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On solvability of focal boundary value problems for higher order functional differential equations with integral restrictions

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Abstract. Sharp conditions are obtained for the unique solvability of focal boundary value problems for higher-order functional differential equations under integral restrictions on functional operators. In terms of the norm of the functional operator, unimprovable conditions for the unique solvability of the boundary value problem are established in the explicit form. If these conditions are not fulfilled, then there exists a positive bounded operator with a given norm such that the focal boundary value problem with this operator is not uniquely solvable. In the symmetric case, some estimates of the best constants in the solvability conditions are given. Comparison with existing results is also performed.

Keywords: functional differential equations, focal boundary value problem, unique solvability.

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1 Introduction

We consider here boundary value problems

$$\begin{cases} (-1)^{(n-k)} x^{(n)}(t) + (Tx)(t) = f(t), \quad t \in [0,1], \\ x^{(i)}(0) = 0, \quad i = 0, \dots, k-1, \\ x^{(j)}(1) = 0, \quad j = k, \dots, n-1, \end{cases} \tag{1.1}$$

where $n \in \{2, 3, \dots\}$, $k \in \{1, 2, \dots, n-1\}$, $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is a linear bounded operator, $\mathbf{C}[0,1]$ and $\mathbf{L}[0,1]$ are the space of real continuous and integrable functions (respectively) with the standard norms, $f \in \mathbf{L}[0,1]$. A real absolutely continuous function with absolutely continuous derivatives up to $(n-1)$-th order which satisfies the boundary conditions from (1.1) and satisfies the functional differential equation from (1.1) almost everywhere on $[0,1]$ is called a solution to problem (1.1).

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The boundary value problems with such kind of boundary conditions are called focal ones. The solvability of such problems for linear and non-linear functional differential equations occupies a special place in many studies of physical, chemical, and biological processes (see, for example, [1,2,7,14,31,37] end references there).

The focal problem for the ordinary differential equation

$$\begin{cases} (-1)^{(n-k)}x^{(n)}(t) = f(t), \quad t \in [0,1], \\ x^{(i)}(0) = 0, \quad i = 0,\dots,k-1, \\ x^{(j)}(1) = 0, \quad j = k,\dots,n-1, \end{cases}$$

has a unique solution $x(t) = \int_0^1 G(t,s)f(s)\,ds$, $t \in [0,1]$, where Green's function $G(t,s)$ is defined by the equality [20]

$$G(t,s) = \frac{1}{(n-k-1)!}\frac{1}{(k-1)!}\int_0^{\min(t,s)} (s-\tau)^{n-k-1}(t-\tau)^{k-1}\,d\tau, \quad t,s \in [0,1]. \tag{1.2}$$

Note, that the function $G(t,s)$ is an oscillating kernel by the Kalafaty–Gantmacher–Krein Theorem [17] (see also [18,19,22,34]), therefore, in particular, the inequality

$$\begin{vmatrix} G(\tau_1,s_1) & G(\tau_1,s_2) \\ G(\tau_2,s_1) & G(\tau_2,s_2) \end{vmatrix} > 0 \tag{1.3}$$

holds for all $0 < \tau_1 < \tau_2 \le 1$, $0 < s_1 < s_2 \le 1$. Problem (1.1) enjoys the Fredholm property [8, Ch. 2]. Thus, if the homogeneous problem has only a trivial solution, then problem (1.1) has a unique solution for all $f \in \mathbf{L}[0,1]$.

Obviously, boundary value problem (1.1) is equivalent to the equation

$$x(t) = -\int_0^1 G(t,s)(Tx)(s)\,ds + \int_0^1 G(t,s)f(s)\,ds, \quad t \in [0,1]. \tag{1.4}$$

Applying some fixed point theorems, for example, the classical methods for estimating the norm of the operator $G : \mathbf{C}[0,1] \to \mathbf{C}[0,1]$ defined by the equality

$$(Gx)(t) = -\int_0^1 G(t,s)(Tx)(s)\,ds, \quad t \in [0,1],$$

one can obtain various unique solvability conditions for problem (1.1).

Conditions for the solvability of focal boundary value problems for higher-order differential equations were obtained in the works by R. Agarwal [1,4], R. Agarwal and I. Kiguradze [3], and others [5,6,15,20,21,23,28,29,31,32,35,36,38]. As for those conditions as applied to the linear higher-order functional differential equations, among the results related to the norm of the operator T, the author does not know of any that would significantly improve the following.

Denote

$$\widetilde{\mathcal{T}}_{n,k} \equiv (n-1)(n-k-1)!(k-1)!$$

Proposition 1.1. *Problem* (1.1) *is uniquely solvable if*

$$\|T\|_{\mathbf{C}\to\mathbf{L}} \le \widetilde{\mathcal{T}}_{n,k}. \tag{1.5}$$

Proof. We have

$$G(1,1) = \frac{1}{\widetilde{\mathcal{T}}_{n,k}} > G(t,s) \geq 0$$

for all $(t,s) \in [0,1] \times [0,1]$, $(t,s) \neq (1,1)$. Therefore, if the condition of the statement is fulfilled, then for any non-zero solution x to equation (1.4) for $f \equiv 0$ the following inequalities hold:

$$\begin{aligned} |x(t)| &= \left| \int_0^1 G(t,s)(Tx)(s)\,ds \right| < G(1,1) \int_0^1 |(Tx)(s)|\,ds \\ &\leq G(1,1)\|\,T\|_{\mathbf{C}\to\mathbf{L}}\|\,x\|_{\mathbf{C}} \leq \|\,x\|_{\mathbf{C}} \quad \text{for all} \quad t \in [0,1]. \end{aligned}$$

