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Rectifiability of orbits for two-dimensional nonautonomous differential systems

Dedicated to Professor Hiroyuki Usami on the occasion of his sixtieth birthday

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Abstract. The present study is concerned with the rectifiability of orbits for the two-dimensional nonautonomous differential systems. Criteria are given whether the orbit has a finite length (rectifiable) or not (nonrectifiable). The global attractivity of the zero solution is also discussed. In the linear case, a necessary and sufficient condition can be obtained. Some examples and numerical simulations are presented to explain the results.

Keywords: rectifiability, global attractivity, two-dimensional nonautonomous system.

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1 Introduction

We consider the two-dimensional nonautonomous differential system

$$\begin{aligned} x' &= -e(t)x + f(t)y - p(t)x\left(x^2+y^2\right)^{\lambda}, \\ y' &= -g(t)x - h(t)y - q(t)y\left(x^2+y^2\right)^{\lambda}, \end{aligned} \tag{1.1}$$

where e, f, g, h, p and q are continuous for $t \geq t_0$, and $\lambda > 0$. Since the right hand side of this system is continuously differentiable with respect to (x, y), so it satisfies the Lipschitz condition. Therefore, the local existence and uniqueness of solutions of (1.1) are guaranteed for the initial-value problem. We can show that, for each $t_0 \in \mathbf{R}$ and $(x_0, y_0) \in \mathbf{R}^2$, the initial value problem (1.1) with $(x(t_0), y(t_0)) = (x_0, y_0)$ has a unique solution on $[t_0, \infty)$ under some conditions (this fact will be shown in Lemma 3.4.). We denote it by $(x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$. Clearly, (1.1) has the zero solution $(x(t), y(t)) \equiv (0, 0)$. Throughout this paper, $\|(x, y)\|$ means

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the Euclidean norm of (x,y); that is, $\|(x,y)\| := \sqrt{x^2+y^2}$. Here, let us give a definition about the zero solution of (1.1). The zero solution of (1.1) is said to be *globally attractive* if

$$\lim_{t\to\infty} \|(x(t;t_1,x_0,y_0), y(t;t_1,x_0,y_0))\| = 0$$

for any $t_1 \in [t_0,\infty)$ and any $(x_0,y_0) \in \mathbf{R}^2$. Now rewrite $(x(t;t_0,x_0,y_0), y(t;t_0,x_0,y_0))$ by $(x(t),y(t))$. We define the *orbit* of $(x(t),y(t))$ by

$$\Gamma_{(t_0,x,y)} := \{(x(t),y(t)) \in \mathbf{R}^2 : t \geq t_0\}.$$

The orbit $\Gamma_{(t_0,x,y)}$ is said to be *simple* if $(x(t_1),y(t_1)) \neq (x(t_2),y(t_2))$ for any t_1, $t_2 \in [t_0,\infty)$ with $t_1 \neq t_2$. Now, we assume that the zero solution of (1.1) is globally attractive. The simple orbit $\Gamma_{(t_0,x,y)}$ is said to be *rectifiable* if the length of $\Gamma_{(t_0,x,y)}$ is finite, that is,

$$\lim_{t\to\infty} \int_{t_0}^{t} \|(x'(s),y'(s))\| ds < \infty.$$

Otherwise, it is said to be *nonrectifiable*.

When $\lambda = 1$ and $e(t) = h(t) = a_0$, $f(t) = g(t) = 1$, $p(t) = q(t) = 1$ for all $t \geq t_0$, system (1.1) reduces to the planar nonlinear differential system

$$\begin{aligned} x' &= y - x\left(x^2+y^2+a_0\right), \\ y' &= -x - y\left(x^2+y^2+a_0\right). \end{aligned} \tag{1.2}$$

For every solution $(x(t),y(t))$ of (1.2), using the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$, then we have

$$\begin{aligned} r' &= -r\left(r^2+a_0\right), \\ \theta' &= -1. \end{aligned}$$

From $\theta' = -1$, every orbit $\Gamma_{(t_0,x,y)}$ of (1.2) is rotating in a clockwise direction. Moreover, if we suppose $a_0 \geq 0$, then $r' \leq -r^3$, so that

$$r(t) \leq \frac{1}{\sqrt{2(t-t_0)+r^{-2}(t_0)}} \leq \frac{1}{\sqrt{2(t-t_0)}}$$

for $t \geq t_0$. This says that $a_0 \geq 0$ implies that the zero solution of (1.2) is globally attractive. Hence, every orbit $\Gamma_{(t_0,x,y)}$ of (1.2) is a spiral.

Remark 1.1. Since (1.2) is an autonomous system and $r' \leq -r^3$, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to any nontrivial solution $(x(t),y(t))$ of (1.2) is simple.

Milišić, Žubrinić and Županović [10] studied rectifiability for more general autonomous differential systems based on planar system (1.2). Theorem 8 given in [10] and the above mentioned facts imply the following.

Theorem A. *Let $(x(t),y(t))$ be any nontrivial solution of* (1.2)*. Suppose that $a_0 \geq 0$ holds. Then the zero solution of* (1.2) *is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t),y(t))$ is simple, and (i) and (ii) below hold:*

(i) if $a_0 > 0$, then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if $a_0 = 0$, then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

Remark 1.2. Milišić, Žubrinić and Županović [10] and Žubrinić and Županović [23, 24] dealt with the rectifiability and the fractal analysis of spiral orbits (or trajectories) of some autonomous systems including (1.2). They dealt with more general, but autonomous systems. This study focuses on the rectifiability of the nonautonomous systems.

For simplicity, we denote

$$\begin{aligned} \alpha_1(t) &:= \min\{e(t), h(t)\} - \frac{|f(t)-g(t)|}{2}, \quad \beta_1(t) := \min\{p(t), q(t)\}, \\ \alpha_2(t) &:= \max\{e(t), h(t)\} + \frac{|f(t)-g(t)|}{2}, \quad \beta_2(t) := \max\{p(t), q(t)\}, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \gamma_1(t) &:= -\max\{f(t), g(t)\} - \frac{|e(t)-h(t)|}{2} - \frac{|p(t)-q(t)|}{2}, \\ \gamma_2(t) &:= -\min\{f(t), g(t)\} + \frac{|e(t)-h(t)|}{2} + \frac{|p(t)-q(t)|}{2}. \end{aligned} \tag{1.4}$$

If $e(t) \equiv h(t)$, $f(t) \equiv g(t)$ and $p(t) \equiv q(t)$, then

$$\alpha_1(t) = \alpha_2(t) = e(t), \quad \beta_1(t) = \beta_2(t) = p(t) \quad \text{and} \quad \gamma_1(t) = \gamma_2(t) = -f(t)$$

for $t \geq t_0$. Moreover, for each $c > 0$, we denote

$$\rho_i(t; c) := \exp\left(2\lambda \int_{t_0}^{t} \alpha_i(s)ds\right)\left(c + 2\lambda \int_{t_0}^{t} \beta_i(s) \exp\left(-2\lambda \int_{t_0}^{s} \alpha_i(\tau)d\tau\right) ds\right), \quad i = 1, 2. \tag{1.5}$$

The first main result in this paper is as follows.

Theorem 1.3. *Let $(x(t), y(t))$ be any nontrivial solution of (1.1). Suppose that*

$$\alpha_1(t) \geq 0, \ \beta_1(t) \geq 0 \quad \text{for} \ \ t \geq t_0, \tag{1.6}$$

$$\alpha_1(t) + \beta_1(t) > 0 \quad \text{for} \ \ t \geq t_0, \tag{1.7}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \alpha_1(s)ds = \infty \quad \text{or} \quad \lim_{t\to\infty} \int_{t_0}^{t} \beta_1(s)ds = \infty. \tag{1.8}$$

Then the zero solution of (1.1) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple, and (i), (ii) and (iii) below hold:

(i) if $\alpha_1(t) > 0$ for $t \geq t_0$, and

$$\limsup_{t\to\infty} \frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)} < \infty, \tag{1.9}$$

then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if $0 < \lambda < 1/2$ and

$$\limsup_{t\to\infty} \frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t;c) + \beta_1(t)} < \infty \quad \text{for each} \ \ c > 0, \tag{1.10}$$

then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(iii) if $\lambda \geq 1/2$ *and*

$$\liminf_{t\to\infty} \frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t)\rho_2(t;c) + \beta_2(t)} > 0 \quad \text{for each } c > 0, \tag{1.11}$$

then the orbit $\Gamma_{(t_0,x,y)}$ *is nonrectifiable.*

Using Theorem 1.3 we get the following result, immediately.

