



Impulsive boundary value problems for nonlinear implicit Caputo-exponential type fractional differential equations

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. This paper deals with existence and uniqueness of solutions to a class of impulsive boundary value problem for nonlinear implicit fractional differential equations involving the Caputo-exponential fractional derivative. The existence results are based on Schaefer's fixed point theorem and the uniqueness result is established via Banach's contraction principle. Two examples are given to illustrate the main results.

Keywords: boundary value problem, Caputo-exponential fractional derivative, implicit fractional differential equations, existence, fixed point, impulses.

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1 Introduction

The fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer orders. Fractional differential equations arise in various fields of science and engineering. Indeed, we can find numerous applications in control theory of dynamical systems, chaotic dynamics, fractals, optics, and signal processing, fluid flow, viscoelasticity, polymer science, rheology, physics, chemistry, biology, astrophysics, cosmology, thermodynamics, mechanics, and other fields. For further details and applications, see, for example, [8, 24, 28, 29]. For some fundamental results on the theory of fractional calculus and fractional ordinary and partial differential equations, we refer to the reader to the books [1, 2, 21, 25, 35], the articles [5, 6, 17], and the references therein.

Impulsive differential equations describe observed evolution processes of several real world phenomena in a natural manner, and exhibit several new phenomena such as noncontinuity and merging of solutions, rhythmical beating, etc. Dynamic processes associated with

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sudden changes in their states are governed by impulsive differential equations. This theory models many phenomena in control theory, population dynamics, medicine, and economics. Recently, fractional differential equations with impulse effects have also received considerable attention, for example, the monographs by Abbas *et al.* [3] Benchohra *et al.* [13], Lakshmikantham *et al.* [26], Samoilenko and Perestyuk [30], and the papers of Benchohra *et al.* [9, 16, 19], Chang *et al.* [20], Henderson *et al.* [23], and Wang *et al.* [32], as well as the references cited therein.

On the other hand, boundary value problems for fractional differential equations have received considerable attention because they occur in the mathematical modeling of a variety of physical processes; see for example [6, 7, 11, 12, 34]. In [10, 14, 15, 18], the authors give existence and uniqueness results for some classes of implicit fractional order differential equations.

Recently, in [27, 31] the authors introduce the exponential fractional calculus and give some existence and uniqueness results for solutions of initial and boundary value problems for fractional differential equations involving Caputo-exponential fractional derivatives (as defined in the next section).

The main goal of this paper is to study existence and uniqueness results for solutions to a more general class of impulsive boundary value problem (BVP for short) given by the following nonlinear implicit fractional-order differential equation:

$${}^c D_{t_k}^\alpha \omega(t) = f(t, \omega(t), {}^c D_{t_k}^\alpha \omega(t)), \quad \text{for each } t \in J_k \subseteq J, \quad k = 0, 1, \dots, m, \quad (1.1)$$

$$\Delta \omega|_{t=t_k} = I_k(\omega(t_k^-)), \quad k = 1, \dots, m, \quad (1.2)$$

$$c_1 \omega(a) + c_2 \omega(b) = c_3, \quad (1.3)$$

where $a = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, ${}^c D_{a^+}^\alpha$ denotes the Caputo-exponential fractional derivative of order α , $0 < \alpha \leq 1$, $J = [a, b]$, $J_0 = [a, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, $f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function, c_1, c_2, c_3 are real constants with $c_1 + c_2 \neq 0$, $\Delta \omega|_{t=t_k} = \omega(t_k^+) - \omega(t_k^-)$, and $\omega(t_k^+) = \lim_{h \rightarrow 0^+} \omega(t_k + h)$ and $\omega(t_k^-) = \lim_{h \rightarrow 0^-} \omega(t_k + h)$ represent the right and left hand limits of $\omega(t)$ at $t = t_k$, respectively.

The present paper is organized as follows. In Section 2, some notations are introduced and we recall some preliminary concepts about Caputo-exponential fractional derivatives and some auxiliary results. In Section 3, two results on the impulsive boundary value problem (1.1)–(1.3) are presented: the first one is based on the Banach contraction principle and the second one on Schaefer's fixed point theorem. In the last section, we give two examples to illustrate the applicability of our main results.

2 Preliminaries

In this section, we introduce notations, definitions, and lemmas that are useful in the next section. Let $J := [a, b]$ such that $a < b$. By $C := C(J, \mathbb{R})$ we denote the Banach space of all continuous functions ω from J into \mathbb{R} with the supremum norm

$$\|\omega\|_\infty = \sup_{t \in J} |\omega(t)|.$$

As usual, $AC(J)$ denote the space of absolutely continuous function from J into \mathbb{R} . We denote by $AC_e^n(J)$ the space

$$AC_e^n(J) := \left\{ \omega : J \rightarrow \mathbb{R} : {}^c D^{n-1} \omega(t) \in AC(J), {}^c D = e^{-t} \frac{d}{dt} \right\},$$

where $n = [\alpha] + 1$, with $[\alpha]$ the integer part of α .

In particular, if $0 < \alpha \leq 1$, then $n = 1$ and $AC_e^1(J) := AC_e(J)$.

Definition 2.1 ([27, 31]). The exponential fractional integral of order $\alpha > 0$ of a function $h \in L^1(J, E)$ is defined by

$$({}^e I_a^\alpha h)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (e^t - e^s)^{\alpha-1} h(s) e^s ds, \quad \text{for each } t \in J,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt, \quad \xi > 0.$$

Definition 2.2 ([27, 31]). Let $\alpha > 0$ and $h \in AC_e^n(J)$. The exponential fractional derivatives of Caputo type of order α is defined by

$$({}^e D_a^\alpha h)(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t (e^t - e^s)^{n-\alpha-1} \left(e^{-s} \frac{d}{ds} \right)^n h(s) \frac{ds}{e^{-s}}, \quad \text{for each } t \in J,$$

where $n = [\alpha] + 1$. In particular, if $\alpha = 0$, then

$$\left({}^e D_{(\cdot)}^0 h \right)(t) := h(t).$$

Lemma 2.3 ([27, 31]). Let $\alpha > 0$, $n = [\alpha] + 1$, and $h \in AC_e^n(J)$. Then we have the formula

$${}^e I_a^\alpha ({}^e D_a^\alpha h)(t) = h(t) - \sum_{k=0}^{n-1} \frac{(e^s - e^a)^k}{k!} {}^e D^k h(a).$$

Lemma 2.4. Let $\alpha > 0$, and $h \in AC_e^n(J)$. Then the differential equation

$${}^e D_a^\alpha h(t) = 0$$

has the solution

$$h(t) = \eta_0 + \eta_1(e^s - e^a) + \eta_2(e^s - e^a)^2 + \dots + \eta_{n-1}(e^s - e^a)^{n-1},$$

where $\eta_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, and $n = [\alpha] + 1$.

