

# Estimating the Dimension of the Subfield Subcodes of Hermitian Codes

Sabira El Khalfaoui<sup>a</sup> and Gábor P. Nagy<sup>b</sup>

## Abstract

In this paper, we study the behavior of the true dimension of the subfield subcodes of Hermitian codes. Our motivation is to use these classes of linear codes to improve the parameters of the McEliece cryptosystem, such as key size and security level. The McEliece scheme is one of the promising alternative cryptographic schemes to the current public key schemes since in the last four decades, they resisted all known quantum computing attacks. By computing and analyzing a data collection of true dimensions of subfield subcodes, we concluded that they can be estimated by the extreme value distribution function.

**Keywords:** AG code, Hermitian code, subfield subcode, extreme value distribution

## 1 Introduction

Recently, there has been a big amount of research addressed to quantum computers that use quantum mechanical techniques to solve hard computational problems in mathematics [2]. The existence of these powerful machines threaten many of the public-key cryptosystem that are widely in use. Combined with Shor's algorithms [38], this would risk the confidentiality and integrity of today's digital communications. Post-quantum cryptography aims to construct and develop cryptosystems that resist against quantum computing attacks.

McEliece [28] introduced the first code-based public-key cryptosystem in 1978, where he employed error correcting codes to generate the public and private key with security relying on two aspects: NP-completeness of decoding linear codes and the distinguishing of the chosen codes. The original McEliece scheme was constructed with binary Goppa codes which are subfield subcodes of generalized

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<sup>a</sup>Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary. E-mail: [sabira@math.u-szeged.hu](mailto:sabira@math.u-szeged.hu), ORCID: <https://orcid.org/0000-0002-1792-2947>.

<sup>b</sup>Department of Algebra, Budapest University of Technology and Economics, Egry József utca 1, H-1111 Budapest, Hungary and Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary. E-mail: [nagy@math.u-szeged.hu](mailto:nagy@math.u-szeged.hu), ORCID: <https://orcid.org/0000-0002-9558-4197>.

Reed-Solomon codes. Even today, this proposal represents a good candidate for post-quantum cryptography [1]. There have been several attempts to find appropriate classes of codes and their parameters, which give rise to a secure and effective cryptosystem, for more details see [31, 27]. In this paper, we study the possibility of the application of subfield subcodes of Hermitian codes in the McEliece scheme. More precisely, we do the first step by investigating the true dimension of these codes for a broad spectrum of parameters, for partial results see [13, 34]. Our main observation is that the true dimension of subfield subcodes of Hermitian codes can be estimated by the extreme value distribution function.

In the literature, several attacks have been proposed against McEliece cryptosystem in general, and against McEliece systems based on AG codes, see [3, 27, 6]. Attacks can be divided into two classes: structural, or key recovery attacks, aimed at recovering the secret code, and decoding, or message recovery attacks, aimed at decrypting the transmitted ciphertext. The generic decoding attack against the McEliece scheme is the information set decoding (ISD) algorithm. The most recent and most effective structural attack against AG code based McEliece systems is the Schur product distinguisher.

The structure of this paper is as follows. In section 2, we review the necessary background to define subfield subcodes, algebraic geometry codes and Hermitian codes. In section 3, we introduce some tools borrowed from statistics in order to handle our computed data on the true dimension of subfield subcodes of Hermitian codes, the latter being presented in section 4. Our main result is Proposition 1 in section 5 which shows the excellent fitting properties of the extreme value distribution to our measurements. In section 6, we applied this estimate to study the development of the key size of Hermitian subfield subcodes.

## 2 Backgrounds, formulas

In this section, we give an overview on subfield subcodes, AG codes and some of their properties, for more details the reader is referred to the monographs [17, 40, 41]. Our terminology on coding theory is standard, see [40, 18]. In particular, by an  $\mathbb{F}_q$ -linear code of length  $n$ , we mean a linear subspace of  $\mathbb{F}_q^n$ .

### 2.1 Subfield subcodes

Let  $h$  be a positive integer and  $r, q$  be prime powers with  $q = r^h$ . Then  $\mathbb{F}_r$  is a subfield of  $\mathbb{F}_q$  and the field extension  $\mathbb{F}_q/\mathbb{F}_r$  has degree  $h$ . Let  $C$  be an  $\mathbb{F}_q$ -linear code of length  $n$  and dimension  $k$ . The  $\mathbb{F}_q/\mathbb{F}_r$  subfield subcode of  $C$  is defined by

$$C|_{\mathbb{F}_r} = C \cap \mathbb{F}_r^n.$$

The trace polynomial  $\text{Tr}(x) = x + x^r + \cdots + x^{r^{h-1}}$  defines a map  $\mathbb{F}_q \rightarrow \mathbb{F}_r$ , which can be extended to a map  $\mathbb{F}_q^n \rightarrow \mathbb{F}_r^n$  component wise. The trace code of the linear code  $C$  is

$$\text{Tr}(C) = \{\text{Tr}(c) \mid c \in C\}.$$

Clearly, both the subfield subcode and the trace code are  $\mathbb{F}_r$ -linear codes of length  $n$ . However, it is in general very hard to determine the true dimension of these new codes. The fascinating result given by Delsarte [8] in 1975 plays a key role for studying the class of the subfield subcodes of linear codes. It established a closed link between subfield subcodes and trace codes:

$$(C|_{\mathbb{F}_r})^\perp = \text{Tr}(C^\perp).$$

Véron [44] used this equation to give the exact dimension formula

$$\dim_{\mathbb{F}_r}(C|_{\mathbb{F}_r}) = n - h(n - k) + \dim_{\mathbb{F}_r} \ker(\text{Tr}). \tag{1}$$