Corollary 1.4. *Let* $(x(t), y(t))$ *be any nontrivial solution of* (1.1)*. Let* $(x(t), y(t))$ *be any nontrivial solution of* (1.1)*. Suppose that* (1.6)*,* (1.7) *and* (1.8) *hold. Then the zero solution of* (1.1) *is globally attractive, the orbit* $\Gamma_{(t_0,x,y)}$ *corresponding to* $(x(t), y(t))$ *is simple, and (i), (ii) and (iii) below hold:*

(i) if $\alpha_1(t) > 0$ *for* $t \geq t_0$*, and* (1.9)*, then the orbit* $\Gamma_{(t_0,x,y)}$ *is rectifiable;*

(ii) if $0 < \lambda < 1/2$ *and* $\beta_1(t) > 0$ *for* $t \geq t_0$*, and*

$$\limsup_{t\to\infty} \frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\beta_1(t)} < \infty, \tag{1.12}$$

then the orbit $\Gamma_{(t_0,x,y)}$ *is rectifiable;*

(iii) if $\lambda \geq 1/2$ *and* $\alpha_2(t) = 0$ *for* $t \geq t_0$*, and*

$$\liminf_{t\to\infty} \frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\beta_2(t)} > 0, \tag{1.13}$$

then the orbit $\Gamma_{(t_0,x,y)}$ *is nonrectifiable.*

Corollary 1.4 is expressed in a form which does not include the functions ρ_1 and ρ_2.

If $e(t) = h(t) = a_0 \geq 0$, $f(t) = g(t) = 1$, $p(t) = q(t) = 1$ for all $t \geq t_0$, then system (1.1) reduces to the planar system

$$\begin{aligned} x' &= -a_0 x + y - x\left(x^2 + y^2\right)^{\lambda}, \\ y' &= -x - a_0 y - y\left(x^2 + y^2\right)^{\lambda}. \end{aligned} \tag{1.14}$$

In this case, we know that $\alpha_1(t) = \alpha_2(t) = a_0$, $\beta_1(t) = \beta_2(t) = 1$ and $\gamma_1(t) = \gamma_2(t) = -1$ for all $t \geq t_0$. Then (1.6), (1.7), (1.8), (1.12) and (1.13) hold. If $a_0 > 0$ then (i) in Corollary 1.4 holds. Hence, we get the following result, immediately.

Corollary 1.5. *Let* $(x(t), y(t))$ *be any nontrivial solution of* (1.14)*. Suppose that* $a_0 \geq 0$ *holds. Then the zero solution of* (1.14) *is globally attractive, the orbit* $\Gamma_{(t_0,x,y)}$ *corresponding to* $(x(t), y(t))$ *is simple, and (i), (ii) and (iii) below hold:*

(i) if $a_0 > 0$*, then the orbit* $\Gamma_{(t_0,x,y)}$ *is rectifiable;*

(ii) if $a_0 = 0$ *and* $0 < \lambda < 1/2$*, then the orbit* $\Gamma_{(t_0,x,y)}$ *is rectifiable;*

(iii) if $a_0 = 0$ *and* $\lambda \geq 1/2$*, then the orbit* $\Gamma_{(t_0,x,y)}$ *is nonrectifiable.*

Remark 1.6. From Corollary 1.5, Theorem A is easily obtained.

Figures 1.1–1.4 below show that the orbits corresponding to the nontrivial solution $(x(t), y(t))$ of (1.14) with $(x(0), y(0)) = (0.9, 0)$. We choose a_0 and λ as follows: $a_0 = 0.1$ and $\lambda = 1$ in Fig. 1.1; $a_0 = 0$ and $\lambda = 0.1$ in Fig. 1.2; $a_0 = 0$ and $\lambda = 0.5$ in Fig. 1.3; $a_0 = 0$ and $\lambda = 0.9$ in Fig. 1.4.

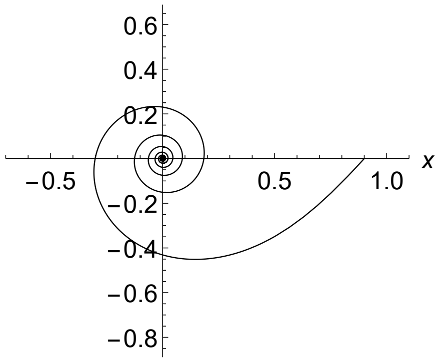

Figure 1.1: $a_0 = 0.1$, $\lambda = 1$; rectifiable.

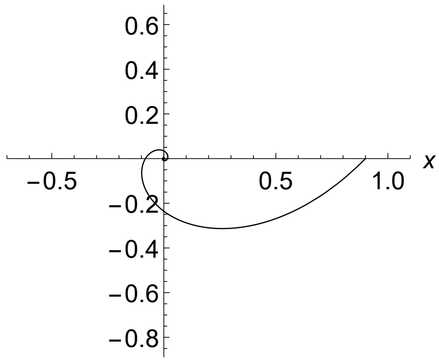

Figure 1.2: $a_0 = 0$, $\lambda = 0.1$; rectifiable.

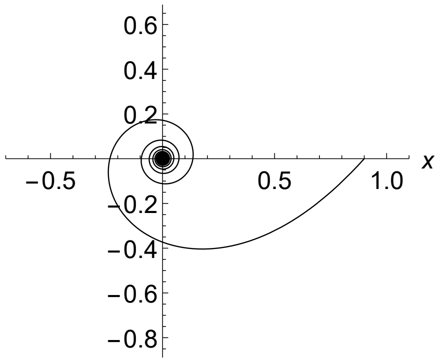

Figure 1.3: $a_0 = 0$, $\lambda = 0.5$; nonrectifiable.

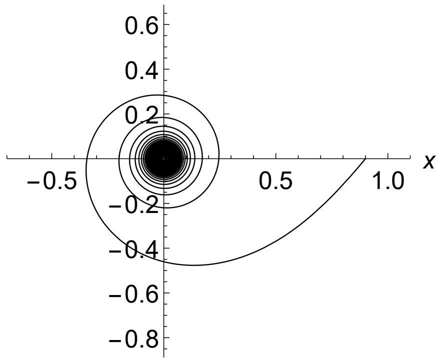

Figure 1.4: $a_0 = 0$, $\lambda = 0.9$; nonrectifiable.

The second main result in this paper is as follows.

Theorem 1.7. *Let $(x(t), y(t))$ be any nontrivial solution of (1.1). Suppose that (1.6), (1.7) and (1.8) hold. Then the zero solution of (1.1) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple, and (i) and (ii) below hold:*

(i) if

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{\sqrt{\left[\alpha_2(s)+\beta_2(s)(\rho_1(s;c))^{-1}\right]^2+(\max\{|\gamma_1(s)|,|\gamma_2(s)|\})^2}}{(\rho_1(s;c))^{\frac{1}{2\lambda}}}ds<\infty \tag{1.15}$$

for each $c > 0$, then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{\sqrt{\left[\alpha_1(s)+\beta_1(s)(\rho_2(s;c))^{-1}\right]^2+(\max\{\gamma_1(s),-\gamma_2(s),0\})^2}}{(\rho_2(s;c))^{\frac{1}{2\lambda}}}ds=\infty \tag{1.16}$$

for each $c > 0$, then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

If $p(t) \equiv q(t) \equiv 0$, then system (1.1) reduces to the two-dimensional linear differential system

$$\begin{aligned} x' &= -e(t)x + f(t)y, \\ y' &= -g(t)x - h(t)y. \end{aligned} \tag{1.17}$$

Note that $\beta_1(t) \equiv \beta_2(t) \equiv 0$. For this linear system, using Theorem 1.7, we obtain the following corollary.