Since the continuous function $|x(t)|$ has a maximal value at a corresponding point $t^\star \in [0,1]$, the inequality $|x(t^\star)| < \|x\|_{\mathbf{C}}$ is impossible. It follows that the homogeneous boundary value problem has only the trivial solution. Therefore, the Fredholm boundary value problem (1.1) is uniquely solvable. □

Examples show that the constant in the right-hand side of inequality (1.5) is unimpovable. Let us define a linear bounded operator $T_\theta : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$, $\theta \in (0,1)$, by the equality

$$(T_\theta x)(t) = \begin{cases} 0, & t \in [0,\theta], \\ -\frac{x(1)}{\int_\theta^1 G(1,s)\,ds}, & t \in (\theta,1]. \end{cases}$$

Homogeneous problem (1.1) for $T = T_\theta$ and $f \equiv 0$ has a non-trivial solution

$$x(t) = \int_\theta^1 G(t,s)\,ds, \quad t \in [0,1].$$

Therefore, this problem isn't uniquely solvable. Since

$$\lim_{\theta\to 1-} \|\,T_\theta\|_{\mathbf{C}\to\mathbf{L}} = \lim_{\theta\to 1-} \frac{1-\theta}{\int_\theta^1 G(1,s)\,ds} = \widetilde{\mathcal{T}}_{n,k},$$

for every $\varepsilon > 0$ there exists a linear bounded operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with $\|\,T\|_{\mathbf{C}\to\mathbf{L}} = \widetilde{\mathcal{T}}_{n,k} + \varepsilon$ such that problem (1.1) isn't uniquely solvable.

However, it was shown in [24–26] that for certain monotone functional operators and for some boundary value problems, the solvability conditions based on contraction mapping principle can be essentially weakened.

An operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is called positive if it maps non-negative functions from $\mathbf{C}[0,1]$ to almost everywhere non-negative functions from $\mathbf{L}[0,1]$. The norm of such an operator is defined by the equality $\|\,T\|_{\mathbf{C}\to\mathbf{L}} = \int_0^1 (T\mathbf{1})(t)\,dt$, where $\mathbf{1}(t) = 1$, $t \in [0,1]$, is the unit function. For $p \in \mathbf{L}[0,1]$ and a measurable function $h : [0,1] \to [0,1]$, the operator

$$(Tx)(t) = p(t)x(h(t)), \quad t \in [0,1],$$

is positive if the function $p \in \mathbf{L}[0,1]$ is non-negative. Its norm equals $\|\,T\|_{\mathbf{C}\to\mathbf{L}} = \int_0^1 p(t)\,dt$.

This work is devoted to weakening the solvability conditions (1.5) for problem (1.1) with positive linear operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$. We obtain a necessary and sufficient condition for the focal boundary value problem (1.1) to be uniquely solvable for all positive operators T with a given norm.

For some other boundary value problems, similar unimprovable conditions are obtained by R. Hakl, A. Lomtatidze, S. Mukhigulashvili, B. Půža, J. Šremr, and others [10,16,24–27,30].

2 Main results

Theorem 2.1. *Let a non-negative number $\mathcal{T}$ be given. Problem (1.1) is uniquely solvable for all positive linear operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with norm $\mathcal{T}$ if and only if*

$$\mathcal{T} \leq \min_{0<t<1,\ 0<s<1} \frac{G(t,1)+G(1,s)+2\sqrt{G(t,s)G(1,1)}}{G(t,s)G(1,1)-G(t,1)G(1,s)} \equiv \mathcal{T}_{n,k}.$$

Taking into account (1.3), the constants $\mathcal{T}_{n,k}$ are well-defined. Green's function $G(t,s)$ has explicit representation (1.2), therefore, the best constant $\mathcal{T}_{n,k}$ from the solvability conditions can be easily calculated approximately. Note, since Green's functions of corresponding problems are symmetric, we have

$$\mathcal{T}_{n,k} = \mathcal{T}_{n,n-k}.$$

In some cases, the constants are calculated exactly. In particular, $\mathcal{T}_{2,1}$, $\mathcal{T}_{4,2}$, $\mathcal{T}_{6,3}$ are obtained in Example 3.3, and the constant $\mathcal{T}_{3,1}$ is obtained in Example 3.9. For even n in Theorem 3.2, the constants $\mathcal{T}_{n,n/2}$ are represented using one-dimensional minimization. In Corollaries 3.5, 3.6, asymptotically unimprovable estimates for $\mathcal{T}_{n,n/2}$ are obtained.

The proof of Theorem 2.1 is based on the following assertion [11, Theorem 2.28, p. 106] (see also a similar proof in [12]).

Proposition 2.2 ([11,12])**.** *Let $\mathcal{T}$ be a non-negative number. Problem (1.1) is uniquely solvable for all positive linear operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with norm $\mathcal{T}$ if and only if for all numbers c, d, τ_1, τ_2, $\mathcal{T}_1$, $\mathcal{T}_2$ satisfying the conditions*

$$c,d \in [0,1], \quad 0 \leq \tau_1 \leq \tau_2 \leq 1, \quad \mathcal{T}_1 \geq 0, \quad \mathcal{T}_2 \geq 0, \quad \mathcal{T}_1 + \mathcal{T}_2 \leq \mathcal{T}, \tag{2.1}$$

the inequality

$$\begin{aligned}\Delta &\equiv \Delta(\tau_1,\tau_2,c,d,\mathcal{T}_1,\mathcal{T}_2)\\ &\equiv 1+\mathcal{T}_1 G(\tau_1,c)+\mathcal{T}_2 G(\tau_2,d)+\mathcal{T}_1\mathcal{T}_2(G(\tau_1,c)G(\tau_2,d)-G(\tau_2,c)G(\tau_1,d)) \geq 0\end{aligned} \tag{2.2}$$

holds.

