of the energy functional driven by the differential operator in the left-hand side of the problem (P_λ).

$$\begin{aligned} S(u) &= \int_\Omega S_0(x,|\nabla u|)dx \\ &\geq \frac{1}{\omega}\int_\Omega \phi(x,|\nabla u|)|\nabla u|^2 + \psi(x,|\nabla u|)|\nabla u|^2 dx \\ &\geq \frac{1}{\omega}\int_\Omega c\left(|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}\right)dx \\ &\geq \frac{c}{\omega}\left(\int_\Omega |\nabla u|^{p_1(x)}dx + \int_\Omega |\nabla u|^{p_2(x)}dx\right). \end{aligned} \tag{3.2}$$

Taking account of the relation (3.1) we have the following continuous embeddings

$$\begin{aligned} W = W_0^{1,p_2(x)}(\Omega) &\hookrightarrow W_0^{1,p_1(x)}(\Omega) \\ W_0^{1,p_1(x)}(\Omega) &\hookrightarrow L^{q(x)}(\Omega). \end{aligned}$$

Therefore we have the following inequalities

$$|u|_{q(x)} < C_1\|u\|_{p_1(x)} \tag{3.3}$$

$$\|u\|_{p_1(x)} < C_2\|u\|, \tag{3.4}$$

where $C_1 > 0$, $C_2 > 0$ are some constants.

Combining (3.3) and (3.4) we obtain

$$|u|_{q(x)} < C_1\|u\|_{p_1(x)} < C_1 \cdot C_2\|u\|.$$

Now, let $r \in (0,1)$ be fixed such that $r < \min\left\{\frac{1}{C_1C_2}, \frac{1}{C_1}\right\}$, therefore we have that

$$\begin{aligned} \|u\|_{p_1(x)} &< 1, \\ |u|_{q(x)} &< 1, \end{aligned} \quad \text{for all } u \in W, \text{ with } \|u\| = r. \tag{3.5}$$

Moreover, using the properties described by relations (2.2) and (3.2), we obtain that

$$\begin{aligned} S(u) &\geq \frac{c}{\omega}\left(\|u\|_{p_1(x)}^{p_1^+} + \|u\|^{p_2^+}\right) \\ &\geq \frac{c}{\omega}\|u\|^{p_2^+}. \end{aligned} \tag{3.6}$$

We proceed now to compute the second part of our energy functional, driven by the reaction term, using assumptions (R_1) and (R_3) we obtain that:

$$\begin{aligned} R(u) &= \int_\Omega F(x,u)dx \\ &\leq \int_\Omega \frac{m(x)}{q(x)}|u|^{q(x)}dx \\ &\leq \frac{\|m\|_\infty}{q^-}\int_\Omega |u|^{q(x)}dx. \end{aligned} \tag{3.7}$$

Taking account of relation (3.5) and the property described by (2.2) we have that

$$\int_\Omega |u|^{q(x)}dx < |u|_{q(x)}^{q^-}.$$

Using the continuous embedding for variable exponent Lebesgue and Sobolev spaces dictated by hypothesis (3.1) and relation (3.7) we obtain that

$$R(u) \leq \frac{\|m\|_\infty}{q^-}(C_1 \cdot C_2)^{q^-}\|u\|^{q^-}. \tag{3.8}$$

Hence taking account of (3.6) and (3.8) we have that:

$$\begin{aligned} T_\lambda(u) &= S(u) - \lambda R(u) \\ &\geq \frac{c}{\omega}\|u\|^{p_2^+} - \lambda\frac{\|m\|_\infty}{q^-}(C_1 \cdot C_2)^{q^-}\|u\|^{q^-} \\ &= \frac{c}{\omega}r^{p_2^+} - \lambda\frac{\|m\|_\infty}{q^-}C_3^{q^-}r^{q^-} \\ &= r^{q^-}\left(\frac{c}{\omega}r^{p_2^+ - q^-} - \lambda\frac{\|m\|_\infty}{q^-}C_3^{q^-}\right), \end{aligned} \tag{3.9}$$

where $C_3 = C_1 \cdot C_2$.

Using the inequality (3.9) we find that for every

$$\lambda \in \left(0, \frac{c}{\omega}r^{p_2^+ - q^-} \cdot \frac{q^-}{C_3^{q^-}\|m\|_\infty}\right)$$

we can find a constant $\delta = \delta\Big(\frac{c}{\omega} r^{p_2^+ - q^-} \cdot \frac{q^-}{C_3^{q^-}\|m\|_\infty}\Big) > 0$ such that

$$T_\lambda(u) \geq \delta > 0$$

for any $u \in W$, with $\|u\| = r$.

Hence the proposition is proved. □

Remark 3.6. So, further on we will denote $\lambda_{\phi,\psi}$ by the quantity

$$\lambda_{\phi,\psi} = \frac{c}{\omega} r^{p_2^+ - q^-} \cdot \frac{q^-}{C_3^{q^-}\|m\|_\infty}. \tag{3.10}$$

Remark 3.7. We also can observe that our energy functional satisfies one of the geometric hypotheses of the mountain pass theorem, that is the existence of a mountain near the origin.

Proposition 3.8. *There exists $h \in W$, with $h > 0$ such that*

$$T_\lambda(th) < 0,$$

provided by a $t > 0$ sufficiently small.

Proof. We proceed first to compute the part of the energy functional which is driven by the double-phase operator from the left-hand side of the problem (P_λ).

Using (S_2), Hölder's inequality for variable exponent Lebesgue and Sobolev spaces and the fact that $t \in (0,1)$ is sufficiently small, we have that

$$\begin{aligned} S(th) &\leq 2C_\phi |v_1|_{p_1'(x)} \|th\|_{p_1(x)}^{p_1^-} + \frac{\xi}{p_1^-}\|th\|_{p_1(x)}^{p_1^-} + 2C_\psi |v_2|_{p_2'(x)}\|th\|^{p_2^-} + \frac{\xi}{p_2^-}\|th\|^{p_2^-} \\ &\leq t^{p_1^-}\tilde{C}_1, \end{aligned} \tag{3.11}$$

where $C_\phi, C_\psi > 0$ are two constants that depend on the potential functions ϕ, ψ and on the continuous embeddings

$$W_0^{1,p_1(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$$
$$W_0^{1,p_2(x)}(\Omega) \hookrightarrow W_0^{1,p_1(x)}(\Omega),$$

and

$$\tilde{C}_1 = \left(2C_\phi |v_1|_{p_1'(x)} + \frac{\xi}{p_1^-}\right)\|h\|_{p_1(x)}^{p_1^-} + \left(2C_\psi |v_2|_{p_2'(x)} + \frac{\xi}{p_2^-}\right)\|h\|^{p_2^-}. \tag{3.12}$$

In what follows we will compute the second part of the energy functional.