In particular, we have the trace bound

$$\dim_{\mathbb{F}_r}(C|_{\mathbb{F}_r}) \geq n - h(n - k). \tag{2}$$

## 2.2 Algebraic geometry codes

In this section, we give an overview on the construction of algebraic geometry (AG) codes, which is a version of V.D. Goppa’s original construction. We note that there are many ways to produce linear codes from algebraic curves. Also we give some details on the properties, parameters and duality of AG codes. AG codes are linear codes that use algebraic curves and finite fields for their construction. The construction can be done by evaluating functions (elements of the function field) or by computing residues of differentials. Our notation and terminology on algebraic plane curves over finite fields, their function fields, divisors and Riemann-Roch spaces are standard, see for instance [17, 29, 41].

Let  $q$  be a prime power and  $\mathbb{F}_q$  be the finite field of order  $q$ . Let  $\mathcal{X}$  be an algebraic curve i.e. an affine or projective variety of dimension one, which is absolutely irreducible and nonsingular and whose defining equations are (homogeneous) polynomials with coefficients in  $\mathbb{F}_q$ . Let  $g$  be the genus of  $\mathcal{X}$  and denote by  $\mathbb{F}_q(\mathcal{X})$  the function field of  $\mathcal{X}$ . For a divisor  $D$  of  $\mathbb{F}_q(\mathcal{X})$ , the Riemann-Roch space is

$$\mathcal{L}(D) = \{f \in \mathbb{F}_q(\mathcal{X}) \mid (f) \succeq -D\} \cup \{0\},$$

where  $(f)$  is the principal divisor of  $f$ . The dimension  $\ell(D)$  of  $\mathcal{L}(D)$  is given by the Riemann-Roch Theorem [41]\*Theorem 1.1.15:

$$\ell(D) = \ell(W - D) + \deg D - g + 1, \tag{3}$$

where  $W$  is a canonical divisor of  $\mathbb{F}_q(\mathcal{X})$ .

Let  $G$  and  $D$  be two divisors of  $\mathbb{F}_q(\mathcal{X})$  such that  $D = P_1 + \dots + P_n$  is the sum of  $n$  distinct rational places of  $\mathbb{F}_q(\mathcal{X})$  and  $P_i \notin \text{supp}(G)$  for any  $i$ . With these data, two types of algebraic geometry codes can be constructed:

$$\begin{aligned} C_L(D, G) &= \{(f(P_1), \dots, f(P_n)) \mid f \in \mathcal{L}(G)\}, \\ C_\Omega(D, G) &= \{(\text{res}_{P_1}(\omega), \dots, \text{res}_{P_n}(\omega)) \mid \omega \in \Omega(G - D)\}. \end{aligned}$$

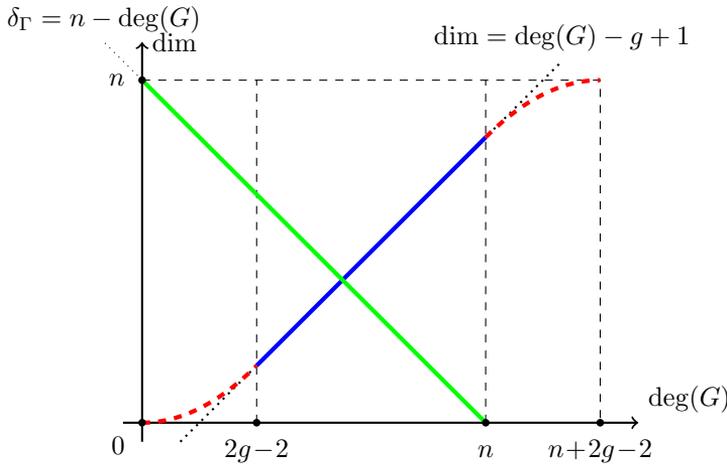


Figure 1: Dimension and designed minimum distance of AG codes

The codes  $C_L(D, G)$  and  $C_\Omega(D, G)$  are called the *functional* and the *differential codes*, respectively. These two codes are dual to each other. Moreover, the differential code  $C_\Omega(D, G)$  is equivalent with the functional code  $C_L(D, W + D - G)$ . In particular, they have the same dimension and minimum distance, even though this equivalence does not preserve all important properties of the code. The formula

$$k = \ell(G) - \ell(G - D)$$

for the dimension  $k$  of  $C_L(D, G)$  follows from the Riemann-Roch Theorem, which also provides a lower bound  $\delta_\Gamma = n - \deg(G)$  for its minimum distance. The integer  $\delta_\Gamma$  is called the *Goppa designed minimum distance* of the AG code.