Proof of Theorem 2.1. We will use Proposition 2.2. Let

$$R \equiv G(\tau_1,c)G(\tau_2,d)-G(\tau_2,c)G(\tau_1,d).$$

If $R \geq 0$, then $\Delta = 1+\mathcal{T}_1 G(\tau_1,c)+\mathcal{T}_2 G(\tau_2,d)+\mathcal{T}_1\mathcal{T}_2 R > 0$.

Let further $R < 0$ and $0 < \tau_1 < \tau_2 < 1$. From (1.3) and $R < 0$ it follows that

$$0 < d < c \leq 1. \tag{2.3}$$

For fixed points τ_1, τ_2, c, d, and $\mathcal{T}_1$, Δ takes its minimum at $\mathcal{T}_2 = \mathcal{T} - \mathcal{T}_1$ or at $\mathcal{T}_2 = 0$. In the latter case, $\Delta = 1+\mathcal{T}_1 G(\tau_1,c) \geq 1$.

Thus, the inequality (2.2) should be verified only at $\mathcal{T}_2 = \mathcal{T} - \mathcal{T}_1$ for all $\mathcal{T}_1 \in [0,\mathcal{T}]$. In this case, we have

$$\begin{aligned}\Delta &\equiv \Delta(\tau_1,\tau_2,c,d,\mathcal{T}_1)\\ &\equiv 1+\mathcal{T}_1 G(\tau_1,c)+(1-\mathcal{T}_1)G(\tau_2,d)+\mathcal{T}_1(1-\mathcal{T}_1)R\\ &= -\mathcal{T}_1^2 R+\mathcal{T}_1(G(\tau_1,c)-G(\tau_2,d)+\mathcal{T}R)+1+\mathcal{T}G(\tau_2,d).\end{aligned}$$

Let us find the minimum of this value in the variable $\mathcal{T}_1$ at fixed values of other variables.

Denote $B \equiv G(\tau_1, c) - G(\tau_2, d)$.

If $|B/R| > \mathcal{T}$, then the value Δ takes its minimum on $\mathcal{T}_1 \in [0, \mathcal{T}]$ at $\mathcal{T}_1 = 0$ or $\mathcal{T}_1 = \mathcal{T}$. In the first case, we have $\Delta = 1 + \mathcal{T}G(\tau_2, d) \geq 1$, in the second one, $\Delta = 1 + \mathcal{T}G(\tau_1, c) \geq 1$.

If $|B/R| \leq \mathcal{T}$, then the minimum of Δ occurs at

$$\mathcal{T}_1 = \frac{G(\tau_1, c) - G(\tau_2, d) + \mathcal{T}R}{2R} \equiv \frac{\mathcal{T} + B/R}{2}.$$

This minimum value is equal to

$$\Delta_{\min} = \frac{R}{4}\mathcal{T}^2 + \mathcal{T}\left(\frac{B}{2} + G(\tau_2, d)\right) + 1 + \frac{B^2}{4R},$$

therefore, $\Delta_{\min} \geq 0$ if and only if the following inequalities hold:

$$Q(\tau_1, \tau_2, c, d) \leq \mathcal{T} \leq S(\tau_1, \tau_2, c, d),$$

where

$$Q(\tau_1, \tau_2, c, d) \equiv \frac{G(\tau_1, c) + G(\tau_2, d) - 2\sqrt{G(\tau_1, d)G(\tau_2, c)}}{|R|},$$

$$S(\tau_1, \tau_2, c, d) \equiv \frac{G(\tau_1, c) + G(\tau_2, d) + 2\sqrt{G(\tau_1, d)G(\tau_2, c)}}{|R|}.$$

From the inequality (1.3) for $s_1 = d$ and $s_2 = c$ it follows that

$$\frac{G(\tau_1, c) + G(\tau_2, d) - 2\sqrt{G(\tau_1, d)G(\tau_2, c)}}{|R|} \leq \frac{|G(\tau_1, c) - G(\tau_2, d)|}{|R|} \leq \frac{|B|}{|R|} \leq \mathcal{T}.$$

Therefore, inequality (2.2) is satisfied for all parameters satisfying the conditions (2.1) if and only if

$$\mathcal{T} \leq \min_{\substack{0 \leq \tau_1 \leq \tau_2 \leq 1 \\ c,d \in [0,1],\ R<0}} S(\tau_1, \tau_2, d, c) \equiv \widetilde{\mathcal{T}}.$$

Since (2.3), we have

$$\widetilde{\mathcal{T}} = \min_{\substack{0 < \tau_1 < \tau_2 \leq 1 \\ 0 < d < c \leq 1}} S(\tau_1, \tau_2, d, c).$$

Our aim is to simplify the expression for evaluating $\widetilde{\mathcal{T}}$.