Using hypotheses (R_1), (R_3) and (R_4) there exists a constant $C_F > 0$ such that $F(x,u) \geq \frac{1}{C_F}|u|^{q(x)}$, with $C_F \geq \frac{q^+}{m^-}$, where $m^- = \min\{m(x) : x \in \overline{\Omega}, m(x) \neq 0\}$.

Let us consider $C_F = \frac{q^+}{m^-} + 1$, and so we have that

$$F(x,u) \geq \frac{m^-}{q^+ + m^-}|u|^{q(x)}. \tag{3.13}$$

Hypothesis (3.1) implies the fact that $q^- < p_1^-$. Let $\alpha_0 > 0$ be such that

$$q^- + \alpha_0 < p_1^-.$$

Since $q \in C(\overline{\Omega})$ we obtain the fact that there exists an open set $\Omega_0 \subset \Omega$ such that

$$|q(x) - q^-| < \alpha_0 \quad \text{for all } x \in \Omega_0,$$

therefore we can say that

$$q(x) < q^- + \alpha_0 < p_1^- \quad \text{for all } x \in \Omega_0.$$

Consider $h \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}(h) \supset \overline{\Omega}_0$, $h(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \le h \le 1$ in Ω.

Now taking account of relation (3.13) one have that

$$\begin{aligned} R(th) &= \int_\Omega F(x, th)dx \\ &\ge \frac{m^-}{q^+ + m^-} \int_\Omega t^{q(x)} |h|^{q(x)} dx \\ &\ge \frac{m^-}{q^+ + m^-} t^{q^- + \alpha_0} \int_{\Omega_0} |h|^{q(x)} dx. \end{aligned} \tag{3.14}$$

Now combining relations (3.11) and (3.14) we obtain that

$$T_\lambda(th) \le \tilde{C}_1 t^{p_1^-} - \lambda t^{q^- + \alpha_0} \frac{m^-}{q^+ + m^-} \int_{\Omega_0} |h|^{q(x)} dx. \tag{3.15}$$

Hence, taking account of relation (3.15) we obtain that

$$T_\lambda(th) < 0$$

provided by $t < s^{\frac{1}{p_1^- - q^- - \alpha_0}}$, where

$$0 < s < \min \left\{ 1, \frac{\lambda \tilde{C}_2}{\tilde{C}_1} \right\}$$

with $\tilde{C}_2 = \frac{m^-}{q^+ + m^-} \int_{\Omega_0} |h|^{q(x)} dx$ and $\tilde{C}_1$ as defined by relation (3.12).

Now taking account of the fact that

$$\int_{\Omega_0} |h|^{q(x)} dx \le \int_\Omega |h|^{q(x)} dx \le \int_\Omega |h|^{q^-} dx,$$

and by the continuous embedding $W \hookrightarrow L^{q^-}(\Omega)$, and the properties of the modular function for variable exponent Lebesgue space (relations (2.2)–(2.5)) we can affirm that

$$\|h\| > 0 \quad \text{and} \quad \int_\Omega |\nabla h|^{p_1(x)} dx > 0, \qquad \int_\Omega |\nabla h|^{p_2(x)} dx > 0,$$

and this completes the proof of our proposition. □

Remark 3.9. We can observe that our energy functional does not satisfy the second geometrical condition of the mountain pass theorem, in the sense that there exists a valley near the origin, but it is not as far away as required. Hence the mountain pass theorem can not be applied at this moment, but it can be applied if we impose some additional conditions on the growing behavior of the reaction term. We will analyze this fact later on this paper.

4 Multiple types of solutions

We can state now our first result.

Theorem 4.1. *Assume that condition* (3.1) *is satisfied and hypotheses* (S_1)–(S_4), $(R_1), (R_3), (R_4)$ *hold true. Then for* $p_2^+ < N$, *for all* $x \in \overline{\Omega}$, *there exists* $\lambda_{\phi,\psi} > 0$ *such that any* λ *with* $0 < \lambda < \lambda_{\phi,\psi}$ *is an eigenvalue for problem* (P_λ).

Proof. We proceed now to prove our first result. Let $\lambda_{\phi,\psi}$ be as declared in relation (3.10) and consider $\lambda \in (0, \lambda_{\phi,\psi})$. In what follows we will denote by $B(0,r) = \{u \in W : \|u\| < r\}$ the ball centered in the origin with r radius from W.

Using Proposition 3.5, we have that

$$\inf_{u \in \partial B(0,r)} T_\lambda(u) > 0. \tag{4.1}$$

Also by Proposition 3.8 we have that there exists $h \in W$ such that $T_\lambda(th) < 0$, provided by $t > 0$ sufficiently small. Furthermore by relation (3.3), (3.4) and (2.2) we have that

$$T_\lambda(u) \geq \frac{c}{\omega}\|u\|^{p_2^+} - \lambda \frac{\|m\|_\infty}{q^-} C_3^{q^-} \|u\|^{q^-}.$$

Therefore we can say that there exists a constant c_0 such that

$$-\infty < c_0 := \inf_{B(0,r)} T_\lambda < 0.$$

Taking account of the above relations let $\varepsilon > 0$ be such that $\varepsilon < \inf_{\partial B(0,r)} T_\lambda - \inf_{B(0,r)} T_\lambda$, by applying the Ekeland's variational principle ([9]) to the functional $T_\lambda : \overline{B(0,r)} \to \mathbb{R}$ we obtain the existence of a function $u_\varepsilon \in \overline{B(0,r)}$ such that

$$\begin{aligned} &T_\lambda(u_\varepsilon) \leq \inf_{B(0,r)} T_\lambda + \varepsilon \\ &T_\lambda(u_\varepsilon) \leq T_\lambda(u) + \varepsilon\|u - u_\varepsilon\|, \quad u \neq u_\varepsilon. \end{aligned}$$