Lemma 2.5. Let $\alpha > 0$, and $h \in AC_e^n(J)$. Then

$${}^e I_a^\alpha ({}^e D_a^\alpha h)(t) = h(t) + \eta_0 + \eta_1(e^s - e^a) + \eta_2(e^s - e^a)^2 + \dots + \eta_{n-1}(e^s - e^a)^{n-1},$$

for some $\eta_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, and $n = [\alpha] + 1$.

Theorem 2.6 ([22] (Banach's fixed point theorem)). Let C be a non-empty closed subset of a Banach space X ; then any contraction mapping F of C into itself has a unique fixed point.

Theorem 2.7 ([22] (Schaefer's fixed point theorem)). Let X be a Banach space and $\Theta : X \rightarrow X$ be a completely continuous operator. If the set

$$\varepsilon = \{ \omega \in X : \omega = \lambda \Theta \omega, \text{ for some } \lambda \in (0, 1) \}$$

is bounded, then Θ has fixed point.

3 Main results

Consider the set of functions

$$PC(J, \mathbb{R}) = \{ \omega : J \rightarrow \mathbb{R} \mid \omega \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, \dots, m, \text{ and there exist } \omega(t_k^+) \text{ and } \omega(t_k^-), k = 1, \dots, m, \text{ with } \omega(t_k^-) = \omega(t_k) \}.$$

This set, together with the norm

$$\|\omega\|_{PC} = \sup_{t \in J} |\omega(t)|,$$

is a Banach space. Let $J_0 = [a, t_1]$ and $J_k = (t_k, t_{k+1}]$ for $k = 1, \dots, m$.

Now, let us start by defining what we mean by a solution of the problem (1.1)–(1.3).

Definition 3.1. A function $\omega \in PC(J, \mathbb{R}) \cap (\cup_{k=0}^m AC_c(J_k, \mathbb{R}))$ is said to be a solution of (1.1)–(1.3) if ω satisfies the equation ${}^c D_{a^+}^\alpha \omega(t) = f(t, \omega(t), {}^c D_{a^+}^\alpha \omega(t))$, on J_k and the conditions

$$\begin{aligned} \Delta \omega|_{t=t_k} &= I_k(\omega(t_k^-)), \quad \text{for } k = 1, \dots, m, \\ c_1 \omega(a) + c_2 \omega(b) &= c_3. \end{aligned}$$

To prove the existence of solutions to (1.1)–(1.3), we need the following auxiliary lemmas.

Lemma 3.2. Let $0 < \alpha \leq 1$ and let $\varphi : J \rightarrow \mathbb{R}$ be continuous. A function ω is a solution of the integral equation

$$\omega(t) = \begin{cases} \frac{-1}{c_1+c_2} \left[c_2 \sum_{i=1}^m I_i(\omega(t_i^-)) + c_2 \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds \right. \\ \left. + c_2 \int_{t_m}^b (e^b - e^s)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds - c_3 \right] + \int_a^t (e^t - e^s)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds, & \text{if } t \in [a, t_1], \\ \frac{-1}{c_1+c_2} \left[c_2 \sum_{i=1}^m I_i(\omega(t_i^-)) + c_2 \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds \right. \\ \left. + c_2 \int_{t_m}^b (e^b - e^s)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds - c_3 \right] + \sum_{i=1}^k I_i(\omega(t_i^-)) & \text{if } t \in (t_k, t_{k+1}], \\ + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds + \int_{t_k}^t (e^t - e^s)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds, & \end{cases} \quad (3.1)$$

where $k = 1, \dots, m$, if and only if, ω is a solution of the fractional BVP

$${}^c D_{t_k^+}^\alpha \omega(t) = \varphi(t), \quad t \in J_k, \quad (3.2)$$

$$\Delta \omega|_{t=t_k} = I_k(\omega(t_k^-)), \quad \text{for } k = 1, \dots, m, \quad (3.3)$$

$$c_1 \omega(a) + c_2 \omega(b) = c_3. \quad (3.4)$$

Proof. Assume that ω satisfies (3.2)–(3.4). If $t \in [a, t_1]$, then

$${}^c D_a^\alpha \omega(t) = \varphi(t).$$

By Lemma 2.5,

$$\varpi(t) = \eta_0 + {}^e I_a^\alpha \varphi(t) = \eta_0 + \frac{1}{\Gamma(\alpha)} \int_a^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds.$$

If $t \in (t_1, t_2]$, then by Lemma 2.5 we obtain

$$\begin{aligned} \varpi(t) &= \varpi(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds \\ &= \Delta\varpi|_{t=t_1} + \varpi(t_1^-) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds \\ &= I_1(\varpi(t_1^-)) + \left[\eta_0 + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} (e^{t_1} - e^s)^{\alpha-1} \varphi(s) e^s ds \right] \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds \\ &= \eta_0 + I_1(\varpi(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} (e^{t_1} - e^s)^{\alpha-1} \varphi(s) e^s ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds. \end{aligned}$$

If $t \in (t_2, t_3]$, then by Lemma 2.5 we have

$$\begin{aligned} \varpi(t) &= \varpi(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds \\ &= \Delta\varpi|_{t=t_2} + \varpi(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds \\ &= I_2(\varpi(t_2^-)) + \left[\eta_0 + I_1(\varpi(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} (e^{t_1} - e^s)^{\alpha-1} \varphi(s) e^s ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (e^{t_2} - e^s)^{\alpha-1} \varphi(s) e^s ds \right] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds \\ &= \eta_0 + [I_1(\varpi(t_1^-)) + I_2(\varpi(t_2^-))] + \left[\frac{1}{\Gamma(\alpha)} \int_a^{t_1} (e^{t_1} - e^s)^{\alpha-1} \varphi(s) e^s ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (e^{t_2} - e^s)^{\alpha-1} \varphi(s) e^s ds \right] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds. \end{aligned}$$

Repeating this process, the solution $\varpi(t)$ for $t \in (t_k, t_{k+1}]$, where $k = 1, \dots, m$, can be written as

$$\begin{aligned} \varpi(t) &= \eta_0 + \sum_{i=1}^k I_i(\varpi(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} \varphi(s) e^s ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds. \end{aligned}$$

