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Multiplicity of nodal solutions for fourth order equation with clamped beam boundary conditions

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. In this paper, we study the global structure of nodal solutions of

$$\begin{cases} u''''(x) = \lambda h(x) f(u(x)), & 0 < x < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

where $\lambda > 0$ is a parameter, $h \in C([0,1],(0,\infty))$, $f \in C(\mathbb{R})$ and $sf(s) > 0$ for $|s| > 0$. We show the existence of S-shaped component of nodal solutions for the above problem. The proof is based on the bifurcation technique.

Keywords: clamped beam, fourth order equations, connected component, nodal solutions, bifurcation.

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1 Introduction

The deformations of an elastic beam whose both ends clamped are described by the fourth order problem

$$\begin{aligned} u''''(x) = \lambda h(x) f(u(x)), \qquad x \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $f \in C(\mathbb{R})$, $f(0) = 0$, $sf(s) > 0$ for all $s \neq 0$ and $h \in C([0,1],(0,\infty))$.

Existence and multiplicity of solutions of (1.1) have been extensively studied by several authors [1,3,6,10,11,14,18,21,22]. For examples, Agarwal and Chow [1] studied the existence of solutions of (1.1) by contraction mapping and iterative methods. Cabada and Enguiça [3] developed the method of lower and upper solutions to show the existence and multiplicity of solutions. Pei and Chang [14] proved the existence of symmetric positive solutions by using

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a monotone iterative technique. Yao [21], Zhai, Song and Han [22] established the existence and multiplicity of solutions via the fixed point theorem in cone.

Recently, Sim and Tanaka [19] were concerned with the existence of three positive solutions for the p-Laplacian problem

$$\begin{cases} -(|y'|^{p-2}y')' = \lambda a(x)f(y), & x \in (0,1), \\ y(0) = y(1) = 0, \end{cases}$$

by employing a bifurcation technique, where the nonlinearity f is asymptotic linear near 0 and sublinear near ∞. They obtained an S-shaped unbounded continuum (which grows to the right from the initial point, to the left at some point and to the right near $\lambda = \infty$). The proof of their main result heavily depends on the Sturm comparison theorem [20]. For other related results on the existence and multiplicity of solutions of fourth order problems, see Li and Gao [12] and Li [13].

Motivated by the above work, we shall study the existence of S-shaped unbounded continua of nodal solutions of fourth order problems (1.1). However, it seems hard to follow this argument in [19, Lemma 3.2] directly for fourth order problem since the Sturm comparison theorem is not available for the fourth order problems, and the nodal solution of (1.1) is not concave down in $[0,1]$.

Let $Y = C[0,1]$ with the norm

$$\|u\|_\infty = \max_{t\in[0,1]} |u(t)|.$$

Let $E = \{u \in C^3[0,1] : \ u(0) = u(1) = u'(0) = u'(1) = 0\}$ with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty, \|u'''\|_\infty\}.$$

Let S_k^+ denote the set of functions in E which have exactly $k-1$ simple zeros in $(0,1)$ and are positive near $t=0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. They are disjoint and open in E. Finally, let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$.

We shall make use of the following assumptions

(A1) $h \in C[0,1]$ with $0 < h_* \le h(x) \le h^*$ on $[0,1]$ for some $h_*, h^* \in (0,\infty)$;

(A2) $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing, and there exists $s_0 > 0$ such that

$$f_* := \inf_{0<s\le s_0} \frac{f(s)}{s} < \sup_{0<s\le s_0} \frac{f(s)}{s} =: f^*$$

with

$$0 < f_* < f^* < \infty;$$

(A3) there exist $\alpha > 0$, $f_0 := \lim_{|s|\to 0} \frac{f(s)}{s} \in (0,\infty)$ and $f_1 > 0$ such that

$$\lim_{|s|\to 0} \frac{f(s) - f_0 s}{s|s|^\alpha} = -f_1;$$

(A4) $f(0) = 0$, $sf(s) > 0$ for $s \ne 0$, $f_\infty := \lim_{|s|\to\infty} \frac{f(s)}{s} = 0$.

Remark 1.1. Typical modal of f which satisfies (A3) is the following

$$\hat{f}(s) = \begin{cases} 2s - s^2, & s \geq 0, \\ 2s + s^2, & s < 0, \end{cases}$$

where $f_0 = 2,\ f_1 = 1$ and $\alpha = 1$.