Corollary 1.8. *Let $(x(t), y(t))$ be any nontrivial solution of (1.17). Suppose that*

$$\alpha_1(t) > 0 \quad \textit{for} \quad t \geq t_0,$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \alpha_1(s)ds = \infty.$$

Then the zero solution of (1.17) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple, and (i) and (ii) below hold:

(i) if

$$\lim_{t\to\infty} \int_{t_0}^{t} \sqrt{\alpha_2^2(s) + (\max\{|\gamma_1(s)|, |\gamma_2(s)|\})^2} \exp\left(-\int_{t_0}^{s} \alpha_1(\tau)d\tau\right) ds < \infty,$$

then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if

$$\lim_{t\to\infty} \int_{t_0}^{t} \sqrt{\alpha_1^2(s) + (\max\{\gamma_1(s), -\gamma_2(s), 0\})^2} \exp\left(-\int_{t_0}^{s} \alpha_2(\tau)d\tau\right) ds = \infty,$$

then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

In particular, if $e(t) \equiv h(t)$, $f(t) \equiv g(t)$ then we have the two-dimensional linear differential system

$$\begin{aligned} x' &= -e(t)x + f(t)y, \\ y' &= -f(t)x - e(t)y. \end{aligned} \tag{1.18}$$

In this case, we know that $\alpha_1(t) \equiv \alpha_2(t) \equiv e(t)$, $\beta_1(t) \equiv \beta_2(t) \equiv 0$ and $\gamma_1(t) \equiv \gamma_2(t) \equiv -f(t)$. We can establish the following result by Corollary 1.8.

Corollary 1.9. *Let $(x(t), y(t))$ be any nontrivial solution of (1.18). Suppose that*

$$e(t) > 0 \quad \textit{for} \quad t \geq t_0, \tag{1.19}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} e(s)ds = \infty. \tag{1.20}$$

Then the zero solution of (1.18) is attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple, and the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable if and only if

$$\lim_{t\to\infty} \int_{t_0}^{t} \sqrt{e^2(s) + f^2(s)} \exp\left(-\int_{t_0}^{s} e(\tau)d\tau\right) ds < \infty. \tag{1.21}$$

Remark 1.10. It is well known that the local attractivity and the global attractivity are equivalent in the linear case (see [1, 20–22]). Hence, the attractivity of (1.18) means the global attractivity.

Consider the two-dimensional nonautonomous linear system

$$\begin{aligned} x' &= -\frac{1}{t}x + t^{\sigma}y, \\ y' &= -t^{\sigma}x - \frac{1}{t}y, \end{aligned} \tag{1.22}$$

where $\sigma \in \mathbf{R}$ and $t \geq 1$. Then assumptions (1.19) and (1.20) are easily satisfied. By Corollary 1.9, the zero solution of (1.22) is attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple. Moreover, we can see that the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable if and only if $\sigma < 0$ (The conditions of Corollary 1.9 will be confirmed in Section 5).

Remark 1.11. Our result on the rectifiability of orbits (or trajectories) of (1.22) is the same as one that the special case of the result given by Naito, Pašić and Tanaka [12, Example 5.2]. Note here that they dealt with half-linear systems. On the other hand, as related research, the rectifiability results of the authors [13, 14] can be mentioned, but note that this study has no inclusion relation with them. Moreover, we can find many results on the rectifiability and the fractal analysis of the systems and equations. For example, the reader is referred to [4–7, 9, 11, 15–19].

In the next section, we will discuss the rectifiability for more general systems under the assumption that the zero solution is globally attractive, and the orbit $\Gamma_{(t_0,x,y)}$ is simple. In Section 3, the simplicity and the global attractivity for (1.1) are considered. In Section 4, we prove Theorems 1.3 and 1.7. In Section 5, some examples and numerical simulations are presented.

2 Rectifiability

In this section, we consider the two-dimensional nonautonomous differential system

$$\begin{aligned} x' &= F_1(t, x, y), \\ y' &= F_2(t, x, y), \end{aligned} \tag{2.1}$$

where F_1 and F_2 are continuously differentiable with respect to (x, y), and satisfying

$$(F_1(t, 0, 0), F_2(t, 0, 0)) \equiv (0, 0).$$

For every solution $(x(t), y(t))$ of (2.1), we introduce the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$. Then we obtain

$$\begin{aligned} r' &= G_1(t, r, \theta), \\ r\theta' &= G_2(t, r, \theta), \end{aligned} \tag{2.2}$$

where G_1 and G_2 are defined by

$$G_1(t, r, \theta) = \cos\theta F_1(t, r\cos\theta, r\sin\theta) + \sin\theta F_2(t, r\cos\theta, r\sin\theta) \tag{2.3}$$

and

$$G_2(t, r, \theta) = \cos\theta F_2(t, r\cos\theta, r\sin\theta) - \sin\theta F_1(t, r\cos\theta, r\sin\theta). \tag{2.4}$$

The obtained result is as follows.

Theorem 2.1. *Let G_1 and G_2 be the functions given by* (2.3) *and* (2.4)*, respectively. Let $(x(t), y(t))$ be any nontrivial solution of* (2.1) *on $[t_0, \infty)$. Suppose that the zero solution of* (2.1) *is globally attractive, and the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple. Then, (i) and (ii) below hold:*

(i) if there exist an $\bar{r} > 0$ and a continuous function $h : (0, \bar{r}) \to (0, \infty)$ such that

$$\|(G_1(t, r, \theta), G_2(t, r, \theta))\| \le -h(r)G_1(t, r, \theta), \quad (t, r, \theta) \in [t_0, \infty) \times (0, \bar{r}) \times \mathbf{R}, \tag{2.5}$$

and

$$\lim_{r\to+0} \int_r^{\bar{r}} h(\eta)d\eta < \infty, \tag{2.6}$$

then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if there exist an $\bar{r} > 0$ and a continuous function $h : (0, \bar{r}) \to (0, \infty)$ such that

$$\|(G_1(t, r, \theta), G_2(t, r, \theta))\| \ge -h(r)G_1(t, r, \theta), \quad (t, r, \theta) \in [t_0, \infty) \times (0, \bar{r}) \times \mathbf{R}, \tag{2.7}$$

and

$$\lim_{r\to+0} \int_r^{\bar{r}} h(\eta)d\eta = \infty, \tag{2.8}$$

then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

Proof. Let $(x(t), y(t))$ be any nontrivial solution of (2.1). Define the functions r and θ by

$$x(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t)$$

for $t \ge t_0$, where

$$r(t) = \|(x(t), y(t))\|.$$

Then $(r(t), \theta(t))$ is a solution to (2.2). Since the existence and uniqueness of solutions of (2.1) are guaranteed for the initial-value problem, the zero solution $(x(t), y(t)) \equiv (0, 0)$ is unique. Thus, $r(t) > 0$ for $t \ge t_0$. This together with the global attractivity of (2.1) implies that $\lim_{t\to\infty} r(t) = 0$, and there exists a $T > 0$ such that

$$r(t) \in (0, \bar{r}) \tag{2.9}$$

for $t \ge t_0 + T$.

Now, we consider case (i). Using (2.5) and (2.9), we have

$$\begin{aligned}
\|(x'(t), y'(t))\| &= \|(F_1(t, x(t), y(t)), F_2(t, x(t), y(t)))\| \\
&= \sqrt{(\cos\theta F_1 + \sin\theta F_2)^2 + (\cos\theta F_2 - \sin\theta F_1)^2} \\
&= \|(G_1(t, r(t), \theta(t)), G_2(t, r(t), \theta(t)))\| \\
&\le -h(r(t))G_1(t, r(t), \theta(t)) = -h(r(t))r'(t)
\end{aligned}$$

for $t \ge t_0 + T$. Since $h(r)$ is a positive continuous function on $(0, \bar{r})$, and (2.9) holds, we see that

$$\begin{aligned}
\int_{t_0+T}^{t} \|(x'(s), y'(s))\|ds &\le -\int_{t_0+T}^{t} h(r(s))r'(s)ds = \int_{r(t)}^{r(t_0+T)} h(\eta)d\eta \\
&\le \int_{r(t)}^{\bar{r}} h(\eta)d\eta
\end{aligned}$$

for $t \geq t_0 + T$. Therefore, we have

$$\begin{aligned}\int_{t_0}^{t} \|(x'(s), y'(s))\| ds &= \int_{t_0}^{t_0+T} \|(x'(s), y'(s))\| ds + \int_{t_0+T}^{t} \|(x'(s), y'(s))\| ds \\ &\leq \int_{t_0}^{t_0+T} \|(x'(s), y'(s))\| ds + \int_{r(t)}^{\bar{r}} h(\eta) d\eta\end{aligned}$$

for $t \geq t_0 + T$. Using (2.6), (2.9) with $\lim_{t\to\infty} r(t) = 0$, we conclude that

$$\lim_{t\to\infty} \int_{t_0}^{t} \|(x'(s), y'(s))\| ds < \infty.$$

Hence, the simple orbit $\Gamma_{(t_0,x,y)}$ is rectifiable.