It is clear that

$$\varpi(a) = \eta_0$$

and

$$\begin{aligned}\omega(b) &= \eta_0 + \sum_{i=1}^m I_i(\omega(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} \varphi(s) e^s ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_m}^b (e^b - e^s)^{\alpha-1} \varphi(s) e^s ds.\end{aligned}$$

Hence, by applying the boundary conditions $c_1\omega(a) + c_2\omega(b) = c_3$, we see that

$$\begin{aligned}c_3 &= \eta_0(c_1 + c_2) + c_2 \sum_{i=1}^m I_i(\omega(t_i^-)) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} \varphi(s) e^s ds \\ &\quad + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^b (e^b - e^s)^{\alpha-1} \varphi(s) e^s ds.\end{aligned}$$

Then,

$$\begin{aligned}\eta_0 &= \frac{-1}{c_1 + c_2} \left[c_2 \sum_{i=1}^m I_i(\omega(t_i^-)) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} \varphi(s) e^s ds \right. \\ &\quad \left. + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^b (e^b - e^s)^{\alpha-1} \varphi(s) e^s ds - c_3 \right].\end{aligned}$$

Thus, if $t \in (t_k, t_{k+1}]$, where $k = 1, \dots, m$, then

$$\begin{aligned}\omega(t) &= \frac{-1}{c_1 + c_2} \left[c_2 \sum_{i=1}^m I_i(\omega(t_i^-)) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} \varphi(s) e^s ds \right. \\ &\quad \left. + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^b (e^b - e^s)^{\alpha-1} \varphi(s) e^s ds - c_3 \right] + \sum_{i=1}^k I_i(\omega(t_i^-)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} \varphi(s) e^s ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds.\end{aligned}$$

Conversely, assume that ω satisfies the impulsive fractional integral equation (3.1).

If $t \in [a, t_1]$ then $c_1\omega(a) + c_2\omega(b) = c_3$, and using the fact that ${}^e D_a^\alpha$ is the left inverse of ${}^e I_a^\alpha$ gives

$${}^e D_a^\alpha \omega(t) = \varphi(t), \quad \text{for each } t \in [a, t_1].$$

If $t \in (t_k, t_{k+1}]$ for $k = 1, \dots, m$, then, by using the fact that ${}^e D_{t_k}^\alpha C = 0$, where C is a constant, and ${}^e D_{t_k}^\alpha$ is the left inverse of ${}^e I_{t_k}^\alpha$, we have

$${}^e D_{t_k}^\alpha \omega(t) = \varphi(t), \quad \text{for each } t \in (t_k, t_{k+1}].$$

Also, we can easily show that

$$\Delta\omega|_{t=t_k} = I_k(\omega(t_k^-)), \quad k = 1, \dots, m. \quad \square$$

Now, we state and prove our first existence result for the problem (1.1)–(1.3); it is based on the Banach contraction principle. The following hypotheses will be used in the sequel.

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

(H2) There exist constants $k_1 > 0$ and $0 < k_2 < 1$ such that

$$|f(t, \varpi_1, \omega_1) - f(t, \varpi_2, \omega_2)| \leq k_1 |\varpi_1 - \varpi_2| + k_2 |\omega_1 - \omega_2|,$$

for any $\varpi_1, \varpi_2, \omega_1, \omega_2 \in \mathbb{R}$ and $t \in J$.

(H3) There exists a constant $\zeta > 0$ such that

$$|I_k(\varpi_1) - I_k(\varpi_2)| \leq \zeta |\varpi_1 - \varpi_2|,$$

for each $\varpi_1, \varpi_2 \in \mathbb{R}$ and $k = 1, 2, \dots, m$.

Set

$$\gamma = \frac{k_1}{1 - k_2}, \quad \mu_1 = \frac{|c_2|}{|c_1 + c_2|} + 1 \quad \text{and} \quad \mu_2 = \frac{\gamma(m+1)(e^b - e^a)^\alpha}{\Gamma(\alpha + 1)}.$$

Theorem 3.3. Assume that (H1)–(H3) are satisfied. If

$$\mu_1(m\zeta + \mu_2) < 1, \tag{3.5}$$

then the boundary value problem (1.1)–(1.3) has a unique solution on J .

Proof. To transform the problem (1.1)–(1.3) into a fixed point problem, consider the operator $\Theta : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R})$ defined by

$$\begin{aligned} \Theta(\varpi)(t) = & \frac{-1}{c_1 + c_2} \left[c_2 \sum_{i=1}^m I_i(\varpi(t_i^-)) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} \varphi(s) e^s ds \right. \\ & \left. + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^b (e^b - e^s)^{\alpha-1} \varphi(s) e^s ds - c_3 \right] + \sum_{a < t_k < t} I_k(\varpi(t_k^-)) \\ & + \frac{1}{\Gamma(\alpha)} \sum_{a < t_k < t} \int_{t_{k-1}}^{t_k} (e^{t_k} - e^s)^{\alpha-1} \varphi(s) e^s ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds, \end{aligned} \tag{3.6}$$

where $\varphi \in C(J, \mathbb{R})$ satisfies

$$\varphi(t) = f(t, \varpi(t), \varphi(t)).$$

It is clear that solutions of problem (1.1)–(1.3) are the fixed points of the operator Θ . Now, for $\varpi_1, \varpi_2 \in PC(J, \mathbb{R})$ and for each $t \in J$, we have

$$\begin{aligned} |\Theta(\varpi_1)(t) - \Theta(\varpi_2)(t)| \leq & \frac{|c_2|}{|c_1 + c_2|} \left[\sum_{i=1}^m |I_i(\varpi_1(t_i^-)) - I_i(\varpi_2(t_i^-))| \right. \\ & + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} e^s |\varphi_1(s) - \varphi_2(s)| ds \\ & \left. + \frac{1}{\Gamma(\alpha)} \int_{t_m}^b (e^b - e^s)^{\alpha-1} e^s |\varphi_1(s) - \varphi_2(s)| ds \right] \\ & + \sum_{a < t_k < t} |I_k(\varpi_1(t_k^-)) - I_k(\varpi_2(t_k^-))| \\ & + \frac{1}{\Gamma(\alpha)} \sum_{a < t_k < t} \int_{t_{k-1}}^{t_k} (e^{t_k} - e^s)^{\alpha-1} e^s |\varphi_1(s) - \varphi_2(s)| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (e^t - e^s)^{\alpha-1} e^s |\varphi_1(s) - \varphi_2(s)| ds, \end{aligned}$$

where $\varphi_1, \varphi_2 \in C(J, \mathbb{R})$ are such that

$$\varphi_1(t) = f(t, \omega_1(t), \varphi_1(t)) \quad \text{and} \quad \varphi_2(t) = f(t, \omega_2(t), \varphi_2(t)).$$