The rest of the paper is organized as follows. In Section 2, we state and prove several preliminary results on the nodal solutions (λ, u) of (1.1) with $\|u\|_\infty = s_0$ and state a method of lower and upper solutions due to Cabada [3]. In Section 3, we state our main result and show the existence of bifurcation from some eigenvalue for the corresponding problem according to the standard argument and the rightward direction of bifurcation. Section 4 is devoted to show the change of direction of bifurcation. Finally in Section 5 we show an *a-priori* bound of solutions for (1.1) and complete the proof of Theorem 3.2.

2 Preliminaries

The following result is a special case of Leighton and Nehari [11, Theorem 5.2]

Lemma 2.1. *Let $p, p_1 : [a,b] \to (0,\infty)$ be two continuous functions with*

$$p(x) \leq p_1(x), \qquad x \in [a,b]. \tag{2.1}$$

Let

$$y'''' - p(x)y = 0, \qquad x \in [a,b], \tag{2.2}$$
$$y_1'''' - p_1(x)y_1 = 0, \qquad x \in [a,b]. \tag{2.3}$$

If

$$y(a) = y_1(a) = y(b) = y_1(b) = 0$$

and the number of zeros of $y(x)$ and $y_1(x)$ in $[a,b]$ is denoted by n and n' $(n \geq 4)$, respectively, then

$$n' \geq n - 1.$$

Lemma 2.2. *Let $k \geq 4$ and $\nu \in \{+,-\}$. Let (A2) hold. If*

$$h_* f_* > 0, \tag{2.4}$$

then there exists $\Lambda > 0$, such that for any solution $(\lambda, u) \in \mathbb{R}^+ \times S_k^\nu$ of (1.1) with $\|u\|_\infty = s_0$, one has

$$\lambda \leq \Lambda := \frac{\gamma_{k+2}}{h_* f_*}, \tag{2.5}$$

where γ_{k+2} is the $(k+2)$-th eigenvalue of the linear problem

$$\begin{aligned} v'''' &= \gamma v(x), \qquad x \in (0,1), \\ v(0) &= v(1) = v'(0) = v'(1) = 0, \end{aligned} \tag{2.6}$$

which is simple, and its corresponding eigenfunction ϕ_{k+2} has $k+1$ zeros in $(0,1)$.

Proof. Assume on the contrary that $\lambda > \Lambda$. Combining this with $\Lambda := \frac{\gamma_{k+2}}{h_* f_*}$ and using

$$u''''(x) = \lambda h(x)\frac{f(u(x))}{u(x)}u(x), \qquad x \in (0,1),$$
$$u(0) = u(1) = u'(0) = u'(1) = 0$$

and the fact

$$\lambda h(x)\frac{f(u(x))}{u(x)} > \gamma_{k+2}, \qquad x \in [0,1],$$

it deduces that $u \in S_{j+1}^{\nu}$ for some $j \geq k+1$. However, this contradicts the fact $u \in S_k^{\nu}$. □

Lemma 2.3. *Let*

$$M := \max\{\lambda h(x) f(s) : x \in [0,1],\, s \in [0, s_0],\, 0 \leq \lambda \leq \Lambda\}. \tag{2.7}$$

Then for any solution $(\eta, u) \in \mathbb{R}^+ \times S_k^+$ of (1.1) with $\|u\|_\infty = s_0$, one has

$$\|u'\|_\infty \leq M. \tag{2.8}$$

Proof. It follows from the equation in (1.1) and (2.7) that

$$\|u''''\|_\infty \leq M,$$

which together with the boundary value conditions in (1.1) imply the desired result. □

Let

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$$

be the zeros on u in $[0,1]$. Let x_j be such that

$$|u(x_j)| = \max\{|u(t)| : t \in [t_j, t_{j+1}]\}, \qquad j \in \{0, 1, \cdots, k-1\}.$$

Lemma 2.4. *Let*

$$|u(x_{j_0})| = \|u\|_\infty = s_0.$$

Then

$$t_{j_0+1} - t_{j_0} \geq \frac{2s_0}{M}.$$

Proof. We only deal with the case that $u(x_{j_0}) = \|u\|_\infty = s_0$. The other can be treated by the similar method.

Consider the lines

$$y - u(x_{j_0}) = M(t - x_{j_0}), \qquad y - u(x_{j_0}) = -M(t - x_{j_0}).$$

They intersect on the horizontal axis at

$$\left(x_{j_0} - \frac{u(x_{j_0})}{M}, 0\right), \qquad \left(x_{j_0} + \frac{u(x_{j_0})}{M}, 0\right),$$

respectively. Thus, it follows from this and (2.8) that

$$\left(x_{j_0} - \frac{u(x_{j_0})}{M}, x_{j_0} + \frac{u(x_{j_0})}{M}\right) \subset (t_{j_0}, t_{j_0+1}).$$ □

