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Analysis of an age-structured dengue model with multiple strains and cross immunity

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Abstract. Dengue fever is a typical mosquito-borne infectious disease, and four strains of it are currently found. Clinical medical research has shown that the infected person can provide life-long immunity against the strain after recovering from infection with one strain, but only provide partial and temporary immunity against other strains. On the basis of the complexity of transmission and the diversity of pathogens, in this paper, a multi-strain dengue transmission model with latency age and cross immunity age is proposed. We discuss the well-posedness of this model and give the terms of the basic reproduction number $\mathcal{R}_0 = \max\{\mathcal{R}_1,\ \mathcal{R}_2\}$, where $\mathcal{R}_i$ is the basic reproduction number of strain i $(i = 1, 2)$. Particularly, we obtain that the model always has a unique disease-free equilibrium P_0 which is locally stable for $\mathcal{R}_0 < 1$. And same time, an explicit condition of the global asymptotic stability of P_0 is obtained by constructing a suitable Lyapunov functional. Furthermore, we also shown that if $\mathcal{R}_i > 1$, the strain-i dominant equilibrium P_i is locally stable for $\mathcal{R}_j < \mathcal{R}_i^*$ $(i,\ j = 1,\ 2,\ i \neq j)$. Additionally, the threshold criteria on the uniformly persistence, the existence and global asymptotically stability of coexistence equilibrium are also obtained. Finally, these theoretical results and interesting conclusions are illustrated with some numerical simulations.

Keywords: dengue fever, age-structured model, cross immunity, uniform persistence, stability.

2020 Mathematics Subject Classification: 35E99, 92D30.

1 Introduction

Dengue is a vector-borne disease which was first described in 1779, and is common in more than 100 countries around the world [16]. Dengue viruses are spread to humans through the bite of an infected female mosquito (mainly *Aedes aegypti* and *Aedes albopictus*, which are known as the principal vector of Zika, chikungunya, and other viruses). In recent decades, the global incidence of dengue fever has increased dramatically and about half the world's population is now at risk. Each year, up to 400 million infections occur particularly in tropical and sub-tropical regions [1]. Due to its high morbidity and mortality, the World Health Organization has identified dengue as one of ten threats to global health in 2019 [44]. In order to understand

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the mechanism of dengue fever transmission, a lot of mathematical models have been used to analyze its epidemiological characteristics [10,12,20,24,26,30]. For example, Esteva et al. [10] proposed an ordinary differential equations for the transmission of dengue fever with variable human population size, found three threshold parameters that control the development of this disease and the growth of the human population. Lee et al. [24] formulated a two-patch model to assess the impact of dengue transmission dynamics in heterogeneous environments, and found that reducing traffic is likely to take a host-vector system into the world of manageable outbreaks.

It is well know that dengue fever is caused by the dengue virus, which contains four different but closely relevant serotypes (DEN1-DEN4), for more details, see [9,11,43]. Medical statistic results show that recovery from infection with one virus provides lifelong immunity to that virus, but just temporal cross immunity to the other viruses. Subsequent infection with other viruses increases the risk of severe dengue (including Dengue Hemorrhagic Fever and Dengue Shock Syndrome) which can be life-threatening [43]. According to the diversity and transmission mechanism of dengue fever virus, some multi-strain dengue fever models have been established to investigate the effect of immunological interactions between heterotypic infections on disease dynamics. One example can be found in Ref. [9], Esteva et al. proposed a multi-strain dengue fever model, where the authors assumed that the primary infection with a specific strain changes the probability of being infected by a heterologous strain. Another example is that Feng et al. [11] established a multi-strain dengue fever model and found that there exists competitive exclusion phenomenon between different strains. More research can be found in [9,11,17,19,27,29,32,34,41,42] and the references therein. Of course, there is still a lot of research that has not been mentioned, and the research continues.

The patterns of transmission, infectivity and latent period of infectious diseases play an important role in the process of transmission. It is well known that the period for individuals in latent compartment is different from one to one, which depends on individuals situation. For dengue fever, the period for individuals in latent compartment varies from 3 to 14 days and its distributions usually peak around their mean [3,7]. And for tuberculosis, the latent period for individuals in latent compartment may take months, years or even decades. Therefore, several epidemic models with latent age (time since entry into latent compartment) have been proposed by many famous experts and scholars [5,21,37,40]. Particularly, Wang et al. [37] proposed an SVEIR epidemic model with age-dependent vaccination and latency, found that the latency age not only impacts on the basic reproduction number but also could affect the values of the endemic steady state. They also showed that the introduction of age structure may change the dynamics of the corresponding model without age structure. Additionally, recent studies [3,15] pointed out cross immunity starts immediately after the primary infectious period and prevents individuals from becoming infected by another strain for a period ranging from 6 months to 9 months, even to lifelong. To the best of our knowledge, there is currently no work on the effect of cross immunity age on the dynamics of dengue fever model.