We illustrate the behavior of the dimension  $k$  of  $C_L(D, G)$  depending on the degree of the divisor  $G$  by Figure 1. In fact, (3) implies the exact value  $k = \deg(G) - g + 1$  provided  $2g - 2 < \deg(G) < n$ . Furthermore, if  $\deg(G) > n + 2g - 2$ , then  $k = n$ . In the intervals  $[0, 2g - 2]$ , and  $[n, n + 2g - 2]$ , the dimension depends on the specific structure of the divisor  $G$ .

### 2.3 On the decoding of AG codes

Algebraic geometry codes are a generalization of Reed-Solomon codes, then it is not extraordinary that they benefit from similar decoding algorithms. The work on the decoding of AG codes seems to begin in 1986 when Driencourt gave a first decoding algorithm for codes on elliptic curves of characteristic 2 [9] correcting  $\lfloor (\delta_\Gamma - 1)/2 \rfloor$  errors. By generalizing the work of Arimoto and Peterson [33] on employing a locator polynomial to decode Reed-Solomon codes, Justesen, Larsen, Jensen, Havemose and Høhold published [21] in 1989 a decoding algorithm for a larger class of AG codes, which can correct up to  $\lfloor (\delta_\Gamma - g - 1)/2 \rfloor$  errors, moreover

in improved version [20] the error capability is increased to  $\lfloor (\delta_\Gamma - g/2 - 1)/2 \rfloor$ . This method was generalized to arbitrary curves by Skorobogatov and Vladut [39], and independently by Krachkovskii [26], then extended by Duursma [10, 12] to correct  $\lfloor (\delta_\Gamma - 1)/2 \rfloor - \sigma$  errors, where  $\sigma$  is the Clifford defect of the curve [12]\*Definition 3.7 (is approximately  $g/4$ ). In 1993, Feng and Rao [15] gave a majority voting scheme allowing a decoding up to  $\lfloor (\delta_\Gamma - 1)/2 \rfloor$  errors. Duursma generalized this result to all AG codes [11]. An efficient algorithm was described by Sakata, Justesen, Madelung, Jensen and Høhold in [35] using a multidimensional generalization of Massey-Berlekamp algorithm done by Sakata [36]. Kirfel and Pellikaan [22] noticed that one can decode beyond  $\lfloor (\delta_\Gamma - 1)/2 \rfloor$  errors for 1-point AG codes by studying the Weierstrass semigroup. The reader can refer to [18, 19, 32] for more details on decoding methods.

### 2.4 Hermitian codes

The classes of AG codes we study in this paper are defined over the Hermitian curve [41]\*VI.3.6 and VI.4.3. Let  $\mathbb{F}_q$  be a finite field and define the Hermitian curve  $\mathcal{H}_q$  by the affine equation  $Y^q + Y = X^{q+1}$ . Notice that  $\mathcal{H}_q$  is defined over  $\mathbb{F}_{q^2}$ , that is, its rational points are points of the projective plane  $PG(2, q^2)$ , satisfying the homogeneous equation  $Y^q Z + YZ^q = X^{q+1}$ . With respect to the line  $Z = 0$  at infinity,  $\mathcal{H}_q$  has one infinite point  $P_\infty = (0 : 1 : 0)$  and  $q^3$  affine rational points  $P_1, \dots, P_{q^3}$ . As usual, we also look at the curve  $\mathcal{H}_q$  as the smooth curve defined over the algebraic closure  $\overline{\mathbb{F}_{q^2}}$ . Then, there is a one-to-one correspondence between the points of  $\mathcal{H}_q$  and the places of the function field  $\overline{\mathbb{F}_{q^2}}(\mathcal{H}_q)$  of  $\mathcal{H}_q$ .

With a Hermitian code we mean a functional AG code of the form  $C_L(D, G)$ , where the divisor  $D$  is defined as the sum  $P_1 + \dots + P_{q^3}$  affine rational points of  $\mathcal{H}_q$ . In our investigations, the divisor  $G$  can take two forms. In the *1-point case*, we set  $G = sP_\infty$  with integer  $s$ . In the *degree 3 case*, we put  $G = sP$ , where  $P$  is a place of degree 3. Let  $P_1, P_2, P_3$  be the extensions of  $P$  in the constant field extension of  $\mathbb{F}_{q^2}(\mathcal{H}_q)$  of degree 3. Then  $P_1, P_2, P_3$  are degree one places of  $\mathbb{F}_{q^6}(\mathcal{H}_q)$  and, up to labeling the indices,  $P_{j+1} = \text{Frob}(P_j)$  where  $\text{Frob}$  is the  $q^2$ -th Frobenius map and the indices are taken modulo 3. Also,  $P$  may be identified with the  $\mathbb{F}_{q^2}$ -rational divisor  $P_1 + P_2 + P_3$  of  $\mathbb{F}_{q^6}(\mathcal{H}_q)$ . Functional AG codes of the form  $C_L(D, sP_\infty)$  and  $C_L(D, sP)$  will be called 1-point Hermitian codes, and Hermitian codes over a degree 3 place, respectively. In the 1-point case, the basis of the Riemann-Roch space  $\mathcal{L}(sP_\infty)$  can be given explicitly by [40]:

$$\mathcal{M}(s) := \{x^i y^j \mid 0 \leq i \leq q^2 - 1, 0 \leq j \leq q - 1, qi + (q + 1)j \leq s\}.$$