Lemma 2.5. *Let (λ, u) be a S_k^ν-solution with $\|u\|_\infty = s_0$. Let*

$$|u(x_0)| = \max_{t_{j_0} \le x \le t_{j_0+1}} |u(x)|.$$

Then

$$\left[x_0 - \frac{s_0}{M}, x_0 + \frac{s_0}{M}\right] \subset (t_{j_0}, t_{j_0+1}),$$

$$\min\left\{|u(t)| : t \in \left[x_0 - \frac{s_0}{2M}, x_0 + \frac{s_0}{2M}\right]\right\} \ge \frac{1}{2}\|u\|_\infty. \tag{2.9}$$

Proof. We only deal with the case $u(x_0) > 0$. The other case can be treated by the similar way.

Using the fact

$$u(t) \ge u(x_0) + M(t - x_0), \qquad t \in \left[x_0 - \frac{s_0}{2M}, x_0\right],$$
$$u(t) \ge u(x_0) - M(t - x_0), \qquad t \in \left[x_0, x_0 + \frac{s_0}{2M}\right],$$

and the similar argument in the proof of Lemma 2.4, we may get the desired result. □

Definition 2.6. We say that $\alpha \in C^4[a,b]$ is a lower solution of

$$\begin{aligned} y'''' &= g(x, y), \qquad x \in (a,b), \\ y(a) &= y(b) = y'(a) = y'(b) = 0, \end{aligned} \tag{2.10}$$

if

$$\begin{aligned} &\alpha''''(x) \le g(x, \alpha(x)), \\ &\alpha(a) \le 0, \quad \alpha(b) \le 0, \quad \alpha'(a) \le 0, \quad \alpha'(b) \ge 0. \end{aligned} \tag{2.11}$$

We say that $\beta \in C^4[a,b]$ is an upper solution of (2.10) if β satisfies the reversed inequalities of the definition of lower solution.

Let us consider the following inequality that will appear later:

$$g(x, \alpha(x)) \le g(x, u) \le g(x, \beta(x)), \qquad \alpha(x) \le u \le \beta(x). \tag{2.12}$$

Lemma 2.7 (Cabada [3, Theorem 4.2])**.** *Suppose that $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and α, β are respectively a lower and an upper solution of* (2.10)*. If $\alpha \le \beta$ and* (2.12) *holds, then there exists a solution $u(x)$ of* (2.10) *such that*

$$\alpha(x) \le u(x) \le \beta(x), \qquad x \in [a,b].$$

3 Rightward bifurcation

Let μ_k be the k-th eigenvalue of

$$\begin{aligned} y'''' &= \mu h(x) y, \qquad x \in (0,1), \\ y(0) &= y(1) = y'(0) = y'(1) = 0. \end{aligned}$$

Then its corresponding eigenfunction φ_k has exactly $k-1$ simple zeros in $(0,1)$, see Elias [8, Corollary 2 and Theorems 1 and 3].

To state our main result, we need to make the following assumption which will guarantee that any S_k^ν-solution u with $\|u\|_\infty = s_0$ implies $\lambda < \frac{\mu_k}{f_0}$, see the proof of Lemma 4.2 below.

(A5) Let $k \geq 4$ and

$$\frac{\mu_k}{f_0} h_* \min_{|s| \in [\frac{1}{2}s_0, s_0]} \frac{f(s)}{s} > \chi_3,$$

where χ_k is the k-th eigenvalue of

$$\begin{aligned} &y'''' = \chi y, \qquad x \in \left(0, \frac{s_0}{M}\right), \\ &y(0) = y\left(\frac{s_0}{M}\right) = y'(0) = y'\left(\frac{s_0}{M}\right) = 0. \end{aligned} \tag{3.1}$$

Remark 3.1. As we mentioned above, to show the existence of three nodal solutions, we shall employ a bifurcation technique. Indeed, under (A3) we have an unbounded connected component which is bifurcating from μ_k/f_0. Conditions (A1), (A3) and (A4) push the direction of bifurcation to the right near $u = 0$. Since Conditions (A5) and (A4) mean that $f(s)/s$ is large enough in $[s_0/2, s_0]$ and sublinear near ∞, respectively, it is natural to expect that the bifurcation curve (λ, u) grows to the right from the initial point $(\mu_k/f_0, 0)$, to the left at some point and to the right near $\lambda = \infty$.