Next, we consider case (ii). From (2.7) and (2.9), we have

$$\|(x'(t), y'(t))\| \geq -h(r(t))G_1(t, r(t), \theta(t)) = -h(r(t))r'(t)$$

for $t \geq t_0 + T$. Since $h(r)$ is a positive continuous function on $(0, \bar{r})$, and (2.9) holds, we see that

$$\begin{aligned}\int_{t_0}^{t} \|(x'(s), y'(s))\| ds &\geq \int_{t_0+T}^{t} \|(x'(s), y'(s))\| ds \\ &\geq -\int_{t_0+T}^{t} h(r(s))r'(s) ds = \int_{r(t)}^{r(t_0+T)} h(\eta) d\eta \\ &= \int_{r(t)}^{\bar{r}} h(\eta) d\eta - \int_{r(t_0+T)}^{\bar{r}} h(\eta) d\eta\end{aligned}$$

for $t \geq t_0 + T$. From (2.8), (2.9) with $\lim_{t\to\infty} r(t) = 0$, we get

$$\lim_{t\to\infty} \int_{t_0}^{t} \|(x'(s), y'(s))\| ds = \infty.$$

Consequently, the simple orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable. This completes the proof. □

For our main system (1.1), we find that

$$\begin{aligned}F_1(t, x, y) &= -e(t)x + f(t)y - p(t)x \left(x^2 + y^2\right)^{\lambda}, \\ F_2(t, x, y) &= -g(t)x - h(t)y - q(t)y \left(x^2 + y^2\right)^{\lambda},\end{aligned}$$

and

$$\begin{aligned}G_1(t, r, \theta) = &-\left(e(t)\cos^2\theta + h(t)\sin^2\theta\right) r + (f(t) - g(t)) r \sin\theta\cos\theta \\ &- \left(p(t)\cos^2\theta + q(t)\sin^2\theta\right) r^{2\lambda+1}, \\ G_2(t, r, \theta) = &-\left(g(t)\cos^2\theta + f(t)\sin^2\theta\right) r + (e(t) - h(t)) r \sin\theta\cos\theta \\ &+ (p(t) - q(t)) r^{2\lambda+1} \sin\theta\cos\theta.\end{aligned} \tag{2.10}$$

3 Simplicity and global attractivity

In this section, we deal with the simplicity and the global attractivity for our main system (1.1). First, we give two lemmas.

Lemma 3.1. *Let G_1 be the function given in (2.10). Then*

$$G_1(t,r,\theta) \le -\left(\alpha_1(t) + \beta_1(t)r^{2\lambda}\right) r$$

holds for $t \ge t_0$ and $r \in [0,\infty)$, where α_1 and β_1 are given in (1.3).

Proof. By (2.10), we get

$$\begin{aligned} G_1(t,r,\theta) &\le -\min\{e(t),h(t)\}r + \frac{|f(t)-g(t)|}{2}r - \min\{p(t),q(t)\}r^{2\lambda+1} \\ &= -\left(\alpha_1(t) + \beta_1(t)r^{2\lambda}\right) r \end{aligned}$$

for $t \ge t_0$ and $r \in [0,\infty)$. □

Lemma 3.2. *Suppose that (1.6) and (1.7) hold. Then*

$$\left(\alpha_1(t) + \beta_1(t)r^{2\lambda}\right) r > 0$$

holds for $t \ge t_0$ and $r \in (0,\infty)$, where α_1 and β_1 are given in (1.3).

Proof. By way of contradiction, we suppose that there exists a $t_1 \ge t_0$ such that

$$\left(\alpha_1(t_1) + \beta_1(t_1)r^{2\lambda}\right) r \le 0.$$

From (1.6) and $r \in (0,\infty)$, we have

$$\alpha_1(t_1) + \beta_1(t_1)r^{2\lambda} = 0.$$

This together with (1.6) says that $\alpha_1(t_1) = \beta_1(t_1) = 0$. However, this contradicts assumption (1.7). □

We now consider the simplicity of the nontrivial solutions to (1.1). The obtained result is as follows.

Lemma 3.3. *Let $(x(t),y(t))$ be a nontrivial solution of (1.1). Suppose that (1.6) and (1.7) hold. Then the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t),y(t))$ is simple.*

Proof. Let $(x(t),y(t))$ be a nontrivial solution of (1.1). Assume to the contrary that there exist t_1, $t_2 \in [t_0,\infty)$ such that $t_1 < t_2$ with $(x(t_1),y(t_1)) = (x(t_2),y(t_2))$. Let $(r(t),\theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t),y(t))$. Then $r(t_1) = r(t_2)$ holds. Since $(x(t),y(t))$ is a nontrivial solution and the zero solution is unique, we know that $r(t) > 0$ for all $t \ge t_0$. From Lemmas 3.1 and 3.2, we see that $r'(t) < 0$ for $t \ge t_0$. Integrating this inequality from t_1 to t_2, we obtain

$$r(t_2) - r(t_1) = \int_{t_1}^{t_2} r'(t)dt < 0.$$

This is a contradiction. Consequently, $\Gamma_{(t_0,x,y)}$ is a simple orbit. □

We will give an important inequality.

Lemma 3.4. *Let $(x(t), y(t))$ be a nontrivial solution of (1.1) with the initial condition $(x(t_0), y(t_0)) = (x_0, y_0)$. Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Suppose that $\beta_1(t) \geq 0$ holds for $t \geq t_0$. Then $(x(t), y(t))$ exists on $[t_0, \infty)$ and is the unique solution of (1.1) with $(x(t_0), y(t_0)) = (x_0, y_0)$, and the inequality*

$$0 < r(t) \leq \exp\left(-\int_{t_0}^{t} \alpha_1(s)ds\right)\left(r^{-2\lambda}(t_0) + 2\lambda\int_{t_0}^{t} \beta_1(s)\exp\left(-2\lambda\int_{t_0}^{s}\alpha_1(\tau)d\tau\right)ds\right)^{-\frac{1}{2\lambda}} \tag{3.1}$$

holds for $t \geq t_0$, where α_1 and β_1 are given in (1.3).

Proof. Let $(x(t), y(t))$ be a nontrivial solution of (1.1) with $(x(t_0), y(t_0)) = (x_0, y_0)$. Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Let $I \subset [t_0, \infty)$ be the maximal interval of the existence of $(x(t), y(t))$. Then $r(t) > 0$ holds for $t \in I$, from the uniqueness of the zero solution. Using Lemma 3.1, we have

$$r'(t) \leq -\left(\alpha_1(t) + \beta_1(t)r^{2\lambda}(t)\right)r(t)$$

for $t \in I$. Set $z(t) := r^{-2\lambda}(t)$. Then, it follows from the above inequality and $r(t) > 0$ that

$$z'(t) = -2\lambda r^{-2\lambda-1}(t)r'(t) \geq 2\lambda r^{-2\lambda}(t)\left(\alpha_1(t) + \beta_1(t)r^{2\lambda}(t)\right) = 2\lambda\alpha_1(t)z(t) + 2\lambda\beta_1(t)$$

for $t \in I$. Hence

$$\left(\exp\left(-2\lambda\int_{t_0}^{t}\alpha_1(s)ds\right)z(t)\right)' \geq 2\lambda\beta_1(t)\exp\left(-2\lambda\int_{t_0}^{t}\alpha_1(s)ds\right)$$

for $t \in I$. Integrating this inequality from t_0 to t, we get

$$\exp\left(-2\lambda\int_{t_0}^{t}\alpha_1(s)ds\right)z(t) \geq z(t_0) + 2\lambda\int_{t_0}^{t}\beta_1(s)\exp\left(-2\lambda\int_{t_0}^{s}\alpha_1(\tau)d\tau\right)ds,$$

and so that

$$r^{-2\lambda}(t) = z(t) \geq \exp\left(2\lambda\int_{t_0}^{t}\alpha_1(s)ds\right)\left(r^{-2\lambda}(t_0) + 2\lambda\int_{t_0}^{t}\beta_1(s)\exp\left(-2\lambda\int_{t_0}^{s}\alpha_1(\tau)d\tau\right)ds\right)$$

for $t \in I$. Therefore, if $\beta_1(t) \geq 0$ for $t \geq t_0$, then we obtain (3.1) for $t \in I$.

