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Asymptotic phase for flows with exponentially stable partially hyperbolic invariant manifolds

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Abstract. We consider an autonomous system admitting an invariant manifold $\mathcal{M}$. The following questions are discussed: (i) what are the conditions ensuring exponential stability of the invariant manifold? (ii) does every motion attracting by $\mathcal{M}$ tend to some motion on $\mathcal{M}$ (i.e. have an asymptotic phase)? (iii) what is the geometrical structure of the set formed by orbits approaching a given orbit? We get an answer to (i) in terms of Lyapunov functions omitting the assumption that the normal bundle of $\mathcal{M}$ is trivial. An affirmative answer to (ii) is obtained for invariant manifold $\mathcal{M}$ with partially hyperbolic structure of tangent bundle. In this case, the existence of asymptotic phase is obtained under new conditions involving contraction rates of the linearized flow in normal and tangential to $\mathcal{M}$ directions. To answer the question (iii), we show that a neighborhood of $\mathcal{M}$ has a structure of invariant foliation each leaf of which corresponds to motions with common asymptotic phase. In contrast to theory of cascades, our technique exploits the classical Lyapunov–Perron method of integral equations.

Keywords: invariant manifold, exponential stability, asymptotic phase, partially hyperbolic dynamical system

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1 Introduction

It is well known that, under quite general conditions, motions of dissipative dynamical system evolve towards attracting invariant sets. One may reasonably expect that the behavior of system on attracting set adequately displays main asymptotic properties of system motions in the whole phase space. It is important to note that in many cases the dimension of attracting set such as, e.g., fixed point, limit cycle, invariant torus, strange or chaotic attractor, is essentially lower than the dimension of the total phase space. This circumstance can help us to simplify the qualitative analysis of the system under consideration.

Nevertheless we should keep in mind that there are cases where no motion starting outside the attracting invariant set exhibits the same long time behavior as a motion on the set. As an

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example, consider the polynomial planar system

$$\begin{aligned}\dot{x} &= x(1-x^2-y^2)^3 - y(1+x^2+y^2),\\ \dot{y} &= x(1+x^2+y^2) + y(1-x^2-y^2)^3\end{aligned}$$

which in polar coordinates $(\varphi|\mathrm{mod} 2\pi, r)$ takes the form

$$\begin{aligned}\dot{\varphi} &= 1+r^2,\\ \dot{r} &= r\left(1-r^2\right)^3.\end{aligned}$$

The limit cycle of the system given by $r = 1$ attracts all the orbits except the equilibrium $(0,0)$. Let $\varphi(t;\varphi_0,r_0)$ be the φ-coordinate of the motion starting at point $(r_0\cos\varphi_0, r_0\sin\varphi_0)$. Obviously, $\varphi(t;\varphi_*,1) = 2t+\varphi_*$, but if $r_0 \notin \{0,1\}$, then it is not hard to show that

$$\lim_{t\to\infty} |\varphi(t;\varphi_0,r_0) - \varphi(t;\varphi_*,1)| = \infty \qquad \forall\,\{\varphi_0,\varphi_*\} \subset [0,2\pi),$$

meaning that there is no motion starting outside the cycle and asymptotic to a motion on the cycle (for another examples with non-polynomial planar systems we refer the reader to [11, 14]).

Let $\{\chi^t(\cdot) : \mathfrak{M} \mapsto \mathfrak{M}\}_{t\in\mathbb{R}}$ (resp. $\{\chi^t(\cdot) : \mathfrak{M} \mapsto \mathfrak{M}\}_{t\in\mathbb{Z}}$) be a flow (resp. a cascade) on a metric space $\mathfrak{M}$ with metric $\varrho(\cdot,\cdot)$, and let there exists a χ^t-invariant set $\mathcal{A} \subset \mathfrak{M}$. It is said that a motion $t \mapsto \chi^t(x)$ attracted by $\mathcal{A}$ has an *asymptotic phase* if there exists $z \in \mathcal{A}$ such that

$$\varrho\left(\chi^t(x), \chi^t(z)\right) \to 0, \qquad t \to \infty.$$

The following problem arises: what are the conditions ensuring the existence of asymptotic phase? The answer to this problem is rather important, especially in the case where $\mathcal{A}$ is an attractor with a basin $\mathfrak{B}$. In fact, the existence of asymptotic phase for every $x \in \mathfrak{B}$ guarantees that the flow restricted to attractor $\mathcal{A}$ faithfully describes the long-time behavior of the motions starting in $\mathfrak{B}$.

The above problem was studied in a series of papers. The most complete examination concerns the case where the attracting set is a closed orbit [7, 11, 12, 14, 19, 31]. For more general situation, it is known that if $\mathcal{A}$ is an isolated compact invariant *hyperbolic* set of a cascade, then every motion which is asymptotic to such a set has an asymptotic phase [21, 26]. N. Fenichel [16] established the existence and uniqueness of asymptotic phase for a cascade possessing exponentially stable overflowing invariant manifold with, so-called, expanding structure. A. M. Samojlenko [28] and W. A. Coppel [13] studied the problem for the case of exponentially stable invariant torus. B. Aulbach [4] proved the existence of asymptotic phase for motions approaching a normally hyperbolic invariant manifold under assumption that the latter carries a parallel flow. In [8], A. A. Bogolyubov and Yu. A. Il'in established the existence of asymptotic phase for non-exponentially stable invariant torus under some quite restrictive hypotheses concerning the corresponding system (however the authors do not use the notion of asymptotic phase explicitly).

As was pointed out in [4, 10], standard conditions ensuring the existence of asymptotic phases for motions approaching an invariant set $\mathcal{A}$, involve the requirement that the exponential rate of contraction in the normal to $\mathcal{A}$ direction is greater than that along $\mathcal{A}$ (see, e.g., [6, 16, 28]). Analogous conditions usually appear in the perturbation theory of invariant manifolds (see, e.g. [15, 17, 23, 27, 29] and references therein).

One of the main goals of the present paper is to show that the aforementioned requirement can be weakened in the presence of more accurate information about the character of the flow within the invariant manifold. We consider an autonomous system in $\mathbb{R}^n$ admitting an invariant manifold $\mathcal{M}$ satisfying the following condition of *partial hyperbolicity in the broad sense* [9, 20]: the tangent co-cycle generated by the associated linearized system (system in variations) splits the tangent bundle $T\mathcal{M}$ into a Whitney sum of two invariant sub-bundles V^s and $V^\star$ such that the maximal Lyapunov exponent corresponding to V^s does not exceed some negative number $-\nu$, while the minimal Lyapunov exponent corresponding to $V^\star$ is not less then $-\sigma \in (-\nu, 0)$. (In an important particular case, where the restriction of the flow on $\mathcal{M}$ is an Anosov type dynamical system, the tangent bundle splits into Whitney sum $T\mathcal{M} = V^s \oplus V^c \oplus V^u$ of invariant sub-bundles: stable V^s, center V^c, and unstable V^u. Then $V^\star = V^c \oplus V^u$ and one can consider that $\sigma = 0$.)

It should be stressed that a priori we do not require that $\mathcal{M}$ is a partially hyperbolic set as a subset of the whole space $\mathbb{R}^n$, in particular, the Whitney sum of V^s and normal bundle of $\mathcal{M}$ need not be invariant. Nevertheless, we prove that if the decay rate of solutions of linearized system in normal to $\mathcal{M}$ direction is characterized by a Lyapunov exponent $-\gamma < 0$, then the inequality $\lambda := \min\{\nu, \gamma\} > \sigma$ guarantees both the partial hyperbolicity of $\mathcal{M}$ and the existence of asymptotic phase for all motions starting in a neighborhood of $\mathcal{M}$. Thus, we need not require any additional inequalities involving ν and γ, meaning that our result cover the case $\nu > \gamma$ which, to our knowledge, was excluded in preceding papers concerning the asymptotic phase.

If there holds the inequality $\nu \geq \gamma$, then in contrast to [16], we cannot be sure that the asymptotic phase is unique. The reason lies in the geometrical structure of a neighborhood of $\mathcal{M}$. Namely, let $\mathcal{W}(z)$ be the stable manifold for a point $z \in \mathcal{M}$ [26, p. 88] (i.e. $\mathcal{W}(z)$ is the set of points $x \in \mathbb{R}^n$ such that $\left\|\chi^t(x) - \chi^t(z)\right\| = O\left(\mathrm{e}^{-\lambda t}\right)$, $t \to \infty$). In our case, we cannot exclude that $\mathcal{W}(z_1) = \mathcal{W}(z_2)$ for different points $z_1 \neq z_2$. As a consequence, when proving that every motion starting in a neighborhood of the invariant manifold $\mathcal{M}$ has an asymptotic phase, we are not able to apply the theorem on invariance of domain as in [16] . Our proof is based on the Brouwer fixed point theorem.

In contrast to the technique developed for cascades, e.g., in [16,21–23,26], our main results concerning theory of asymptotic phase are obtained by exploiting the classical Lyapunov–Perron method of integral equations. With this in mind, and targeting on the rather general readers audience we intentionally provide independent proofs of some facts on the invariant manifolds theory already known to specialists in the field. Hope that this will not cause serious objection from experts on the issue.

The present paper is organized as follows. In Section 2, we consider an autonomous non-linear system possessing invariant manifold $\mathcal{M}$ and in terms of Lyapunov functions establish conditions ensuring that $\mathcal{M}$ is exponentially stable. In Section 3, we formulate the main conditions concerning the co-cycle $\left\{X^t\right\}$ generated by system in variations. These include the aforementioned partial hyperbolicity condition of $\left\{X^t\right\}$ on $T\mathcal{M}$ and decay rate condition for $\left\{X^t\right\}$ in normal to $\mathcal{M}$ direction. Next we show that there do exists a X^t-invariant splitting of $T\mathbb{R}^n$ along $\mathcal{M}$ into a direct Whitney sum $W \oplus V^*$ of tangent sub-bundle $V^\star \subset T\mathcal{M}$ and a complementary exponentially stable sub-bundle W. Thus, actually, under the conditions imposed, $\mathcal{M}$ turns out to be a partially hyperbolic subset of $\mathbb{R}^n$ in the sense of [20, Definition 2.1, p. 8]. Due to this circumstance, for any orbit $\mathcal{O}(z) \subset \mathcal{M}$, there is a local stable invariant manifold through $\mathcal{O}(z)$ tangent to W along this orbit. Each motion starting at this invariant manifold exponentially approaches a motion on $\mathcal{O}(z)$ as $t \to \infty$ (see Section 4). In Section 5, we prove the main theorem which states that the union of all local stable invariant manifolds form an open neighborhood of $\mathcal{M}$. The global geometrical aspects of the exposed theory and some generalizations are discussed in Sections 6 and 7. Finally, in Section 8, we apply the main theorem to a system defined on cotangent bundle of a compact homogeneous space $\mathrm{SL}(2;\mathbb{R})/\Gamma$.

2 Exponential stability of invariant manifold

Let v be a C^2-vector field in a domain $\mathcal{D}$ of the space $\mathbb{R}^n$ endowed with the standard scalar product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$. Assume that the vector field v is complete, i.e.the corresponding autonomous system

$$\dot{x} = v(x) \tag{2.1}$$

generates the flow$\{\chi^t(\cdot) : \mathcal{D} \mapsto \mathcal{D}\}_{t\in\mathbb{R}}$, and let this system possesses an m-dimensional compact χ^t-invariant C^2-sub-manifold $\mathcal{M} \overset{\iota}{\hookrightarrow} \mathcal{D}$, where $\iota(\cdot) : \mathcal{M} \mapsto \mathbb{R}^n$ stands for an isometric inclusion map.

Introduce some notations. Denote by $N_z\mathcal{M}$ the orthogonal complement of the tangent space $T_z\mathcal{M}$ at $z \in \mathcal{M}$. For the sake of simplifying notations, it will be convenient for us to identify $T_z\mathbb{R}^n$ with $\mathbb{R}^n$and to treat both $T_z\mathcal{M}$ and $N_z\mathcal{M}$ as linear sub-spaces of $\mathbb{R}^n$. Thus, for any given $z \in \mathcal{M}$, we have $T_z\mathbb{R}^n = T_z\mathcal{M} \oplus N_z\mathcal{M}$, and the vector bundle $\coprod_{z\in\mathcal{M}} T_z\mathbb{R}^n$ splits into Whitney sum of the tangent and normal sub-bundles

$$\coprod_{z\in\mathcal{M}} T_z\mathbb{R}^n = T\mathcal{M} \oplus N\mathcal{M}, \qquad T\mathcal{M} := \coprod_{z\in\mathcal{M}} T_z\mathcal{M}, \qquad N\mathcal{M} := \coprod_{z\in\mathcal{M}} N_z\mathcal{M}.$$

Let $\pi : T\mathcal{M} \oplus N\mathcal{M} \mapsto \mathcal{M}$ stands for the natural vector bundle projection. As is well known, there exists sufficiently small $r > 0$ such that the set $N\mathcal{M}_r = \{\xi \in N\mathcal{M} : \|\xi\| < r\}$ can be identified with a tubular neighborhood of $\mathcal{M}$. Namely, the mapping $N\mathcal{M}_r \ni \xi \mapsto z + \xi \in \mathbb{R}^n$, where $z = \pi(\xi)$, define a natural embedding $N\mathcal{M}_r \hookrightarrow \mathbb{R}^n$. Let the vector bundle mappings $P_N : T\mathcal{M} \oplus N\mathcal{M} \mapsto N\mathcal{M}$ and $P_T : T\mathcal{M} \oplus N\mathcal{M} \mapsto T\mathcal{M}$ stand for the orthogonal projections onto $N\mathcal{M}$ and $T\mathcal{M}$ respectively.