For $0 \leq \tau_1 \leq \tau_2 \leq 1$, $0 < d < c \leq 1$, we prove that

$$S'_{\tau_2}(\tau_1, \tau_2, d, c) = \frac{1}{R^2}\left(\frac{G'_{\tau_2}(\tau_2, d)}{G(\tau_2, d)}A - \frac{G'_{\tau_2}(\tau_2, c)}{G(\tau_2, c)}B\right) \leq 0, \tag{2.4}$$

where

$$A = G(\tau_1, c)^2 G(\tau_2, d) + G(\tau_1, d)G(\tau_2, d)G(\tau_2, c) + 2G(\tau_1, c)G(\tau_2, d)\sqrt{G(\tau_1, d)G(\tau_2, c)},$$

$$\begin{aligned} B = {} & G(\tau_1, c)G(\tau_1, d)G(\tau_2, c) + G(\tau_1, d)G(\tau_2, d)G(\tau_2, c) \\ & + (G(\tau_1, c)G(\tau_2, d) + G(\tau_1, d)G(\tau_2, c))\sqrt{G(\tau_1, d)G(\tau_2, c)}. \end{aligned}$$

Since the function $G(t,s)$ is an oscillating kernel, we easily see that $B \geq A \geq 0$. Indeed, we have

$$B - A = (G(\tau_1,c) + \sqrt{G(\tau_1,d)G(\tau_2,c)})(G(\tau_1,d)G(\tau_2,c) - G(\tau_1,c)G(\tau_2,d)) \geq 0.$$

Let us prove that for each $t \in (0,1]$ the function $\frac{G'_t(t,s)}{G(t,s)}$ does not decrease in the second argument for $s \in (0,1]$. It suffices to show that for all $0 < t_1 < t_2 \leq 1$, $0 < s_1 < s_2 \leq 1$, the inequality

$$\frac{G(t_2,s_2) - G(t_1,s_2)}{G(t_1,s_2)} \geq \frac{G(t_2,s_1) - G(t_1,s_1)}{G(t_1,s_1)}$$

holds. This inequality is a direct consequence of the inequality (1.3). It follows that inequality $B \geq A$ implies inequality (2.4).

Similarly, it is verified that for $0 \leq \tau_1 \leq \tau_2 \leq 1$, $0 < d < c \leq 1$, the inequality

$$S'_c(\tau_1,\tau_2,d,c) \leq 0 \tag{2.5}$$

holds. From (2.4) and (2.5) it follows that in (2.5) the value $\widetilde{\mathcal{T}}$ has the minimum point at $\tau_2 = 1$ and $c = 1$. This implies the assertion of the theorem. □

3 Consequences

For calculating the constants $\mathcal{T}_{n,n/2}$, we need the following lemma, a technical proof of which was carried out in the paper [13].

Lemma 3.1. *Let $n = 2k$. Then the function*

$$M(t,s) = \sqrt{G(t,s)G(1,1)} - \sqrt{G(t,1)G(s,1)}, \quad t,s \in [0,1],$$

has its maximum value at $t = s$.

Let us show that for even n to calculate the constants $\mathcal{T}_{n,n/2}$, it is sufficient to solve an one-dimensional optimization problem.

Theorem 3.2. *Let a non-negative number $\mathcal{T}$ and $n = 2k$ be given. Problem (1.1) is uniquely solvable for all positive linear operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with the norm $\mathcal{T}$ if and only if*

$$\mathcal{T} \leq \frac{2\left((n/2-1)!\right)^2}{\max\limits_{0<t<1}\left(\frac{t^{(n-1)/2}}{n-1} - \int_0^t (t-\tau)^{n/2-1}(1-\tau)^{n/2-1}\,d\tau\right)} \equiv \mathcal{T}_{n,n/2}. \tag{3.1}$$

Proof. Let us use the Theorem 2.1. We have

$$\frac{G(t,1) + G(1,s) + 2\sqrt{G(t,s)G(1,1)}}{G(t,s)G(1,1) - G(t,1)G(1,s)}$$
$$= \frac{(\sqrt{G(t,1)} - \sqrt{G(1,s)})^2}{G(t,s)G(1,1) - G(t,1)G(1,s)} + \frac{2}{\sqrt{G(t,s)G(1,1)} - \sqrt{G(t,1)G(1,s)}}.$$

It follows that if $t_0 = s_0$ and the point $(t,s) = (t_0,s_0)$ is the minimum point of the function

$$\frac{2}{\sqrt{G(t,s)G(1,1)} - \sqrt{G(t,1)G(1,s)}},$$

then the minimum of the value expressing the exact estimate of the norm of the operator T under the conditions of the Theorem 2.1 will be taken at this point.

Lemma 3.1 implies that for $n = 2k$ the minimum under the conditions of Theorem 2.1 is taken namely at $s = t$. Calculating $G(t,t)$ and $G(t,1)$ using representation (1.2), we obtain the assertion of the theorem. □