By (H2), we have

$$\begin{aligned} |\varphi_1(s) - \varphi_2(s)| &= |f(t, \omega_1(t), \varphi_1(t)) - f(t, \omega_2(t), \varphi_2(t))| \\ &\leq k_1 |\omega_1(t) - \omega_2(t)| + k_2 |\varphi_1(t) - \varphi_2(t)|, \end{aligned}$$

so

$$|\varphi_1(s) - \varphi_2(s)| \leq \gamma |\omega_1(s) - \omega_2(s)|. \quad (3.7)$$

Hence, for each $t \in J$,

$$\begin{aligned} |\Theta(\omega_1)(t) - \Theta(\omega_2)(t)| &\leq \frac{|c_2|}{|c_1 + c_2|} \left[\sum_{k=1}^m \zeta |\omega_1(t_k^-) - \omega_2(t_k^-)| \right. \\ &\quad + \frac{\gamma}{\Gamma(\alpha)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (e^{t_k} - e^s)^{\alpha-1} e^s |\omega_1(s) - \omega_2(s)| ds \\ &\quad \left. + \frac{\gamma}{\Gamma(\alpha)} \int_{t_m}^b (e^b - e^s)^{\alpha-1} e^s |\omega_1(s) - \omega_2(s)| ds \right] \\ &\quad + \sum_{i=1}^m \zeta |\omega_1(t_i^-) - \omega_2(t_i^-)| \\ &\quad + \frac{\gamma}{\Gamma(\alpha)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (e^{t_k} - e^s)^{\alpha-1} e^s |\omega_1(s) - \omega_2(s)| ds \\ &\quad + \frac{\gamma}{\Gamma(\alpha)} \int_{t_k}^t (e^t - e^s)^{\alpha-1} e^s |\omega_1(s) - \omega_2(s)| ds \\ &\leq \frac{|c_2|}{|c_1 + c_2|} \left[m\zeta + \frac{\gamma m (e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} + \frac{\gamma (e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right] \|\omega_1 - \omega_2\|_{PC} \\ &\quad + \left[m\zeta + \frac{\gamma m (e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} + \frac{\gamma (e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right] \|\omega_1 - \omega_2\|_{PC} \\ &= \left(\frac{|c_2|}{|c_1 + c_2|} + 1 \right) \left[m\zeta + \frac{\gamma (m + 1) (e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right] \|\omega_1 - \omega_2\|_{PC}. \end{aligned}$$

Thus,

$$\|\Theta(\omega_1) - \Theta(\omega_2)\|_{PC} \leq \mu_1 (m\zeta + \mu_2) \|\omega_1 - \omega_2\|_{PC}.$$

By (3.5), the operator Θ is a contraction. Hence, by Banach's contraction principle, Θ has a unique fixed point that is a unique solution of (1.1)–(1.3). \square

Our second existence result is based on Schaefer's fixed point theorem (Theorem 2.7 above). Let us introduce the following condition:

(H4) There exist constants $\tilde{\zeta}, \tilde{I} > 0$ such that

$$|I_k(\omega)| \leq \tilde{\zeta} |\omega| + \tilde{I},$$

for each $\omega \in \mathbb{R}$ and $k = 1, 2, \dots, m$.

Notice that (H4) is weaker than condition (H3).

Theorem 3.4. *Assume that conditions (H1), (H2), and (H4) hold. If*

$$\mu_1 \left(m\tilde{\xi} + \mu_2 \right) < 1, \quad (3.8)$$

then the problem (1.1)–(1.3) has at least one solution on J .

Proof. We shall use Schaefer's fixed point theorem to prove that Θ , defined by (3.6), has at least one fixed point on J . The proof will be given in several steps.

Step 1: Θ is continuous. Let $\{v_n\}$ be a sequence such that $v_n \rightarrow v$ in $PC(J, \mathbb{R})$. Then, for each $t \in J$,

$$\begin{aligned} |\Theta(v_n)(t) - \Theta(v)(t)| &\leq \frac{|c_2|}{|c_1 + c_2|} \left[\sum_{i=1}^m |I_i(v_n(t_i^-)) - I_i(v(t_i^-))| \right. \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds \\ &\quad + \left. \frac{1}{\Gamma(\alpha)} \int_{t_m}^b (e^b - e^s)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds \right] \\ &\quad + \sum_{a < t_k < t} |I_k(v_n(t_k^-)) - I_k(v(t_k^-))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{a < t_k < t} \int_{t_{k-1}}^{t_k} (e^{t_k} - e^s)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (e^t - e^s)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds, \end{aligned} \quad (3.9)$$

where $\varphi_n, \varphi \in C(J, E)$ satisfy

$$\varphi_n(t) = f(t, v_n(t), \varphi_n(t)) \quad \text{and} \quad \varphi(t) = f(t, v(t), \varphi(t)).$$

By (H2), we have

$$\begin{aligned} |\varphi_n(t) - \varphi(t)| &= |f(t, v_n(t), \varphi_n(t)) - f(t, v(t), \varphi(t))| \\ &\leq k_1 |v_n(t) - v(t)| + k_2 |\varphi_n(t) - \varphi(t)|. \end{aligned}$$

Then,

$$|\varphi_n(t) - \varphi(t)| \leq \gamma |v_n(t) - v(t)|.$$

Since $v_n \rightarrow v$, we have $\varphi_n(t) \rightarrow \varphi(t)$ as $n \rightarrow \infty$ for each $t \in J$. Let $\delta > 0$ be such that, for each $t \in J$, we have $|\varphi_n(t)| \leq \delta$ and $|\varphi(t)| \leq \delta$. Then,

$$\begin{aligned} (e^t - e^s)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| &\leq (e^t - e^s)^{\alpha-1} e^s [|\varphi_n(s)| + |\varphi(s)|] \\ &\leq 2\delta (e^t - e^s)^{\alpha-1} e^s \end{aligned}$$

and

$$\begin{aligned} (e^{t_k} - e^s)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| &\leq (e^{t_k} - e^s)^{\alpha-1} e^s [|\varphi_n(s)| + |\varphi(s)|] \\ &\leq 2\delta (e^{t_k} - e^s)^{\alpha-1} e^s. \end{aligned}$$

For each $t \in J$, the functions $s \rightarrow 2\delta(e^t - e^s)^{\alpha-1}e^s$ and $s \rightarrow 2\delta(e^{t_k} - e^s)^{\alpha-1}e^s$ are integrable on $[a, t]$. Then, the Lebesgue dominated convergence theorem and (3.9) imply that

$$|\Theta(v_n)(t) - \Theta(v)(t)| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and so

$$\|\Theta(u_n) - \Theta(u)\|_{PC} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, Θ is continuous.