There naturally arise problems concerning the behavior of the flow in a neighborhood of $\mathcal{M}$, in particular the stability problem of $\mathcal{M}$. The first step in solving the latter is to study the so-called normal co-cycle generated by the system in variations w.r.t. a given motion $t \mapsto \chi^t(x)$ of a point $x \in \mathcal{D}$

$$\dot{y} = v'\left(\chi^t(x)\right) y. \tag{2.2}$$

As is well known, the group property of the flow, $\chi^{t+\tau}(\cdot) = \chi^t \circ \chi^\tau(\cdot)$ for all $t, \tau \in \mathbb{R}$, implies the co-cycle property of the the corresponding evolution operator

$$X^t(x) := \frac{\partial\chi^t(x)}{\partial x},$$

namely

$$X^{t+\tau}(x) = X^t\left(\chi^\tau(x)\right) X^\tau(x), \qquad X^{-\tau}\left(\chi^\tau(x)\right) = \left[X^\tau(x)\right]^{-1} \qquad \forall t,\tau \in \mathbb{R},\ \forall x \in \mathcal{D}, \tag{2.3}$$

and the χ^t-invariance of $\mathcal{M}$ implies the X^t-equivariance of fibers of vector bundle $T\mathcal{M} \oplus N\mathcal{M}$ and its sub-bundle $T\mathcal{M}$, meaning that for each $z \in \mathcal{M}$ and $t \in \mathbb{R}$ there hold

$$\begin{gathered} X^t(z)\left(T\mathcal{M} \oplus N\mathcal{M}\right)|_z = \left(T\mathcal{M} \oplus N\mathcal{M}\right)|_{\chi^t(z)}, \\ X^t(z)\, T_z\mathcal{M} = T_{\chi^t(z)}\mathcal{M}. \end{gathered} \tag{2.4}$$

In other words, the linear co-cycle $\{X^t\}_{t\in\mathbb{R}}$ over the flow $\{\chi^t(\cdot) : \mathcal{M} \mapsto \mathcal{M}\}_{t\in\mathbb{R}}$ defines a one-parameter family of automorphisms both of $T\mathcal{M} \oplus N\mathcal{M}$ and $T\mathcal{M}$. As a result, we obtain

$$X^t P_T = P_T X^t P_T, \qquad P_N X^t = P_N X^t (P_N + P_T) = P_N X^t P_N. \tag{2.5}$$

Note that the fibers of $N\mathcal{M}$ need not be X^t-equivariant. At the same time, the one-parameter family of mappings (the normal co-cycle)

$$X_N^t(z) := P_N X^t(z) : N_z\mathcal{M} \mapsto N_{\chi^t(z)}\mathcal{M}, \qquad t \in \mathbb{R},$$

possesses the required property:

$$\begin{aligned} X_N^{t+s}(z) &= P_N(\chi^{t+s}(z))X^{t+s}(z) = P_N\left(\chi^t \circ \chi^s(z)\right) X^t\left(\chi^s(z)\right) X^s(z) \\ &= P_N\left(\chi^t \circ \chi^s(z)\right) X^t\left(\chi^s(z)\right) P_N(\chi^s(z))X^s(z) = X_N^t(\chi^s(z))X_N^s(z). \end{aligned}$$

One can expect that the invariant manifold $\mathcal{M}$ will be stable provided that $\|X_N^t\|$ tends to zero as $t \to \infty$ sufficiently fast. Following [25, 28, 29], to approve the correctness of such a hypothesis, we shall exploit the apparatus of Lyapunov functions. Proposition 2.1 given below is a direct generalization of results [25] obtained for the case where $\mathcal{M}$ is a torus with trivial normal bundle.

Proposition 2.1. *The following statements are equivalent:*

(i) the integral $\int_0^\infty \|X_N^s(z)\|^2\,\mathrm{d}s$ is uniformly convergent w.r.t. z;

(ii) there exist positive constants γ and c_0 such that

$$\|X_N^t(z)\| \le c_0 \mathrm{e}^{-\gamma t} \qquad \forall t \ge 0; \tag{2.6}$$

(iii) there exists a continuous field of positive definite symmetric operators

$$\{S(z) : N_z\mathcal{M} \mapsto N_z\mathcal{M}\}_{z\in\mathcal{M}}$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\rangle = -\|\xi\|^2 \qquad \forall z \in \mathcal{M},\ \forall \xi \in N_z\mathcal{M}. \tag{2.7}$$

Proof. To show that (i)⇒(ii) and (i)⇒(iii), define the continuous field of positive definite symmetric operators on fibers of $N\mathcal{M}$ by

$$\langle S(z)\xi, \xi\rangle := \int_0^\infty \|X_N^s(z)\xi\|^2\,\mathrm{d}s \qquad \forall z \in \mathcal{M},\ \forall \xi \in N_z\mathcal{M}. \tag{2.8}$$

Due to the compactness of $\mathcal{M}$ there are positive constants a and A such that

$$a\|\xi\|^2 \le \langle S(z)\xi, \xi\rangle \le A\|\xi\|^2 \qquad \forall z \in \mathcal{M},\ \forall \xi \in N_z\mathcal{M}. \tag{2.9}$$

Since

$$\begin{aligned} \langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\rangle &= \int_0^\infty \|X_N^s(\chi^t(z))X_N^t(z)\xi\|^2\,\mathrm{d}s \\ &= \int_0^\infty \|X_N^{t+s}(z)\xi\|^2\,\mathrm{d}s = \int_t^\infty \|X_N^s(z)\xi\|^2\,\mathrm{d}s, \end{aligned}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\right\rangle = -\left\|X_N^t(z)\xi\right\|^2 \le -\frac{1}{A}\left\langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\right\rangle. \tag{2.10}$$

Hence,

$$\left\langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\right\rangle \le \mathrm{e}^{-t/A}\left\langle S(z)\xi,\xi\right\rangle \qquad \forall t \ge 0,$$

and thus,

$$\left\|X_N^t(z)\xi\right\|^2 \le \frac{A}{a}\mathrm{e}^{-t/A}\left\|\xi\right\|^2 \qquad \forall t \ge 0.$$

It is obvious, that (2.10) implies (2.7), and (ii)⇒(i).

It remains to show that (iii)⇒(ii). If (2.7) is satisfied, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\right\rangle = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\left\langle S(\chi^{t+s}(z))X_N^{t+s}(z)\xi, X_N^{t+s}(z)\xi\right\rangle$$
$$= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\left\langle S(\chi^s\circ\chi^t(z))X_N^s\left(\chi^t(z)\right)X_N^t(z)\xi, X_N^s\left(\chi^t(z)\right)X_N^t(z)\xi\right\rangle = -\left\|X_N^t(z)\xi\right\|^2.$$

This ensures inequality (2.10), which implies (2.6) with $c_0 = A/a$ and $\gamma = 1/A$. □

As in the case where $\mathcal{M}$ is a torus with trivial normal bundle, the additional requirement of continuous differentiability of $S(\cdot)$ together with (2.7) ensures exponential stability of $\mathcal{M}$.

Proposition 2.2. *Let there exist a continuously differentiable field of positive definite symmetric operators*

$$\{S(z) : N_z\mathcal{M} \mapsto N_z\mathcal{M}\}_{z\in\mathcal{M}}$$

satisfying (2.7). Then the invariant manifold $\mathcal{M}$ is exponentially stable.

Proof. Let $x \in N\mathcal{M}_r$. Then there is a unique representation $x = z(x) + \xi(x)$ where $z(x) \in \mathcal{M}$, $\xi(x) \in N_z\mathcal{M}$. Define the function $V(x) := \langle S(z(x))\xi(x), \xi(x)\rangle$. To calculate the derivative $\dot{V}_v(x)$ of this function along the vector $v(x)$, consider a finite open cover $\bigcup_{i=1}^{I}\mathcal{U}_i$ of $\mathcal{M}$ with the following properties: the restriction of normal bundle to every $\mathcal{U}_i$ is trivial, and there exist compact subsets $\mathcal{K}_i \subset \mathcal{U}_i$, $i = 1,\dots,I$, such that $\bigcup_{i=1}^{I}\mathcal{K}_i = \mathcal{M}$.

Let $\mathcal{U}$ stands for one of the sets $\mathcal{U}_1,\dots,\mathcal{U}_I$ and $\mathcal{K} \in \{\mathcal{K}_1,\dots,\mathcal{K}_I\}$ be the corresponding compact subset, thus $\mathcal{K} \subset \mathcal{U}$. Then there exist C^1-mappings $\nu_k(\cdot) : \mathcal{U} \mapsto N\mathcal{M}$, $k = 1,\dots,n-m$, such that for any $z \in \mathcal{U}$ the vectors $\nu_1(z),\dots,\nu_{n-m}(z)$ form an orthonormal basis of $N_z\mathcal{M}$. Compose the matrix $\mathbf{N}(z)$ of the vectors $\nu_1(z),\dots,\nu_{n-m}(z)$ as columns and denote by $\mathbf{N}^\top(z)$ the transposed matrix. Then $\mathbf{P}_N(z) := \mathbf{N}(z)\mathbf{N}^\top(z)$ and $\mathbf{P}_T(z) := \mathbf{Id} - \mathbf{P}_N(z)$ are matrices of projections $P_N(z)$ and $P_T(z)$ respectively. Now by means of the diffeomorphism

$$\mathcal{U} \times \mathcal{B}_r^{n-m}(0) \ni (z,p) \mapsto z + \mathbf{N}(z)p \in N\mathcal{M}_r \tag{2.11}$$

where $p := (p_1,\dots,p_{n-m})$, $\mathcal{B}_r^{n-m}(0) := \{p : \|p\| < r\}$ and r is sufficiently small, we obtain a system on $\mathcal{U} \times B_r^{n-m}$ induced by system (2.1). Namely, we have

$$\left(\mathbf{Id} + [\mathbf{N}(z)p]_z'\right)\dot{z} + \mathbf{N}(z)\,\dot{p} = v\left(z + \mathbf{N}(z)p\right),$$

and taking into account that $v(z) \perp \nu_i(z)$, $i = 1,\dots,n-m$, the induced system on $\mathcal{U} \times B_r^{n-m}$ takes the form

$$\dot{z} = v(z) + v_1(z,p), \qquad \dot{p} = [\mathbf{A}(z) + \mathbf{A}_1(z,p)]\,p, \tag{2.12}$$

where

$$\mathbf{A}(z) := \mathbf{N}^{\top}(z)\left[\mathbf{J}(z)\,\mathbf{N}(z) - \dot{\mathbf{N}}_{v(z)}(z)\right], \tag{2.13}$$
$$v_1(z,p) := \left(\mathbf{Id} + [\mathbf{N}(z)p]'_z\right)^{-1}\mathbf{P}_T(z)v\left(z+\mathbf{N}(z)p\right) - v(z),$$
$$A_1(z,p)p := \mathbf{N}^{\top}(z)\left(v(z+\mathbf{N}(z)p) - v(z) - \mathbf{J}(z)\mathbf{N}(z)p - [\mathbf{N}(z)p]'_z\, v_1(z,p)\right).$$

Here $\mathbf{J}(z)$ is the Jacobi matrix of the mapping $x \mapsto v(x)$ at the point $x = z$, and $\dot{\mathbf{N}}_{v(z)}(z) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{s=0}\mathbf{N}\left(\chi^t(z)\right)$. It is not hard to see that there exists a constant $C > 0$ such that

$$\|v_1(z,p)\| \le C\,\|p\|\,, \qquad \|A_1(z,p)\| \le C\,\|p\| \qquad \forall z \in \mathcal{K},\ \forall p \in \mathcal{B}_r^{n-m}(0). \tag{2.14}$$

Obviously, since $\mathcal{M}$ is compact, one can choose a common constant C for all $\mathcal{K}_1, \dots, \mathcal{K}_I$.

Note that locally the diffeomorphism (2.11) conjugates the system on $N\mathcal{M}$ generating the normal co-cycle $\{X_N^t\}$ with the system

$$\dot{z} = v(z)\,, \qquad \dot{p} = \mathbf{A}(z)p.$$

Hence, for $\xi = \mathbf{N}(z)p$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}&\langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\rangle\\ &= \left(\langle S(z)\mathbf{N}(z)p, \mathbf{N}(z)p\rangle\right)'_p\mathbf{A}(z)p + \left(\langle S(z)\mathbf{N}(z)p, \mathbf{N}(z)p\rangle\right)'_z v(z) = -\|\xi\|^2,\end{aligned}$$

and thus

$$\dot{V}_v(x) = -\|\xi\|^2 + \left(\langle S(z)\mathbf{N}(z)p, \mathbf{N}(z)p\rangle\right)'_p\mathbf{A}_1(z,p)p + \left(\langle S(z)\mathbf{N}(z)p, \mathbf{N}(z)p\rangle\right)'_z v_1(z,p).$$

Since $\|\xi\| = \|p\|$ and there are positive constants A and a such that $S(z)$ satisfies (2.9), then on account of (2.14) there holds the inequality

$$\dot{V}_v(x) \le -\frac{1}{2}\|\xi\|^2 \le -\frac{1}{2A}V(x) \qquad \forall x \in N\mathcal{M}_r$$

provided that r is sufficiently small. By means of the last inequality one can show in a standard way that there exists $\delta \in (0,r)$ such that $\|\chi^t(x) - \pi\left(\chi^t(x)\right)\|$ tends to zero with exponential rate as $t \to \infty$ provided that $x \in N\mathcal{M}_\delta$. □

3 Invariant splitting of vector bundle along invariant manifold

Let us agree on the following. Hereinafter, if $\xi \in T\mathcal{M} \oplus N\mathcal{M}$ and $z = \pi(\xi)$, then $X^t\xi := X^t(z)\,\xi$, and $X^tX^\tau\xi := X^t\left(\chi^\tau(z)\right)X^\tau(z)\xi$ for all $t, \tau \in \mathbb{R}$.