Arguing the shape of bifurcation we have the following

Theorem 3.2. *Assume that (A1)–(A5) hold. Let $\nu \in \{+, -\}$. Then there exist $\lambda_* \in (0, \mu_k/f_0)$ and $\lambda^* > \mu_k/f_0$, such that*

(i) (1.1) *has at least one S_k^ν-solution if $\lambda = \lambda_*$;*

(ii) (1.1) *has at least two S_k^ν-solutions if $\lambda_* < \lambda \leq \mu_k/f_0$;*

(iii) (1.1) *has at least three S_k^ν-solutions if $\mu_k/f_0 < \lambda < \lambda^*$;*

(iv) (1.1) *has at least two S_k^ν-solutions if $\lambda = \lambda^*$;*

(v) (1.1) *has at least one S_k^ν-solution if $\lambda > \lambda^*$.*

In the rest of this section, we show a global bifurcation phenomena from the trivial branch with the rightward direction of bifurcation. Rewriting (1.1) by

$$\begin{aligned} &u''''(x) = \lambda h(x) f_0 u(x) + \lambda h(x)[f(u(x)) - f_0 u(x)], \qquad x \in (0,1), \\ &u(0) = u(1) = u'(0) = u'(1) = 0, \end{aligned} \tag{3.2}$$

and using Dancer [7, Theorem 2] and following the similar arguments in the proof of [5, Theorem 3.2], we have

Lemma 3.3. *Assume that (A1)–(A4) hold. Then for each $\nu \in \{+, -\}$, there exists an unbounded continuum C_k^ν which is bifurcating from $(\mu_k/f_0, 0)$ for* (1.1). *Moreover, if $(\lambda, u) \in C_k^\nu$, then u is a S_k^ν-solution for* (1.1).

Lemma 3.4. *Assume that (A1)–(A4) hold. Let u be a S_k^ν-solution of* (1.1). *Then there exists a constant $C > 0$ independent of u such that*

$$\|u'\|_\infty \leq \lambda C \|u\|_\infty.$$

Proof.

$$u(x) = \lambda \int_0^1 G(x,s)h(s)f(u(s))ds, \qquad x \in [0,1].$$

The Green function G can be explicitly given by

$$G(x,s) = \frac{1}{6}\begin{cases} x^2(1-s)^2[(s-x)+2(1-x)s], & 0 \le x \le s \le 1, \\ s^2(1-x)^2[(x-s)+2(1-s)x], & 0 \le s \le x \le 1, \end{cases} \tag{3.3}$$

see Cabada and Enguiça [3]. Thus

$$u'(x) = \lambda \int_0^1 G_x(x,s)h(s)f(u(s))ds, \qquad x \in [0,1], \tag{3.4}$$

Noticing that (A3) and (A4) imply that

$$|f(s)| \le f^{\diamond}|s|, \qquad s \in \mathbb{R} \tag{3.5}$$

for some $f^{\diamond} > 0$, it follows from (3.3), (3.4) and the fact

$$G(x,s) \le 1/4, \quad (x,s) \in [0,1]\times[0,1]; \qquad |G_x(x,t)| \le 1, \quad (x,t) \in [0,1]\times[0,1]$$

that

$$|u'(x)| \le \lambda f^{\diamond} \int_0^1 h(t)dt\|u\|_\infty, \qquad x \in [0,1]. \qquad \square$$

By the same method used in the proof of [19, Lemma 3.3], with obvious changes, we may get the following

Lemma 3.5. *Assume that (A1)–(A4) hold. Let (λ_n, u_n) be a sequence of S_k^ν-solutions to (1.1) which satisfies $\|u_n\|_\infty \to 0$ and $\lambda_n \to \mu_k/f_0$. Let $\varphi_k \in S_k^\nu$ be the eigenfunction corresponding to μ_k which satisfies $\|\varphi_k\|_\infty = 1$. Then there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $u_n/\|u_n\|_\infty$ converges uniformly to φ_k on $[0,1]$.*

Lemma 3.6. *Assume that (A1)–(A4) hold. Then there exists $\delta > 0$ such that $(\lambda, u) \in C_k^\nu$ and $|\lambda - \mu_k/f_0| + \|u\|_\infty \le \delta$ imply $\lambda > \mu_k/f_0$.*

Proof. We only deal with the case that $\nu = +$. The other case can be treated by the similar method.