In the degree 3 case, the basis of

$$\mathcal{L}(sP) = \left\{ \frac{f}{(\ell_1 \ell_2 \ell_3)^u} \mid f \in \mathbb{F}_{q^2}[X, Y], \deg f \leq 3u, v_{P_i}(f) \geq u \right\} \cup \{0\}.$$

can be computed, see [24]. In this formula,  $\ell_i = 0$  is the equation of the tangent line of  $\mathcal{H}_q$  at  $P_i$ , and  $s = u(q + 1) - v, 0 \leq v \leq q$ .

The group  $\text{Aut}(\mathcal{H}_q)$  of all automorphisms of  $\mathcal{H}_q$  is defined over  $\mathbb{F}_{q^2}$ . It is a group of projective linear transformations of  $PG(2, q^2)$ , isomorphic to the projective unitary group  $PGU(3, q)$ . Furthermore,  $\text{Aut}(\mathcal{H}_q)$  acts doubly transitively on the set  $\{P_\infty, P_1, \dots, P_{q^3}\}$  of  $\mathbb{F}_{q^2}$ -rational points. As it was pointed out in [24], the automorphism group of  $\mathcal{H}_q$  acts transitively on the set of degree 3 places of  $\mathbb{F}_{q^2}(\mathcal{H}_q)$ , as well. Hence, the geometry of a degree 3 place is independent on the choice of  $P$ . However, the stabilizer  $G_P$  of  $P$  in  $\text{Aut}(\mathcal{H}_q)$  is not transitive on the set of  $q^3 + 1$  rational points. In fact,  $G_P$  is a cyclic group of order  $q^2 - q + 1$  and the number of  $G_P$ -orbits on the set of rational points is  $q + 1$ . (See [5, 24], where [5]\*Section 4.2 holds for any characteristic.)

### 3 Moments of the extended rate of subfield subcodes

In order to make our notation consistent, we make the following conventions. Let  $\mathcal{X}$  be an algebraic curve over  $\mathbb{F}_q$  and  $D, G$  effective divisors such that the AG code  $C_L(D, G)$  is well defined. Assume that the objects  $\delta$  and  $\gamma$  determine the curve  $\mathcal{X}$  and the divisors  $D, G$  in a unique way. Let  $s$  be an integer and  $\mathbb{F}_r$  be a subfield of  $\mathbb{F}_q$ . Then,

$$C_{\delta,r}^\gamma(s) = C_L(D, sG)|_{\mathbb{F}_r}$$

denotes the  $\mathbb{F}_q/\mathbb{F}_r$  subfield subcode of the AG code  $C_L(D, sG)$ . The length of  $C_{\delta,r}^\gamma(s)$  is  $n = \text{deg}(D)$ .

For the integer  $s$ , let

$$R(s) = R_{\delta,r}^\gamma(s) = \frac{\dim_{\mathbb{F}_r} C_{\delta,r}^\gamma(s)}{n}$$

denote the rate of the subfield subcode  $C_{\delta,r}^\gamma(s)$ . We extend  $R_{\delta,r}^\gamma$  to  $\mathbb{R}$  in the usual way:  $R_{\delta,r}^\gamma(x) = R_{\delta,r}^\gamma(\lfloor x \rfloor)$ .

**Lemma 1.** *Let  $g$  be the genus of  $\mathcal{X}$  and define*

$$\alpha = \left\lceil \frac{n + 2g - 2}{\text{deg}(G)} \right\rceil.$$

*Then  $R(x) = R_{\delta,r}^\gamma(\lfloor x \rfloor)$  is a monotone increasing function, with*

$$R(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq \alpha. \end{cases}$$

*Proof.* If  $s \text{deg}(G) > n + 2g - 2$ , then  $\text{deg}(D + W - G) < 0$ , and

$$C_\Omega(D, G) \cong C_L(D, D + W - G) = \{0\}.$$

Hence, if  $s \geq \alpha$ , then  $C_L(D, sG) = \mathbb{F}_q^n$  and  $C_L(D, sG)|_{\mathbb{F}_r} = \mathbb{F}_r^n$ . □

The following observation has been made in [13]\*Theorem 5.1 for the special case of a one point divisor of a Hermitian curve.

**Lemma 2.** For  $0 \leq x < n/(r \deg(G))$ , we have  $R(x) = 1/n$ .

*Proof.* Let  $s$  be an integer with  $0 \leq s < \frac{n}{r \deg(G)}$ . As the divisor  $sG$  is positive for  $s > 0$ , the constant vectors are in  $C_L(D, sG)|_{\mathbb{F}_r}$  and  $R(s) \geq 1/n$  holds. Assume  $R(s) > 1/n$ , that is, the subfield subcode contains a non constant element  $\mathbf{v} = (f(P_1), \dots, f(P_n))$  with  $f \in \mathcal{L}(sG)$ . Since a function of the form  $f + c$  cannot have more than  $\deg(sG)$  zeros,  $\mathbf{v}$  cannot have the same entry more than  $s \deg(G)$  times. This implies  $r \deg(sG) \geq n$ . □

Lemma 1 implies that we can consider  $R(x)$  as the distribution function of some random variable  $\xi$ , cf. [37]\*Definition 1, Section 2.3.