Therefore we have that

$$T_\lambda(u_\varepsilon) \leq \inf_{B(0,r)} T_\lambda + \varepsilon \leq \inf_{B(0,r)} T_\lambda + \varepsilon < \inf_{\partial B(0,r)} T_\lambda,$$

thus we have obtained that $\|u_\varepsilon\| < r$. Now, let E be the energy functional defined on $\overline{B(0,r)}$ as it follows

$$\begin{aligned} &E : \overline{B(0,r)} \to \mathbb{R} \\ &E(u) = T_\lambda(u) + \varepsilon\|u - u_\varepsilon\|. \end{aligned} \tag{4.2}$$

Now using relation (4.2) we have that

$$E(u_\varepsilon) = T_\lambda(u_\varepsilon) < T_\lambda(u) + \varepsilon\|u - u_\varepsilon\| = E(u), \qquad u \neq u_\varepsilon. \tag{4.3}$$

So far, taking a look at relation (4.3) it turns out that u_ε is a minimum point for E, therefore, using arguments from [2, 12, 17] we have that

$$\frac{E(u_\varepsilon + t\varphi) - E(u_\varepsilon)}{t} \geq 0 \tag{4.4}$$

for $t > 0$ small and every φ, with $\|\varphi\| < 1$.

Relation (4.4) yields the fact that

$$\frac{T_\lambda(u_\varepsilon + t\varphi) - T_\lambda(u_\varepsilon)}{t} + \varepsilon\|\varphi\| \geq 0.$$

We let $t \to 0$ and we obtain that

$$\langle T'_\lambda(u_\varepsilon), \varphi\rangle > -\varepsilon\|\varphi\|$$
$$\langle T'_\lambda(u_\varepsilon), \varphi\rangle > -\varepsilon$$

which yields to the fact that $\|T'_\lambda(u_\varepsilon)\| \leq \varepsilon$.

Therefore we get the existence of a sequence $(v_n)_n \subset B(0, r)$ such that

$$T_\lambda(v_n) \to c_0 \quad \text{and} \quad T'_\lambda(v_n) \to 0. \tag{4.5}$$

Since $(v_n)_n \subset B(0, r)$ it yields that

$$\|v_n\| \leq r, \quad \text{for every } n \in \mathbb{N}, \tag{4.6}$$

hence the sequence $(v_n)_n$ is bounded in W. As a consequence we can find an element v_0 such that (passing eventually to a subsequence)

$$v_n \rightharpoonup v_0 \quad \text{in } W.$$

By the fact that W is compactly embedded in $L^{q(x)}(\Omega)$ we get that $v_n \to v_0$ in $L^{q(x)}(\Omega)$. Using [24, Lemma 21, Chapter 3] and some arguments from the proof of [12, Lemma 3.5] we have that $R'(u)$ is compact therefore we have that

$$\begin{aligned} \lim_{n\to\infty} R(v_n) &= R(v_0) \\ \lim_{n\to\infty} \langle R'(v_n), v_n - v_0\rangle &= 0 \end{aligned} \tag{4.7}$$

It only remains to show that

$$\lim_{n\to\infty} S(v_n) = S(v_0).$$

Using relation (4.5) we have that

$$\lim_{n\to\infty} \langle T'_\lambda(v_n), v_n - v_0\rangle = 0. \tag{4.8}$$

Using (4.7) and (4.8) we can obtain that

$$\lim_{n\to\infty} \langle S'(v_n) - S'(v_0), v_n - v_0\rangle \leq \lim_{n\to\infty} \langle T'_\lambda(v_n), v_n - v_0\rangle = 0,$$

thus using [12, Lemma 3.4] we get that

$$v_n \to v_0 \quad \text{in } W. \tag{4.9}$$

Hence by relations (4.9) and (4.7) combined with relation (4.5) we obtain the fact that

$$T_\lambda(v_0) = c_0 < 0 \quad \text{and} \quad T'_\lambda(v_0) = 0.$$

We conclude by pointing out that v_0 is a nontrivial weak solution of problem (P_λ) and every $\lambda \in (0, \lambda_{\phi,\psi})$ is an eigenvalue of our problem. $\square$

Let us assume now that the hypotheses of Theorem 4.1 are fulfilled and moreover we have more knowledge about the variable growth of the reaction term; namely the following relation holds true:

$$1 < q^- < p_1^- \leq p_1(x) \leq p_1^+ < p_2^- \leq p_2(x) \leq p_2^+ < q^+ < p_1^*(x), \tag{4.10}$$

for all $x \in \Omega$.

Remark 4.2. Taking account of the relation (4.10) we still can not prove the fact that our energy functional T_λ is coercive, so we can not apply the so called Direct Method in the Calculus of Variations in order to point out our eigenvalues. This method have been applied on this types of operators in the following works: [2,12,27].

Using the new information given by relation (4.10) about the growth behavior of the reaction term we can obtain the following property for our energy functional.

Proposition 4.3. *Suppose that hypotheses* (S_1)–(S_4), (R_1)–(R_4) *and* (4.10) *hold true, then we can find some element* $\theta \in W$ *such that*

$$T_\lambda(t\theta) < 0,$$

provided by t *sufficiently large.*

Proof. Using similar arguments as in the proof of Proposition 3.8 and keeping in mind that t is sufficiently large we obtain that

$$\begin{aligned} S(t\theta) &= \int_\Omega S_0(x, |\nabla(t\theta)|)dx \\ &\leq 2\overline{C}_\phi |v_1|_{p_1'(x)} \|t\theta\|_{p_1(x)}^{p_1^+} + \frac{\xi}{p_1^-}\|t\theta\|_{p_1(x)}^{p_1^+} + 2\overline{C}_\psi |v_2|_{p_2'(x)} \|t\theta\|^{p_2^+} + \frac{\xi}{p_2^-}\|t\theta\|^{p_2^+} \\ &\leq \tilde{C}_\theta t^{p_2^+}, \end{aligned} \tag{4.11}$$

where $\tilde{C}_\theta = \left(2\overline{C}_\phi |v_1|_{p_1'(x)} + \frac{\xi}{p_1^-}\right)\|\theta\|_{p_1(x)}^{p_1^+} + \left(2\overline{C}_\psi |v_2|_{p_2'(x)} + \frac{\xi}{p_2^-}\right)\|\theta\|^{p_2^+}$.