Assume that the following conditions are fulfilled:

H1 The tangent bundle $T\mathcal{M}$ splits into a continuous Whitney sum $T\mathcal{M} = V^s \oplus V^\star$ of X^t-invariant vector sub-bundles $V^s = \coprod_{z\in\mathcal{M}} V_z^s$, $V^\star = \coprod_{z\in\mathcal{M}} V_z^\star$ (i.e. fibers of vector bundles $V^s$and $V^\star$are X^t-equivariant), and there exist constants $c_0 \ge 1$, $\nu > 0$, $\sigma \in [0,\nu)$ such that

$$\left\|X^t\xi\right\| \le c_0\mathrm{e}^{-\nu t}\,\|\xi\| \qquad \forall t \ge 0,\ \forall \xi \in V^s, \tag{3.1}$$
$$\left\|X^t\xi\right\| \le c_0\mathrm{e}^{-\sigma t}\,\|\xi\| \qquad \forall t \le 0,\ \forall \xi \in V^\star. \tag{3.2}$$

H2 There exists $\gamma > \sigma$ such that

$$\left\| P_N X^t P_N \right\| \le c_0 \mathrm{e}^{-\gamma t} \qquad \forall t \ge 0.$$

It should be noted that the last inequality actually matches (2.6) and on account of (2.5) implies

$$\left\| P_N X^t \right\| \le c_0 \mathrm{e}^{-\gamma t} \qquad \forall t \ge 0. \tag{3.3}$$

Besides, (3.2) together with (2.3) implies

$$\left\| X^t \xi \right\| \ge c_0^{-1} \mathrm{e}^{-\sigma t} \left\| \xi \right\| \qquad \forall t \ge 0,\ \forall \xi \in V^\star. \tag{3.4}$$

Note also that the sub-bundle $V^\star$ contains 1-D X^t-invariant sub-bundle $V^c := \{\theta v\}_{\theta \in \mathbb{R}}$ generated by the vector field v. Each solution of (2.2) with initial value in V^c is bounded. An important particular case is when $\mathcal{M}$ is hyperbolic, i.e. there is X^t-invariant splitting $V^\star = V^c \oplus V^u$ such that

$$\left\| X^t \xi \right\| \le c_0 \mathrm{e}^{\nu t} \left\| \xi \right\| \qquad \forall t \le 0,\ \forall \xi \in V^u.$$

In this case we consider that $\sigma = 0$.

Define the natural projections

$$P_s : T\mathcal{M} \mapsto V^s, \qquad P_\star : T\mathcal{M} \mapsto V^\star.$$

Since the splitting $V^s \oplus V^\star$ is X^t-invariant, then

$$X^t P_{s,\star} P_T = P_{s,\star} X^t P_T \qquad \forall t \in \mathbb{R}. \tag{3.5}$$

On account of (2.3) and (3.5), we get

$$X^{t-\tau}\left(\chi^\tau(z)\right) = X^t(z) X^{-\tau}\left(\chi^\tau(z)\right) = \left[X^t(z)\right]\left[X^\tau(z)\right]^{-1} \tag{3.6}$$

and thus,

$$2X^t P_{s,\star}\left[X^\tau\right]^{-1} P_T = \left[X^t\right]\left[X^\tau\right]^{-1} P_{s,\star} P_T = X^{t-\tau} P_{s,\star} P_T. \tag{3.7}$$

Now **H1** yields that there exists a positive constant c_1 such that

$$\begin{aligned} \left\| X^t P_s \left[X^\tau\right]^{-1} P_T \right\| &\le c_1 \mathrm{e}^{-\nu(t-\tau)}, \qquad 0 \le \tau \le t, \\ \left\| X^t P_\star \left[X^\tau\right]^{-1} P_T \right\| &\le c_1 \mathrm{e}^{-\sigma(t-\tau)}, \qquad 0 \le t < \tau \end{aligned} \tag{3.8}$$

In what follows, for any $\xi \in T\mathcal{M} \oplus N\mathcal{M}$, we will use the notations

$$\xi_T := P_T \xi, \qquad \xi_N := P_N \xi, \qquad \xi_{s,\star} := P_{s,\star} P_T \xi.$$

Proposition 3.1. *There exists a continuous X^t-invariant splitting of $T\mathcal{M} \oplus N\mathcal{M}$ into a Whitney sum $W \oplus V^\star$ such that $P_N W = N\mathcal{M}$, and there is a positive constant c such that*

$$\left\| X^t \xi \right\| \le c \mathrm{e}^{-\lambda t} \left\| \xi \right\| \qquad \forall t \ge 0,\ \forall \xi \in W \tag{3.9}$$

where $\lambda := \min\{\nu, \gamma\}$.

Proof. Let us construct a sub-bundle of vectors $\xi \in T\mathcal{M} \oplus N\mathcal{M}$, such that $\left\| X^t \xi \right\|$ has a Lyapunov exponent not exceeding $-\lambda$. Since

$$X^t \xi = P_T X^t \xi + P_N X^t \xi,$$

then, on account of (3.3), it remains to deal with $P_T X^t \xi$. Derive an equation for $P_T X^t \xi$. Since

$$P_T X^t \xi = X^t \xi - P_N X^t \xi = X^t \xi - P_N P_N X^t \xi$$

and the map $\mathcal{M} \ni z \mapsto P_N(z)$ is continuously differentiable, then

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} P_T X^t \xi = v' X^t \xi - \frac{\mathrm{d}}{\mathrm{d}t}\left(P_N P_N X^t \xi\right) \qquad \Longrightarrow \\ &\frac{\mathrm{d}}{\mathrm{d}t} P_T X^t \xi = v' P_T X^t \xi + v' P_N X^t \xi - \left(P_N' v\right) P_N X^t \xi - P_N \frac{\mathrm{d}}{\mathrm{d}t}\left(P_N X^t \xi\right). \end{aligned}$$

In view of (2.5), we get

$$P_T \frac{\mathrm{d}}{\mathrm{d}t} P_T X^t \xi = P_T v' P_T X^t \xi + P_T \left(v' - P_N' v\right) P_N X^t \xi_N.$$

Recall that, for a given vector field $\mathbb{R} \ni t \mapsto \eta(t) \in T_{z(t)}\mathcal{M}$ along a curve $z(\cdot) : \mathbb{R} \mapsto \mathcal{M}$ and for any $t \in \mathbb{R}$, the vector $P_T \dot{\eta}(t)$ is nothing else but the covariant derivative $\nabla_{\dot{z}} \eta(t)$ at point $z(t)$. Hence, for every ξ such that $\pi(\xi) = z$, the vector field $\eta(t;\xi) := P_T X^t \xi$ along the curve $t \mapsto \chi^t(z)$ is a unique solution of the initial problem

$$\nabla_{\dot{z}} \eta = P_T v'\left(\chi^t(z)\right) \eta + P_T Q(t)\, \xi_N, \qquad \eta(0) = \xi_T, \tag{3.10}$$

where the vector bundle homomorphism $Q(t)$ is defined by

$$Q(t)\,\xi = \left[v'\left(\chi^t(z)\right) - P_N'\left(\chi^t(z)\right) v\left(\chi^t(z)\right)\right] P_N X^t(z)\,\xi \qquad \forall \xi \in T_z\mathcal{M} \oplus N_z\mathcal{M}. \tag{3.11}$$

It turns out that the set of solutions of problem (3.10), which we are interested in, is given by

$$\eta(t;\xi) = X^t \xi_s + \int_0^\infty \Gamma(t,\tau)\, P_T Q(\tau)\, \xi_N \mathrm{d}\tau \tag{3.12}$$

where $\xi_s \in V^s$ is taken at will and

$$\Gamma(t,\tau) := \begin{cases} X^t P_s \left[X^\tau\right]^{-1}, & 0 \le \tau \le t \\ -X^t P_\star \left[X^\tau\right]^{-1}, & 0 \le t < \tau. \end{cases}$$

In fact, taking into account (3.8), one can choose a constant $c_2 > 0$ such that

$$\begin{aligned} \left\| X^t P_s \left[X^\tau\right]^{-1} P_T Q(\tau) \right\| &\le c_2 \mathrm{e}^{-\nu t + (\nu - \gamma)\tau}, \qquad 0 \le \tau \le t, \\ \left\| X^t P_\star \left[X^\tau\right]^{-1} P_T Q(\tau) \right\| &\le c_2 \mathrm{e}^{-\sigma t + (\sigma - \gamma)\tau}, \qquad 0 \le t < \tau. \end{aligned}$$

Hence, there exists a positive constant $c_3 > 0$ such that

$$\|\eta(t;\xi)\| \le \left\|X^t \xi_s\right\| + \int_0^\infty \|\Gamma(t,\tau)\, P_T Q(\tau)\, P_N\| \,\mathrm{d}\tau\, \|\xi\| \le c_3 \mathrm{e}^{-\lambda t} \|\xi\|, \qquad t \ge 0.$$

By means of direct calculations, one can easily verify that $\eta(\cdot;\xi)$ is a unique solution of the initial problem for linear inhomogeneous system

$$\dot{y} = v'\left(\chi^t(z)\right) y + P_T Q(t)\, \xi_N, \qquad y(0) = \xi_T \in T_x\mathcal{M}, \tag{3.13}$$

where

$$\xi_T = \xi_s + \Xi\xi_N, \quad \Xi\xi := -\int_0^\infty P_\star [X^s]^{-1} P_T Q(s) P_N \xi \mathrm{d}s. \tag{3.14}$$

Since $P_T\eta(t;\xi) \equiv \eta(t;\xi)$, then $\eta(\cdot;\xi)$ satisfies both (3.13) and (3.10).

Hence, for arbitrary ξ_s, ξ_N, we have found $\xi = \xi_s + \Xi\xi_N + \xi_N$ such that

$$X^t\xi = P_T X^t \xi + P_N X^t \xi_N = \eta(t;\xi) + P_N X^t \xi_N$$

and thus,

$$\left\|X^t\xi\right\| \le (c_3 + c_0)\,\mathrm{e}^{-\lambda t}\,\|\xi\| \qquad \forall t \ge 0.$$

Now it is naturally to define the projection

$$\Pi := P_s P_T + \Xi + P_N,$$

and the corresponding sub-bundle

$$W := \Pi\,(T\mathcal{M} \oplus N\mathcal{M})\,.$$

The uniform convergence of integral (3.14) ensures that the splitting $W \oplus V^*$ is continuous. One can easily verify that Π has the projection property $\Pi^2 = \Pi$. Besides, $P_N W = P_N N\mathcal{M} = N\mathcal{M}$.

It remains to verify that the splitting $W \oplus V^\star$ is X^t-invariant. Note that if $\xi \notin W$, than on account of (3.4) the Lyapunov exponent of $\left\|X^t\xi\right\|$ exceeds $-\lambda$. Since,

$$\left\|X^t X^\tau \xi\right\| = \left\|X^{t+\tau}\xi\right\| \le (c_3 + c_0)\,\mathrm{e}^{-\lambda(t+\tau)}\,\|\xi\|$$

for any $\xi \in W$, $\tau \in \mathbb{R}$ and $t \ge -\tau$, then the Lyapunov exponent of $\left\|X^t X^\tau\xi\right\|$ does not exceed $-\lambda$. Hence, $X^\tau\xi \in W$ for all $\tau \in \mathbb{R}$, provided that $\xi \in W$. Thus $X^t W \subseteq W$, and since X^t is non-degenerate, then $X^t W = W$. As a consequence, $\Pi X^t\xi = X^t \Pi\xi$ for any $\xi \in W$, but since both W and V^* are X^t-invariant, than the above equality holds true for any $\xi \in T\mathcal{M} \oplus N\mathcal{M}$. This yields that $\mathrm{Id} - \Pi$ commutes with X^t as well:

$$(\mathrm{Id} - \Pi)\,X^t\xi = X^t\xi - \Pi X^t \xi = X^t(\xi - \Pi\xi) = X^t(\mathrm{Id} - \Pi)\,\xi \qquad \forall \xi \in T\mathcal{M} \oplus N\mathcal{M}. \qquad \square$$

Corollary 3.2. *There is a constant $K > 0$ such that the following inequalities hold true:*

$$\left\|X^t \Pi\,[X^\tau]^{-1}\right\| \le K\mathrm{e}^{-\lambda(t-\tau)}, \qquad 0 \le \tau \le t,$$
$$\left\|X^t(\mathrm{Id} - \Pi)\,[X^\tau]^{-1}\right\| \le K\mathrm{e}^{-\sigma(t-\tau)}, \qquad 0 \le t < \tau.$$

4 Existence of local exponentially stable set for a given orbit

After introducing the new variable y by

$$x = \chi^t(z) + y,$$

system (2.1) takes the form

$$\dot{y} = v'\left(\chi^t(z)\right) y + w(t, z, y) \tag{4.1}$$

where
$$w(t,z,y) := v\left(\chi^t(z)+y\right) - v\left(\chi^t(z)\right) - v'\left(\chi^t(z)\right)y,$$
and $z \in \mathcal{M}$ is considered as a parameter. From now on throughout this section, we do not show explicitly the variable z among arguments of mappings whenever it does not cause a confusion.

In order to apply the Lyapunov–Perron method of integral equations, introduce the Green function
$$G(t,\tau) := \begin{cases} X^t\Pi\left[X^\tau\right]^{-1}, & 0 \le \tau \le t, \\ X^t\left(\Pi\text{-}\mathrm{Id}\right)\left[X^\tau\right]^{-1}, & 0 \le t < \tau \end{cases}$$
and use the following standard statement.

Proposition 4.1. *A mapping $y(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}^n$ with upper Lyapunov exponent not exceeding $-\lambda$ is a solution of (4.1) if and only if there is $\zeta \in W \cap \pi^{-1}(z)$ such that $y(\cdot) = y(\cdot,\zeta)$ satisfies the integral equation*
$$y(t,\zeta) = X^t\zeta + \int_0^\infty G(t,\tau)w(\tau,y(\tau,\zeta))\mathrm{d}\tau =: \mathcal{G}[y](t,\zeta), \tag{4.2}$$
as well as the condition $\Pi y(0,\zeta) = \zeta$.