**Lemma 3.** Let  $R(x)$  be the extended rate function of a class of subfield subcodes  $C_L(D, sG)|_{\mathbb{F}_r}$ . Define the integer  $\alpha$  as in Lemma 1. Let  $\xi$  be a random variable with distribution function  $R(x)$ . Then

$$E(\xi) = \sum_{s=0}^{\alpha} 1 - R(s), \quad E(\xi^2) = \sum_{s=0}^{\alpha} (2s + 1)(1 - R(s)).$$

*Proof.* This follows from [37]\*Section 2.6, Corollary 2. □

**Remark 1.** Considered as a distribution function,  $R_{\delta,r}^{\gamma}(s)$  has an expectation  $E_{\delta,r}^{\gamma}$ , a variance  $\text{Var}_{\delta,r}^{\gamma}$  and a standard deviation  $D_{\delta,r}^{\gamma}$ . These constants can be computed from the true dimensions of the subfield subcodes using Lemma 3 and the well known formulas for random variables.

## 4 Computed true dimensions of Hermitian subfield subcodes

Let  $q$  be a prime power. We say that the object  $\delta = q$  determines the Hermitian curve  $\mathcal{H}_q$  over  $\mathbb{F}_{q^2}$ , together with the divisor  $D$  which is the sum of affine rational points of  $\mathcal{H}_q$ . The objects  $\gamma = 1\text{-pt}$  or  $\gamma = \text{deg-3}$  determine the divisor  $G$  to be equal either to the rational infinite place  $P_{\infty}$ , or the degree 3 Hermitian place  $P$ , respectively. That being said, for any integer  $s$  and subfield  $\mathbb{F}_r$  of  $\mathbb{F}_{q^2}$ , the Hermitian subfield subcodes

$$C_{q,r}^{1\text{-pt}}(s) = C_L(D, sP_{\infty})|_{\mathbb{F}_r}, \quad C_{q,r}^{\text{deg-3}}(s) = C_L(D, sP)|_{\mathbb{F}_r}$$

are well defined and consistent with the notation of section 3. These codes are  $\mathbb{F}_r$ -linear codes of length  $n = q^3$ .

Let  $R_{q,r}^{1\text{-pt}}(s)$  and  $R_{q,r}^{\text{deg-3}}(s)$  be the true rates of the codes  $C_{q,r}^{1\text{-pt}}(s)$  and  $C_{q,r}^{\text{deg-3}}(s)$ . Using the GAP [16] package `HERmitian` [30], we have been able to compute the true dimension values of the codes  $C_{q,q}^{1\text{-pt}}(s)$ ,  $C_{q,q}^{\text{deg-3}}(s)$  for

$$q \in \{2, 3, 4, 5, 7, 8, 9, 11, 13\}$$

and the binary codes  $C_{q,2}^{1\text{-pt}}(s)$ ,  $C_{q,2}^{\text{deg-3}}(s)$  for

$$q \in \{2, 4, 8, 16\}.$$

(Cf. [13] for preliminary results on explicit computation of subfield subcodes of Hermitian 1-point codes.)

As given in Lemma 3, we computed the expectations  $E_{q,q}^{1\text{-pt}}$ ,  $E_{q,2}^{1\text{-pt}}$ ,  $E_{q,q}^{\text{deg-3}}$ ,  $E_{q,2}^{\text{deg-3}}$ , the variances  $\text{Var}_{q,q}^{1\text{-pt}}$ ,  $\text{Var}_{q,2}^{1\text{-pt}}$ ,  $\text{Var}_{q,q}^{\text{deg-3}}$ ,  $\text{Var}_{q,2}^{\text{deg-3}}$ , and the standard deviations  $D_{q,r}^{1\text{-pt}}$ ,  $D_{q,2}^{1\text{-pt}}$ ,  $D_{q,q}^{\text{deg-3}}$ ,  $D_{q,2}^{\text{deg-3}}$  for these true rates. The numerical results are shown in Table 1 for  $q = 3, 4, 5, 7, 8, 9, 11, 13$  and  $r = q$ , and in Table 2 for  $q = 2, 4, 8, 16$  and  $r = 2$ . In Figure 2, we present the ratios  $E_{q,r}^\gamma \text{deg}(G)/n$  and  $D_{q,r}^\gamma \text{deg}(G)/n$ , where  $\gamma \in \{1\text{-pt}, \text{deg-3}\}$ . While our data sets are small, these figures motivate the following open problem.