Hypothesis (4.10) implies that $p_2^+ < q^+$. Thinking similarly as in the proof of Proposition 3.8 we obtain the existence of a constant $\alpha_1 > 0$ such that $p_2^+ + \alpha_1 < q^+$. By the fact that $p_2, q \in C(\overline{\Omega})$ it follows that there exists an open set $\Omega_1 \subset \Omega$ such that $|q^+ - q(x)| < \alpha_1$ for all $x \in \Omega_1$. Therefore we obtain that

$$p_2^+ < q^+ - \alpha_1 < q(x) \tag{4.12}$$

for all $x \in \Omega_1$.

Now let $\theta \in C_0^\infty(\Omega)$ by such that $\mathrm{supp}(\theta) \supset \overline{\Omega}_1$, $\theta(x) = 1$ for all $x \in \overline{\Omega}_1$ and $0 \leq \theta \leq 1$ in Ω, taking account of relation (3.13) combined with hypothesis (R_2) we have that

$$F(x, t\theta) \geq \frac{m^-}{\eta + m^-}|t\theta|^{q(x)}.$$

Therefore by relation (4.12) and the properties of θ described before we obtain that

$$\begin{aligned} R(t\theta) &\geq \frac{m^-}{\eta + m^-}\int_\Omega t^{q(x)}|\theta|^{q(x)}dx \\ &\geq \frac{m^-}{\eta + m^-} t^{q^+ - \alpha_1} \int_{\Omega_1} |\theta|^{q(x)}dx. \end{aligned} \tag{4.13}$$

Hence taking use of relations (4.11) and (4.13) we obtain that

$$T_\lambda(t\theta) \leq t^{p_2^+} \tilde{C}_\theta - \frac{m^-}{\eta + m^-} t^{q^+ - \alpha_1} \int_{\Omega_1} |\theta|^{q(x)} dx.$$

Letting $t \to \infty$ and keeping in mind that $p_2^+ < q^+ - \alpha_1$ we have that

$$\lim_{t\to\infty} T_\lambda(t\theta) = -\infty.$$

Reasoning as in the end of the proof of Proposition 3.8 we have that $\|\theta\| > 0$, $\|\theta\|_{p_1(x)} > 0$ and so our proof is complete. □

Remark 4.4. Comparing the results of Proposition 3.8 and Proposition 4.3, we can observe that for the new growth conditions imposed by relation (4.10) the energy functional T_λ fulfills the second geometrical condition of the mountain pass theorem, namely we can find a valley far away of the origin as required.

In order to obtain our second result we need to require a slightly more restrictive condition (S_4), namely:

(S_4') $0 \leq [\phi(x,|t|) + \psi(x,|t|)]\,|t|^2 \leq p_2^+ S_0(x,|t|)$, for all $x \in \overline{\Omega}$, all $t \in \mathbb{R}^N$.

Of course we can observe that (S_4') implies (S_4).

We state now our second result.

Theorem 4.5. *Assume that condition (4.10) holds true and hypotheses (S_1)–(S_3), (S_4'), (R_1)–(R_4) are fulfilled. Then for every $\lambda \in (0, \lambda_{\phi,\psi})$ the problem (P_λ) has a mountain pass type solution.*

Proof. Taking account of Propositions 3.5 and 4.3, we have that our energy functional has a mountain pass geometry.

Since $T_\lambda(0) = 0$, employing the mountain pass theorem we obtain the existence of a sequence $(w_n)_n \subset W$ such that

$$T_\lambda(w_n) \to c_1 > 0 \quad \text{and} \quad T_\lambda'(w_n) \to 0 \quad \text{in} \quad W^{-1,p_2'(x)}(\Omega) \quad \text{as } n \to \infty, \tag{4.14}$$

namely a Palais–Smale sequence for the energy level c_1.

By the fact that R' is compact and S' is of type (S_+), using the fact that the space W is reflexive it suffices to prove that $(w_n)_n$ is bounded in W. To this end we argue by contradiction and suppose that $\|w_n\| \to \infty$ (passing eventually to a subsequence).

Using hypotheses (S_4'), (R_2) and the fact that we assumed $\|w_n\| \to \infty$ we obtain that

$$\begin{aligned} T_\lambda(w_n) - \frac{1}{\eta}\langle T_\lambda'(w_n), w_n\rangle &= \int_\Omega S_0(x,|\nabla w_n|) - \frac{1}{\eta}\left[\phi(x,|\nabla w_n|)\nabla w_n + \psi(x,|\nabla w_n|)\nabla w_n\right]\nabla w_n dx \\ &\quad + \lambda \int_\Omega \left[\frac{1}{\eta} f(x,w_n)w_n - F(x,w_n)\right] dx \\ &\geq \int_\Omega \left(1 - \frac{p_2^+}{\eta}\right) S_0(x,|\nabla w_n|)dx + \lambda \int_\Omega \left[\frac{1}{\eta} f(x,w_n)w_n - F(x,w_n)\right] dx. \end{aligned}$$

Let us define now

$$C_A = \sup\left\{\left|\frac{1}{\eta} f(x,z)z - F(x,z)\right| : x \in \overline{\Omega}, |z| \leq A\right\}.$$

Hence by assumption (R_2) we have that

$$\left(1-\frac{p_2^+}{\eta}\right)\int_\Omega S_0(x,|\nabla w_n|)dx \le T_\lambda(w_n)-\frac{1}{\eta}\langle T'_\lambda(w_n),w_n\rangle$$
$$-\lambda\int_{\{x\in\Omega:\,|w_n(x)|>A\}}\left[\frac{1}{\eta}f(x,w_n)w_n-F(x,w_n)\right]dx+\lambda C_A|\Omega|$$
$$\le T_\lambda(w_n)-\frac{1}{\eta}\langle T'_\lambda(w_n),w_n\rangle+\lambda C_A|\Omega|,$$

where by $|\Omega|$ we denotes the Lebesgue measure of the domain Ω.