Proof. Note that Corollary 3.2 together with inequality $\lambda > \sigma$ yields
$$\begin{aligned}\int_0^\infty \mathrm{e}^{-2\lambda\tau}\left\|G(t,\tau)\right\|\mathrm{d}\tau &\le K\mathrm{e}^{-\lambda t}\left[\int_0^t \mathrm{e}^{-\lambda\tau}\mathrm{d}\tau + \mathrm{e}^{(\lambda-\sigma)t}\int_t^\infty \mathrm{e}^{(\sigma-\lambda)\tau}\mathrm{e}^{-\lambda\tau}\mathrm{d}\tau\right] \\ &\le K\mathrm{e}^{-\lambda t}\int_0^\infty \mathrm{e}^{-\lambda\tau}\mathrm{d}\tau \le \frac{K}{\lambda}\mathrm{e}^{-\lambda t},\end{aligned} \tag{4.3}$$
and since v is C^2-vector field, then $\|w(t,z,y)\| = O\big(\|y\|^2\big)$ as $\|y\| \to 0$. If now $y(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}^n$ is a solution of (4.1) with upper Lyapunov exponent not exceeding $-\lambda$, then by means of direct calculations it is not hard to verify that
$$\tilde{y}(t) := \int_0^\infty G(t,\tau)w(\tau,y(\tau))\mathrm{d}\tau = O\big(\mathrm{e}^{-\lambda t}\big), \quad t \to \infty,$$
is a solution of the linear non-homogeneous system
$$\dot{y} = v'\left(\chi^t(z)\right)y + w(t,z,y(t)).$$
The last one has the solution $t \mapsto y(t) = O\left(\mathrm{e}^{-\lambda t}\right)$, $t \to \infty$, as well. Hence, there exists $\zeta \in W \cap \pi^{-1}(z)$ such that $y(t) - \tilde{y}(t) = X^t\zeta$. From $\Pi\tilde{y}(0) = 0$ it follows that $\Pi y(0) = \zeta$.

Vice versa, by means of direct calculations one can easily verify that any solution $t \mapsto y(t,\zeta) = O\left(\mathrm{e}^{-\lambda t}\right), t \to \infty$, of (4.2) is a solution of (4.1) such that $\Pi y(0,\zeta) = \zeta$. □

By means of the mapping Id $+\Xi$ (see (3.14)), we define an isomorphic image of $N\mathcal{M}_r$ as
$$U_r := (\mathrm{Id}+\Xi)\left(N\mathcal{M}_r\right) \equiv \bigcup_{\zeta \in N\mathcal{M}_r}\left\{\zeta + \Xi\zeta\right\}.$$
Note that $P_N U_r = N\mathcal{M}_r$, and if we introduce the set
$$W_r := \left\{\zeta \in W : \|\zeta\| < r\right\},$$

then $U_r = \{\zeta \in W_r : P_s P_T \zeta = 0\}$.

Let $C(\mathbb{R}_+ \times W_r \mapsto \mathbb{R}^n; \|\cdot\|_\lambda)$ stands for a Banach space of mappings endowed with the norm

$$\|\cdot\|_\lambda := \sup_{(t,\zeta)\in\mathbb{R}_+\times W_r} e^{\lambda t}\|\cdot\|.$$

For a constant $C > 0$, define the closed subset

$$\mathcal{Y}_{r,C} := \left\{ y(\cdot,\cdot) \in C(\mathbb{R}_+ \times W_r \mapsto \mathbb{R}^n; \|\cdot\|_\lambda) : \left\|y(t,\zeta) - X^t\zeta\right\| \le Ce^{-\lambda t}\|\zeta\|^2 \right\}.$$

Proposition 4.2. *There exist positive numbers r and C such that:*

(i) *equation (4.1) has a unique solution $y_*(\cdot,\cdot) \in \mathcal{Y}_{r,C}$;*

(ii) *the mapping $y_*(\cdot,\cdot)$ has a continuous derivative along every fiber $W(z) := W \cap \pi^{-1}(z)$, $z \in \mathcal{M}$.*

Proof. One can prove assertion (i) in a standard way by means of the Banach contraction principle. For the sake of completeness, we present here some essential details.

Firstly, impose conditions on r, C ensuring inclusion $\mathcal{G}[\mathcal{Y}_{r,C}] \subset \mathcal{Y}_{r,C}$. Since v is C^2-vector field, then there is a constant $C_w > 0$ such that

$$\|w(t,y,z)\| \le \frac{C_w}{2}\|y\|^2, \qquad \left\|w'_y(t,y,z)\right\| \le C_w\|y\|, \qquad \left\|w''_{yy}(t,y,z)\right\| \le C_w \tag{4.4}$$

for all $(t,z) \in \mathbb{R}\times\mathcal{M}$, $\|y\| \le 1$. Now, on account of (4.3), for any $y(\cdot,\cdot) \in \mathcal{Y}_{r,C}$, we obtain

$$\Pi\mathcal{G}[y](0,\zeta) = \Pi\zeta = \zeta,$$

$$\left\|\mathcal{G}[y](t,\zeta) - X^t\zeta\right\| \le \frac{KC_w}{2\lambda}(c + Cr)^2 e^{-\lambda t}\|\zeta\|^2 \le Ce^{-\lambda t}\|\zeta\|^2$$

provided that

$$cr + Cr^2 < 1, \qquad \frac{KC_w}{2\lambda}(c + Cr)^2 \le C.$$

If we set $C := 2KC_w c^2/\lambda$ then it is sufficient to require that r is small enough to satisfy the inequalities

$$2cr < 1, \qquad Cr \le c. \tag{4.5}$$

Now let us find conditions under which $\mathcal{G}[\cdot]$ is a contraction mapping in $\mathcal{Y}_{r,C}$. Since

$$\begin{aligned}\|w(t,y_1) - w(t,y_2)\| &\le \left\|\int_0^1 \left[v'\left(\chi^t(z) + \theta y_1 + (1-\theta)y_2\right) - v'\left(\chi^t(z)\right)\right] d\theta\right\| \|y_1 - y_2\| \\ &\le \frac{C_w}{2}(\|y_1\| + \|y_2\|)\|y_1 - y_2\| \qquad \forall y_1, y_2 : \|y_1\|, \|y_2\| \le 1,\end{aligned}$$

then for every $y_1(\cdot,\cdot), y_2(\cdot,\cdot) \in \mathcal{Y}_{r,C}$ we obtain

$$\|\mathcal{G}[y_1](t,\zeta) - \mathcal{G}[y_2](t,\zeta)\|_\lambda \le \frac{KC_w}{\lambda}(cr + Cr^2)\|y_1(\cdot,\cdot) - y_2(\cdot,\cdot)\|_\lambda. \tag{4.6}$$

The inequality

$$\frac{KC_w}{\lambda}(cr + Cr^2) \le \frac{1}{2}, \tag{4.7}$$

ensures that $\mathcal{G}[\cdot]$ is a contraction in $\mathcal{Y}_{r,C}$ and then, by the Banach contraction principle, equation (4.2) has a unique solution $y_*(\cdot,\cdot) \in \mathcal{Y}_{r,C}$. Taking into account (4.5) and definition of C, to satisfy (4.7) it is sufficient to replace the second inequality in (4.5) with $2Cr \leq c$. This completes the proof of assertion (i).

To prove (ii), firstly observe that every point $z_0 \in \mathcal{M}$ has a neighborhood $\mathcal{N}(z_0) \subset \mathcal{M}$ such that $\pi^{-1}(\mathcal{N}(z_0)) \cap W_r$ is homeomorphic to $\mathcal{N}(z_0) \times \mathcal{B}_r^k(0)$ where $k = \dim W$ and $\mathcal{B}_r^k(0) \subset \mathbb{R}^k$ is a ball of radius r centered at the origin. So, we regard $y_*(\cdot,\cdot)$ as a mapping with domain $\mathcal{N}(z_0) \times \mathcal{B}_r^k(0)$. Now for $\rho \in (0,r)$, $\delta \in (0, r-\rho)$ and unit vector $e \in \mathbb{R}^k$, consider a family of mappings $\left\{u_s(\cdot,\cdot;e) : \mathbb{R}_+ \times \mathcal{N}(z_0) \times \mathcal{B}_\rho^k(0) \mapsto \mathbb{R}^n\right\}_{s\in[-\delta,\delta]\setminus\{0\}}$ defined by

$$u_s(t,\zeta;e) := \frac{1}{s}\left[y_*(t,\zeta+se) - y_*(t,\zeta)\right]$$

(recall that we agreed not to show explicitly the dependence on z). We aim to establish the existence of

$$\partial_e y_*(t,\zeta) := \lim_{s\to 0} u_s(t,\zeta;e)$$

and show that $\partial_e y_*(\cdot,\zeta)$ is a solution of the linear integral equation

$$u(t,\zeta;e) = X^t e + \int_0^\infty G(t,\tau)\, w_y'(\tau, y_*(\tau,\zeta))\, u(\tau,\zeta;e)\, \mathrm{d}\tau. \tag{4.8}$$

Similarly to the previous reasoning, introduce the Banach space

$$\mathfrak{B} := \mathrm{C}\left(\mathbb{R}_+ \times \mathcal{N}(z_0) \times \mathcal{B}_\rho^k(0) \mapsto \mathbb{R}^n; \|\cdot\|_\lambda\right)$$

endowed with the norm

$$\|\cdot\|_\lambda := \sup\left\{\mathrm{e}^{\lambda t}\|\cdot\| : (t,z,\zeta) \in \mathbb{R}_+ \times \mathcal{N}(z_0) \times \mathcal{B}_\rho^k(0)\right\}.$$

On account of (4.4), (4.3) and (4.7), one can easily obtain the estimate

$$\int_0^\infty \left\|G(t,\tau)\, w_y'(\tau, y_*(\tau,\zeta))\right\| \mathrm{e}^{-\lambda\tau}\mathrm{d}\tau \leq \mathrm{e}^{-\lambda t}\frac{KC_w}{\lambda}\left(c\|\zeta\| + C\|\zeta\|^2\right) \leq \frac{1}{2}\mathrm{e}^{-\lambda t} \tag{4.9}$$

which allows us to apply the Banach contraction principle and prove that (4.8) has a unique solution $u_*(\cdot,\cdot;e) \in \mathfrak{B}$ satisfying

$$\|u_*(\cdot,\cdot;e)\|_\lambda \leq 2c.$$

Besides, by means of (4.6) and (4.7) we obtain

$$\|u_s(\cdot,\cdot;e)\|_\lambda \leq c + \frac{1}{2}\|u_s(\cdot,\cdot;e)\|_\lambda \quad \Longrightarrow \quad \|u_s(\cdot,\cdot;e)\|_\lambda \leq 2c.$$

Next, we have

$$\begin{aligned}\|u_s(t,\zeta;e) - u_*(t,\zeta;e)\| \leq & \int_0^\infty \left\|G(t,\tau)\, w_y'(\tau, y_*(\tau,\zeta))\right\| \|u_s(\tau,\zeta;e) - u_0(\tau,\zeta;e)\|\, \mathrm{d}\tau \\ & + \int_0^\infty \|G(t,\tau)\, H(\tau,\zeta,s;e)\| \, \|u_s(\tau,\zeta;e)\|\, \mathrm{d}\tau\end{aligned}$$

where

$$H(\tau,\zeta,s;e) := \int_0^1 \left[w_y'(\theta y_*(\tau,\zeta+se) + (1-\theta)y_*(\tau,\zeta)) - w_y'(\tau, y_*(\tau,\zeta))\right] \mathrm{d}\theta.$$

Since
$$\|H(\tau,\zeta,s;e)\| \le \frac{C_w}{2}\|y_*(\tau,\zeta+se)-y_*(\tau,\zeta)\|$$
and (4.9) yields
$$\sup_{t\ge 0} \mathrm{e}^{\lambda t}\int_0^\infty \left\|G(t,\tau)\,w_y'(\tau,y_*(\tau,\zeta))\right\|\|u_s(\tau,\zeta;e)-u_0(\tau,\zeta;e)\|\,\mathrm{d}\tau$$
$$\le \frac{1}{2}\sup_{t\ge 0}\mathrm{e}^{\lambda t}\|u_s(\tau,\zeta;e)-u_0(\tau,\zeta;e)\|,$$
then, on account of (4.3), we obtain
$$\lim_{s\to 0}\sup_{t\ge 0}\mathrm{e}^{\lambda t}\|u_s(t,\zeta;e)-u_0(t,\zeta;e)\|$$
$$\le cC_w\lim_{s\to 0}\sup_{t\ge 0}\left\|\mathrm{e}^{\lambda t}\int_0^\infty \mathrm{e}^{-2\lambda\tau}\|G(t,\tau)\|\left[\mathrm{e}^{\lambda\tau}\|y_*(\tau,\zeta+se)-y_*(\tau,\zeta)\|\right]\mathrm{d}\tau\right\|$$
$$\le cC_wK\lim_{s\to 0}\int_0^\infty \mathrm{e}^{-\lambda\tau}\left[\mathrm{e}^{\lambda\tau}\|y_*(\tau,\zeta+se)-y_*(\tau,\zeta)\|\right]\mathrm{d}\tau$$
$$\le cC_wK\lim_{T\to\infty}\lim_{s\to 0}\left[\int_0^T\|y_*(\tau,\zeta+se)-y_*(\tau,\zeta)\|\,\mathrm{d}\tau+\frac{4e^{-\lambda T}}{\lambda}\|y_*(\cdot,\cdot)\|_\lambda\right]=0.$$
This completes the proof of assertion (ii). □

Corollary 4.3. *For all $(t,\zeta)\in\mathbb{R}_+\times W_r$ and every unite vector $e\in W_r\cap\pi^{-1}(z)$, where $z:=\pi(\zeta)$, the following inequalities hold:*
$$\|\partial_e y_*(t,\zeta)\|\le 2c\mathrm{e}^{-\lambda t},\qquad \left\|\partial_e y_*(t,\zeta)-X^t e\right\|\le \mathrm{e}^{-\lambda t}\frac{2cKC_w}{\lambda}\|\zeta\|.$$

Proposition 4.4. *Let C, C_w and r be the constants specified according to Proposition 4.2. If $y(\cdot):\mathbb{R}_+\mapsto\mathbb{R}^n$ is a solution of (4.1) such that $\sup_{t\in\mathbb{R}_+}\mathrm{e}^{\lambda t}\|y(t)\|\le\min\{\lambda/(KC_w),1\}$ and $\zeta:=\Pi y(0)\in W_r$, then*
$$\left\|y(t)-X^t\zeta\right\|\le C\mathrm{e}^{-\lambda t}\|\zeta\|^2\qquad\forall t\ge 0,$$
and thus, $y(t)\equiv y_*(t,\zeta)$.