**Problem 1.** *Are there constants  $c_1, c_2 > 0$  such that*

$$E_{q,q}^{1\text{-pt}} \approx E_{q,q}^{\text{deg-3}} \approx c_1 q^3 / \text{deg}(G), \quad D_{q,q}^{1\text{-pt}} \approx D_{q,q}^{\text{deg-3}} \approx c_2 q^3 / \text{deg}(G),$$

where  $a \approx b$  means  $a/b \rightarrow 1$  with  $q \rightarrow \infty$ .

**Remark 2.** Our data suggests that for small  $q$ , the choice  $c_1 = 0.75$  and  $c_2 = 0.2$  is sound.

Table 1: Expectations and variances for Hermitian  $\mathbb{F}_{q^2}/\mathbb{F}_q$  subfield subcodes

$q$	1-point codes		Codes over a degree 3 place	
	Expectation	Variance	Expectation	Variance
3	20.15	53.46	7.63	4.09
4	48.66	246.79	17.77	16.02
5	95.04	841.16	33.37	60.18
7	259.10	5 553.32	88.99	503.78
8	385.49	11 862.84	131.61	1 106.63
9	546.30	23 541.65	186.22	2 206.21
11	992.73	74 679.83	336.49	7 262.13
13	1 631.29	197 675.07	550.94	19 807.94

Table 2: Expectations and variances for Hermitian  $\mathbb{F}_{q^2}/\mathbb{F}_2$  subfield subcodes

$q$	1-point codes		Codes over a degree 3 place	
	Expectation	Variance	Expectation	Variance
2	5.38	6.48	2.12	0.86
4	54.86	164.96	20.38	10.52
8	458.22	4 838.52	162.50	216.32
16	3 698.92	195 390.48	1 303.40	6 029.44

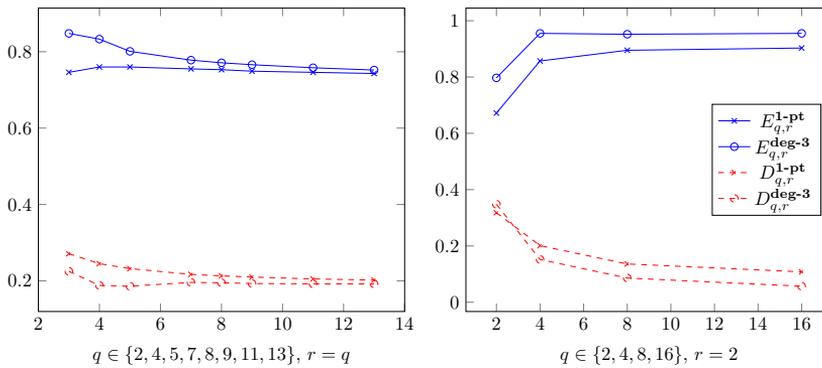


Figure 2: The ratios of expectations and standard deviations to  $n/\deg(G)$

## 5 Distribution fitting

In general, no explicit formula is known for the true dimension of subfield subcodes of AG codes. Our goal was to use the method of distribution fitting in order to study the behavior of these true dimensions in the case of subfield subcodes of Hermitian codes.

As in the previous sections, we used the notation  $\mathcal{H}_q$  for the Hermitian curve over  $\mathbb{F}_{q^2}$ ,  $P_\infty, P$  for the places of degree 1 and 3,  $D$  and  $G \in \{P_\infty, P\}$  for the divisors, and  $C_{q,r}^\gamma(s)$ ,  $\gamma \in \{1\text{-pt}, \text{deg-3}\}$ , for the  $\mathbb{F}_{q^2}/\mathbb{F}_r$  subfield subcodes  $C_L(D, sG)|_{\mathbb{F}_r}$ . Then, with fixed  $q, r$  and  $\gamma \in \{1\text{-pt}, \text{deg-3}\}$  the dimensions of the subfield subcodes are given by the extended rate function  $R_{q,r}^\gamma(x)$ .

$$R_{q,q}^{1-pt}(x), \quad R_{q,2}^{1-pt}(x), \quad R_{q,q}^{\text{deg-3}}(x), \quad R_{q,2}^{\text{deg-3}}(x).$$

Our goal was to consider these functions as distribution functions and fit some well known probability distribution functions to our experimental rate function  $R(x)$ .

We obtained numerical results by using the distribution fitting methods offered by MATLAB's Statistics and Machine Learning Toolbox [43]. The technique MLE (Maximum Likelihood Estimation) is a method for estimating the parameters of a

probability distribution from a data set. The method finds the parameter values maximizing the logarithm of the likelihood function [14]. In order to compare different distributions for a given data set, one can use the log-likelihood values for a ranking. This is implemented MATLAB's `fitmethis` function [7]. Notice that `fitmethis` also computes the AIC value for each distribution, which stands for Akaike Information Criterion, that measures the quality of a model (distribution) versus the other models. It has the formula

$$AIC = 2l - 2\log(\hat{L})$$

where  $l$  is the number of parameters and  $\hat{L}$  is the maximum values of the likelihood function. In the case of AIC, smaller values correspond to better fitting distributions (see [23]).