Since we supposed that $\|w_n\|\to\infty$, for a sufficiently large n we get that $\|w_n\|>1$, and by assumptions (S_3), (S'_4) and relation (2.3) we have that

$$\left(1-\frac{p_2^+}{\eta}\right)\frac{c}{p_2^+}\|w_n\|^{p_2^-}\le T_\lambda(w_n)+\frac{1}{\eta}\|T'_\lambda(w_n)\|_{W^{-1,p'_2(x)}(\Omega)}\cdot\|w_n\|+\lambda C_A|\Omega|.$$

Now by the fact that $\eta>p_2^+$ and $p_2^->1$ we obtain a contradiction.

Therefore we have proved that there exists a Palais–Smale sequence for the energy level $c_1>0$, which is bounded. So passing eventually to a subsequence (labeled for the ease of writing with the same notation) $(w_n)_n$ and taking account the fact that the space W is reflexive we can find an element w_0 such that $w_n\rightharpoonup w_0$ in W. Now, with the same arguments as in the final part of the proof for Theorem 4.1 we have that

$$T_\lambda(w_0)=c_1>0\quad\text{and}\quad T'_\lambda(w_0)=0.$$

Hence for every $\lambda\in(0,\lambda_{\phi,\psi})$ we can find a mountain pass type solution of the problem (P_λ). □

In the final part of this section we will present our last existence result, that is, the existence of infinitely many high-energy weak solutions of the problem (P_λ).

In order to prove our last result we first remind the following result.

Lemma 4.6 ([10]). *Let W be a reflexive and separable Banach space, then there are $\{e_j\}\subset W$ and $\{e_j^*\}\subset W^*$ such that*

$$W=\overline{\operatorname{span}\{e_j:\ j=1,2,\dots\}}\quad\textit{and}\quad W^*=\overline{\operatorname{span}\{e_j^*:\ j=1,2,\dots\}}$$

with

$$\langle e_i^*,e_j\rangle=\begin{cases}1, & \text{if } i=j\\ 0, & \text{if } i\neq j.\end{cases}$$

For the simplicity of the notation we will take use of the following:

$$W_j=\operatorname{span}\{e_j\},\qquad Y_k=\bigoplus_{j=1}^{k}W_j,\qquad Z_k=\bigoplus_{j=k}^{\infty}W_j.$$

We state now our multiplicity result.

Theorem 4.7. *Suppose that hypotheses (S_1)–(S_3), (S'_4), (R_1)–(R_4) and relation (4.10) hold true. If $f(x,-z)=-f(x,z)$ for almost all $x\in\Omega$, all $z\in\mathbb{R}$ and $\lambda>0$, then the problem (P_λ) admits a sequence of solutions $(\pm u_n)_n$ such that $T_\lambda(u_n)\to\infty$ as $n\to\infty$.*

Proof. In order to point out the sequence of solutions for the problem (P_λ) we will reveal the fact that our energy functional T_λ possesses a sequence $(\pm u_n)_n \subset W$ of critical points with higher and higher energies. To this end we have to prove the fact that functional T_λ is an even functional, and there are some constants $\gamma_k > \vartheta_k > 0$ such that for $k \in \mathbb{N}$ large enough:

(i) $\inf\{T_\lambda(u) : u \in Z_k, \|u\| = \vartheta_k\} \to \infty$ as $k \to \infty$

(ii) $\max\{T_\lambda(u) : u \in Y_k, \|u\| = \gamma_k\} \leq 0$

(iii) T_λ satisfies the Palais–Smale condition for every $c > 0$.

As the energy functional T_λ is even and with the same arguments as in the proof of Theorem 4.5 we can prove that T_λ satisfies the Palais–Smale condition for $c > 0$, it only remains to verify condition (i) and (ii).

Verification of (i): Let $a_k := \sup\{|u|_{q(x)} : \|u\| = 1, u \in Z_k\}$. From a straightforward computation, taking use of [25, proof of Theorem 3.2] we obtain that $a_k \to 0$ as $k \to \infty$.

Let $u \in Z_k$ with $\|u\| = \vartheta_k > 1$, where ϑ_k will be specified later. By hypothesis (S_3), (S_4') and (2.3) we obtain that

$$\begin{aligned}
T_\lambda(u) &= \int_\Omega S_0(x, |\nabla u|)dx - \lambda \int_\Omega F(x,u)dx \\
&\geq \frac{c}{p_2^+}\left(\int_\Omega |\nabla u|^{p_1(x)}dx + \|u\|^{p_2^-}\right) - \lambda \int_\Omega F(x,u)dx \\
&\geq \frac{c}{p_2^+}\|u\|^{p_2^-} - \frac{\lambda\|m\|_\infty}{q^-}\int_\Omega |u|^{q(x)}dx \\
&\text{(using hypothesis } (R_3)) \\
&\geq \frac{c}{p_2^+}\|u\|^{p_2^-} - \frac{\lambda\|m\|_\infty}{q^-}\max\left\{|u|_{q(x)}^{q^-}, |u|_{q(x)}^{q^+}\right\}.
\end{aligned}$$

Taking account of the continuous embedding $W \hookrightarrow L^{q(x)}(\Omega)$, we obtain that:

$$\begin{aligned}
T_\lambda(u) &\geq \frac{c}{p_2^+}\|u\|^{p_2^-} - \frac{\lambda\|m\|_\infty}{q^-}\max\left\{C_3^{q^-}\|u\|^{q^-}, C_3^{q^+}\|u\|^{q^+}\right\} \\
&\geq \frac{c}{p_2^+}\|u\|^{p_2^-} - \frac{\lambda\|m\|_\infty C_3^q}{q^-}\|u\|^{q^+} \\
&\left(\text{where } C_3^q = \max\left\{C_3^{q^-}, C_3^{q^+}\right\}\right) \\
&\geq \frac{c}{p_2^+}\|u\|^{p_2^-} - \frac{\lambda\|m\|_\infty C_3^q}{q^-}a_k^{q^+}\|u\|^{q^+}.
\end{aligned}$$

Due to a straightforward computation, we can choose

$$\vartheta_k = \left(\frac{\lambda\|m\|_\infty C_3^q}{q^-}\cdot\frac{p_2^+}{c}a_k^{p_2^+}\right)^{\frac{1}{p_2^- - q^+}}. \tag{4.15}$$

It is easy to remark that by the fact that $p_2^- < q^+$ and $a_k \to 0$ as $k \to \infty$ we obtain $\vartheta_k \to \infty$. Now taking ϑ_k as defined in relation (4.15) we obtain that

$$T_\lambda(u) \to \infty \quad \text{as } k \to \infty,$$

and so condition (i) is verified.