Example 3.3. Under the conditions of the Theorem 3.2 for $n = 2$, $n = 4$, and $n = 6$ the values $\mathcal{T}_{n,n/2}$ are calculated exactly. We have

$$\mathcal{T}_{2,1} = 8$$

(the maximum in the representation of $\mathcal{T}_{2,1}$ (3.1) occurs at $t_2 = 1/4$);

$$\mathcal{T}_{4,2} = 66 + 30\sqrt{5},$$

(the maximum in the representation of $\mathcal{T}_{4,2}$ (3.1) occurs at $t_4 = \frac{3-\sqrt{5}}{2}$);

$$\mathcal{T}_{6,3} = 120\frac{2t_6^3 - 10t_6^2 + 20t_6 + 12\sqrt{t_6}}{t_6^3(1-t_6)(t_6^4 - 9t_6^3 + 36t_6^2 - 64t_6^1 + 36)} \approx 2610,$$

where the point of the maximum t_6 in representation (3.1) of $\mathcal{T}_{6,3}$ is defined by the equalities

$$t_6 = ((C - 1 - \sqrt{27 - C^2 + 22/C})/4)^2 \approx 0.49,$$

$$C = \sqrt{2(124 + 4\sqrt{97})^{1/3} + 9 + 48(124 + 4\sqrt{97})^{-1/3}}.$$

Remark 3.4. Apparently only the constant

$$\mathcal{T}_{2,1} = 8 \tag{3.2}$$

was previously known. In particular, equality (3.2) follows from the results of the work [33] on the solvability of two-dimensional systems functional differential equations. The solvability conditions associated with the rest of the found constants $\mathcal{T}_{n,k}$ are new.

For even $n \geq 8$, we obtain sufficient conditions for solvability (lower bounds for the constants $\mathcal{T}_{n,n/2}$).

Corollary 3.5. *Let $n = 2k \geq 8$ and a linear operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be positive. If*

$$\|T\|_{\mathbf{C}\to\mathbf{L}} \leq \frac{(n^2-9)(n^2-1)\left((n/2-1)!\right)^2}{3 + (n-2)\left(\frac{n-7}{n-3}\right)^{\frac{n+1}{2}}},$$

then problem (1.1) is uniquely solvable.

Corollary 3.6. *Let $n = 2k \geq 8$ and a linear operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be positive. If*

$$\|T\|_{\mathbf{C}\to\mathbf{L}} \leq e^2(n-3)^3\left((n/2-1)!\right)^2, \tag{3.3}$$

then problem (1.1) is uniquely solvable.

Remark 3.7. In (3.3), the constant e^2 and the exponent 3 are sharp.

Proof of Corollary 3.5. Let us introduce the notation

$$y_n(t) \equiv \frac{t^{(n-1)/2}}{n-1} - \int_0^t (t-\tau)^{n/2-1}(1-\tau)^{n/2-1}\,d\tau,$$

$$\Upsilon_n \equiv \max_{0<t<1} y_n(t), \quad \mathcal{T}_n \equiv \mathcal{T}_{n,n/2}.$$

By Theorem 3.2, it is obvious that

$$\mathcal{T}_n \equiv \frac{2\left((n/2-1)!\right)^2}{\Upsilon_n}.$$

We obtain the estimate $\widehat{\Upsilon}_n \geq \Upsilon_n$. Then

$$\mathcal{T}_n \geq \widehat{\mathcal{T}}_n \equiv \frac{2\left((n/2-1)!\right)^2}{\widehat{\Upsilon}_n},$$

therefore, the condition $\mathcal{T} \leq \widehat{\mathcal{T}}_n$ ensures the unique solvability of the problem (1.1) for each positive operator T with given norm $\mathcal{T}$.

It is convenient to present y_n' using the hypergeometric function ${}_2F_1$ [9, p. 69]:

$$\begin{aligned} y_n'(t) &= \frac{t^{(n-3)/2}}{2} - (n/2-1)\int_0^t (t-\tau)^{n/2-2}(1-\tau)^{n/2-1}\,d\tau \\ &= \frac{t^{(n-3)/2}}{2} - (n/2-1)t^{n/2-1}\int_0^1 (1-\theta)^{n/2-2}(1-t\theta)^{n/2-1}\,d\theta \\ &= \frac{t^{(n-3)/2}}{2} - t^{n/2-1}\,{}_2F_1(1-n/2,1;n/2;t) \equiv \frac{t^{(n-3)/2}}{2} z_n(t), \end{aligned} \tag{3.4}$$

where

$$z_n(t) \equiv 1 - 2\sqrt{t}\,{}_2F_1(1,1-n/2;n/2;t).$$

Further, for the hypergeometric function, the following properties will be used (it is obvious that in our case the hypergeometric function is a polynomial, moreover, we only need real parameters and a real argument). [9, p. 71–72] :

$$\frac{d^m}{dt^m}\,{}_2F_1(a,b;c;t) = \frac{(a)_m(b)_m}{(c)_m}\,{}_2F_1(a+m,b+m;c+m;t), \quad t \in [0,1],$$

$$(a)_m = a(a+1)\cdot\ldots\cdot(a+m-1), \quad m = 1,2,3,\ldots, \ (a)_0 = 1,$$

$${}_2F_1(a,b;c;t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{\theta^{b-1}(1-\theta)^{c-b-1}}{(1-t\theta)^{c-b-1}}\,\theta \quad t \in [0,1],\ c > b > 0. \tag{3.5}$$

Estimating $z_n(t)$, we obtain an approximation for $y_n'(t)$. Let

$$\widehat{z}_n(t) \equiv (t-1)\left(\frac{1}{2(n-3)} + \frac{t-1}{8}\right).$$

Lemma 3.8. *For every $n \geq 8$, the inequality*

$$z_n(t) \geq \widehat{z}_n(t), \quad t \in [0,1], \tag{3.6}$$

holds.