Step 2: Θ maps bounded sets into bounded sets in $PC(J, \mathbb{R})$. It suffices to show that for any $\bar{\delta} > 0$, there exists a positive constant $\bar{\ell}$ such that, for any $v \in B_{\bar{\delta}} = \{v \in PC(J, \mathbb{R}) : \|v\|_{PC} \leq \bar{\delta}\}$, we have $\|\Theta(v)\|_{PC} \leq \bar{\ell}$. Now for each $t \in J$,

$$\begin{aligned} |\Theta(v)(t)| &\leq \frac{|c_2|}{|c_1 + c_2|} \left[\sum_{i=1}^m |I_i(v(t_i^-))| + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} e^s |\varphi(s)| ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_m}^b (e^b - e^s)^{\alpha-1} e^s |\varphi(s)| ds \right] + \frac{|c_3|}{|c_1 + c_2|} + \sum_{a < t_k < t} |I_k(v(t_k^-))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{a < t_k < t} \int_{t_{k-1}}^{t_k} (e^{t_k} - e^s)^{\alpha-1} e^s |\varphi(s)| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (e^t - e^s)^{\alpha-1} e^s |\varphi(s)| ds, \end{aligned} \quad (3.10)$$

where $\varphi \in C(J, \mathbb{R})$ satisfies

$$\varphi(t) = f(t, v(t), \varphi(t)).$$

By (H2), for each $t \in J$ we have

$$\begin{aligned} |\varphi(t)| &= |f(t, v(t), \varphi(t)) - f(t, 0, 0) + f(t, 0, 0)| \\ &\leq |f(t, v(t), \varphi(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq k_1 |v| + k_2 |\varphi(t)| + \tilde{f}. \end{aligned}$$

Thus,

$$|\varphi(t)| \leq \gamma |v| + \frac{\tilde{f}}{1 - k_2}. \quad (3.11)$$

From this and (3.10), for any $v \in B_{\bar{\delta}}$, we have

$$\begin{aligned} |\Theta(v)(t)| &\leq \frac{|c_2|}{|c_1 + c_2|} \left[m (\tilde{\xi} |v| + \tilde{I}) + m \left(\gamma |v| + \frac{\tilde{f}}{1 - k_2} \right) \frac{(e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right. \\ &\quad \left. + \left(\gamma |v| + \frac{\tilde{f}}{1 - k_2} \right) \frac{(e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right] + \frac{|c_3|}{|c_1 + c_2|} + m (\tilde{\xi} |v| + \tilde{I}) \\ &\quad + m \left(\gamma |v| + \frac{\tilde{f}}{1 - k_2} \right) \frac{(e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} + \left(\gamma |v| + \frac{\tilde{f}}{1 - k_2} \right) \frac{(e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \\ &= \left(\frac{|c_2|}{|c_1 + c_2|} + 1 \right) \left[m (\tilde{\xi} |v| + \tilde{I}) + \left(\gamma |v| + \frac{\tilde{f}}{1 - k_2} \right) \frac{(m + 1) (e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right] + \frac{|c_3|}{|c_1 + c_2|} \end{aligned}$$

$$\begin{aligned}
 &\leq \left(\frac{|c_2|}{|c_1 + c_2|} + 1 \right) \left[m \left(\tilde{\xi} \tilde{\delta} + \tilde{I} \right) + \left(\gamma \tilde{\delta} + \frac{\tilde{f}}{1 - k_2} \right) \frac{(m + 1) (e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right] + \frac{|c_3|}{|c_1 + c_2|} \\
 &= \mu_1 \left[m \left(\tilde{\xi} \tilde{\delta} + \tilde{I} \right) + \left(\tilde{\delta} + \frac{\tilde{f}}{k_1} \right) \mu_2 \right] + \frac{|c_3|}{|c_1 + c_2|} \\
 &=: \bar{\ell},
 \end{aligned}$$

which implies that $\|\Theta(v)\|_{PC} \leq \bar{\ell}$.

Step 3: Θ maps bounded sets into equicontinuous sets in $PC(J, \mathbb{R})$. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$, $B_{\bar{\delta}}$ be a bounded set in $PC(J, \mathbb{R})$ as in Step 2, and let $v \in B_{\bar{\delta}}$. Then, we have

$$\begin{aligned}
 &|\Theta(v)(\tau_2) - \Theta(v)(\tau_1)| \\
 &\leq \frac{1}{\Gamma(\alpha)} \int_a^{\tau_1} \left| (e^{\tau_2} - e^s)^{\alpha-1} - (e^{\tau_1} - e^s)^{\alpha-1} \right| e^s |\varphi(s)| ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (e^{\tau_2} - e^s)^{\alpha-1} e^s |\varphi(s)| ds + \sum_{\tau_1 < t_k < \tau_2} |I_k(v(t_k^-))| \\
 &\quad + \frac{1}{\Gamma(\alpha)} \sum_{\tau_1 < t_k < \tau_2} \int_{t_{k-1}}^{t_k} (e^{t_k} - e^s)^{\alpha-1} e^s |\varphi(s)| ds \\
 &\leq \left(\gamma |v| + \frac{\tilde{f}}{(1 - k_2)} \right) \frac{1}{\Gamma(\alpha + 1)} \left[(e^{\tau_1} - e^a)^\alpha - (e^{\tau_2} - e^a)^\alpha \right. \\
 &\quad \left. + 2(e^{\tau_2} - e^{\tau_1})^\alpha \right] + (\tau_2 - \tau_1) \left[\left(\tilde{\xi} |v| + \tilde{I} \right) + \left(\gamma |v| + \frac{\tilde{f}}{(1 - k_2)} \right) \frac{(e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right] \\
 &\leq \left(\gamma \tilde{\delta} + \frac{\tilde{f}}{(1 - k_2)} \right) \frac{1}{\Gamma(\alpha + 1)} \left[(e^{\tau_1} - e^a)^\alpha - (e^{\tau_2} - e^a)^\alpha \right. \\
 &\quad \left. + 2(e^{\tau_2} - e^{\tau_1})^\alpha \right] + (\tau_2 - \tau_1) \left[\left(\tilde{\xi} \tilde{\delta} + \tilde{I} \right) + \left(\gamma \tilde{\delta} + \frac{\tilde{f}}{(1 - k_2)} \right) \frac{(e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right].
 \end{aligned}$$

As $\tau_1 \rightarrow \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of the steps 1 to 3 together with the Ascoli–Arzelà theorem, we conclude that $\Theta : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R})$ is completely continuous.