Verification of (ii): Let $u \in Y_k$ and $\|u\| = \gamma_k > 1$, where γ_k will be defined later. Using hypothesis (S_2) we get that

$$T_\lambda(u) \leq 2C_{\gamma,\phi}|v_1|_{p_1'(x)} \max\left\{\|u\|_{p_1(x)}^{p_1^-}, \|u\|_{p_1(x)}^{p_1^+}\right\} + \frac{\xi}{p_1^-} \max\left\{\|u\|_{p_1(x)}^{p_1^-}, \|u\|_{p_1(x)}^{p_1^+}\right\}$$
$$+ 2C_{\gamma,\psi}|v_2|_{p_2'(x)}\|u\|^{p_2^+} + \frac{\xi}{p_2^-}\|u\|^{p_2^+} - \lambda \int_\Omega F(x,u)dx,$$

where $C_{\gamma,\phi} > 0$ and $C_{\gamma,\psi} > 0$ are some constants.

Taking account of relation (3.4) we obtain that

$$T_\lambda(u) \leq \tilde{C}_\gamma \|u\|^{p_2^+} - \lambda \int_\Omega F(x,u)dx, \tag{4.16}$$

where $\tilde{C}_\gamma = \left(2C_{\gamma,\phi}|v_1|_{p_1'(x)}C_{p_1} + \frac{\xi}{p_1^-}C_{p_1}\right) + \left(2C_{\gamma,\psi}|v_2|_{p_2'(x)} + \frac{\xi}{p_2^-}\right)$ and $C_{p_1} = \max\left\{C_2^{p_1^-}, C_2^{p_1^+}\right\}$.

Using hypothesis (R_2) we can find two constants $C_4 > 0$ and $C_5 > 0$ such that

$$F(x,z) \geq C_4|z|^\eta - C_5. \tag{4.17}$$

In what follows using relations (4.16) and (4.17) we obtain that

$$T_\lambda(u) \leq \tilde{C}_\gamma \|u\|^{p_2^+} - \lambda C_4 \int_\Omega |u|^\eta dx + \lambda C_5 |\Omega|.$$

Taking use by the fact that we work on a finite dimensional space ($\dim Y_k < \infty$), by the fact that the assumption (R_2) implies that $\eta > p_2^+$ and $|\Omega| < \infty$ we obtain:

$$T_\lambda(u) \leq \tilde{C}_\gamma \|u\|^{p_2^+} + C_6|\Omega| - C_7\|u\|^\eta$$

for some constants $C_6 > 0$, $C_7 > 0$.

Now letting $\|u\| \to \infty$ we have that

$$\lim_{\|u\| \to \infty} T_\lambda(u) = -\infty. \tag{4.18}$$

Choosing $\gamma_k > \vartheta_k > 0$ and keeping in mind relation (4.18) we obtain that

$$\max\{T_\lambda(u) : u \in Y_k, \|u\| = \gamma_k\} \leq 0,$$

for every γ_k large enough.

In order to complete our proof we only have to apply the fountain theorem (for more details we refer to [10, Theorem 4.8], [25, Theorem 6.1], [28, Lemma 3.3]) and the proof is fulfilled. □

As the definition of our double-phase differential operator is general, in what follows we will give some specific examples in order to illustrate our results.

Example 4.8. Consider the following weight coefficient functions $a, b : \Omega \to \mathbb{R}$, with $a, b \in L^\infty(\Omega)_+$ for all $x \in \Omega$. Suppose there exist a constant $C_{a,b} > 0$ such that $a(x), b(x) \geq C_{a,b}$ for all $x \in \Omega$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function which satisfy the assumptions $(R_1) - (R_4)$, (4.10) then the results of Theorems 4.1, 4.5 hold true for the following class of Dirichlet problems:

$$\begin{cases} -\operatorname{div}\left[a(x)|\nabla u|^{p_1(x)-2}\nabla u + b(x)|\nabla u|^{p_2(x)-2}\nabla u\right] = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{cases}$$

It is easy to check the fact that our differential operator satisfy hypotheses (S_1)–(S_3), (S_4'). Moreover if the reaction function f is odd in respect to the second argument (that is, $f(x,-z) = -f(x,z)$) then Theorem 4.7 holds also true.

Example 4.9. As we stated in the first section of this paper our potential functions ϕ and ψ generalize the following type of differential operator

$$A(x,|z|) = \left(1 + \frac{|z|^{p(x)}}{\sqrt{1+|z|^{2p(x)}}}\right)|z|^{p(x)-2} \tag{4.19}$$

corresponding to the differential operator which describes the capillary phenomenon, so we obtain the following class of double-phase problems:

$$\begin{cases} -\operatorname{div}\left[\left(|\nabla u|^{p_1(x)-2} + \frac{|\nabla u|^{2p_1(x)-2}}{\left(1+|\nabla u|^{2p_1(x)}\right)^{1/2}}\right)\nabla u \right. \\ \qquad\qquad \left. + \left(|\nabla u|^{p_2(x)-2} + \frac{|\nabla u|^{2p_2(x)-2}}{\left(1+|\nabla u|^{2p_2(x)}\right)^{1/2}}\right)\nabla u\right] = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{cases}$$

If hypotheses (4.10), (R_1)–(R_4) hold true, then the results of Theorems 4.1 and 4.5 hold true for this class of problems. Moreover if the reaction term is odd in respect with the second argument (that is, $f(x,-z) = -f(x,z)$) for all $(x,z) \in \Omega \times \mathbb{R}$ then this class of problems admits infinitely many nontrivial weak solutions with higher and higher energies.