In our comparisons, we restricted ourselves to parametric distributions having at most two parameters, that is, we used MATLAB's `fitmethis` to compare the log-likelihood values of the following distributions: normal, exponential, gamma, logistic, uniform, extreme value, Rayleigh, beta, Nakagami, Rician, inverse Gaussian, Birnbaum-Saunders, log-logistic, log-normal and Weibull. We can summarize the results as follows:

- Proposition 1.**
1. *The best fitting distribution was the extreme value distribution for  $R_{q,q}^{1-pt}(x)$ ,  $q \in \{4, 5, 7, 8, 9, 11, 13\}$ , for  $R_{q,q}^{deg-3}(x)$ ,  $q \in \{7, 8, 9, 11, 13\}$ , and for  $R_{8,2}^{1-pt}(x)$ ,  $R_{16,2}^{1-pt}(x)$ ,  $R_{4,2}^{deg-3}(x)$ ,  $R_{8,2}^{deg-3}(x)$ , and  $R_{16,2}^{deg-3}(x)$ .*
  2. *For the missing cases  $R_{2,2}^{1-pt}(x)$ ,  $R_{3,3}^{1-pt}(x)$ ,  $R_{2,2}^{deg-3}(x)$ ,  $R_{3,3}^{deg-3}(x)$ ,  $R_{4,4}^{deg-3}(x)$ , and  $R_{5,5}^{deg-3}(x)$ , the best fitting distribution was the gamma distribution.*
  3. *The second best fitting distribution was the extreme value distribution for  $R_{3,3}^{1-pt}(x)$ ,  $R_{3,3}^{deg-3}(x)$ ,  $R_{4,4}^{deg-3}(x)$ ,  $R_{5,5}^{deg-3}(x)$ .*

Our results show that for  $q \geq 3$ , among the two-parameter distributions considered, the extreme value distribution function is a good estimation of the rate function of subfield subcodes of Hermitian codes.

The extreme value distribution is also referred to as Gumbel or type 1 Fisher-Tippet distribution. In probability theory, these are the limiting distributions of the minimum of a large number of unbounded identically distributed random variables. The extreme value distribution function is

$$F(x; \alpha, \beta) = 1 - \exp\left(-\exp\left(\frac{x - \alpha}{\beta}\right)\right),$$

with location parameter  $\alpha \in \mathbb{R}$  and a scale parameter  $\beta > 0$ . The mean  $\mu$  and the variance  $\sigma^2$  are

$$\mu = \alpha - \beta\gamma, \quad \sigma^2 = \frac{\pi^2}{6}\beta^2,$$

where

$$\gamma = \int_1^\infty \left(-\frac{1}{x} + \frac{1}{[x]}\right) dx \approx 0.57721566490153$$

is the Euler-Mascheroni constant, see [25]\*Section 1.4. With given empirical mean and variance of the data series, the parameters can be computed by

$$\alpha = \mu + \frac{\sqrt{6}\gamma}{\pi}\sigma, \quad \beta = \frac{\sqrt{6}}{\pi}\sigma.$$

In Figure 3 we visualized the fitting of the extreme value distribution function to our experimental results on the true dimension of subfield subcodes of Hermitian codes.

The occurrence of the extreme value distribution in the context of subfield subcodes of AG codes may be somewhat surprising and we cannot give a simple mathematical explanation for this. However, the rank of random matrices over finite fields is known to be related to the class of Gumbel type distributions, see Cooper’s result [4]\*Theorem 2 for the theoretical background. This theory has been applied to parameter estimates of random erasure codes by Studholme and Blake [42].

## 6 Application: Estimating the key size of McEliece Cryptosystem

The largest (but not the only) part of the public key of the McEliece cryptosystem is the matrix  $A$  which defines the underlying error correction code.  $A$  is either the  $n \times k$  generator matrix, or the  $n \times (n - k)$  parity check matrix. In either case,  $A$  may be assumed to be in standard form, which means that the public key is given by  $k(n - k)$  elements of  $\mathbb{F}_r$ . Hence, the key size is

$$\log_2(r)k(n - k).$$

Hence, for a fixed field  $\mathbb{F}_r$  and length  $n$ , the key size is propotional to  $R(1 - R)$ , see [31]. The true values of  $R_{q,r}^\gamma(s)(1 - R_{q,r}^\gamma(s))$  can be estimated by  $F(x)(1 - F(x))$ , where  $F(x)$  is the extreme value distribution function, see Figure 4.

## 7 Conclusion and future work

The main goal of this study was to establish an approximating formula of the true dimension of the subfield subcodes of Hermitian codes. We conducted an experimental study to analyze the datasets of the true dimension of the  $\mathbb{F}_r$ -linear codes  $C_{q,r}^{1-pt}(s)$ ,  $C_{q,r}^{deg-3}(s)$  for  $q \in \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16\}$ ,  $r = 2$  or  $r = q$ , and  $s$  is an integer parameter running from 0 to  $q^3 + (q + 1)(q - 2)$ . This analysis helped us to derive new properties of their structure and led to an approach that might be useful for further research and applications.

Theoretically, the main contribution of this work is a collection of formulas of statistical flavour, such as moments of the extended rate function for subcodes of Hermitian codes.