Proof. It suffices to show that

$$H_n(t) \equiv {}_2F_1(1, 1-n/2; n/2; t) \le \frac{1+(1-t)\left(\frac{1}{2(n-3)}+\frac{t-1}{8}\right)}{2\sqrt{t}} \equiv Z_n(t), \quad t \in (0,1]. \tag{3.7}$$

We have

$$Z_n(1) = H_n(1) = 1/2, \quad Z_n'(1) = H_n'(1) = -\frac{n-2}{4(n-3)}, \quad Z_n''(1) = H_n'(1) = \frac{n-2}{4(n-3)}.$$

To prove (3.7), it is now sufficient to prove that for all $t \in (0,1]$

$$H_n'''(t) = \frac{6\,(1-\frac{n}{2})_3}{(\frac{n}{2})_3}\,{}_2F_1(4, 4-n/2; n/2+3; t) \ge Z_n'''(t) = \frac{3(n(t^2+2t-35)+3t^2-10t+85)}{128(n-3)t^{7/2}}.$$

It remains to verify the chain of the inequalities

$$H_n'''(t) \ge w_0(t) \ge w_1(t) \ge w_2(t) \ge Z_n'''(t), \quad t \in (0,1], \tag{3.8}$$

where

$$w_0(t) \equiv H_n'''(0) + t(H_n'''(1) - H_n'''(0)),$$

$$H_n'''(0) = -6\frac{(n/2-3)(n/2-2)(n/2-1)}{(n/2)(n/2+1)(n/2+2)}, \quad H_n'''(1) = -6\frac{(n/2-1)^2(n/2-2)(n/2-3)}{(n-5)(n-4)(n-3)(n-2)},$$

$$w_1(t) \equiv \frac{45}{8}t - 6, \quad w_2(t) = \frac{3(t^2+2t-35)}{128t^{7/2}}.$$

To prove the first inequality in (3.8), we use the equality [9, p. 71]

$$H_n^{(m)}(t) = \frac{(1-n/2)_m(1)_m}{(n/2)_m}\,{}_2F_1(1-n/2+m, 1+m; n/2+m; t),$$

from which it follows that the sign of the function $H_n^{(m)}(t)$ coincides with $(-1)^m$, in particular, for $m = 3$, $m = 4$, $m = 5$ (it is also taken into account that due to the integral representation (3.5) [9, p. 72] the function ${}_2F_1(1-n/2+m, 1+m; n/2+m; t)$ is non-negative. The rest inequalities can be verified directly. □

Define the function $\widehat{y}_n$ by the equality

$$\widehat{y}_n(t) \equiv -\frac{1}{2}\int_t^1 s^{\frac{n-3}{2}}\widehat{z}_n(s)\,ds, t \in (0,1].$$

It is clear that $\widehat{y}_n(1) = y_n(1) = 0$. From (3.4) and (3.6) it follows that

$$\widehat{y}_n(t) \ge y_n(t), \quad t \in [0,1].$$

Its maximum $\widehat{Y}_n \ge Y_n$ the function $\widehat{y}_n(t)$ takes at the point $t_n \in (0,1)$ defined by the equality

$$\widehat{y}_n'(t_n) = \frac{t_n^{\frac{n-3}{2}}}{2}\widehat{z}_n(t_n) = 0.$$

therefore, we get

$$t_n = \frac{n-7}{n-3},$$

$$\widehat{Y}_n = \int_{t_n}^1 \frac{s^{\frac{n-3}{2}}}{2}(1-s)\left(\frac{1}{2(n-3)}+\frac{s-1}{8}\right)ds = \frac{6+2(n-2)\left(\frac{n-7}{n-3}\right)^{n/2+1/2}}{(n^2-9)(n^2-1)}.$$

This implies the assertion of Corollary 3.5. □

Proof of Corollary 3.6. It is easy to see that

$$\lim_{n\to\infty}(n-3)^3\widehat{Y}_n = \frac{2}{e^2}.$$

Moreover, $(n-3)^3\widehat{Y}_n < \frac{2}{e^2}$ for all $n \geq 8$. Thus, the statement of Corollary 3.6 is also true. □

Example 3.9. Consider problem (1.1) for the third-order equation for $k = 1$

$$\begin{cases} x'''(t) + (Tx)(t) = f(t), \quad t \in [0,1], \\ x(0) = 0, \\ x'(1) = 0, \quad x''(1) = 0, \end{cases} \tag{3.9}$$

By Theorem 2.1 problems (3.9) is uniquely solvable for all positive linear operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with the norm $\mathcal{T}$ if and only if

$$\mathcal{T} \leq \min_{0<s\leq t<1} 2\frac{t^2 - s^2 + 2s + 2\sqrt{(2t-s)s}}{s(1-t)(2t-s-st)} = 6(3+2\sqrt{3}) \approxeq 38.8.$$

Note, the minimum occurs at $s = (3-\sqrt{3})/6$, $t = (3-\sqrt{3})/3$.

For each $\varepsilon > 0$, there is a positive operator with the norm $6(3+2\sqrt{3}) + \varepsilon$, for which problem (3.9) is not uniquely solvable.

Proposition 1.1 only allows us to claim that problems (3.9) is uniquely solvable if the norm of the operator T is less than or equal to two.

Example 3.10. It is clear that the constant $\mathcal{T}_{n,k}$ from the necessary and sufficient conditions of Theorem 2.1 is equal or greater than the constants $\widetilde{\mathcal{T}}_{n,k}$ from Preposition 1.1. With the help of approximate computation, we make the following table containing the integer parts of the quotients $\mathcal{T}_{n,k}/\widetilde{\mathcal{T}}_{n,k}$, which shows how the classical results are improved by Theorem 2.1:

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$n=2$	8				
$n=3$	19				
$n=4$	31	44			
$n=5$	42	75			
$n=6$	54	109	130		
$n=7$	66	145	190		
$n=8$	78	184	255	275	
$n=9$	90	226	326	366	
$n=10$	101	269	404	464	481

Every element of this table shows approximately how many times the conditions of Theorem 2.1 are weaker than in Proposition 1.1 for given n and k, and gives a sufficient solvability conditions for corresponding problem (1.1). Formulate, for example, one such sufficient condition.