Using the above inequality and $\beta_1(t) \geq 0$ for $t \geq t_0$, we have

$$r^{-2\lambda}(t) \geq \exp\left(2\lambda\int_{t_0}^{t}\alpha_1(s)ds\right)r^{-2\lambda}(t_0),$$

and thus,

$$0 < \|(x(t), y(t))\| \leq \|(x_0, y_0)\|\exp\left(-\int_{t_0}^{t}\alpha_1(s)ds\right) \quad \text{for } t \in I. \tag{3.2}$$

This inequality means that $I = [t_0, \infty)$, that is, any nontrivial solution of (1.1) exists on $[t_0, \infty)$ by a standard argument of a general theory on ordinary differential equations. Consequently, the initial value problem (1.1) with $(x(t_0), y(t_0)) = (x_0, y_0)$ has a unique solution on $[t_0, \infty)$. □

Next, we consider the global attractivity for (1.1). Assuming a stronger condition, we can get stronger stability. The zero solution is said to be *globally exponentially stable* if there exists a $k > 0$ and, for any $\eta > 0$, there exists a $\delta(\eta) > 0$ such that $t_1 \in \mathbf{R}$ with $t_1 \geq t_0$ and $\|(x_0, y_0)\| < \eta$ imply

$$\|(x(t; t_1, x_0, y_0), y(t; t_1, x_0, y_0))\| \leq \delta(\eta)\|(x_0, y_0)\|e^{-k(t-t_1)}$$

for all $t \geq t_1$. The following lemma is established.

Lemma 3.5. *Suppose that* (1.6) *and* (1.8) *hold, where* α_1 *and* β_1 *are given in* (1.3)*. Then the zero solution of* (1.1) *is globally attractive. In particular, if there exists an* $\underline{a} > 0$ *such that*

$$\alpha_1(s) \geq \underline{a} \quad \textit{for } t \geq t_0, \tag{3.3}$$

then the zero solution of (1.1) *is globally exponentially stable.*

Proof. Let t_1 satisfy $t_1 \geq t_0$. Let $(x(t), y(t))$ be any nontrivial solution of (1.1) with $(x(t_1), y(t_1)) = (x_0, y_0)$. Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Using Lemma 3.4, we have inequality (3.1) for $t \geq t_1$.

Now we consider the case $\lim_{t\to\infty} \int_{t_0}^t \alpha_1(s)ds < \infty$. This together with (1.8) yields

$$\lim_{t\to\infty} \int_{t_0}^t \beta_1(s)ds = \infty.$$

Let $L := \lim_{t\to\infty} \int_{t_0}^t \alpha_1(s)ds \geq 0$. Using this and (3.1), we obtain

$$0 < \|(x(t), y(t))\| = r(t) \leq \frac{1}{\left(r^{-2\lambda}(t_1) + 2\lambda e^{-2\lambda L} \int_{t_1}^t \beta_1(s)ds\right)^{\frac{1}{2\lambda}}} < \frac{1}{\left(2\lambda e^{-2\lambda L} \int_{t_1}^t \beta_1(s)ds\right)^{\frac{1}{2\lambda}}}$$

for $t \geq t_1$. Hence, any nontrivial solution of (1.1) tends to $(0, 0)$ as $t \to \infty$. That is, the zero solution of (1.1) is globally attractive.

Next we consider the case $\lim_{t\to\infty} \int_{t_0}^t \alpha_1(s)ds = \infty$. Then, by assumption (1.6), we obtain inequality (3.2). Therefore, the zero solution of (1.1) is globally attractive. Moreover, if we suppose condition (3.3), then inequality (3.2) implies global exponential stability. This completes the proof. □

Remark 3.6. If $\alpha_1(t) \equiv 0$ then, it does not imply the (global) exponential stability for (1.1). For example, we consider the case $\lambda = 1$, $e(t) = h(t) = 0$ and $f(t) = g(t) = 1$ and $p(t) = q(t) = 1$ for $t \geq t_0$. That is, $\alpha_1(t) = \alpha_2(t) = 0$, $\beta_1(t) = \beta_2(t) = 1$ and $\gamma_1(t) = \gamma_2(t) = -1$ for $t \geq t_0$. From (2.2) and (2.10), we have

$$r' = -r^3.$$

Solving this equation, we get

$$r(t) = \frac{1}{\sqrt{2(t - t_0) + r^{-2}(t_0)}}$$

for $t \geq t_0$. Thus, the zero solution is not exponentially stable. Although not described here in detail, we can see that the zero solution of this system is uniformly asymptotically stable. It is well known that the exponential stability implies the uniform asymptotic stability; the uniform asymptotic stability implies the asymptotic stability (the zero solution is attractive and stable). If (1.1) is a periodic or autonomous system, then the asymptotic stability and the uniform asymptotic stability are equivalent. For example, see [2, 3, 8, 21, 22]. Moreover, if (1.1) is a linear system, the uniform asymptotic stability and the exponential stability are equivalent. For example, the reader is referred to [3, 21, 22] and the references cited therein. In general, our main equations are nonautonomous and nonlinear, so their stabilities are often different.

4 Proofs of the main theorems

Before proving the main theorems, we give three lemmas.

Lemma 4.1. *Let G_1 be the function given in (2.10). Then*

$$G_1(t,r,\theta) \geq -\left(\alpha_2(t) + \beta_2(t)r^{2\lambda}\right) r \tag{4.1}$$

holds for $t \geq t_0$ and $r \in [0, \infty)$, where α_2 and β_2 are given in (1.3).

Proof. By (2.10), we get

$$\begin{aligned} G_1(t,r,\theta) &\geq -\max\{e(t), h(t)\}r - \frac{|f(t) - g(t)|}{2}r - \max\{p(t), q(t)\}r^{2\lambda+1} \\ &= -\left(\alpha_2(t) + \beta_2(t)r^{2\lambda}\right) r \end{aligned}$$

for $t \geq t_0$ and $r \in [0, \infty)$. □

Lemma 4.2. *Let $(x(t), y(t))$ be any nontrivial solution of (1.1). Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Suppose that $\beta_2(t) \geq 0$ holds for $t \geq t_0$. Then the inequality*

$$r(t) \geq \exp\left(-\int_{t_0}^{t} \alpha_2(s)ds\right) \left(r^{-2\lambda}(t_0) + 2\lambda \int_{t_0}^{t} \beta_2(s) \exp\left(-2\lambda \int_{t_0}^{s} \alpha_2(\tau)d\tau\right) ds\right)^{-\frac{1}{2\lambda}} \tag{4.2}$$

holds for $t \geq t_0$, where α_2 and β_2 are given in (1.3).

Proof. Let $(x(t), y(t))$ be any nontrivial solution of (1.1). Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Using Lemma 4.1, we have

$$r'(t) \geq -\left(\alpha_2(t) + \beta_2(t)r^{2\lambda}(t)\right) r(t)$$

for $t \geq t_0$. Set $z(t) := r^{-2\lambda}(t)$. Then, it follows from the above inequality and $r(t) > 0$ that

$$z'(t) \leq 2\lambda\alpha_2(t)z(t) + 2\lambda\beta_2(t)$$

for $t \geq t_0$. Hence

$$\left(\exp\left(-2\lambda \int_{t_0}^{t} \alpha_2(s)ds\right) z(t)\right)' \leq 2\lambda\beta_2(t) \exp\left(-2\lambda \int_{t_0}^{t} \alpha_2(s)ds\right)$$

for $t \geq t_0$. Integrating this inequality from t_0 to t, we get

$$r^{-2\lambda}(t) \leq \exp\left(2\lambda \int_{t_0}^{t} \alpha_2(s)ds\right) \left(r^{-2\lambda}(t_0) + 2\lambda \int_{t_0}^{t} \beta_2(s) \exp\left(-2\lambda \int_{t_0}^{s} \alpha_2(\tau)d\tau\right) ds\right)$$

for $t \geq t_0$. Therefore, if $\beta_2(t) \geq 0$ for $t \geq t_0$, then we obtain the inequality in Lemma 4.2. □

Lemma 4.3. *Let G_2 be the function given in (2.10). Then*

$$\gamma_1(t)r \leq G_2(t,r,\theta) \leq \gamma_2(t)r \tag{4.3}$$

holds for $t \geq t_0$ and $r \in [0, 1)$, where γ_1 and γ_2 are given by (1.4).