By simple computations we could verify that the potential function of type A from relation (4.19) satisfies the assumptions (S_1)–(S_3), (S_4'). For a thorough proof of the validity of our example we can associate the following energy functional to our problem $E_\lambda : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} E_\lambda(u) = &\int_\Omega \frac{1}{p_1(x)}\left[|\nabla u|^{p_1(x)} + \left(1+|\nabla u|^{2p_1(x)}\right)^{1/2}\right]dx \\ &+ \int_\Omega \frac{1}{p_2(x)}\left[|\nabla u|^{p_2(x)} + \left(1+|\nabla u|^{2p_2(x)}\right)^{1/2}\right]dx - \lambda\int_\Omega F(x,u)dx \end{aligned}$$

and recalculate the computations for this functional energy.

In what follows we will construct an example of a reaction function for our problems. As we can observe by relation (4.10) the reaction term variable growth is very general and in order to use an explicit defined nonlinearity in the right-hand side of the problem (P_λ) we have to impose some eloquent conditions.

By relation (4.10) we have that $1 < q(x) < p_1^*(x)$ for all $x \in \overline{\Omega}$. Similarly with the details used in the proof of Proposition 3.8 and 4.3 we may find some functions $r_1 : \overline{\Omega}_0 \to (1, \infty)$ such that $r_1(x) = q(x)$ for all $x \in \overline{\Omega}_0$ and $r_2 : \overline{\Omega}_1 \to (1, \infty)$ such that $r_2(x) = q(x)$ for all $x \in \overline{\Omega}_1$ (where $\overline{\Omega}_0 \cap \overline{\Omega}_1 = \emptyset$). So by relation (4.10) we can state that

$$1 < r_1^- \leq r_1^+ < p_1^- \leq p_1^+ < p_2^- \leq p_2^+ < r_2^- \leq r_2^+ < p_1^*(x)$$

for all $x \in \overline{\Omega}$, where

$$r_1^- = \min_{x \in \overline{\Omega}_0} r_1(x) \text{ and } r_1^+ = \max_{x \in \overline{\Omega}_0} r_1(x)$$

and

$$r_2^- = \min_{x \in \overline{\Omega}_1} r_2(x) \text{ and } r_2^+ = \max_{x \in \overline{\Omega}_1} r_2(x).$$

So our reaction function may be defined as

$$f(x,z) = \begin{cases} m(x)|z|^{r_1(x)-2}z, & \text{if } x \in \overline{\Omega_0}, \\ m(x)|z|^{r_2(x)-2}z, & \text{if } x \in \overline{\Omega_1}. \end{cases}$$

We can deduce the fact that $f(x,z)$ has a $|z|^{r_1(x)-1}$ growth near the origin and $|z|^{r_2(x)-1}$ growth near $+\infty$. For more details we refer to the proof of Proposition 3.8 and 4.3 and to [15, Lemma 2]. Also some good examples of this type of reaction nonlinearity can be find in [16]. This last restriction is necessary in order that our function to satisfy the Ambrosetti–Rabinowitz type condition (R_2) (for example we could take $\eta = r_2^-$).

We also need to impose some particular conditions on the weight function $m : \overline{\Omega} \to [0, \infty)$:

(m_1) $m \in L^\infty(\Omega)$;

(m_2) there exist a constant m^- such that $m(x) \geq m^- > 0$ for all $x \in \overline{\Omega}_0 \cup \overline{\Omega}_1$ and $m(x) = 0$ for all $x \in \Omega \setminus (\overline{\Omega}_0 \cup \overline{\Omega}_1)$.

Remark 4.10. For this particular restrictions we can observe that function f defined as above satisfies hypotheses (R_1)–(R_4) and so, our existence theorems hold true.

5 Final remarks

(i) For every $\lambda \in (0, \lambda_{\phi,\psi})$ problem (P_λ) has at least two different solutions. Indeed, suppose that solutions given by Theorem 4.1 and Theorem 4.5 coincide ($v_0 = w_0$), we get that

$$T_\lambda(w_0) = c_1 > 0 > c_0 = T_\lambda(v_0),$$

which is a contradiction. So the multiplicity of every eigenvalue $\lambda \in (0, \lambda_{\phi,\psi})$ is at least two.

(ii) We point out that hypothesis (R_2) plays a crucial role in the proof of our results. This hypothesis is an Ambrosetti–Rabinowitz type condition which implies that our reaction function $f(x, \cdot)$ has at least a $(\eta - 1)$-polynomial growth near $+\infty$.

(iii) Theorems 4.5 and 4.7 have a strong dependency on hypothesis (R_2) whilst the results of Theorem 4.1 hold true using only the weaker hypothesis (R_3).

We may consider the following functions:

- $f_1(x,z) = |z|^{\eta-1}$
- $f_2(x,z) = |z|^{p_2^+-1}\ln(1+|z|)$
- $f_3(x,z) = |z|^{q(x)-2}z$

Only the function f_1 satisfy the Ambrosetti–Rabinowitz condition. We can also remark the fact that the results of Theorem 4.1 hold true if $f(x,u) = m(x)|u|^{q(x)-2}u$, with m defined as in relation (R_1).

(iv) For our results to hold true we can not use superlinear nonlinearities with slower growth near $+\infty$. This type on nonlinearity is represented by function f_2.

(v) It is easy to observe the fact that we have a strong connection between the first and the second type of solutions, whilst the third type of solutions (high-energy solutions) does not depend on the condition that parameter $\lambda \in (0, \lambda_{\phi,\psi})$ but only on the fact that $\lambda > 0$.

(vi) Moreover using hypothesis (4.10) instead of (3.1) we can find some information about the existence of some ground state solutions of problem (P_λ) (that is, solutions which minimizes the functional of the action in the set of all weak solutions), which lie on some Nehari manifold. To this end we refer to [1,26].