Step 4: *A priori bounds.* It remain to show that the set

$$\varepsilon = \{v \in PC(J, \mathbb{R}) : v = \lambda \Theta(v), \text{ for some } \lambda \in (0, 1)\}$$

is bounded. Let $v \in \varepsilon$; then $v = \lambda \Theta(v)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$, we have

$$\begin{aligned}
 v(t) &= \frac{-\lambda}{c_1 + c_2} \left[c_2 \sum_{i=1}^m I_i(v(t_i^-)) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (e^{t_i} - e^s)^{\alpha-1} \varphi(s) e^s ds \right. \\
 &\quad \left. + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^b (e^b - e^s)^{\alpha-1} \varphi(s) e^s ds - c_3 \right] + \lambda \sum_{a < t_k < t} I_k(v(t_k^-)) \\
 &\quad + \frac{\lambda}{\Gamma(\alpha)} \sum_{a < t_k < t} \int_{t_{k-1}}^{t_k} (e^{t_k} - e^s)^{\alpha-1} \varphi(s) e^s ds + \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^t (e^t - e^s)^{\alpha-1} \varphi(s) e^s ds.
 \end{aligned}$$

From (3.11) and (H4), for each $t \in J$, we obtain

$$\begin{aligned}
|v(t)| &\leq \frac{|c_2|}{|c_1 + c_2|} \left[m \left(\tilde{\xi} |v| + \tilde{I} \right) + m \left(\gamma |v| + \frac{\tilde{f}}{1 - k_2} \right) \frac{(e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right. \\
&\quad \left. + \left(\gamma |v| + \frac{\tilde{f}}{1 - k_2} \right) \frac{(e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right] + \frac{|c_3|}{|c_1 + c_2|} + m \left(\tilde{\xi} |v| + \tilde{I} \right) \\
&\quad + m \left(\gamma |v| + \frac{\tilde{f}}{1 - k_2} \right) \frac{(e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} + \left(\gamma |v| + \frac{\tilde{f}}{1 - k_2} \right) \frac{(e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \\
&= \left(\frac{|c_2|}{|c_1 + c_2|} + 1 \right) \left[m \left(\tilde{\xi} |v| + \tilde{I} \right) + \left(\gamma |v| + \frac{\tilde{f}}{1 - k_2} \right) \frac{(m + 1) (e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right] + \frac{|c_3|}{|c_1 + c_2|} \\
&\leq \left(\frac{|c_2|}{|c_1 + c_2|} + 1 \right) \left(m \tilde{\xi} + \frac{\gamma (m + 1) (e^b - e^a)^\alpha}{\Gamma(\alpha + 1)} \right) |v| \\
&\quad + \left(\frac{|c_2|}{|c_1 + c_2|} + 1 \right) \left(m \tilde{I} + \frac{\tilde{f} (m + 1) (e^b - e^a)^\alpha}{(1 - k_2) \Gamma(\alpha + 1)} \right) + \frac{|c_3|}{|c_1 + c_2|} \\
&\leq \mu_1 \left(m \tilde{\xi} + \mu_2 \right) |v| + \mu_1 \left(m \tilde{I} + \frac{\tilde{f} \mu_2}{k_1} \right) + \frac{|c_3|}{|c_1 + c_2|}.
\end{aligned}$$

Thus,

$$\left[1 - \mu_1 \left(m \tilde{\xi} + \mu_2 \right) \right] \|v\|_{PC} \leq \mu_1 \left(m \tilde{I} + \frac{\tilde{f} \mu_2}{k_1} \right) + \frac{|c_3|}{|c_1 + c_2|}.$$

By using condition (3.8), it follows that

$$\|v\|_{PC} \leq \frac{\left[\mu_1 \left(m \tilde{I} + \frac{\tilde{f} \mu_2}{k_1} \right) + \frac{|c_3|}{|c_1 + c_2|} \right]}{\left[1 - \mu_1 \left(m \tilde{\xi} + \mu_2 \right) \right]} =: \bar{M}.$$

This shows that the set ε is bounded. As a consequence of Schaefer's fixed point theorem, Θ has at least one fixed point which in turn is a solution of (1.1)–(1.3). \square

Remark 3.5. Often times using different techniques of proof for the same type of result necessitates requiring different hypotheses. It interesting to point out here that we have also been able to obtain both Theorems 3.3 and 3.4 above with no changes in conditions by using the Nonlinear Alternative of Leray–Schauder type.

Remark 3.6. Our results for the boundary value problem (1.1)–(1.3) remain true for the following cases:

- Initial value problem: $c_1 = 1$, $c_2 = 0$ and c_3 arbitrary.
- Terminal value problem: $c_1 = 0$, $c_2 = 1$ and c_3 arbitrary.
- Anti-periodic problem: $c_1 = c_2 \neq 0$ and $c_3 = 0$.

However, our results are not applicable to the periodic problem, i.e., the case $c_1 = 1$, $c_2 = -1$, and $c_3 = 0$.

4 Examples

In this section, we will give two examples to illustrate our main results.

Example 4.1. Consider the impulsive boundary value problem for the nonlinear implicit fractional differential equation

$${}^c D_{t_k}^{\frac{1}{2}} \varpi(t) = \frac{e^{-\sqrt{t+9}} \sin t}{7(t^2 + 1) \left(\sqrt{3} + |\varpi(t)| + \left| {}^c D_{t_k}^{\frac{1}{2}} \varpi(t) \right| \right)}, \quad \text{for each } t \in J_0 \cup J_1, \quad (4.1)$$

$$\Delta \varpi|_{t=\frac{\pi}{2}} = \frac{2|\varpi(\frac{\pi}{2}^-)|}{3 + |\varpi(\frac{\pi}{2}^-)|}, \quad (4.2)$$

$$\varpi(0) + \varpi(\pi) = 13, \quad (4.3)$$

where $J_0 = [0, \frac{\pi}{2}]$, $J_1 = (\frac{\pi}{2}, \pi]$, $m = 1$, $\alpha = \frac{1}{2}$, $a = 0$, $b = \pi$, $c_1 = c_2 = 1$, $c_3 = 13$,

$$f(t, \varpi, \omega) = \frac{e^{-\sqrt{t+9}} \sin t}{7(t^2 + 1) \left(\sqrt{3} + |\varpi| + |\omega| \right)},$$

and

$$I_1(\varpi) = \frac{|\varpi|}{19 + |\varpi|}.$$

Now, for each $t \in [0, \pi]$ and for any $\varpi_1, \varpi_2, \omega_1, \omega_2 \in \mathbb{R}$, we can show that