Proof. By (2.10) and $r \in [0,1)$, we obtain

$$\begin{aligned} G_2(t,r,\theta) &\geq -\max\{f(t),g(t)\}r - \frac{|e(t)-h(t)|}{2}r - \frac{|p(t)-q(t)|}{2}r^{2\lambda+1} \\ &\geq \gamma_1(t)r \end{aligned}$$

and

$$\begin{aligned} G_2(t,r,\theta) &\leq -\min\{f(t),g(t)\}r + \frac{|e(t)-h(t)|}{2}r + \frac{|p(t)-q(t)|}{2}r^{2\lambda+1} \\ &\leq \gamma_2(t)r. \end{aligned}$$

Thus, (4.3) holds. □

Now, we will prove the main theorems.

Proof of Theorem 1.3. From Lemma 3.5, the zero solution of (1.1) is globally attractive. Let $(x(t),y(t))$ be any nontrivial solution of (1.1). By Lemma 3.3, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t),y(t))$ is simple. Let $x = r\cos\theta$, $y = r\sin\theta$. Then we have (2.3) and (2.4). By Lemmas 3.1, 3.2, 4.1 and 4.3, the inequalities

$$0 < \left(\alpha_1(t) + \beta_1(t)r^{2\lambda}\right) r \leq |G_1(t,r,\theta)| = -G_1(t,r,\theta) \leq \left(\alpha_2(t) + \beta_2(t)r^{2\lambda}\right) r, \tag{4.4}$$

and

$$\max\{\gamma_1(t), -\gamma_2(t), 0\}r \leq |G_2(t,r,\theta)| \leq \max\{|\gamma_1(t)|, |\gamma_2(t)|\}r \tag{4.5}$$

hold for $t \geq t_0$ and $r \in (0,1)$. Therefore, we obtain

$$\frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t) + \beta_2(t)r^{2\lambda}} \leq \left|\frac{G_2(t,r,\theta)}{G_1(t,r,\theta)}\right| \leq \frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t) + \beta_1(t)r^{2\lambda}} \tag{4.6}$$

for $t \geq t_0$ and $r \in (0,1)$.

First, we consider case (i). Suppose that $\alpha_1(t) > 0$ for $t \geq t_0$, and (1.9), that is, there exists a $\mu > 0$ and a $t_1 \geq t_0$ such that

$$\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)} \leq \mu$$

holds for $t \geq t_1$. By (1.6), $\beta_1(t) \geq 0$ for $t \geq t_0$. This together with the above inequality implies

$$\sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t) + \beta_1(t)r^{2\lambda}}\right)^2} \leq \sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)}\right)^2} \leq \sqrt{1+\mu^2}$$

for $t \geq t_1$. Moreover, we can choose an $M_1 \geq \sqrt{1+\mu^2}$ such that

$$\sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t) + \beta_1(t)r^{2\lambda}}\right)^2} \leq M_1$$

for $t_0 \leq t \leq t_1$. Using these inequalities and (4.6), we have

$$\|(G_1(t,r,\theta), G_2(t,r,\theta))\| \leq -\sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t) + \beta_1(t)r^{2\lambda}}\right)^2} G_1(t,r,\theta) \leq -M_1 G_1(t,r,\theta)$$

for $t \geq t_0$ and $r \in (0,1)$, so that we get (2.5) with $\bar{r} = 1$ and $h(r) = M_1$. By

$$\lim_{r \to +0} \int_r^1 h(\eta) d\eta = M_1,$$

we have (2.6). Consequently, the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable.

Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Before proving cases (ii) and (iii), we will discuss some properties of $r(t)$. By the global attractivity for (1.1), there exits a $t_1 \geq t_0$ such that

$$0 < r(t) \leq 1 \quad \text{for} \quad t \geq t_1.$$

From Lemmas 3.4 and 4.2, we have

$$\rho_1(t; c_0) \leq r^{-2\lambda}(t) \leq \rho_2(t; c_0) \quad \text{for some} \quad c_0 > 0, \tag{4.7}$$

and for $t \geq t_0$, where ρ_1 and ρ_2 are given by (1.5). This together with (4.6) implies that

$$\frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t)\rho_2(t; c_0) + \beta_2(t)} r^{-2\lambda}(t) \leq \left| \frac{G_2(t, r(t), \theta(t))}{G_1(t, r(t), \theta(t))} \right| \leq \frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t; c_0) + \beta_1(t)} r^{-2\lambda}(t) \tag{4.8}$$

for $t \geq t_1$.

Now, we consider case (ii). Suppose that $0 < \lambda < 1/2$ and (1.10) hold, that is, there exists a $\mu > 0$ and a $t_2 \geq t_1$ such that

$$\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t; c) + \beta_1(t)} \leq \mu$$

holds for $t \geq t_2$. By (4.8), we have

$$\begin{aligned}
\sqrt{1 + \left| \frac{G_2(t, r(t), \theta(t))}{G_1(t, r(t), \theta(t))} \right|^2} &\leq \sqrt{r^{4\lambda}(t) + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t; c_0) + \beta_1(t)} \right)^2} r^{-2\lambda}(t) \\
&\leq \sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t; c_0) + \beta_1(t)} \right)^2} r^{-2\lambda}(t) \\
&\leq \sqrt{1 + \mu^2 r^{-2\lambda}(t)}
\end{aligned}$$

for $t \geq t_2$. Moreover, we can choose an $M_2 \geq \sqrt{1 + \mu^2}$ such that

$$\sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t; c_0) + \beta_1(t)} \right)^2} \leq M_2$$

for $t_0 \leq t \leq t_2$. Therefore, we see that

$$\begin{aligned}
\|(x'(t), y'(t))\| = \|(F_1(t, x(t), y(t)), F_2(t, x(t), y(t)))\| &= \|(G_1(t, r(t), \theta(t)), G_2(t, r(t), \theta(t)))\| \\
&= \sqrt{1 + \left| \frac{G_2(t, r(t), \theta(t))}{G_1(t, r(t), \theta(t))} \right|^2} |G_1(t, r(t), \theta(t))| \\
&\leq M_2 r^{-2\lambda}(t) |G_1(t, r(t), \theta(t))| = -M_2 r^{-2\lambda}(t) r'(t)
\end{aligned}$$

holds for $t \geq t_0$. Integrating this inequality, we obtain

$$\int_{t_0}^t \|(x'(s), y'(s))\| ds \leq M_2 \int_{r(t)}^{r(t_0)} \eta^{-2\lambda} d\eta = \frac{M_2}{1 - 2\lambda} \left(r^{1-2\lambda}(t_0) - r^{1-2\lambda}(t) \right) < \frac{M_2 r^{1-2\lambda}(t_0)}{1 - 2\lambda}$$

for $t \geq t_0$. Hence, we conclude that the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable.