Proposition 3.11. *For $n = 10$ and $k = 1$ problem (1.1) is uniquely solvable if $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is a linear positive operator and $\int_0^1 (T\mathbf{1})(t)\,dt \leq 101 \cdot 9!$. There exists a linear positive operator T with $\int_0^1 (T\mathbf{1})(t)\,dt \geq 102 \cdot 9!$ such that problem (1.1) isn't uniquely solvable.*

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References

[1] R. P. Agarwal, *Focal boundary value problems for differential and difference equations*, Mathematics and its Applications, Vol. 436, Kluwer Academic Publishers, Dordrecht, 1998. MR1619877; Zbl 0914.34001

[2] R. P. Agarwal, D. O'Regan, P. J. Y. Wong, *Positive solutions of differential, difference and integral equations*, Kluwer Academic Publishers, Dordrecht, 1999. MR1680024; Zbl 1157.34301

[3] R. P. Agarwal, I. Kiguradze, Two-point boundary value problems for higher-order linear differential equations with strong singularities, *Bound. Value Probl.* **2006**, 2006:83910, 32 pp. https://doi.org/10.1155/BVP/2006/83910; MR2211396; Zbl 1137.34006

[4] R. P. Agarwal, D. O'Regan, Superlinear higher order boundary value problems, *Aequationes Math.* **57**(1999), No. 2–3, 233–240. https://doi.org/10.1007/s000100050080; MR1689196; Zbl 0931.34011

[5] M. J. Alves, S. Labovskiy, On monotone solutions and a self-adjoint spectral problem for a functional-differential equation of even order, *Electron. J. Qual. Theory Differ. Equ.* **59**(2019), No. 2, 1–14. https://doi.org/10.14232/ejqtde.2019.1.59; MR3991108; Zbl 1438.34225

[6] D. R. Anderson, R. I. Avery, A fourth-order four-point right focal boundary value problem, *Rocky Mountain J. Math.* **36**(2006), No. 2, 367–380. https://doi.org/10.1216/rmjm/1181069456; MR2234809; Zbl 1137.34008

[7] D. R. Anderson, T. O. Anderson, M. M. Kleber, Green's function and existence of solutions for a functional focal differential equation, *Electron. J. Differ. Equ.* **2006**, No. 12, 1–14. MR2198925; Zbl 1112.34041

[8] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina, *Introduction to the theory of functional differential equations: methods and applications*, Contemporary Mathematics and Its Applications, Vol. 3, Hindawi Publishing Corporation, New York–Cairo, 2007. MR2319815; Zbl 1202.34002

[9] H. Bateman, A. Erdelyi, *Higher transcendental functions. Vol. 1.*, McGraw-Hill Book Company, New York, 1953. MR0058756; Zbl 0051.30303

[10] E. I. Bravyi, Solvability of the periodic problem for higher-order linear functional differential equations, *Differ. Equ.* **51**(2015), No. 5, 571–585. https://doi.org/10.1134/S0012266115050018; MR3374833; Zbl 1345.34119

[11] E. I. Bravyi, *Solvability of boundary value problems for linear functional differential equations* (in Russian), Rerular and Chaotic dynamic, Moscow–Izhevsk, 2011.

[12] E. I. Bravyi, Solvability of the Cauchy problem for higher-order linear functional differential equations, *Differ. Equ.* **48**(2012), No. 4, 465–476. https://doi.org/10.1134/S0012266112040015; MR3177178; Zbl 1253.34071

[13] E. Bravyi, On the maximum of a function connected with the Green function of a focal boundary value problem, *Funct. Differ. Equ.* **27**(2020), No. 3–4, 71–83. https://doi.org/10.26351/FDE/27/3-4/3

[14] J. M. Davis, K. R. Prasad, W. K. C. Yin, Nonlinear eigenvalue problems involving two classes of functional differential equations, *Houston J. Math.* **26**(2000), No. 3, 597–608. MR1811944; Zbl 1013.34078

[15] A. Domoshnitsky, G. Landsman, Semi-nonoscillation intervals in analysis of sign constancy of Green's functions of Dirichlet, Neumann, and focal impulsive problems, *Adv. Difference Equ.* **2017**, No. 81, 14 pp. https://doi.org/10.1186/s13662-017-1134-1; MR3625857; Zbl 1422.34229

[16] R. Hakl, A. Lomtatidze, J. Šremr, *Some boundary value problems for first order scalar functional differential equations*, Mathematica, Vol. 10, Masaryk University, Brno, 2002. Zbl 1048.34004

[17] P. Kalafaty, Sur les fonctions de Green des équations différentielles ordinaires (in French) [Green functions of ordinary differential equations], *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **26**(1940), 526–530. MR0003314; Zbl 0023.04004

[18] S. Karlin, Total positivity, interpolation by splines, and Green's functions of differential operators, *J. Approximation Theory* **4**(1971), No. 1, 91–112. https://doi.org/10.1016/0021-9045(71)90041-4; MR0275022; Zbl 0228.41002