Proof. By Proposition 4.1 $y(\cdot)$ satisfies integral equation (4.4) with $\zeta=\Pi y(0)$. Then on account of (4.3) and (4.4) we have
$$\sup_{t\in\mathbb{R}_+}\mathrm{e}^{\lambda t}\|y(t)\|\le c\|\zeta\|+\frac{KC_w}{2\lambda}\min\{\lambda/(KC_w),1\}\sup_{t\in\mathbb{R}_+}\mathrm{e}^{\lambda t}\|y(t)\|,$$
and thus
$$\sup_{t\in\mathbb{R}_+}\mathrm{e}^{\lambda t}\|y(t)\|\le 2c\|\zeta\|\le 2cr<1.$$
This inequality, in its turn, implies
$$\left\|y(t)-X^t\zeta\right\|\le\frac{2KC_wc^2}{\lambda}\mathrm{e}^{-\lambda t}\|\zeta\|^2=C\|\zeta\|^2\qquad\forall t\ge 0.$$
To end the proof it remains only to refer to assertion (i) from Proposition 4.2 which ensures the uniqueness of $y_*(\cdot,\cdot)$. □

Define the sets

$$\begin{aligned} W_r(z) &:= W_r \cap \pi^{-1}(z), \qquad \mathcal{W}_r(z) := z + y_*(0, W_r(z)) \\ U_r(z) &:= U_r \cap \pi^{-1}(z), \qquad \mathcal{U}_r(z) := z + y_*(0, U_r(z)). \end{aligned}$$

Corollary 4.3 yields the following

Proposition 4.5. *There is a sufficiently small positive r such that the sets $\mathcal{W}_r(z)$ and $\mathcal{U}_r(z)$ are differentiable manifolds diffeomorphic to $W_r(z)$ and $P_N U_r(z)$ respectively. Besides, $T_z\mathcal{W}_r(z) = W_r(z)$, $T_z\mathcal{U}_r(z) = U_r(z)$.*

Let $z_0 \in \mathcal{M}$ be a given point and $\mathcal{O}(z_0) := \bigcup_{\tau\in\mathbb{R}} \{\chi^\tau(z_0)\}$ stands for its orbit.

Definition 4.6. We say that the set $\mathcal{W}_r(\mathcal{O}(z_0)) = \bigcup_{z\in\mathcal{O}(z_0)} \mathcal{W}_r(z)$ is a local λ-stable set of the orbit $\mathcal{O}(z_0)$.

Theorem 4.7. *Let system (2.1) satisfy conditions* **H1**, **H2**. *Then there exist positive constants r, C, T and $\rho \in (0, r]$ such that for every $z_0 \in \mathcal{M}$, the set $\mathcal{W}_r(\mathcal{O}(z_0))$ has the following properties:*

(a) for every $x \in \mathcal{W}_r(\mathcal{O}(z_0))$, there exist $z \in \mathcal{O}(z_0)$ and $\zeta \in W_r(z)$ such that

$$\left\|\chi^t(x) - \chi^t(z) - X^t\zeta\right\| \le Ce^{-\lambda t}\|\zeta\|^2 \qquad \forall t \ge 0,$$

and thus, the motion $t \mapsto \chi^t(x)$ has an asymptotic phase;

(b) there hold the inclusions

$$\chi^t(\mathcal{W}_\rho(z)) \subset \mathcal{W}_r(\chi^t(z)) \qquad \forall t \ge 0, \quad \text{and} \quad \chi^t(\mathcal{W}_r(z)) \subset \mathcal{W}_r(\chi^t(z)) \qquad \forall t \ge T;$$

(c) if in addition the vector field v has no singular points on $\mathcal{M}$, then for every $z \in \mathcal{O}(z_0)$, there is a sufficiently small arc $\mathcal{O}_\delta(z) := \bigcup_{|\tau|<\delta}\{\chi^\tau(z)\}$, $0 < \delta \ll 1$, such that that $\mathcal{W}_r(z_1) \cap \mathcal{W}_r(z_2) = \varnothing$ for any $z_1, z_2 \in \mathcal{O}_\delta(z)$, and the set $\mathcal{W}_r(\mathcal{O}(z_0))$ is an immersed into $\mathbb{R}^n$ topological manifold.

Proof. Let r and C be specified via Proposition 4.2, and let $x \in \mathcal{W}_r(z)$ for some $z \in \mathcal{O}(z_0)$. Then there is $\zeta \in W_r(z)$ such that $x = z + y_*(0,\zeta)$ is an initial value for the solution $t \mapsto \chi^t(z) + y_*(t,\zeta)$ of system (2.1). Hence, $\chi^t(x) \equiv \chi^t(z) + y_*(t,\zeta)$, and now (a) is a direct consequence of Proposition 4.2.

Now we proceed to (b). Let $\rho \in (0, r]$ and $x \in \mathcal{W}_\rho(z)$. Then there is $\zeta \in W_\rho(z)$ such that $x = z + y_*(0,\zeta)$, $\Pi y_*(0,\zeta) = \zeta$ and

$$\chi^s(x) = \chi^s(z) + y_*(s,\zeta) = \chi^s(z) + \Pi y_*(s,\zeta) + (\mathrm{Id} - \Pi)\, y_*(s,\zeta).$$

Put $\zeta^s := \Pi y_*(s,\zeta)$. By the definition of Π, we have $\zeta^s \in W(\chi^s(z))$, and by means of estimates from the proof of Proposition 4.2 we obtain,

$$\|\zeta^s\| = \left\| X^s\zeta + \int_0^s X^s\Pi\,[X^\tau]^{-1} w(\tau, y_*(\tau,\zeta))\,\mathrm{d}\tau \right\| \le e^{-\lambda s}\left(c\rho + C\rho^2\right).$$

Hence, if $\rho \in (0, \rho_0)$, where ρ_0 is small enough to satisfy $c\rho_0 + C\rho_0^2 \le r$, then $\zeta^s \in W_r(\chi^s(z))$ for all $s \ge 0$. And if $\rho = r$, then $\zeta^s \in W_r(\chi^s(z))$ for all $s \ge T$, provided that T is large enough to satisfy $e^{-\lambda T}(cr + Cr^2) \le r$. Besides, property (a) implies

$$\begin{aligned} \left\|\chi^t \circ \chi^s(x) - \chi^t \circ \chi^s(z)\right\| &\le \left\|\chi^{t+s}(x) - \chi^{t+s}(z) - X^{t+s}\zeta\right\| + \left\|X^{t+s}\zeta\right\| \\ &\le e^{-\lambda(t+s)}\left(C\|\zeta\|^2 + c\|\zeta\|\right). \end{aligned}$$

Hence, if $\rho \in (0, \rho_0)$, whereρ_0 is small enough to satisfy additional condition $c\rho_0 + C\rho_0^2 \le \lambda / KC_w$, then

$$\left\| \chi^t \circ \chi^s(x) - \chi^t \circ \chi^s(z) \right\| \le \mathrm{e}^{-\lambda t} \frac{\lambda}{KC_\omega} \qquad \forall t \ge 0, \ \forall s \ge 0;$$

and if $\rho = r$ then

$$\left\| \chi^t \circ \chi^s(x) - \chi^t \circ \chi^s(z) \right\| \le \mathrm{e}^{-\lambda t} \frac{\lambda}{KC_\omega} \qquad \forall t \ge 0, \ \forall s \ge T,$$

provided that T is large enough to ensure $\mathrm{e}^{-\lambda T}(cr + Cr^2) \le \lambda / (KC_\omega)$. Now the mapping $t \mapsto \chi^t \circ \chi^s(x) - \chi^t \circ \chi^s(z)$, as a solution of (4.1), satisfies conditions of Proposition 4.4 where z and ζ should be replaced with $\chi^s(z)$ and ζ^s respectively. Hence,

$$\chi^t \circ \chi^s(x) - \chi^t \circ \chi^s(z) = y_*(t, \zeta^s),$$

and thus, $\chi^s(x) = \chi^s(z) + y_*(0, \zeta^s)$. As a consequence,

$$\chi^s(x) \in \begin{cases} \mathcal{W}_r(\chi^s(z)) & \forall s \ge 0 \text{ if } x \in \mathcal{W}_\rho(z) \text{ and } \rho \in (0, \rho_0); \\ \mathcal{W}_r(\chi^s(z)) & \forall s \ge T \text{ if } x \in \mathcal{W}_r(z). \end{cases}$$

Finally, let us prove (c) by reasoning ad absurdum. Suppose that for every $z \in \mathcal{O}(z_0)$ there is no $\delta > 0$ such that $\mathcal{W}_r(z_1) \cap \mathcal{W}_r(z_2) = \varnothing$ for any pair of different points $z_1, z_2 \in \mathcal{O}_\delta(z)$. Then there exist sequences $\{t_{1,k}\}_{k\in\mathbb{N}}$, $\{t_{2,k}\}_{k\in\mathbb{N}}$ such that $t_{i,k} \to 0$, $k \to \infty$, $i \in \{1,2\}$, $t_{1,k} > t_{2,k}$, as well as the sequence

$$\left\{ x_k \in \mathcal{W}_r\left(\chi^{t_{1,k}}(z)\right) \cap \mathcal{W}_r\left(\chi^{t_{2,k}}(z)\right) \right\}_{k\in\mathbb{N}}.$$

Now, for any $k \in \mathbb{N}$, we obtain

$$\left\| \chi^{t+t_{1,k}}(z) - \chi^{t+t_{2,k}}(z) \right\| \le \left\| \chi^t(x_k) - \chi^t \circ \chi^{t_{1,k}}(z) \right\| + \left\| \chi^t(x_k) - \chi^t \circ \chi^{t_{2,k}}(z) \right\| \to 0, \qquad t \to \infty,$$

and thus,

$$\lim_{t\to\infty} \left\| \chi^{T_k} \circ \chi^t(z) - \chi^t(z) \right\| = 0 \tag{4.10}$$

where $T_k = t_{1,k} - t_{2,k} \ne 0$. Since $\mathcal{M}$ is compact, then ω-limit set of $\mathcal{O}(z)$ contains at least one point, e.g. $z_* \in \mathcal{M}$, and (4.10) implies that z_* is T_k -periodic for all $k \in \mathbb{N}$. But, as is easily seen, from $T_k \to 0$ it follows that $v(z_*) = 0$, and we arrive at contradiction.

Now we see that the continuous mapping $\mathcal{O}(z_0) \ni z \mapsto \mathcal{W}_r(z)$ is locally one-to-one. Since each $\mathcal{W}_r(z)$ is diffeomorphic to an open d-dimensional ball of Euclidean space (Proposition (4.5)), and $\mathcal{W}_r(\mathcal{O}(z_0))$ is given by the equation

$$x = \chi^t(z_0) + y_*(0, \zeta), \qquad \zeta \in W_r\left(\chi^t(z_0)\right), \ t \in \mathbb{R},$$

then $\mathcal{W}_r(\mathcal{O}(z_0))$ is an immersed $(d+1)$-dimensional topological manifold. □

Remark 4.8. On contrary to [16], Theorem 4.7 do not guarantee that $\mathcal{W}_r(z_1) \cap \mathcal{W}_r(z_2) = \varnothing$ for any pair of different points $z_1, z_2 \in \mathcal{M}$.

Corollary 4.9. *If there exist $z_1, z_2 \in \mathcal{O}(z_0)$ such that $\mathcal{W}_r(z_1) \cap \mathcal{W}_r(z_2) \ne \varnothing$, then ω-limit set of $\mathcal{O}(z_0)$ contains at least one closed orbit.*

5 Existence of asymptotic phase

Now we are in position to prove the following theorem on the existence of asymptotic phase.

Theorem 5.1. *Let system* (2.1) *satisfy conditions* **H1**, **H2**. *Then there exists* $\varepsilon > 0$ *such that a motion* $t \mapsto \chi^t(x)$ *has the asymptotic phase, provided that* $\mathcal{O}(x) \cap N\mathcal{M}_\varepsilon \neq \varnothing$.

Proof. Let x_0 be a point in the tubular neighborhood $N\mathcal{M}_\varepsilon \subset N\mathcal{M}_r$, where r is specified in Theorem 4.7. Then $x_0 = z_0 + \xi_0$, where $z_0 \in \mathcal{M}$, $\xi_0 \in N_{z_0}\mathcal{M}$, $\|\xi_0\| < \varepsilon$. We have to show that if $\varepsilon \in (0, r)$ is sufficiently small, then there exists $z(x_0) \in \mathcal{M}$ such that $x_0 \in \mathcal{U}_r(z(x_0))$ (see Proposition 4.5 concerning $\mathcal{U}_r(z)$). Since $\mathcal{U}_r(z(x_0)) \subset \mathcal{W}_r(z(x_0))$, then by Theorem 4.7 the above inclusion implies

$$\left\|\chi^t(x_0) - \chi^t(z(x_0))\right\| \to 0, \qquad t \to \infty,$$

meaning that the motion $t \mapsto \chi^t(x_0)$ has the asymptotic phase.