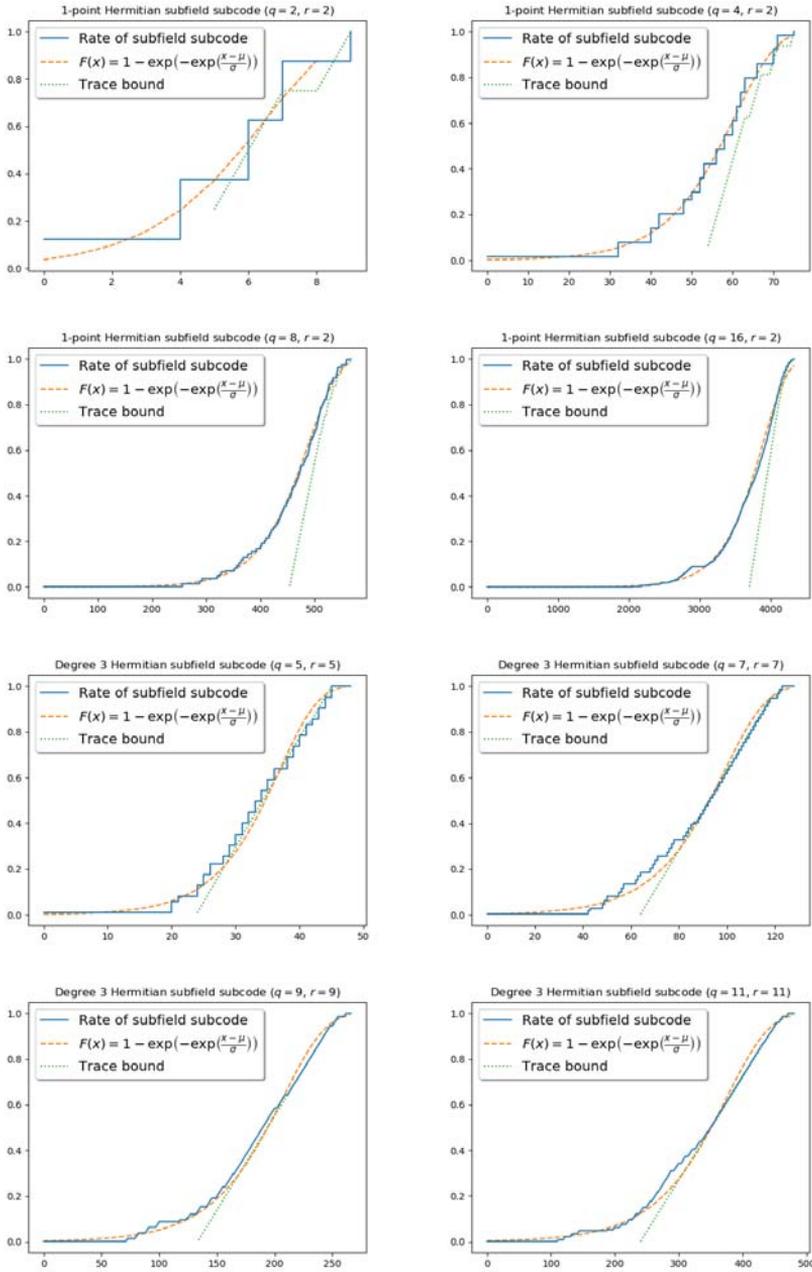


Figure 3: Estimating the extended rate function by extreme value distribution for subfield subcodes Hermitian codes

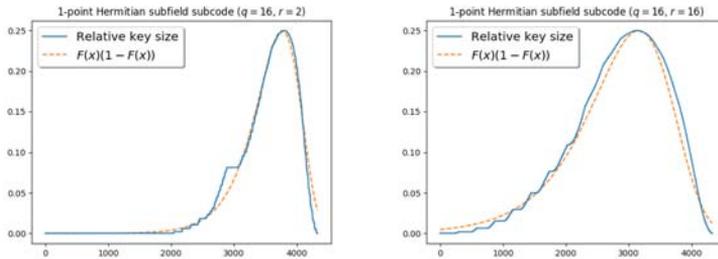


Figure 4: Estimating the key size  $n^2R(1 - R)$

From a statistical perspective, the main result is the comparison of the fitting of our datasets of true dimensions to well known distribution functions of MATLAB’s Statistics and Machine Learning Toolbox, using the method of `fitmethis`.

We found that the extreme value distribution is the best fitting one for  $q > 5$  and the second best fitting distribution for smaller values of  $q$ . Also the gamma and the normal distributions have good fitting properties. Our proposal is to use the extreme value distribution function to estimate the true dimension of subfield subcodes of Hermitian codes. In the last section of this paper, we applied this formula to give an approximation for the key size of the McEliece scheme, depending on the parameter  $s$ .

In the future, we aim to replace Goppa codes in McEliece’s original version with a family of codes that permit to reduce the public key size and to increase the code rate by maintaining a given level of security. Therefore, we intend to analyze the McEliece cryptosystem based on subclasses of subfield subcodes of Hermitian codes. Our future work will include experiments, simulations, and security and cryptanalysis of the McEliece scheme in terms of its public key size and other parameters. The measurements are based on attacks with supposed lowest complexity, e.g. information set decoding or the Schur product distinguisher.

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