Assume to the contrary that there exists a sequence $\{(\beta_n, u_n)\}$ such that $(\beta_n, u_n) \in C_k^+$, $\beta_n \to \mu_k/f_0$, $\|u_n\|_\infty \to 0$ and $\beta_n \le \mu_k/f_0$. By Lemma 3.5, there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $u_n/\|u_n\|_\infty$ converges uniformly to φ_k on $[0,1]$. Multiplying the equation of (1.1) with $(\lambda, u) = (\beta_n, u_n)$ by u_n and integrating it over $[0,1]$, we obtain

$$\beta_n \int_0^1 h(x)f(u_n(x))u_n(x)dx = \int_0^1 |u_n''(x)|^2dx,$$

and accordingly,

$$\beta_n \int_0^1 h(x)\frac{f(u_n(x))}{\|u_n\|_\infty}\frac{u_n(x)}{\|u_n\|_\infty}dx = \int_0^1 \frac{|u_n''(x)|^2}{\|u_n\|_\infty^2}dx. \tag{3.6}$$

From Lemma 3.5, after taking a subsequence and relabeling if necessary, $u_n/\|u_n\|_\infty$ converges to φ_k in $C[0,1]$.

$$\int_0^1 |\varphi_k''(x)|^2dx = \mu_k \int_0^1 h(x)|\varphi_k(x)|^2dx,$$

it follows that

$$\beta_n \int_0^1 h(x)\frac{f(u_n(x))}{\|u_n\|_\infty}\frac{u_n(x)}{\|u_n\|_\infty}dx = \mu_k \int_0^1 h(x)\frac{|u_n(x)|^2}{\|u_n\|_\infty^2}dx - \zeta(n),$$
$$\beta_n \int_0^1 h(x)f(u_n(x))u_n(x)dx = \mu_k \int_0^1 h(x)|u_n(x)|^2dx - \zeta(n)\|u_n\|_\infty^2$$

with a function $\zeta : \mathbb{N} \to \mathbb{R}$ satisfying

$$\lim_{n\to\infty} \zeta(n) = 0. \tag{3.7}$$

That is

$$\begin{aligned}\int_0^1 h(x)\frac{f(u_n(x)) - f_0 u_n(x)}{|u_n(x)|^\alpha u_n(x)}\left|\frac{u_n(x)}{\|u_n\|_\infty}\right|^{2+\alpha} dx \\ = \frac{\beta_n}{\|u_n\|_\infty^\alpha}\left[\frac{\mu_k - f_0}{\beta_n}\int_0^1 h(x)\left|\frac{u_n(x)}{\|u_n\|_\infty}\right|^2 dx - \zeta(n)\right].\end{aligned} \tag{3.8}$$

Lebesgue's dominated convergence theorem and condition (A3) imply that

$$\int_0^1 h(x)\frac{f(u_n(x)) - f_0 u_n}{|u_n(x)|^\alpha u_n(x)}\left|\frac{u_n(x)}{\|u_n\|_\infty}\right|^{2+\alpha} dx \to -f_1\int_0^1 h(x)|\varphi_k|^{2+\alpha}dx < 0$$

and

$$\int_0^1 h(x)\left|\frac{u_n(x)}{\|u_n\|_\infty}\right|^2 dx \to \int_0^1 h(x)|\varphi_k|^2 dx > 0.$$

This contradicts with $\beta_n \le \mu_k/f_0$. □

4 Direction turn of bifurcation

In this section, we will show that

$$C_k^\nu \cap \{(\lambda, w) : (\lambda, w) \in (\mu_k/f_0, \infty) \times E \text{ with } \|w\|_\infty = s_0\} = \emptyset.$$

In other word, there exists a "barrier strip" for C_k^ν. From Lemmas 2.4–2.5, we obtain

Lemma 4.1. *Assume that (A1)–(A4) hold. Let u be a S_k^ν-solution of (1.1) with $\|u\|_\infty = s_0$. Then there exists $I_u := (\alpha_u, \beta_u)$, such that*

$$\begin{aligned} u(\alpha_u) = u(\beta_u) = 0,& \\ \beta_u - \alpha_u \ge \frac{2s_0}{M},& \\ |u| > 0 \ \text{ in } I_u, \quad \|u\|_\infty = u(t_0) \quad &\text{for some } t_0 \in (\alpha_u, \beta_u). \end{aligned} \tag{4.1}$$

$$\frac{1}{2}\|u\|_\infty \le |u(x)| \le \|u\|_\infty, \qquad x \in \left[x_0 - \frac{s_0}{2M}, x_0 + \frac{s_0}{2M}\right] =: [a, b]. \tag{4.2}$$

Lemma 4.2. *Assume that (A1)–(A5) hold. Let $(\lambda, u) \in C_k^\nu$ be such that $\|u\|_\infty = s_0$. Then $\lambda < \mu_k/f_0$.*

Proof. Let u be a S_k^ν-solution of (1.1) with $\|u\|_\infty = s_0$. By Lemma 4.1,

$$\frac{1}{2}\|u\|_\infty \le |u(x)| \le \|u\|_\infty, \qquad x \in [a, b] = J_u. \tag{4.3}$$