Let us prove the existence of $z(x_0)$. Note that if r is sufficiently small, then there are local coordinates in $N\mathcal{M}_r$

$$(q_1, \ldots, q_m, p_1, \ldots, p_{n-m}) = (q, p), \qquad m := \dim \mathcal{M},$$

with the following properties: (i) the coordinates of z_0 are $(0, 0)$; (ii) the manifold $\mathcal{M}$ is given by a local equation $x = z(q)$, where $z(\cdot)$ is a C^1-mapping defined in a neighborhood of $q = 0$; (iii) the columns of the matrix $\mathbf{T}(0)$, where $\mathbf{T}(q) := \left[\frac{\partial z_i(q)}{\partial q_j}\right]_{i=1,j=1}^{n,\ m}$, are pairwise orthogonal unit vectors; (iv) if (q, p) are local coordinates of a point $x \in N\mathcal{M}_r$, then

$$x = z(q) + \mathbf{N}(q)p,$$

in particular $x_0 = z(0) + \mathbf{N}(0)p_0$, where $\mathbf{N}(q)$ is $n \times (n-m)$-matrix whose columns are unit vectors pairwise mutually orthogonal, and orthogonal to $\mathcal{M}$ at $z(q)$ as well, thus $\mathbf{N}^\top(q)\mathbf{T}(q) = 0$*; (v) both mappings $q \mapsto \mathbf{N}(q)$ and $q \mapsto \mathbf{T}(q)$ are continuous in a neighborhood of 0.

Having analyzed the mapping $y_*(0, \cdot)$, one can make a conclusion that the manifold $\mathcal{U}_r(z)$ is given by the equation

$$x = z(q) + \mathbf{N}(q)\, p + \mathbf{T}(q)\left[\mathbf{L}(q) + \mathbf{M}(q, p)\right] p,$$

where $\mathbf{L}(q)$ and $\mathbf{M}(q, p)$ are $m \times (n-m)$-matrices with continuous elements, and $\|\mathbf{M}(q, p)\| \to 0$ as $\|p\| \to \mathbf{0}$. Now for a given p_0 such that $\|p_0\| < \varepsilon \ll 1$, we have to solve the equation

$$z(q) + \mathbf{N}(q)\, p + \mathbf{T}(q)\left[\mathbf{L}(q) + \mathbf{M}(q, p)\right] p = z(0) + \mathbf{N}(0)p_0.$$

Since $z(q) - z(0) = \left[\mathbf{T}(0) + \mathbf{T}_1(q)\right] \mathbf{q}$, where $\|\mathbf{T}_1(q)\| = o(1)$, $\|q\| \to 0$, then the above equation can be represented in the form

$$\mathbf{T}(0)\left[q + \mathbf{L}(0)p\right] + \mathbf{N}(0)p = \mathbf{F}(q, p) + \mathbf{N}(0)p_0$$

where

$$\mathbf{F}(q, p) := \left[\mathbf{T}(0)\,\mathbf{L}(0) - \mathbf{T}(q)\,\mathbf{L}(q) + \mathbf{N}(0) - \mathbf{N}(q) - \mathbf{M}(q, p)\right] p - \mathbf{T}_1(q)q.$$

*Here we use the notation $\mathbf{N}(q)$ instead of $\mathbf{N}(z(q))$ where $\mathbf{N}(z)$ is the matrix defined in Section 2.

Note, that $\|\mathbf{F}(q,p)\| = o(\|q\|+\|p\|)$, $\|q\|+\|p\| \to 0$. After the change of variables $q = u - \mathbf{L}(0)p$, on account that the matrix $[\mathbf{T}(0); \mathbf{N}(0)]$ is orthogonal, we arrive at the equation

$$\begin{pmatrix} u \\ p \end{pmatrix} = \mathbf{H}(u,p) + \begin{pmatrix} 0 \\ p_0 \end{pmatrix}, \tag{5.1}$$

where $\mathbf{H}(u,p) = [\mathbf{T}(0); \mathbf{N}(0)]^\top \mathbf{F}(q,p)\big|_{q=u-\mathbf{L}(0)p}$. It is obvious that $\|\mathbf{H}(u,p)\| = o(\|u\|+\|p\|)$, $\|u\|+\|p\| \to 0$. Now we are in position to apply the Brouwer fixed point theorem. Namely, let $D_\varepsilon := \left\{ (u,p)^\top : \|u\|^2 + \|p\|^2 \le \varepsilon^2 \right\}$. If $0 < \varepsilon \ll 1$ and $\|p_0\| \le \varepsilon$, then $\max_{(u,p)\in D_{2\varepsilon}} \|\mathbf{H}(u,p)\| \le \varepsilon$. Hence,

$$D_{2\varepsilon} \ni \begin{pmatrix} u \\ p \end{pmatrix} \mapsto \mathbf{H}(u,p) + \begin{pmatrix} 0 \\ p_0 \end{pmatrix} \in D_{2\varepsilon},$$

and by the Brouwer fixed point theorem equation (5.1) has at least one solution. □

6 Global λ-stable sets of orbits on $\mathcal{M}$

In this section we analyze the geometrical structure of sets formed by motions approaching a given orbit with exponential rate and having asymptotic phases.

Definition 6.1. For a given $z \in \mathcal{M}$ the set

$$\mathcal{W}(z) := \left\{ x \in \mathcal{D} : \|\chi^t(x) - \chi^t(z)\| = O\left(e^{-\lambda t}\right),\ t \to 0 \right\}$$

is said to be (a global) λ-stable set of the point z. For a given $z_0 \in \mathcal{M}$, the set $\mathcal{W}(\mathcal{O}(z_0)) := \bigcup_{z\in\mathcal{O}(z_0)} \mathcal{W}(z)$ is said to be (a global) λ-stable set of the orbit $\mathcal{O}(z_0)$.

Theorem 6.2. *Let system (2.1) satisfy conditions* **H1**, **H2**, *and let r and T be specified according to Theorem 4.7. Then for every $z_0 \in \mathcal{M}$ the λ-stable set for the orbit $\mathcal{O}(z_0)$ has the following properties:*

(a) the λ-stable set of any $z \in \mathcal{O}(z_0)$ is an immersed into $\mathbb{R}^n$ differentiable manifold admitting the representation $\mathcal{W}(z) = \bigcup_{k\in\mathbb{Z}_+} \chi^{-kT}\left(\mathcal{W}_r\left(\chi^{kT}(z)\right)\right)$;

(b) the foliation of $\mathcal{W}(\mathcal{O}(z_0))$ by the family of manifolds $\{\mathcal{W}(z)\}_{z\in\mathcal{O}(z_0)}$ is χ^t-invariant, meaning that

$$\chi^t(\mathcal{W}(z)) = \mathcal{W}\left(\chi^t(z)\right) \qquad \forall t \in \mathbb{R};$$

(c) if the vector field v does not have singular points on $\mathcal{M}$, then for every $z \in \mathcal{O}(z_0)$ there is a sufficiently small arc $\mathcal{O}_\delta(z)$, $0 < \delta \ll 1$, such that $\mathcal{W}(z_1) \cap \mathcal{W}(z_2) = \varnothing$ for any $z_1, z_2 \in \mathcal{O}_\delta(z)$, and $\mathcal{W}(\mathcal{O}(z_0))$ is an immersed into $\mathbb{R}^n$ topological manifold.

Proof. Let $z \in \mathcal{O}(z_0)$ and $x \in \mathcal{W}(z)$. By the definition of $\mathcal{W}(z)$,

$$R = R(x,z) := \sup_{t\ge 0} e^{\lambda t} \|\chi^t(x) - \chi^t(z)\| < \infty.$$

For a $k \in \mathbb{Z}_+$, define

$$x_k := \chi^{kT}(x), \qquad z_k := \chi^{kT}(z), \qquad \zeta_k := \Pi(x_k - z_k), \qquad y_k(t) := \chi^t(x_k) - \chi^t(z_k).$$

Then

$$\|\zeta_k\| \le \sup_{z\in\mathcal{O}(z_0)} \|\Pi\|\, \mathrm{e}^{-kT} R \le \max_{z\in\mathcal{M}} \|\Pi\|\, \mathrm{e}^{-kT} R,$$
$$\|y_k(t)\| = \left\|\chi^{t+kT}(x) - \chi^{t+kT}\right\| \le R\mathrm{e}^{-\lambda(t+kT)} \qquad \forall t \ge 0,$$

and by Proposition 4.1 we have

$$\zeta_k \in W_r(z_k), \qquad y_k(t) = y_*(t, \zeta_k),$$

provided that k is sufficiently large. Hence,

$$x_k = z_k + y_*(0, \zeta_k) \in \mathcal{W}_r(z_k) \quad \Longrightarrow \quad x \in \chi^{-kT}\mathcal{W}_r\left(\chi^{kT}(z)\right),$$

and the last inclusion implies the required representation for $\mathcal{W}(z)$.

By Theorem 4.7, for any $z \in \mathcal{O}(z_0)$, the set $\mathcal{W}_r(z)$ is a differentiable manifold and $\mathcal{W}_r(z) \subset \chi^{-T}\mathcal{W}_r(\chi^T(z))$. Hence,

$$\chi^{-kT}\mathcal{W}_r\left(\chi^{kT}(z)\right) \subset \chi^{-kT}\circ\chi^{-T}\mathcal{W}_r\left(\chi^T\circ\chi^{kT}(z)\right) = \chi^{-(k+1)T}\mathcal{W}_r\left(\chi^{(k+1)T}(z)\right),$$

and thus,

$$\mathcal{W}_r(z) \subset \cdots \chi^{-kT}\mathcal{W}_r\left(\chi^{kT}(z)\right) \subset \chi^{-(k+1)T}\mathcal{W}_r\left(\chi^{(k+1)T}(z)\right) \subset \cdots = \mathcal{W}(z).$$

Now it is obvious that $\mathcal{W}(z)$ is a differentiable manifold. The proof of (a) is complete.

We proceed to assertion (b). Let $x \in \mathcal{W}(z)$ and $t \in \mathbb{R}$. Then there are $i, k \in \mathbb{Z}_+$ such that $x \in \chi^{-kT}\mathcal{W}_r(\chi^{kT}(z))$ and $iT + t \ge T$. Now we obtain

$$\begin{aligned}\chi^t(x) \in \chi^{t-kT}\left(\mathcal{W}_r\left(\chi^{kT}(z)\right)\right) &= \chi^{-(k+i)T}\circ\chi^{iT+t}\left(\mathcal{W}_r\left(\chi^{kT}(z)\right)\right)\\ &\subset \chi^{-(k+i)T}\left(\mathcal{W}_r\left(\chi^{(k+i)T}\circ\chi^t(z)\right)\right) \subset \mathcal{W}\left(\chi^t(z)\right).\end{aligned}$$

Hence, $\chi^t(\mathcal{W}(z)) \subset \mathcal{W}(\chi^t(z))$ for all $t \in \mathbb{R}$. But then $\mathcal{W}(\chi^{-t}(z)) \subset \chi^{-t}(\mathcal{W}(z))$ for all $-t \in \mathbb{R}$. This completes the proof of (b).

The proof of assertion (c) is the same as in Theorem 4.7. □

Corollary 6.3. *A λ-stable set of $\mathcal{O}(z_0)$ is generated by λ-stable set of z_0, i.e.*

$$\mathcal{W}(\mathcal{O}(z_0)) = \bigcup_{t\in\mathbb{R}} \chi^t(\mathcal{W}(z_0)).$$

7 Asymptotic phase for motions attracting by semi-invariant domains

Let us consider a more general case where the system under consideration satisfies conditions like **H1**, **H2** not on the whole invariant manifold, but on some forward χ^t-semi-invariant domain $\mathcal{M}^+ \subset \mathcal{M}$. Namely,

H1$^+$ The tangent bundle $T\mathcal{M}^+$ splits into a continuous Whitney sum $T\mathcal{M}^+ = V^s \oplus V^\star$ of forward X^t-semi-invariant vector sub-bundles $V^s = \coprod_{z\in\mathcal{M}^+} V^s_z$, $V^\star = \coprod_{z\in\mathcal{M}^+} V^\star_z$, and there exist constants $c_0 \geq 1$, $\nu > 0$, $\sigma \in [0,\nu)$ such that

$$\left\|X^t\xi\right\| \leq c_0 \mathrm{e}^{-\nu t}\left\|\xi\right\| \qquad \forall t\geq 0,\ \forall\xi\in V^s, \tag{7.1}$$
$$\left\|X^t\xi\right\| \geq c_0^{-1} \mathrm{e}^{-\sigma t}\left\|\xi\right\| \qquad \forall t\geq 0,\ \forall\xi\in V^\star. \tag{7.2}$$

H2$^+$ The natural projections

$$P_s : T\mathcal{M}^+ \mapsto V^s, \qquad P_\star : T\mathcal{M}^+ \mapsto V^\star$$

are uniformly bounded.