[19] J. M. Karon, The sign-regularity properties of a class of Green's functions for ordinary differential equations, *J. Differential Equations* **6**(1969), 484–502. https://doi.org/10.1016/0022-0396(69)90005-9; MR0259221; Zbl 0183.36102

[20] M. S. Keener, C. C. Travis, Positive cones and focal points for a class of n-th order differential equations, *Trans. Amer. Math. Soc.* **237**(1978), 331–351. https://doi.org/10.2307/1997625; MR479377; Zbl 0779.34042

[21] C. Kunkel, Singular higher order boundary value problems for ordinary differential equations, *Electron. J. Qual. Theory Differ. Equ.* **2009**, No. 19, 1–11. https://doi.org/10.14232/ejqtde.2009.4.19; MR2558844; Zbl 1204.34027

[22] A. Ju. Levin, G. D. Stepanov, One-dimensional boundary value problems with operators that do not lower the number of sign alternations. II (in Russian), *Sib. Mat. Zh.* **17**(1976), No. 4, 813–830. MR0425685; Zbl 0358.34035

[23] Y. Liu, A survey and new investigation on $(n, n-k)$-type boundary value problems for higher order impulsive fractional differential equations, *Tbilisi Math. J.* **10**(2017), No. 1, 207–248. https://doi.org/10.1515/tmj-2017-0014; MR3630173; Zbl 06707546

[24] A. Lomtatidze, S. Mukhigulashvili, On periodic solutions of second order functional differential equations, *Mem. Differ. Equ. Math. Phys.* **5**(1995), 125–126. Zbl 0866.34054

[25] A. Lomtatidze, S. Mukhigulashvili, On a two-point boundary value problem for second-order functional differential equations. I, *Mem. Differ. Equ. Math. Phys.* **10**(1997), 125–128. Zbl 0908.34050

[26] A. Lomtatidze, S. Mukhigulashvili, On a two-point boundary value problem for second-order functional differential equations. II, *Mem. Differ. Equ. Math. Phys.* **10**(1997), 150–152. Zbl 0939.34511

[27] A. Lomtatidze, S. Mukhigulashvili, *Some two-point boundary value problems for second-order functional-differential equations*, Mathematica, Vol. 8, Masaryk University, Brno, 2000. MR1846770; Zbl 0994.34052

[28] M. Maroun, Positive solutions to an nth order right focal boundary value problem, *Electron. J. Qual. Theory Differ. Equ.* **2007**, No. 4, 1–17. https://doi.org/10.14232/ejqtde.2007.1.4; MR2295682; Zbl 1115.34027

[29] S. Mukhigulashvili, N. Partsvania, Two-point boundary value problems for strongly singular higher-order linear differential equations with deviating arguments, *Electron. J. Qual. Theory Differ. Equ.* **2012**, No. 38, 1–34. https://doi.org/10.1016/S0895-7177(98)00097-1; MR2920961; Zbl 1098.34523

[30] S. Mukhigulashvili, B. Půža, On a periodic boundary value problem for third order linear functional differential equations, *Funct. Differ. Equ.* **14**(2007), No. 3–4, 347–362. MR2323216; Zbl 1138.34030

[31] S. Mukhigulashvili, B. Půža, The focal boundary value problem for strongly singular higher-order nonlinear functional-differential equations, *Bound. Value Probl.* **2015**, 2015:17, 21 pp. https://doi.org/10.1186/s13661-014-0277-1; MR3875442; Zbl 1317.34151

[32] J. Neugebauer, Methods of extending lower order problems to higher order problems in the context of smallest eigenvalue comparisons, *Electron. J. Qual. Theory Differ. Equ.* **2011**, No. 99, 1–16. https://doi.org/10.14232/ejqtde.2011.1.99; MR2869457; Zbl 1340.34131

[33] J. Šremr, Solvability conditions of the Cauchy problem for two-dimensional systems of linear functional differential equations with monotone operators, *Math. Bohem.* **132**(2007), No. 3, 263–295. https://doi.org/10.21136/MB.2007.134126; MR2355659; Zbl 1174.34049

[34] G. D. Stepanov, Effective criteria for the strong sign-regularity and the oscillation property of Green functions of two-point boundary value problems, *Sb. Math.* **188**(1997), No. 11, 1687–1728. https://doi.org/10.4213/sm282; MR1601504; Zbl 0906.34021

[35] S. Umamaheswaram, M. Venkata Rama, Existence theorems for focal boundary value problems, *Differential Integral Equations* **4**(1991), No. 4, 883–889. MR1108067; Zbl 0745.34018

[36] P. J. Y. Wong, R. P. Agarwal, Multiple positive solutions of two-point right focal boundary value problems, *Math. Comput. Modelling* **28**(1998), No. 3, 41–49. https://doi.org/10.1016/S0895-7177(98)00097-1; MR1645544; Zbl 1098.34523

[37] P. J. Y. Wong, On the existence of fixed-sign solutions for a system of generalized right focal problems with deviating arguments, *Discrete Contin. Dyn. Syst.* **2007**, Dynamical systems and differential equations. Proceedings of the 6th AIMS International Conference, suppl., 1042–1051. https://doi.org/10.3934/proc.2007.2007.1042; MR2409942; Zbl 1163.34379

[38] B. Yang, Positive solutions for a fourth order boundary value problem, *Electron. J. Qual. Theory Differ. Equ.* **2005**, No. 3, 1–17. https://doi.org/10.14232/ejqtde.2005.1.3; MR2121804; Zbl 1081.34025