Note that u is a solution of

$$u''''(x) = \lambda h(x)\frac{f(u(x))}{u(x)}u(x), \qquad x \in J_u.$$

Assume on the contrary that

$$\lambda \geq \mu_k/f_0. \tag{4.4}$$

Then for $x \in J_u$, we have from (A5) that

$$\lambda h(x)\frac{f(u(x))}{u(x)} \geq \frac{\mu_k}{f_0}h_* \min_{s\in[s_0/2,s_0]} \frac{f(s)}{s} > \chi_3, \qquad x \in J_u. \tag{4.5}$$

Take

$$\beta(t) := u(t), \qquad t \in J_u;$$
$$\alpha(t) := \epsilon\psi_1(t), \qquad t \in J_u,$$

where ψ_k is the eigenfunction corresponding to the k-th eigenvalue r_k of the problem

$$\begin{aligned} \psi'''' &= r\psi(t), \qquad t \in (a,b), \\ \psi(a) &= \psi(b) = \psi'(a) = \psi'(b) = 0, \end{aligned} \tag{4.6}$$

and $\psi_1(t) > 0$ in (a,b). Since the equations in (3.1) and (4.6) are autonomous,

$$r_1 = \chi_1. \tag{4.7}$$

We claim that

$$\beta'(a) > 0, \qquad \beta'(b) < 0.$$

In fact, let us denote

$$\tilde{\gamma}(x) := \lambda h(x)\frac{f(u(x))}{u(x)} > 0 \quad \text{for } x \in (0,1),$$

and

$$\tilde{\gamma}(0) := \lambda h(0) f_0, \qquad \tilde{\gamma}(1) := \lambda h(1) f_0.$$

Then $\tilde{\gamma} \in C^0[0,1]$ since $f_0 = \lim_{s\to 0} f(s)/s$ exists by (A3). Now, the claim can be easily deduced from Bari and Rynne [2, Lemma 2.1] and Elias [8] and the facts

$$u'''' = \lambda h(x)\frac{f(u(x))}{u(x)}u(x), \qquad x \in (0,1).$$

Obviously, β is an upper solution of

$$\begin{aligned} z''''(x) &= \lambda h(x)\frac{f(u(x))}{u(x)}z(x), \qquad a < x < b, \\ z(a) &= z(b) = z'(a) = z'(b) = 0. \end{aligned} \tag{4.8}$$

From (4.5) and (4.7), it is follows that

$$(\epsilon\psi_1(x))'''' = r_1(\epsilon\psi_1(x)) = \chi_1(\epsilon\psi_1(x)) < \chi_3(\epsilon\psi_1(x)) < \lambda h(x)\frac{f(u(x))}{u(x)}(\epsilon\psi_1(x)), \qquad x \in (a,b).$$

So, α is a lower solution of (4.8).

We may take $\epsilon > 0$ is so small that

$$\alpha(x) \leq \beta(x), \qquad x \in (a,b).$$

Therefore, it follows from Cabada [3, Theorem 4.2] that there exists a solution $y(x)$ of (4.8) such that

$$\alpha(x) \leq y(x) \leq \beta(x). \tag{4.9}$$

On the other hand, $\|y\|_\infty \leq \|u\|_\infty = s_0$ implies that the weight function in (4.8) satisfies

$$\lambda h(x)\frac{f(u(x))}{u(x)} > \chi_3, \qquad x \in (a,b).$$

Combining this with the facts $\psi_3(x-a)$ has exactly two simple zeros in (a,b) and

$$y'''' = \lambda h(x)\frac{f(u(x))}{u(x)}\,y(x), \qquad x \in (a,b),$$

and using Lemma 2.1, it deduces that y has a zero in (a,b). However, this contradicts (4.9). $\square$

5 Second turn and proof of Theorem 3.2

In this section, we shall give a-priori estimate and finalize the proof of Theorem 3.2.

Lemma 5.1. *Assume that (A1)–(A4) hold. Let (λ, u) be a S_k^ν-solution of (1.1). Then there exists $\lambda_* > 0$ such that $\lambda \geq \lambda_*$.*