H3$^+$ There exists $\gamma > \sigma$ such that

$$\left\|P_N X^t P_N\xi\right\| \leq c_0 \mathrm{e}^{-\gamma t}\left\|\xi\right\| \qquad \forall t\geq 0,\ \forall\xi\in T\mathcal{M}^+.$$

It turns out that conditions **H1$^+$**,**H2$^+$** imply a counterpart of inequalities (3.8), namely, there exists a constant c_1^+ such that

$$\left\|X^t P_s\left[X^\tau\right]^{-1} P_T\big|_{T\chi^\tau(\mathcal{M}^+)}\right\| \leq c_1^+\mathrm{e}^{-\nu(t-\tau)}, \qquad 0\leq\tau\leq t,$$
$$\left\|X^t P_\star\left[X^\tau\right]^{-1} P_T\big|_{T\chi^\tau(\mathcal{M}^+)}\right\| \leq c_1^+\mathrm{e}^{-\sigma(t-\tau)}, \qquad 0\leq t<\tau.$$

E.g., derive the last inequality. Let $0\leq t<\tau$. For any $\zeta\in V^\star\big|_{\chi^\tau(\mathcal{M}^+)}$ define $\xi = X^{-t}\zeta$. Then (3.2) implies $\left\|X^{-t}\zeta\right\| \leq c_0\mathrm{e}^{\sigma t}\left\|\zeta\right\|$. Hence,

$$\left\|X^{t-\tau}\zeta\right\| \leq c_0\mathrm{e}^{-\sigma(t-\tau)}\left\|\zeta\right\| \qquad \forall\zeta\in V^\star\big|_{\chi^{\tau-t}(\mathcal{M}^+)},\ 0\leq t\leq\tau.$$

But $V^\star\big|_{\chi^\tau(\mathcal{M}^+)}\subset V^\star\big|_{\chi^{\tau-t}(\mathcal{M}^+)}$, and thus, for any $\eta\in T\chi^\tau\left(\mathcal{M}^+\right)$, we obtain

$$\begin{aligned}\left\|X^t P_\star[X^\tau]^{-1}\eta\right\| &= \left\|X^t X^{-\tau}P_\star\eta\right\| = \left\|X^{t-\tau}P_\star\eta\right\| \leq c_0\mathrm{e}^{-\sigma(t-\tau)}\left\|P_*\eta\right\| \\ &\leq c_1^+\mathrm{e}^{-\sigma(t-\tau)}\left\|\eta\right\|, \qquad 0\leq t\leq\tau.\end{aligned}$$

Now in our case, one can perform all steps analogous to those of Sections 3, 4. We first observe that the projections P_s and $P_\star$ are uniformly bounded in $\mathcal{M}^+$ and satisfy counterparts of inequalities (3.8) with constant c_1^+ instead of c_1. Everywhere in what follows the mapping $\left[X^\tau(z)\right]^{-1}$ will act on $T_{\chi^\tau(z)}\mathcal{M}^+$ with $\tau\geq 0$, $z\in\mathcal{M}^+$. In view of this fact and since

$$P_T Q(\tau)\xi\in T_{\chi^\tau(z)}\mathcal{M}^+ \qquad \forall z\in\mathcal{M}^+,\ \xi\in T_z\mathcal{M}^+,\ \tau\geq 0,$$

then, for all $t\geq 0$, $\eta(t;\xi)$ is correctly defined via (3.12) and satisfies the inequality

$$\left\|\eta(t;\xi)\right\| \leq c_3\mathrm{e}^{-\lambda t}\left\|\xi\right\|$$

with an appropriately redefined constant $c_3 > 0$. Then, in the same way as in the proof of Proposition 3.1, we define the mapping Ξ via (3.14), the projection Π and the corresponding sub-bundle

$$W^+ := \Pi\left(T\mathcal{M}^+ \oplus N\mathcal{M}^+\right).$$

Note that Π is uniformly bounded in $\mathcal{M}^+$. To prove that W^+ is forward semi-invariant, it is sufficient to take into account that the Lyapunov exponent of $\|X^t X^\tau\|$ does not exceed $-\lambda$ for any $\xi \in W^+$ and $t, \tau \in \mathbb{R}_+$ (but, in general case, not for all $\tau \in \mathbb{R}$ and $t \geq -\tau$ as in Proposition 3.1). Now, as in Corollary 3.2, we obtain the estimates for $\|X^t \Pi [X^\tau]^{-1}\|$ and $\|X^t (\mathrm{Id} - \Pi) [X^\tau]^{-1}\|$ with appropriately redefined constant K.

Next, in Section 4, up to Proposition 4.2 we need to replace $\mathcal{M}, W, \mathcal{M}_r$ with $\mathcal{M}^+, W^+, \mathcal{M}_r^+$ respectively. As a consequence, W_r, U_r will be replaced by W_r^+, U_r^+. The proofs of counterparts to Propositions 4.1–4.4 need no changes, except that starting from Proposition 4.2 the constants C and r should be found via the relevant inequalities, e.g. (4.5), (4.7), involving redefined constant K. As a result, for any $z \in \mathcal{M}^+$, we are able to define the sets

$$\mathcal{W}_r^+(z) := z + y_* \left(0, W_r^+(z)\right), \qquad \mathcal{U}_r^+(z) := z + y_* \left(0, U_r^+(z)\right),$$

where

$$W_r^+(z) := W_r^+ \cap \pi^{-1}(z), \qquad U_r^+(z) := U_r^+ \cap \pi^{-1}(z).$$

Since $\mathcal{M}^+$ is forward semi-invariant, then for any $z_0 \in \mathcal{M}^+$, it is natural to define the phase curve

$$\mathcal{O}^+(z_0) := \mathcal{O}(z_0) \cap \mathcal{M}^+$$

which includes all points of orbit $\mathcal{O}(z_0)$ containing in $\mathcal{M}^+$. And since $\mathcal{M}^+$ is open, then there exists $\epsilon(z_0) > 0$ such that

$$\bigcup_{-\epsilon(z_0) < t < \infty} \{\chi^t(z_0)\} \subset \mathcal{O}^+(z_0) \subset \mathcal{M}^+.$$

Finally, we define the local λ-stable set of the phase curve $\mathcal{O}^+(z_0)$ as

$$\mathcal{W}_r\left(\mathcal{O}^+(z_0)\right) := \bigcup_{z \in \mathcal{O}^+(z_0)} \mathcal{W}_r^+(z).$$

Now the assertions (a), (b), (c) of Theorem 4.7 as well as their proofs remain correct for every $z_0 \in \mathcal{M}^+$ after we replace $\mathcal{M}, \mathcal{O}(z_0), W, \mathcal{W}$ with $\mathcal{M}^+, \mathcal{O}^+(z_0), W^+, \mathcal{W}^+$ respectively. As a consequence, we obtain the following counterpart of Theorem 5.1.

Theorem 7.1. *Let system (2.1) satisfy conditions* **H1**$^+$ – **H3**$^+$ *in a forward χ^t-semi-invariant domain $\mathcal{M}^+ \subset \mathcal{M}$. Then there exists $\varepsilon > 0$ such that a motion $t \mapsto \chi^t(x)$ has an asymptotic phase, provided that $\mathcal{O}(x) \cap N\mathcal{M}_\varepsilon^+ \neq \varnothing$, where $N\mathcal{M}_\varepsilon^+$ is a portion of the tubular neighborhood $N\mathcal{M}_\varepsilon$ over $\mathcal{M}^+$.*

Now let us apply Theorem 7.1 to each forward semi-invariant domain $\chi^{-k}(\mathcal{M}^+)$. Then we obtain a sequence of positive numbers $\{\varepsilon_k\}$ and the corresponding sequence of sets $\{N\mathcal{M}_{\varepsilon_k}^+\}$. Next define χ^t-invariant domains in $\mathcal{M}$ and $\mathbb{R}^n$ respectively

$$\mathcal{M}' := \bigcup_{k \in \mathbb{N}} \chi^{-k}(\mathcal{M}^+), \qquad \mathcal{D}' = \bigcup_{t \geq 0} \chi^{-t}\left(\bigcup_{k \in \mathbb{N}} N\mathcal{M}_{\varepsilon_k}^+\right).$$

Finally we arrive at the following result

Theorem 7.2. *Let system (2.1) satisfy conditions* **H1**$^+$–**H3**$^+$ *in a forward χ^t-semi-invariant domain $\mathcal{M}^+ \subset \mathcal{M}$. Then for any $x \in \mathcal{D}'$ the motion $t \mapsto \chi^t(x)$ has an asymptotic phase in $\mathcal{M}'$.*

8 A system on cotangent bundle of a compact homogeneous space $\mathrm{SL}(2;\mathbb{R})/\Gamma$

Consider a (right) homogeneous space $\mathcal{Q} =: \mathfrak{G}/\Gamma := \{q = G\Gamma : G \in \mathfrak{G}\}$ where $\mathfrak{G} := \mathrm{SL}(2;\mathbb{R})$ and Γ is a discrete subgroup of $\mathfrak{G}$ such that $\mathcal{Q}$ is compact. As is well known, homogeneous spaces of such a kind are naturally associated with compact Riemannian surfaces of constant negative curvature, and the geodesic flows on such surfaces are classical examples of Anosov dynamical systems [1,2,5,30]. We aims to apply the results of previous sections to a specific system defined on cotangent bundle $T^*\mathcal{Q}$. To obtain such a system, we first construct an appropriate right-invariant system on cotangent bundle $T^*\mathfrak{G}$ and then factorize it by the right action of the lattice Γ.

Recall that the group $\mathfrak{G}$ generates a Poissonian action on $T^*\mathfrak{G}$ (see [3]). Namely, let Λ be the Liouville 1-form ("$p\mathrm{d}q$"-form) on $T^*\mathfrak{G}$. The exact 2-form $\omega^2 := \mathrm{d}\Lambda$ defines a standard symplectic structure on $T^*\mathfrak{G}$. For any $A \in \mathfrak{g} := \mathrm{sl}(2,\mathbb{R})$, denote by $AG := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{e}^{At}G$ the right-invariant vector field generating the left action of one-parameter subgroup $\{\mathrm{e}^{At}\}$ on $\mathfrak{G}$. There is a natural lift of this action to $T^*\mathfrak{G}$ as the flow of Hamiltonian system with right invariant Hamiltonian function

$$h_a(x) = \Lambda\left(A\pi\left(x\right)\right), \qquad x \in T^*\mathfrak{G},$$

where $\pi : T^*\mathfrak{G} \mapsto \mathfrak{G}$ is the natural projection. For any $A, B \in \mathfrak{g}$, the Poisson bracket of Hamiltonians $h_A(\cdot)$, $h_B(\cdot)$ satisfies

$$\left\{h_A, h_B\right\}(x) := \omega\left(A\pi(x), A\pi(x)\right) = h_{[A,B]}(x), \qquad x \in T^*\mathfrak{G},$$

meaning that $\mathfrak{G}$-action on $T^*\mathfrak{G}$ is Poissonian. Let $m(\cdot) : T^*\mathfrak{G} \mapsto \mathfrak{g}^*$ be the corresponding momentum map, thus $m(x)A := h_A(x)$. Choose a standard base in $\mathfrak{g}$ represented, respectively, by the matrices

$$A_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and define the corresponding components of co-vector $m(x)$ by setting

$$m_k(x) := m(x)A_k, \qquad k \in \{1,2,3\}.$$

Since

$$[A_1, A_2] = -2A_3, \qquad [A_1, A_3] = 2A_2, \qquad [A_2, A_3] = 2A_1,$$

then

$$\{m_1, m_2\} = -2m_3, \qquad \{m_1, m_3\} = 2m_2, \qquad \{m_2, m_3\} = 2m_1 \tag{8.1}$$

The diffeomorphism

$$T^*\mathfrak{G} \ni x \mapsto (m(x), \pi(x)) \in \mathfrak{g}^* \times \mathfrak{G}$$

induces a Poissonian structure on $\mathfrak{g}^* \times \mathfrak{G}$ such that the brackets $\{m_i, m_j\}$ satisfy (8.1), the bracket of any pair of functions $f_i(\cdot), f_j(\cdot) : \mathfrak{G} \mapsto \mathbb{R}$ equals zero, and

$$\{G, m_k\} = A_k G, \tag{8.2}$$

meaning that $\mathfrak{G}$-component of Hamiltonian vector field with Hamiltonian function m_k is the right-invariant vector field $A_k G$.

Now introduce a Hamiltonian of the form

$$H(m) := \frac{1}{2}\sum_{k=1}^{3} \lambda_k m_k^2$$

where $\lambda_1, \lambda_2, \lambda_3$, are given positive numbers. The corresponding Hamiltonian system on $\mathfrak{g}^* \times \mathfrak{G}$ reads

$$\dot{m} = \{m, H(m)\}, \tag{8.3}$$

$$\dot{G} = \sum_{k=1}^{3} \lambda_k m_k A_k G. \tag{8.4}$$

Here

$$\{m, H(m)\} := \begin{pmatrix} \{m_1, H(m)\} \\ \{m_2, H(m)\} \\ \{m_3, H(m)\} \end{pmatrix} \equiv \begin{pmatrix} 2(\lambda_3 - \lambda_2) m_2 m_3 \\ 2(\lambda_1 + \lambda_3) m_1 m_3 \\ -2(\lambda_1 + \lambda_2) m_1 m_2 \end{pmatrix}.$$

Note that $H(m)$ in a standard way defines a right-invariant metrics $\langle \cdot, \cdot \rangle$ on $\mathfrak{G}$, and thus, one can consider system (8.3)–(8.4) as the Hamiltonian form of Lagrangian system for geodesics on $\mathfrak{G}$.

It is easily seen that, except $H(m)$, system (8.3) has an additional first integral (the Casimir function for the Poisson bracket on $\mathfrak{g}^*$)

$$J(m) = m_1^2 - m_2^2 - m_3^2.$$

If we consider sub-system (8.3) in $\mathfrak{g}^*$, then for any constants c_1 and c_2 satisfying

$$\min\{c_1 - \lambda_1 c_2, c_1 + \lambda_2 c_2, c_1 + \lambda_3 c_2\} > 0$$

the set $H^{-1}(c_1) \cap J^{-1}(c_2)$ is a union of two closed phase curves. Let $\mathcal{C}$ be one of such curves, and c_1^0, c_2^0 be the corresponding values of the constants. There is a tubular neighborhood $\mathcal{W}$ of $\mathcal{C}$ that is diffeomorphic to a direct product $D \times \mathbb{S}^1$ where $D \subset \mathbb{R}^2$ is a disc centered at the origin and $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. The set $\mathcal{W}$ is foliated by closed phase curves of system (8.3). By means of an appropriate diffeomorphism $\mu(\cdot) : \mathbb{S}^1 \times D \mapsto \mathcal{W}$ one can introduce an action-angular-type coordinates $(y_1, y_2, \theta|\bmod 1)$ in $D \times \mathbb{S}^1$ in such a way that the following relations are satisfied

$$\begin{aligned} y_1 &= H(m) - c_1^0, & y_2 &= J(m) - c_2^0, \\ H \circ \mu(\theta, y) &= y_1 + c_1^0, & J \circ \mu(\theta, y) &= y_2 + c_2^0, \\ \{\theta, y_1\} &= \omega(c^0 + y), & \{\theta, y_2\} &= 0. \end{aligned}$$

Here $\omega(c) > 0$ is a frequency of periodic motion over the closed phase curve given by the equation $m = \mu(\theta, c)$, and thus, being a component of $H^{-1}(c_1) \cap J^{-1}(c_2)$.