Finally, we consider case (iii). Suppose that $\lambda \geq 1/2$ and (1.11) hold, that is, there exists a $\nu > 0$ and a $t_2 \geq t_1$ such that

$$\frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t)\rho_2(t;c) + \beta_2(t)} \geq \nu$$

holds for $t \geq t_2$. By (4.8), we have

$$\sqrt{1 + \left|\frac{G_2(t,r(t),\theta(t))}{G_1(t,r(t),\theta(t))}\right|^2} > \left|\frac{G_2(t,r(t),\theta(t))}{G_1(t,r(t),\theta(t))}\right| \geq \frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t)\rho_2(t;c_0) + \beta_2(t)} r^{-2\lambda}(t) \geq \nu r^{-2\lambda}(t)$$

for $t \geq t_2$. From this, we see that

$$\begin{aligned} \|(x'(t), y'(t))\| &= \sqrt{1 + \left|\frac{G_2(t,r(t),\theta(t))}{G_1(t,r(t),\theta(t))}\right|^2} |G_1(t,r(t),\theta(t))| \\ &> \nu r^{-2\lambda}(t)|G_1(t,r(t),\theta(t))| = -\nu r^{-2\lambda}(t) r'(t) \end{aligned} \tag{4.9}$$

for $t \geq t_2$. Now, we consider the case $\lambda = 1/2$. Integrating (4.9), we obtain

$$\int_{t_0}^{t} \|(x'(s), y'(s))\| ds \geq -\nu \int_{r(t_2)}^{r(t)} \eta^{-1} d\eta = -\nu \log \frac{r(t)}{r(t_2)}$$

for $t \geq t_2$. Since the zero solution of (1.1) is globally attractive, we conclude that the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable. On the other hand, we consider the case $\lambda > 1/2$. Integrating (4.9), we obtain

$$\int_{t_0}^{t} \|(x'(s), y'(s))\| ds \geq -\nu \int_{r(t_2)}^{r(t)} \eta^{-2\lambda} d\eta = \frac{\nu}{2\lambda - 1}\left(\frac{1}{r^{2\lambda-1}(t)} - \frac{1}{r^{2\lambda-1}(t_2)}\right)$$

for $t \geq t_2$. Consequently, $\Gamma_{(t_0,x,y)}$ is nonrectifiable. This completes the proof of Theorem 1.3. □

Proof of Theorem 1.7. Let $(x(t), y(t))$ be any nontrivial solution of (1.1). From Lemmas 3.3 and 3.5, the zero solution of (1.1) is globally attractive, and the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple. Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Then the global attractivity for (1.1) implies that there exits a $t_1 \geq t_0$ such that

$$0 < r(t) < 1 \quad \text{for} \quad t \geq t_1.$$

From Lemmas 3.4 and 4.2, we have (4.7) for $t \geq t_0$. Using Lemmas 3.1, 3.2, 4.1 and 4.3, we get inequalities (4.4) and (4.5) for $t \geq t_0$ and $r \in (0,1)$. Therefore,

$$\|(G_1(t,r,\theta), G_2(t,r,\theta))\| \leq \sqrt{(\alpha_2(t) + \beta_2(t)r^{2\lambda})^2 + (\max\{|\gamma_1(t)|, |\gamma_2(t)|\})^2 r} \tag{4.10}$$

and

$$\|(G_1(t,r,\theta), G_2(t,r,\theta))\| \geq \sqrt{(\alpha_1(t) + \beta_1(t)r^{2\lambda})^2 + (\max\{\gamma_1(t), -\gamma_2(t), 0\})^2 r} \tag{4.11}$$

for $t \geq t_0$ and $r \in (0,1)$.

First we consider case (i). By (4.7), (4.10) and the fact

$$\|(x'(t), y'(t))\| = \|(F_1(t,x(t),y(t)), F_2(t,x(t),y(t)))\| = \|(G_1(t,r(t),\theta(t)), G_2(t,r(t),\theta(t)))\|,$$

we obtain

$$\|(x'(t),y'(t))\| \le \frac{\sqrt{[\alpha_2(t)+\beta_2(t)(\rho_1(t;c_0))^{-1}]^2+(\max\{|\gamma_1(t)|,|\gamma_2(t)|\})^2}}{(\rho_1(t;c_0))^{\frac{1}{2\lambda}}}$$

for $t \ge t_1$. Hence from (1.15) it follows that $\Gamma_{(t_0,x,y)}$ is rectifiable.

Next we consider case (iii). By (4.7) and (4.11), we obtain

$$\|(x'(t),y'(t))\| \ge \frac{\sqrt{[\alpha_1(t)+\beta_1(t)(\rho_2(t;c_0))^{-1}]^2+(\max\{\gamma_1(t),-\gamma_2(t),0\})^2}}{(\rho_2(t;c_0))^{\frac{1}{2\lambda}}}$$

for $t \ge t_1$. Integrating this inequality and using (1.16), we conclude that $\Gamma_{(t_0,x,y)}$ is nonrectifiable. This completes the proof of Theorem 1.7. □

Using Theorems 1.3 and 1.7, and Lemma 3.5, we can establish the following result.

Theorem 4.4. *Let $(x(t),y(t))$ be any nontrivial solution of (1.1). Suppose that (1.6) and (3.3) hold. Then the zero solution of (1.1) is globally exponentially stable, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t),y(t))$ is simple, and (i), (ii) and (iii) below hold:*

- *(i) if (1.9) holds, then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;*
- *(ii) if (1.15) holds, then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;*
- *(iii) if (1.16) holds, then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.*

Corollary 1.9 and Lemma 3.5 imply the following.

Corollary 4.5. *Let $(x(t),y(t))$ be any nontrivial solution of (1.18). Suppose that there exists an $\underline{e} > 0$ such that*

$$e(t) \ge \underline{e} \quad \text{for} \quad t \ge t_0. \tag{4.12}$$

Then the zero solution of (1.18) is exponentially stable, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t),y(t))$ is simple, and the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable if and only if (1.21) holds.

5 Examples and numerical simulations

In this section we will present some examples and numerical simulations.

Example 5.1. Let $\lambda = 0.5$. Consider the two-dimensional nonautonomous differential system (1.1) with

$$e(t) = h(t) = \frac{1}{t}, \quad f(t) = g(t) = \frac{10\cos t}{t} \quad \text{and} \quad p(t) = q(t) = t. \tag{5.1}$$

Then

$$\alpha_1(t) = \alpha_2(t) = e(t) = \frac{1}{t}, \quad \beta_1(t) = \beta_2(t) = p(t) = t \quad \text{and} \quad \gamma_1(t) = \gamma_2(t) = -f(t) = -\frac{10\cos t}{t}.$$

Hence, assumptions (1.6), (1.7) and (1.8) are easily satisfied. Moreover,

$$\alpha_1(t) = \frac{1}{t} > 0 \quad \text{for} \quad t \ge 1,$$

and
$$\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)} = 10|\cos t| \le 10 \quad \text{for } t \ge 1.$$

By Theorem 1.3 (i), we conclude that the zero solution of (1.1) with (5.1) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ is simple and rectifiable. Fig. 5.1 shows the orbit $\Gamma_{(1,x,y)}$ corresponding to the nontrivial solution $(x(t), y(t))$ of (1.1) with (5.1) and $(x(1), y(1)) = (0.9, 0)$.

Example 5.2. Let $\lambda = 0.1$. Consider the two-dimensional nonautonomous differential system (1.1) with

$$e(t) = h(t) = 0, \quad f(t) = g(t) = \frac{1}{2} + \frac{\cos t}{t} \quad \text{and} \quad p(t) = q(t) = 0.1. \tag{5.2}$$

Then

$$\alpha_1(t) = \alpha_2(t) = 0, \quad \beta_1(t) = \beta_2(t) = 0.1 \quad \text{and} \quad \gamma_1(t) = \gamma_2(t) = -\frac{1}{2} - \frac{\cos t}{t}.$$

Hence, assumptions (1.6), (1.7) and (1.8) are easily satisfied. Moreover,

$$\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\beta_1(t)} = 10\left(\frac{1}{2} + \frac{\cos t}{t}\right) \le 15 \quad \text{for } t \ge 1.$$

By Corollary 1.4 (ii), we conclude that the zero solution of (1.1) with (5.2) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ is simple and rectifiable. Fig. 5.2 shows the orbit $\Gamma_{(1,x,y)}$ corresponding to the nontrivial solution $(x(t), y(t))$ of (1.1) with (5.2) and $(x(1), y(1)) = (0.9, 0)$.

Example 5.3. Let $\lambda = 0.5$. Consider the two-dimensional nonautonomous differential system (1.1) with (5.2). Then

$$\frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\beta_2(t)} \ge \frac{-\gamma_2(t)}{\beta_2(t)} = 10\left(\frac{1}{2} + \frac{\cos t}{t}\right) > \frac{5}{2} \quad \text{for } t \ge 4.$$

By Corollary 1.4 (iii), the zero solution of (1.1) with (5.2) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ is simple and nonrectifiable. Fig. 5.3 shows the orbit $\Gamma_{(1,x,y)}$ corresponding to the nontrivial solution $(x(t), y(t))$ of (1.1) with (5.2) and $(x(1), y(1)) = (0.9, 0)$.