References

[1] A. Azzollini, P. d'Avenia, A. Pomponio, Quasilinear elliptic equations in $\mathbb{R}^N$ via variational methods and Orlicz–Sobolev embeddings, *Calc. Var. Partial Differential Equations* **49**(2014), 197–213. https://doi.org/10.1007/s00526-012-0578-0

[2] S. Baraket, S. Chebbi, N. Chorfi, V. Rădulescu, Non-autonomous eigenvalue problems with variable (p_1, p_2)-growth, *Adv. Nonlinear Stud.* **17**(2017), 781–792. https://doi.org/10.1515/ans-2016-6020; Zbl 1372.35205

[3] P. Baroni, M. Colombo, G. Mingione, Non-autonomous functionals borderline cases and related function classes, *St. Petersburg Math. J.* **27**(2016), 347–379. https://doi.org/10.1090/spmj/1392; MR3570955

[4] M. Boureanu, P. Pucci, V. Radulescu, Multiplicity of solutions for a class of anisotropic elliptic equations with variable exponent, *Complex Var. Elliptic Equ.* **56**(2011), 755–767. https://doi.org/10.1080/17476931003786709; Zbl 1229.35086

[5] N. Chorfi, V. Rădulescu, Standing wave solutions of a quasilinear degenerate Schrödinger equation with unbounded potential, *Electron. J. Qual. Theory Differ. Equ.* **37**(2016), 1–12. https://doi.org/10.14232/ejqtde.2016.1.37; Zbl 1363.35117

[6] M. Colombo, G. Mingione, Bounded minimisers of double phase variational integrals, *Arch. Ration. Mech. Anal.* **218**(2015), 219–273. https://doi.org/10.1007/s00205-015-0859-9; Zbl 1325.49042

[7] M. Colombo, G. Mingione, Calderón–Zygmund estimates and non-uniformly elliptic operators, *J. Funct. Anal* **270**(2016), 1416–1478. https://doi.org/10.1016/j.jfa.2015.06.022; Zbl 06535749

[8] L. Diening, P. Hästö, P. Harjulehto, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Springer Lecture Notes, Vol. 2017, Springer-Verlag, Berlin, 2011. https://doi.org/10.1007/978-3-642-18363-8; Zbl 1222.46002

[9] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47**(1974), 324–353. https://doi.org/10.1016/0022-247X(74)90025-0; Zbl 0286.49015

[10] X. Fan, Q. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.* **52**(2003), 1843–1852. https://doi.org/10.1016/S0362-546X(02)00150-5; Zbl 1146.35353

[11] R. Filippucci, P. Pucci, V. Rădulescu, Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions, *Comm. Partial Differential Equations* **33**(2008), 706–717. https://doi.org/10.1080/03605300701518208; Zbl 1147.35038

[12] I. H. Kim, Y. H. Kim, Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents, *Manuscripta Math.* **147**(2015), 169–191. https://doi.org/10.1007/s00229-014-0718-2; Zbl 1322.35009

[13] P. Marcellini, On the definition and the lower semicontinuity of certain quasiconvex integrals, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **3**(1986), 391–409. https://doi.org/10.1016/S0294-1449(16)30379-1; Zbl 0609.49009

[14] P. Marcellini, Regularity and existence of solutions of elliptic equations with (p,q)-growth conditions, *J. Differential Equations* **90**(1991), 1–30. https://doi.org/10.1016/0022-0396(91)90158-6; Zbl 0724.35043

[15] M. Mihăilescu, G. Moroșanu, Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions, *Applicable Analysis* **89**(2010), No. 2, 257–271. https://doi.org/10.1080/00036810802713826; Zbl 1187.35074

[16] M. Mihailescu, P. Pucci, V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, *J. Math. Anal. Appl.* **340**(2008), 687–698. https://doi.org/10.1016/j.jmaa.2007.09.015; Zbl 1135.35058

[17] M. Mihăilescu, V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.* **135**(2007), 2929–2937. https://doi.org/10.1090/S0002-9939-07-08815-6; MR2317971; Zbl 1146.35067

[18] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer-Verlag, Berlin, 1983. https://doi.org/10.1007/BFb0072210; Zbl 0557.46020

[19] P. Pucci, V. Rădulescu, The impact of the mountain pass theory in nonlinear analysis: a mathematical survey, *Boll. Unione Mat. Ital.* (3) **9**(2010), 543–582. MR2742781

[20] P. Pucci, V. Rădulescu, The maximum principle with lack of monotonicity, *Electron. J. Qual. Theory Differ. Equ.* **2018**, No. 58, 1–11. https://doi.org/10.14232/ejqtde.2018.1.58; Zbl 1413.35198

[21] P. Pucci, J. Serrin, A mountain pass theorem, *J. Differential Equations* **60**(1985), 142–149. https://doi.org/10.1016/0022-0396(85)90125-1; Zbl 0585.58006

[22] P. Pucci, J. Serrin, Extensions of the mountain pass theorem, *J. Funct. Anal.* **59**(1984), 185–210. https://doi.org/10.1016/0022-1236(84)90072-7; Zbl 0564.58012

[23] V. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Anal.* **121**(2015), 336–369. https://doi.org/10.1016/j.na.2014.11.007; Zbl 1321.35030

[24] V. Rădulescu, D. Repovš, *Partial differential equations with variable exponents: variational methods and qualitative analysis*, CRC Press, Taylor & Francis Group, Boca Raton FL, 2015. https://doi.org/10.1201/b18601; MR3379920; Zbl 1343.35003

[25] D. Repovš, Stationary waves of Schrödinger-type equations with variable exponent, *Anal. Appl. (Singap.)* **13**(2015), No. 6, 645–661. https://doi.org/10.1142/S0219530514500420; MR3376930; Zbl 1331.35139

[26] V. F. Uță, Ground state solutions and concentration phenomena in nonlinear eigenvalue problems with variable exponents, *An. Univ. Craiova Ser. Mat. Inform.* **45**(2018), No.1, 122–136. Zbl 07122549

[27] V. F. Uță, Multiple solutions for eigenvalue problems involving an indefinite potential and with $(p_1(x), p_2(x))$ balanced growth, *An. Științ. Univ. ații "Ovidius" Constanța Ser. Mat.* **27** (2019), No. 1, 289–307. https://doi.org/10.2478/auom-2019-0015; Zbl 07089809

[28] M. Zhou, K. Q. Wang, Infinitely many weak solutions for $p(x)$-Laplacian-like problems with sign-changing potential, *Electron. J. Qual. Theory Differ. Equ.* **2020**, No. 10, 1–14. https://doi.org/10.14232/ejqtde.2020.1.10