$$|f(t, \varpi_1, \omega_1) - f(t, \varpi_2, \omega_2)| \leq \frac{1}{21e^3} (|\varpi_1 - \varpi_2| + |\omega_1 - \omega_2|)$$

and

$$|I_1(\varpi_1) - I_1(\varpi_2)| \leq \frac{1}{19} |\varpi_1 - \varpi_2|.$$

Thus, for $k_1 = k_2 = \frac{1}{21e^3}$ and $\zeta = \frac{1}{19}$ we have that

$$\begin{aligned} \mu_1 (m\zeta + \mu_2) &= \left(\frac{|c_2|}{|c_1 + c_2|} + 1 \right) \left[m\zeta + \frac{k_1 (m+1) (e^b - e^a)^\alpha}{(1 - k_2) \Gamma(\alpha + 1)} \right] \\ &= \frac{3}{2} \left[\frac{1}{19} + \frac{\frac{2\sqrt{e^\pi - 1}}{21e^3}}{\left(1 - \frac{1}{21e^3}\right) \Gamma\left(\frac{3}{2}\right)} \right] \\ &= \frac{3}{2} \left[\frac{1}{19} + \frac{4\sqrt{e^\pi - 1}}{(21e^3 - 1) \sqrt{\pi}} \right] \\ &\approx 0.1168003443 \\ &< 1. \end{aligned}$$

Hence, conditions (H1)–(H3) and (3.5) are satisfied. As a consequence of Theorem 3.3, the problem (4.1)–(4.3) has a unique solution on $[0, \pi]$.

Example 4.2. Consider the problem

$${}^c D_{t_k}^{\frac{1}{2}} \varpi(t) = \frac{e^{-\sqrt{t+16}} \left(2 + |\varpi(t)| + \left| {}^c D_{t_k}^{\frac{1}{2}} \varpi(t) \right| \right)}{179(t^2 + 1) \left(1 + |\varpi(t)| + \left| {}^c D_{t_k}^{\frac{1}{2}} \varpi(t) \right| \right)}, \quad \text{for each } t \in J_0 \cup J_1, \quad (4.4)$$

$$\Delta\omega|_{t=\frac{1}{4}} = \frac{5|\omega(\frac{1}{4}^-)| + 1}{20 + |\omega(\frac{1}{4}^-)|}, \quad (4.5)$$

$$\omega(0) = -\omega(1), \quad (4.6)$$

where $J_0 = [0, \frac{1}{4}]$, $J_1 = (\frac{1}{4}, 1]$, $m = 1$, $\alpha = \frac{1}{2}$, $a = 0$, $b = 1$, $c_1 = c_2 = 1$, $c_3 = 0$,

$$f(t, \varpi, \omega) = \frac{e^{-\sqrt{t+16}}(2 + |\varpi| + |\omega|)}{179(t^2 + 1)(1 + |\varpi| + |\omega|)}, \quad \text{for each } t \in J_0 \cup J_1,$$

and

$$I_1(\varpi) = \frac{5|\varpi| + 1}{20 + |\varpi|}.$$

Now, for each $t \in [0, 1]$ and for any $\varpi_1, \varpi_2, \omega_1, \omega_2 \in \mathbb{R}$, we can show that

$$|f(t, \varpi_1, \omega_1) - f(t, \varpi_2, \omega_2)| \leq \frac{1}{179e^4} (|\varpi_1 - \varpi_2| + |\omega_1 - \omega_2|)$$

and

$$|I_1(\varpi)| \leq \frac{1}{4}|\varpi| + \frac{1}{20}.$$

Thus, for $k_1 = k_2 = \frac{1}{179e^4}$ and $\tilde{x}_i = \frac{1}{4}$, we have

$$\begin{aligned} \mu_1(m\tilde{\xi} + \mu_2) &= \left(\frac{|c_2|}{|c_1 + c_2|} + 1 \right) \left[m\tilde{\xi} + \frac{k_1(m+1)(e^b - e^a)^\alpha}{(1-k_2)\Gamma(\alpha+1)} \right] \\ &= \frac{3}{2} \left[\frac{1}{4} + \frac{\frac{2\sqrt{e-1}}{179e^4}}{(1 - \frac{1}{179e^4})\Gamma(\frac{3}{2})} \right] \\ &= \frac{3}{2} \left[\frac{1}{4} + \frac{4\sqrt{e-1}}{(179e^4 - 1)\sqrt{\pi}} \right] \\ &= \frac{3}{8} + \frac{4\sqrt{e-1}}{(179e^4 - 1)\sqrt{\pi}} \\ &\approx 0.3753057 \\ &< 1. \end{aligned}$$

Hence, conditions (H1), (H2), (H4), and (3.8) are satisfied, so by Theorem 3.4 the problem (4.4)–(4.6) has at least one solution on $[0, 1]$.

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References

- [1] S. ABBAS, M. BENCHOHRA, J. R. GRAEF, J. HENDERSON, *Implicit fractional differential and integral equations. Existence and stability*, De Gruyter, Berlin, 2018. [MR3791511](#)