Example 5.4. Let $\lambda = 0.5$. Consider the two-dimensional nonautonomous differential system (1.1) with

$$e(t) = h(t) = \frac{1}{t}, \quad f(t) = g(t) = 2 + \cos t \quad \text{and} \quad p(t) = q(t) = \frac{1}{t^2}. \tag{5.3}$$

Then

$$\alpha_1(t) = \alpha_2(t) = \frac{1}{t}, \quad \beta_1(t) = \beta_2(t) = \frac{1}{t^2} \quad \text{and} \quad \gamma_1(t) = \gamma_2(t) = -2 - \cos t.$$

Hence, assumptions (1.6), (1.7) and (1.8) are easily satisfied. Since

$$\exp\left(2\lambda \int_{t_0}^t \alpha_2(s)ds\right) = \exp\left(\log \frac{t}{t_0}\right) = \frac{t}{t_0}$$

for $t \ge t_0$, we have

$$\rho_2(t; c) = \frac{t}{t_0}\left(c + t_0 \int_{t_0}^t s^{-3} ds\right) = t\left(\frac{c}{t_0} + \frac{1}{2t_0^2} - \frac{1}{2t^2}\right),$$

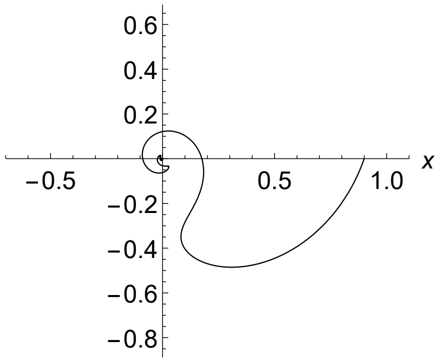

Figure 5.1: Example 5.1; Theorem 1.3 (i); rectifiable.

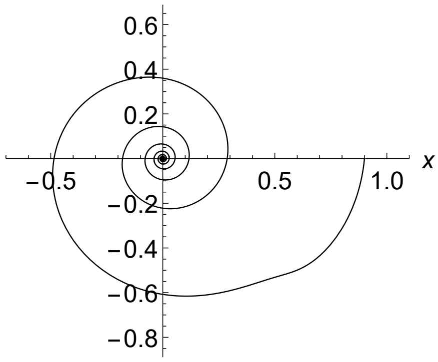

Figure 5.2: Example 5.2; Corollary 1.4 (ii); rectifiable.

and hence

$$\frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t)\rho_2(t;c) + \beta_2(t)} \geq \frac{2+\cos t}{\frac{c}{t_0} + \frac{1}{2t_0^2} + \frac{1}{2t^2}} \geq \frac{1}{\frac{c}{t_0} + \frac{1}{t_0^2}}$$

for $t \geq t_0$. Hence (1.11) is satisfied. By Theorem 1.3 (iii), the zero solution of (1.1) with (5.3) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ is simple and nonrectifiable. Fig. 5.4 shows the orbit $\Gamma_{(1,x,y)}$ corresponding to the nontrivial solution $(x(t), y(t))$ of (1.1) with (5.3) and $(x(1), y(1)) = (0.9, 0)$.

Example 5.5. Consider the two-dimensional nonautonomous linear system (1.18) with

$$e(t) = 1 \quad \text{and} \quad f(t) = e^t. \tag{5.4}$$

Then assumption (4.12) is easily satisfied. It is clear that

$$\int_{t_0}^{t} \sqrt{e^2(s) + f^2(s)} \exp\left(-\int_{t_0}^{s} e(\tau)d\tau\right) ds \geq \int_{t_0}^{t} e^s e^{-s+t_0} ds = e^{t_0}(t - t_0)$$

for all $t \geq t_0$. Hence, by Corollary 4.5 we conclude that the zero solution of (1.18) with (5.4) is exponentially stable, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple and nonrectifiable. Fig. 5.5 shows the orbit $\Gamma_{(1,x,y)}$ corresponding to the nontrivial solution $(x(t), y(t))$ of (1.18) with (5.4) and $(x(1), y(1)) = (0.9, 0)$.

Example 5.6. Consider the two-dimensional nonautonomous linear system (1.22), where $\sigma \in \mathbf{R}$. Then assumptions (1.19) and (1.20) are easily satisfied. By Corollary 1.9 we conclude that the zero solution of (1.22) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple. Moreover, the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable if and only if

$$\lim_{t\to\infty} \int_{t_0}^{t} \sqrt{e^2(s) + f^2(s)} \exp\left(-\int_{t_0}^{s} e(\tau)d\tau\right) ds < \infty.$$

Let

$$\omega(t) := \sqrt{e^2(t) + f^2(t)} \exp\left(-\int_{t_0}^{t} e(s)ds\right)$$

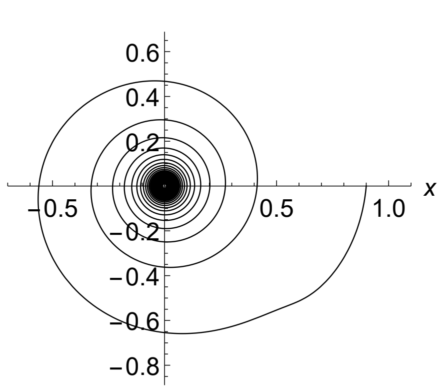

Figure 5.3: Example 5.3; Corollary 1.4 (iii); nonrectifiable.

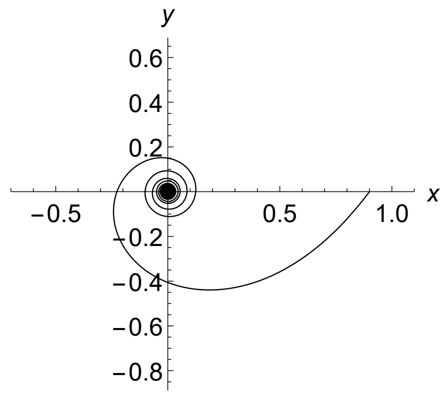

Figure 5.4: Example 5.4; Theorem 1.3 (iii); nonrectifiable.

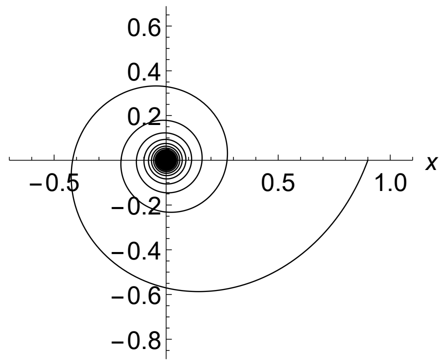

Figure 5.5: Example 5.5; Corollary 4.5; exponentially stable; nonrectifiable.

for all $t \geq 1$. Then we have

$$\omega(t) = t^{-1}\sqrt{t^{-2} + t^{2\sigma}} \tag{5.5}$$

holds for all $t \geq 1$. We will consider the three cases (i) $\sigma \leq -1$, (ii) $-1 < \sigma < 0$ and (iii) $\sigma \geq 0$.

Case (i). Using (5.5), we get

$$\int_1^t \omega(s)ds = \int_1^t s^{-2}\sqrt{1 + s^{2(\sigma+1)}}ds \leq \sqrt{2}\int_1^t s^{-2}ds = -\sqrt{2}(t^{-1} - 1) < \sqrt{2}$$

for all $t \geq 1$. By Theorem 1.9 we see that the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable.

Case (ii). From (5.5), we have

$$\int_1^t \omega(s)ds = \int_1^t s^{\sigma-1}\sqrt{s^{-2(\sigma+1)} + 1}ds \leq \sqrt{2}\int_1^t s^{\sigma-1}ds = \frac{\sqrt{2}}{\sigma}(t^\sigma - 1) < \frac{\sqrt{2}}{-\sigma}$$

for all $t \geq 1$. By Corollary 1.9 we see that the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable.

Case (iii). Using (5.5), we get

$$\int_1^t \omega(s)ds \geq \int_1^t s^{\sigma-1}ds \geq \int_1^t s^{-1}ds = \log t$$

for all $t \geq t_0$. By Corollary 1.9 we see that the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable. Consequently, we can conclude that the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable if and only if $\sigma < 0$.

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