Now, instead of (8.3), consider a system

$$\dot{m} = \{m, H(m)\} + F(m) \tag{8.5}$$

with a perturbation term $F(m)$ under the impact of which the cycle $\mathcal{C}$ becomes asymptotically stable. For example, if we set

$$\begin{aligned} F(m) := &-\varepsilon\left[(H(m) - c_1^0)\,\|\nabla J(m)\|^2 - (J(m) - c_2^0)\,\langle \nabla H(m), \nabla J(m)\rangle\right] \nabla H(m) \\ &-\varepsilon\left[(J(m) - c_2^0)\,\|\nabla H(m)\|^2 - (H(m) - c_1^0)\,\langle \nabla H(m), \nabla J(m)\rangle\right] \nabla J(m) \end{aligned} \tag{8.6}$$

where ε is a (small) positive parameter, then the derivative of the Lyapunov function

$$U(m) := \left(H(m) - c_1^0\right)^2 + \left(J(m) - c_2^0\right)^2$$

by virtue of system (8.5) is

$$\dot{U}(m) = -\varepsilon U(m)\left[\|\nabla H(m)\|^2 \|\nabla J(m)\|^2 - \langle \nabla H(m), \nabla J(m)\rangle^2\right].$$

Hence, both $U(\cdot)$ and $\dot{U}(\cdot)$ vanish along $\mathcal{C}$. Furthermore, since

$$\min_{m\in\mathcal{C}}\left\{\|\nabla H(m)\|^2 \|\nabla J(m)\|^2 - \langle \nabla H(m), \nabla J(m)\rangle^2\right\} > 0,$$

then there is $\varkappa > 0$ such that the inequality

$$\dot{U}(m) \le -\varepsilon\varkappa U(m) \tag{8.7}$$

is satisfied in $\mathcal{W}$, provided that this tubular neighborhood is sufficiently small. This inequality ensures the exponential stability of $\mathcal{C}$ as a limit cycle of perturbed system (8.5). In fact, note that in the coordinates (y, θ) system (8.5) can be presented in the form

$$\dot{y} = P(\theta)y + O(\|y\|^2), \qquad \dot{\theta} = \omega\left(c^0\right) + \left\langle \nabla\omega\left(c^0\right) + b(\theta), y\right\rangle + O(\|y\|^2) \tag{8.8}$$

with a 1-periodic (2×2)-matrix $P(\cdot)$ and 2-vector $b(\cdot)$, and on account of (8.7) the derivative of the function $U \circ \mu\left(\theta, c^0 + y\right) = \|y\|^2$ by virtue of this system does not exceed $-\varepsilon\varkappa\|y\|^2$. Furthermore, the derivative of $\|y\|^2$ by virtue of linearized system

$$\dot{y} = P(\theta)y, \qquad \dot{\theta} = \omega\left(c^0\right) \tag{8.9}$$

is $2\langle P(\theta)y, y\rangle$, and thus, does not exceed $-\varepsilon\varkappa\|y\|^2/2$, provided $\mathcal{W}$ is small enough. (Note that that the last system generates the normal co-cycle associated with the flow on $\mathcal{C}$.) The obtained inequalities imply that y- components of solutions starting in $\mathcal{W}$ of both systems (8.8) and (8.9) vanish with exponential rate as $t \to \infty$.

To find stable limit cycles of system (8.5) in the case where F is a small vector field of general kind one can apply the well developed perturbation theory of periodic solutions (see e.g. [18]).

Now, after we have established that $\mathcal{C}$ is exponentially stable limit cycle of system (8.5), we proceed to analyze the structure of the flow generated by system (8.5) – (8.4) (or, what is the same, of system (8.3) – (8.4)) on its invariant manifold $\mathcal{C} \times \mathfrak{G}$. Since the motion of a point $m_0 = \mu(\theta) := \mu\left(c^0, \theta\right)$ on $\mathcal{C}$ is given by $t \mapsto \mu(\omega t + \theta)$, $\omega := \omega\left(c^0\right)$, we arrive at the linear system with $\frac{1}{\omega}$-periodic coefficients

$$\dot{G} = A(\omega t + \theta)G \tag{8.10}$$

where

$$A(\theta) := \left[\sum_{k=1}^{3} \lambda_k \mu_k(\theta) A_k\right] \tag{8.11}$$

Let $\mathcal{G}^t(\theta)$ stands for a fundamental matrix of (8.10) such that $\mathcal{G}^0(\theta) = \mathrm{Id}$. Thus, the motion of arbitrary point $(\mu(\theta), G) \in \mathcal{C} \times \mathfrak{G}$ in virtue of system (8.3)–(8.4) is governed by the mapping

$$t \mapsto \left(\mu(\omega t + \theta), \mathcal{G}^t(\theta)G\right).$$

Then for any $a \in \mathbb{R}$ and $B \in \mathrm{sl}(2;\mathbb{R})$, the motion of tangent vector $(a\mu'(\theta), BG)$ under the action of the corresponding tangent co-cycle is governed by the mapping

$$t \mapsto \left(a\mu'(\omega t+\theta), a\left[\mathcal{G}^t(\theta)\right]'_\theta G + \mathcal{G}^t(\theta)BG\right). \tag{8.12}$$

The co-cycle property of $\mathcal{G}^t(\theta)$ yields

$$\mathcal{G}^{t+s}(\theta) = \mathcal{G}^t(\omega s+\theta)\mathcal{G}^s(\theta) \quad \Longrightarrow \quad \mathcal{G}^t(\theta) = \mathcal{G}^{t+\theta/\omega}(0)\left[\mathcal{G}^{\theta/\omega}(0)\right]^{-1}, \tag{8.13}$$

$$\frac{\partial}{\partial s}\Big|_{s=0}\mathcal{G}^{t+s}(\theta) = \frac{\partial}{\partial s}\Big|_{s=0}\mathcal{G}^t(\omega s+\theta)\mathcal{G}^s(\theta) \quad \Longrightarrow \quad \frac{\partial}{\partial t}\mathcal{G}^t(\theta) = \omega\left[\mathcal{G}^t(\theta)\right]'_\theta + \mathcal{G}^t(\theta)A(\theta). \tag{8.14}$$

Now we see that the tangent bundle of $\mathcal{C}\times\mathfrak{G}$ splits into two invariant sub-bundles: the first one is spanned by the vector field of the flow $(\omega\mu'(\theta), A(\theta)G)$ (on account of (8.12), (8.14) this fact is also the consequence of (8.12) and (8.14) for $a=\omega$, $B=A(\theta)$), and the second one is naturally identified with the tangent bundle $T\mathfrak{G}$ by the correspondence $(\mu(\theta), G) \mapsto (0, T_G\mathfrak{G})$. Thus, it remains to analyze properties of the tangent co-cycle action on $T_G\mathfrak{G}$.

In what follows, we will focus on the hyperbolic case where the monodromy matrix $M := \mathcal{G}^{1/\omega}(0)$ has real eigenvalues (the Floquet multipliers) $\rho_1 = \rho$ and $\rho_2 = \rho^{-1}$, $|\rho| > 1$. Numerical experiments show that this case actually takes place for an appropriate range of parameters $\lambda_1, \lambda_2, \lambda_3$ and c_0, c_1. E.g., in particular case where $\lambda_1 = 3/2$; $\lambda_2 = 3$; $\lambda_3 = 3/2$, $m_1(0) \equiv 4/5$, $m_2(0) = 1$, $m_3(0) = 0$, and thus, $c_1^0 = 3.96$, $c_2^0 = -0.36$, we obtain

$$M \approx \begin{pmatrix} -6.84081991830724 & 2.57720475804614 \\ -2.57720426780894 & 0.82475266189202 \end{pmatrix}, \qquad \rho \approx -5.84498051556855.$$

By the Floquet theorem, there exists a mapping $\Phi(\cdot) : \mathbb{R} \mapsto \mathrm{SL}(2;\mathbb{R})$ such that

$$\Phi(\theta+1) = \operatorname{sign}\rho\,\Phi(\theta), \qquad \mathcal{G}^t(0) = \Phi(\omega t)\mathrm{e}^{Lt}$$

where $L := \omega \ln(\operatorname{sign}\rho M) \in \mathrm{sl}(2;\mathbb{R})$. Now (8.13) implies

$$\begin{aligned}\mathcal{G}^t(\theta)BG &= \Phi(\omega t+\theta)\mathrm{e}^{Lt}\Phi^{-1}(\theta)BG \\ &= \Phi(\omega t+\theta)\left[\mathrm{e}^{Lt}\Phi^{-1}(\theta)B\Phi(\theta)\,\mathrm{e}^{-Lt}\right]\mathrm{e}^{Lt}\Phi^{-1}(\theta)G.\end{aligned}$$

We see that the properties of the tangent co-cycle action on $T_G\mathfrak{G}$ are completely determined by the adjoint action of the one-parameter sub-group $\{\mathrm{e}^{Lt}\}$ on $\mathfrak{g}$, and thus, by the spectrum of the corresponding operator $\mathrm{ad}_L : \mathfrak{g} \mapsto \mathfrak{g}$, $\mathrm{ad}_L A := LA - AL$. It is not hard to calculate

$$\sigma(\mathrm{ad}_L) = \left\{0, 2\sqrt{-\det L}, -2\sqrt{-\det L}\right\} = \left\{0, \ln\rho^2, -\ln\rho^2\right\}.$$

Now consider a system on $T^*\mathcal{Q}$ obtained by factorization of system (8.5)–(8.4)

$$\dot{m} = \{m, H(m)\} + F(m), \qquad \dot{q} = Q(m,q), \tag{8.15}$$

where $q \in \mathcal{Q}$, and $Q(m,q) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\exp\left(\sum_{k=1}^3 \lambda_k m_k A_k\right)q$. The above reasoning implies that this system has 4-D compact exponentially stable invariant manifold $\mathcal{M} = \mathcal{C}\times\mathcal{Q}$. The tangent bundle $T\mathcal{M}$ admits invariant splitting into a Whitney sum $V^s \oplus V^c \oplus V^u$ of three sub-bundles: 1-D stable V^s, 1-D unstable V^u, and 2-D center V^c (every co-cycle orbit on V^c is bounded). To show that any motion starting close to $\mathcal{M}$ has an asymptotic phase we are going to apply Theorem 5.1. It should be noted that the mentioned theorem concerns systems situated in

Euclidean spaces. Hence, we have to embed system (8.15) into an auxiliary system possessing the same exponentially stable invariant manifold $\mathcal{M}$ and defined in a domain of Euclidean space.

Since $\mathcal{Q}$ is parallelizable, then it is a π-manifold [24], and thus, $\mathcal{Q}$ can be embedded in a Euclidean space $\mathbb{R}^d$ of a sufficiently high dimension d with trivial normal bundle $N\mathcal{Q} \sim \mathcal{Q} \times \mathbb{R}^{d-3}$. Hence, for sufficiently small $\delta > 0$, there exists a diffeomorphism $\psi(\cdot)$ of $\mathcal{Q} \times \mathcal{B}_\delta^{d-3}(0)$** onto a tubular neighborhood of $\mathcal{Q}$ in $\mathbb{R}^d$ such that $\psi(\{q\} \times \mathcal{B}_\delta^{d-3}(0)) \subset N_q\mathcal{Q}$ for any $q \in \mathcal{Q}$. Finally, to obtain the required auxiliary system we embed the system

$$\dot{m} = \{m, H(m)\} + F(m), \qquad \dot{q} = Q(m,q), \qquad y = -y$$

into $\mathbb{R}^3 \times \mathbb{R}^d$ by means of the diffeomorphism $\mathrm{id}(\cdot) \times \psi(\cdot) : \mathbb{R}^3 \times \mathcal{Q} \times \mathcal{B}_\delta^{d-3}(0) \mapsto \mathbb{R}^3 \times \mathbb{R}^d$.

9 Conclusion

In order to simplify our exposition we restrict ourselves to the case where the invariant manifold is situated in Euclidean space. Actually, this is not a serious restriction. If we deal with a system defined on a manifold $\mathfrak{M}$ and $\mathcal{M}$ is an attracting invariant submanifold, than we can apply the same trick as in Section 8. Namely, we have to embed the manifold $\mathfrak{M}$ into Euclidean space of sufficiently high dimension d and to extend the initial system to an auxiliary d-dimensional system such that its domain is a neighborhood of $\mathfrak{M}$ in $\mathbb{R}^d$ and its motions are attracted by $\mathcal{M}$.

It would be interesting to consider the case where the attracting invariant manifold admits a partition into subsets with different types of partial hyperbolicity. Just this case can happen when, for an appropriate perturbation $F(\cdot)$, the attracting manifold of system (8.5) is some level set of the Hamiltonian. We expect that in such a situation, to tackle the problem on the existence of asymptotic phase, the results of Section 7 might be useful.

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**Recall the notation $\mathcal{B}_\delta^{d-3}(0) := \{y \in \mathbb{R}^{d-3} : \|y\| < \delta\}$.

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