- [2] S. ABBAS, M. BENCHOHRA, G. M. N'GUÉRÉKATA, *Topics in fractional differential equations*, Springer, New York, 2012. <https://doi.org/10.1007/978-1-4614-4036-9>; MR2962045
- [3] S. ABBAS, M. BENCHOHRA, G. M. N'GUÉRÉKATA, *Advanced fractional differential and integral equations*, Nova Science Publishers, New York, 2015. MR3309582
- [4] S. ABBAS, M. BENCHOHRA, J. E. LAZREG, A. ALSAEDI, Y. ZHOU, Existence and Ulam stability for fractional differential equations of Hilfer–Hadamard type, *Adv. Difference Equ.* **2017**, No. 180, 14 pp. <https://doi.org/doi.org/10.1186/s13662-017-1231-1>; MR3666455
- [5] S. ABBAS, M. BENCHOHRA, J. E. LAZREG, Y. ZHOU, A survey on Hadamard and Hilfer fractional differential equations: analysis and stability, *Chaos Solitons Fractals* **102**(2017), 47–71. <https://doi.org/10.1016/j.chaos.2017.03.010>; MR3671994
- [6] R. P. AGARWAL, M. BENCHOHRA, S. HAMANI, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.* **109**(2010), 973–1033. <https://doi.org/10.1007/s10440-008-9356-6>; MR2596185
- [7] B. AHMAD, J. J. NIETO, Existence of solutions for impulsive anti-periodic boundary value problems of fractional order, *Taiwanese J. Math.* **15**(2011), 981–993. <https://doi.org/10.11650/twjmath/1500406279>; MR2829892
- [8] D. BALEANU, J. A. T. MACHADO, A. C. J. LUO, *Fractional dynamics and control*, Springer-Verlag, New York, 2012. <https://doi.org/10.1007/978-1-4614-0457-6>
- [9] M. BENCHOHRA, S. BOURIAH, Existence and stability results for nonlinear implicit fractional differential equations with impulses, *Mem. Differ. Equ. Math. Phys.* **69**(2016), 15–31. MR3581036
- [10] M. BENCHOHRA, S. BOURIAH, M. A. DARWISH, Nonlinear boundary value problem for implicit differential equations of fractional order in Banach spaces, *Fixed point theory* **18**(2017), 457–470. <https://doi.org/10.24193/fpt-ro.2017.2.36>; MR3701151
- [11] M. BENCHOHRA, J. R. GRAEF, S. HAMANI, Existence result for boundary value problems with non-linear fractional differential equations, *Appl. Anal.* **87**(2008), 851–863. <https://doi.org/10.1080/00036810802307579>; MR2458962
- [12] M. BENCHOHRA, S. HAMANI, S. K. NTOUYAS, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3**(2008), 1–12. MR2390179
- [13] M. BENCHOHRA, J. HENDERSON, S. K. NTOUYAS, *Impulsive differential equations and inclusions*, Hindawi Publishing Corporation, Contemporary Mathematics and Its Applications, Vol. 2, New York, 2006. <https://doi.org/10.1155/9789775945501>; MR2322133
- [14] M. BENCHOHRA, J. E. LAZREG, Nonlinear fractional implicit differential equations, *Commun. Appl. Anal.* **17**(2013), 471–482. MR3136137
- [15] M. BENCHOHRA, J. E. LAZREG, Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions, *Romanian J. Math. Comput. Sc.* **4**(2014), 60–72. MR3178558

- [16] M. BENCHOHRA, J. E. LAZREG, Existence results for nonlinear implicit fractional differential equations with impulse, *Commun. Appl. Anal.* **19**(2015), 413–426. [MR3363687](#)
- [17] M. BENCHOHRA, J. E. LAZREG, On stability for nonlinear implicit fractional differential equations, *Matematiche (Catania)* **70**(2015), 49–61. <https://doi.org/10.4418/2015.70.2.4>; [MR3437178](#)
- [18] M. BENCHOHRA, J. E. LAZREG, Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative, *Stud. Univ. Babeş-Bolyai Math.* **62**(2017), 27–38. <https://doi.org/10.24193/subbmath.2017.0003>; [MR3627064](#)
- [19] M. BENCHOHRA, B. A. SLIMANI, Existence and uniqueness of solutions to impulsive fractional differential equations, *Electron. J. Differential Equations* **2009**, No. 10, 1–11. [MR2471119](#)
- [20] Y. CHANG, A. ANGURAJ, P. KARTHIKEYAN, Existence results for initial value problems with integral condition for impulsive fractional differential equations. *J. Fract. Calc. Appl.* **2**(2012), 1–10.
- [21] K. DIETHELM, *The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type*, Lecture Notes in Mathematics, Vol. 2004, Springer-Verlag, Berlin, 2010. <https://doi.org/10.1007/978-3-642-14574-2>; [MR2680847](#)
- [22] A. GRANAS, J. DUGUNDJI, *Fixed point theory*, Springer-Verlag, New York, 2003. <https://doi.org/10.1007/978-0-387-21593-8>; [MR1987179](#)
- [23] J. HENDERSON, A. OUAHAB, Impulsive differential inclusions with fractional order, *Comput. Math. Appl.* **59**(2010), 1191–1226. <https://doi.org/10.1016/j.camwa.2009.05.011>; [MR2579483](#)
- [24] R. HERMANN, *Fractional calculus. An introduction for physicists*, World Scientific Publishing Co. Pte. Ltd., 2011. <https://doi.org/10.1142/8072>
- [25] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier Science B.V., Amsterdam, 2006. [MR2218073](#)
- [26] V. LAKSHMIKANTHAM, D. D. BAINOV, P. S. SIMEONOV, *Theory of impulsive differential equations*, Series in Modern Applied Mathematics, Vol. 6, World Scientific, Singapore–London, 1989. <https://doi.org/10.1142/0906>; [MR1082551](#)
- [27] S. K. NTOUYAS, J. TARIBOON, C. SAWADDEE, Nonlocal initial and boundary value problems via fractional calculus with exponential singular kernel, *J. Nonlinear Sci. Appl.* **11** (2018), 1015–1030. <https://doi.org/10.22436/jnsa.011.09.01>; [MR3829317](#)
- [28] I. PETRAS, *Fractional-order nonlinear systems. Modeling, analysis and simulation*, Springer, Heidelberg, Dordrecht, London, New York, 2011. <https://doi.org/10.1007/978-3-642-18101-6>
- [29] P. SAHOO, T. BARMAN, J. P. DAVIM, *Fractal analysis in machining*, Springer, New York, 2011. <https://doi.org/10.1007/978-3-642-17922-8>

- [30] A. M. SAMOILENKO, N. A. PERESTYUK, *Impulsive differential equations*, World Scientific, Singapore, 1995. <https://doi.org/10.1142/9789812798664>; MR1355787
- [31] J. TARIBOON, S. K. NTOUYAS, Initial and boundary value problems via exponential fractional calculus, *Int. Electron. J. Pure Appl. Math.* **12**(2018), 1–19. MR3789347
- [32] G. WANG, B. AHMAD, L. ZHANG, Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions, *Comput. Math. Appl.* **62**(2011), 1389–1397. <https://doi.org/10.1016/j.camwa.2011.04.004>; MR2824726
- [33] W. WEI, X. XIANG, Y. PENG, Nonlinear impulsive integro-differential equations of mixed type and optimal controls, *Optimization: A Journal of Mathematical Programming and Operations Research* **55**(2006), 141–156. <https://doi.org/10.1080/02331930500530401>; MR2221729
- [34] L. ZHANG, G. WANG, Existence of solutions for nonlinear fractional differential equations with impulses and anti-periodic boundary conditions, *Electron. J. Qual. Theory Differ. Equ.* **2011**, No. 7, 1–11. <https://doi.org/10.14232/ejqtde.2011.1.7>
- [35] Y. ZHOU, J.-R. WANG, L. ZHANG, *Basic theory of fractional differential equations*, Second edition, World Scientific, Hackensack, NJ, 2017. <https://doi.org/10.1142/9069>; MR3586266