Proof. Lemma 3.4 implies that (3.2) holds for some constant $C > 0$, which is independent of u. Let $\|u\|_\infty = u(x_0)$. From (3.2) it follows that

$$\|u\|_\infty = |u(x_0)| \leq \int_0^{x_0} |u'(x)|dx \leq \lambda C\|u\|_\infty,$$

that is, $\lambda \geq C^{-1}$. $\square$

Lemma 5.2. *Assume that (A1)–(A4) hold. Let $J = [a_1, b_1]$ be a compact interval in $(0,\infty)$. Then for given $\nu \in \{+,-\}$, there exists $M_J > 0$ such that for all $\lambda \in J$, all possible S_k^ν-solutions u of (1.1) satisfy*

$$\|u\|_\infty \leq M_J. \tag{5.1}$$

Proof. By (A4), we have that for any $\sigma > 0$, there exists $C_\sigma > 0$, such that

$$|f(s)| \leq C_\sigma + \sigma|s|. \tag{5.2}$$

This together with (3.3) imply

$$\begin{aligned}
|u(x)| &= \lambda\left|\int_0^1 G(x,s)h(s)f(u(s))ds\right| \\
&\leq \lambda\left|\int_0^1 G(x,s)h(s)(C_\sigma + \sigma|u(s)|)ds\right| \\
&\leq b_1\left|\int_0^1 G(x,s)h^*(C_\sigma + \sigma|u(s)|)ds\right| \\
&\leq C_1 + \sigma C_2\|u\|_\infty,
\end{aligned} \tag{5.3}$$

where

$$C_1 := b_1 h^* C_\sigma \max\{G(x,s) : (x,s) \in [0,1] \times [0,1]\},$$
$$C_2 := b_1 h^* \max\{G(x,s) : (x,s) \in [0,1] \times [0,1]\}.$$

Take σ so small that $\sigma C_2 < 1$. Then it follows from (5.3) that

$$\|u\|_\infty \le \frac{C_1}{1-\sigma C_2} =: M_J. \qquad \square$$

Lemma 5.3. *Assume that (A1)–(A4) hold. Let C_k^ν be as in Lemma 3.3. Then there exists $\{(\lambda_n, u_n)\}$ such that $(\lambda_n, u_n) \in C_k^\nu$, $\lambda_n \to \infty$ as $n \to \infty$ and $\|u_n\|_\infty \to \infty$.*

Proof. We only deal with the case $\nu = +$. The case $\nu = -$ can be treated by the similar method.

Since C_k^+ is unbounded, there exists $\{(\lambda_n, u_n)\}$ solutions of (1.1) such that $\{(\lambda_n, u_n)\} \subset C_k^+$ and $|\lambda_n| + \|u_n\|_\infty \to \infty$. Lemma 5.1 implies that $\lambda_n > 0$.

Assume on the contrary that there exists sequence $\{(\lambda_n, u_n)\}$ with

$$\|u_n\|_\infty \le M_1, \qquad \forall n \in \mathbb{N}.$$

Then $\lambda_n \to \infty$, and

$$u_n'''' = \lambda_n h(x) \frac{f(u_n)}{u_n} u_n. \tag{5.4}$$

Since

$$h_* \min_{0<s\le M_1} \frac{f(s)}{s} \ge \delta_0 > 0, \tag{5.5}$$

Since $u'''' = 0$ is disconjugate in $[0,1]$ and

$$\lambda_n h(x) \frac{f(u_n)}{u_n} \to \infty \quad \text{uniformly for } x \in [0,1],$$

it follows from the proof of [8, Lemma 4] (see also the remarks in the final paragraph on [8, p. 43], or the proof of Rynne [17, Lemma 3.7]) that u_n has more than k zeros in any given subinterval $I_* \subseteq [0,1]$ if n is large enough. However, this contradicts the fact $u \in S_k^+$. $\square$

Proof of Theorem 3.2. Let C_k^ν be as in Lemma 3.3. We only deal with C_k^+ since the case C_k^- can be treated similarly.

By Lemma 3.6, C_k^+ is bifurcating from $(\mu_k/f_0, 0)$ and goes rightward. Let (λ_n, u_n) be as in Lemma 5.3. Then there exists $(\lambda_0, u_0) \in C_k^+$ such that $\|u_0\|_\infty = s_0$. Lemma 4.2 implies that $\lambda_0 < \mu_k/f_0$.

By Lemmas 3.6, 4.2 and 5.2, it follows that for $\epsilon > 0$ small enough, C_k^+ passes through some points $(\mu_k/f_0 - \epsilon, v_1)$ and $(\mu_k/f_0 + \epsilon, v_2)$ with

$$\|v_1\|_\infty < s_0 < \|v_2\|_\infty.$$

By Lemmas 3.6, 4.2 and 5.2 again, there exist $\underline{\lambda}$ and $\bar{\lambda}$ which satisfy $0 < \underline{\lambda} < \mu_k/f_0 < \bar{\lambda}$ and both (i) and (ii):

(i) if $\lambda \in (\mu_k/f_0, \bar{\lambda}]$, then there exist u and v such that (λ, u), $(\lambda, v) \in C_k^+$ and

$$\|u\|_\infty < \|v\|_\infty < s_0;$$

(ii) if $\underline{\lambda} < \mu_k/f_0$ and $\lambda \in [\underline{\lambda}, \mu_k/f_0]$, then there exist u and v such that (λ, u), $(\lambda, v) \in C_k^+$ and $\|u\|_\infty < s_0 < \|v\|_\infty$.

Define

$$\lambda^* = \sup\{\bar{\lambda} : \bar{\lambda} \text{ satisfies (i)}\}, \qquad \lambda_* = \inf\{\underline{\lambda} : \underline{\lambda} \text{ satisfies (ii)}\}.$$

Then by the standard argument, (1.1) has a S_k^+-solution at $\lambda = \lambda_*$ and $\lambda = \lambda^*$, respectively. Since C_k^+ passes through $(\mu_k/f_0 + \epsilon, v_2)$ and (λ_n, u_n), Lemmas 4.2 and 5.2 show that, for each $\lambda > \mu_k/f_0$, there exists w such that $(\lambda, w) \in C_k^+$ and $\|w\|_\infty > s_0$. This completes the proof. □

Remark 5.4. Let $\rho > 1$ be a positive parameter. Let $g_1 \in C([4,\infty),(0,\infty))$ and $g_2 \in C([1,2],(0,\infty))$ such that

$$g_1(4) = 4\rho + 2, \quad \lim_{|s|\to\infty} \frac{g_1(s)}{s} = 0, \quad g_2(1) = 1, \quad g_2(2) = 2 + 2\rho.$$

Let

$$\hat{f}(s) = \begin{cases} g_1(s), & s \in [4,\infty), \\ \rho s + 2, & s \in [2,4), \\ g_2(s), & s \in (1,2), \\ 2s - s^2, & s \in [0,1], \end{cases}$$

and

$$\tilde{f}(s) = \begin{cases} \hat{f}(s), & s \in [0,\infty), \\ -\hat{f}(-s), & s \in [-\infty, 0). \end{cases}$$

Then $\tilde{f}$ satisfies $(A4)$ and $(A3)$ with $\tilde{f}_0 = 2, \tilde{f}_1 = 1, \alpha = 1$. If we take $s_0 = 4$ and $h(x) \equiv 1$ in $[0,1]$, then $(A5)$ can be rewritten as

$$\frac{\mu_k}{2}\left(\rho + \frac{1}{2}\right) > \chi_3.$$

In order to compute χ_3, we may use (2.5) and (2.7) to find Λ and M, and then use (3.1) to find χ_3. In fact,

$$\chi_3 = \mu_3 \left(\frac{M}{s_0}\right)^4, \qquad \mu_3 \doteq (10.9956)^4 \doteq 14617.6.$$

Therefore, Theorem 3.2 can be used to deal with the case $f = \tilde{f}$ and $h \equiv 1$ if ρ large enough.

Remark 5.5. We may study the oscillating global continua of positive solutions of (1.1) under the conditions

(A6) there exist two positive constant γ^+, γ^- and a sequence $\{\xi_k\} \subset (0,\infty)$ with

$$\xi_{2j-1} < \xi_{2j} < \xi_{2j} < \xi_{2j+1}, \qquad \xi_{2j-1} < \frac{1}{24}\xi_{2j}, \qquad j = 1,2,\ldots; \tag{5.6}$$

such that

$$\frac{f(s)}{s} < \frac{f_0}{(\lambda_1 + \gamma^+ f_0)\int_0^1 \max\{G(t,s) : t \in [0,1]\}h(s)ds}, \qquad s \in (0, \xi_{2j-1}], \tag{5.7}$$

$$\frac{f(s)}{s} > \frac{f_0}{(\lambda_1 - \gamma^- f_0)\eta_0^2 \int_{1/4}^{3/4} \min\{G(t,s) : t \in [1/4,3/4]\}h(s)ds}, \qquad s \in \left[\frac{1}{24}\xi_{2j}, \xi_{2j}\right]. \tag{5.8}$$

Together $f_0 \in (0,\infty)$ with the facts that

$$G(t,s) \geq \frac{1}{24}G(j(s),s), \qquad (t,s) \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

where

$$j(s) = \begin{cases} \frac{1}{3-2s}, & 0 \le s \le \frac{1}{2}, \\ \frac{2s}{1+2s}, & \frac{1}{2} \le s \le 1. \end{cases}$$

By the similar argument in Rynne [16], we may get that for all $\lambda \in \big(\frac{\lambda_1}{f_0} - \gamma^-, \frac{\lambda_1}{f_0} + \gamma^+\big)$, (1.1) has infinitely many positive solutions. Obviously, (5.8) is similar to (A5).

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