Based on the discussion above, it is necessary to incorporate latency age and cross immunity age in the modeling of dengue fever. In this paper, we formulate a multi-strain dengue model with latency age and cross immunity age to assess the effects of latency age and cross immunity age on the transmission of dengue fever. The paper is structured as follows. The model is proposed in Section 2, and the nonnegative, boundedness and smoothness of the solution of this model are presented in Section 3. Section 4 analyzes the existence and stability of the boundary equilibria of model, which includes the disease-free equilibrium and stain dominant equilibrium. In Section 5, the uniform persistence of disease is discussed and the existence of coexistence equilibrium is obtained, and the theoretical results are illustrated with numerical simulations in Section 6. The paper ends with a brief conclusion.

2 Model formulation

Studies have shown that the number of dengue admissions caused by a third and fourth dengue virus infection have relatively few reported cases, accounting for only $0.08\% - -0.80\%$ of the number of cases [14]. Therefore, it is reasonable to consider two strains in our model denote by strain 1 and strain 2, where 1 and 2 can be DEN1–DEN4. The infected individuals are divided into primary infected and secondary infected, and ignore further infections. Let $S(t)$ represent the number of susceptible individuals who are susceptible to both strain 1 and strain 2 at time t. $\widehat{E}_i(t)$, $I_i(t)$ and $R_i(t)$ represent the number of latent, primary infected and recovered individuals with strain $i(i = 1,\ 2)$ at time t, respectively. Likewise, $Y_i(t)$ be the number of secondary infected individuals with strain i after being recovered from strain j $(i,\ j = 1,\ 2,\ j \neq i)$ at time t. Let $R(t)$ represent the number of recovered individuals from secondary infection at time t (to be permanently immune to both strains and hence there is no need to consider the evolution of $R(t)$). At the same time, due to the short length of mosquitoes' life cycle, assuming that a mosquito, once infected, never recovers and no secondary infection occurs. The mosquito population is subdivided into susceptible class $U(t)$, and infectious with strain i class $V_i(t)$ $(i = 1,\ 2)$. Based on the transmission characteristics of dengue fever, we further propose two basic assumptions:

(A1) For latent individuals, the latent age (time since entry into latent class) is denoted by a. Let $E_i(t,a)$ denote the number of strain i latent individuals with latent age a at time t. Then the total number of strain i latent individuals at time t is given by $\widehat{E}_i(t) = \int_0^\infty E_i(t,a)\mathrm{d}a$. The conversion rate at which the latent individuals become infectious depends on the latent age, and is denoted by $\varepsilon_i(a),\ i = 1,\ 2$.

(A2) For recovered individuals, assume that the cross immunity wanes with time. Denote the cross immunity age, i.e., time since entry into recovered class $\widehat{R}_i(i = 1,2)$, by b. Let $R_i(t,b)$ represent the number of the recovered individuals from strain i $(i = 1,2)$ at time t and cross immunity age b. Then the total number of strain i recovered individuals at time t is given by $\widehat{R}_i(t) = \int_0^\infty R_i(t,b)\mathrm{d}b, i = 1,2$. The rate at which the cross immunity wanes of $\widehat{R}_i$ $(i = 1,2)$ depends on cross immunity age, and is denoted by $\theta_j(b), j = 1,2$.

Based on the above assumptions, the model can be written as the following,

$$
\begin{cases}
\dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda_h - \beta_1 S(t)V_1(t) - \beta_2 S(t)V_2(t) - \mu_h S(t), \\
\left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) E_i(t,a) = -(\mu_h + \varepsilon_i(a))E_i(t,a), \\
\dfrac{\mathrm{d}I_i(t)}{\mathrm{d}t} = \displaystyle\int_0^\infty \varepsilon_i(a)E_i(t,a)\mathrm{d}a - (\gamma_i + \mu_h)I_i(t), \\
\left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) R_i(t,b) = -\beta_j\theta_j(b)V_j(t)R_i(t,b) - \mu_h R_i(t,b), \\
\dfrac{\mathrm{d}Y_i(t)}{\mathrm{d}t} = \beta_i V_i(t)\displaystyle\int_0^\infty \theta_i(b)R_j(t,b)\mathrm{d}b - (\gamma_i + d_i + \mu_h)Y_i, \\
\dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \Lambda_m - \alpha_1(I_1(t) + Y_1(t))U(t) - \alpha_2(I_2(t) + Y_2(t))U(t) - \mu_m U(t), \\
\dfrac{\mathrm{d}V_i(t)}{\mathrm{d}t} = \alpha_i(I_i(t) + Y_i(t))U(t) - \mu_m V_i(t), \\
E_i(t,0) = \beta_i S(t)V_i(t),\ R_i(t,0) = \gamma_i I_i(t), \quad i,\ j = 1,\ 2, i \neq j,
\end{cases}
\tag{2.1}
$$

with the initial condition

$$
\begin{aligned}
&S(0)=S_0\geq 0,\quad E_i(0,a)=E_{i0}(a)\geq 0,\quad I_i(0)=I_{i0}\geq 0,\quad R_i(0,b)=R_{i0}(b)\geq 0,\\
&Y_i(0)=Y_{i0}\geq 0,\quad U(0)=U_0\geq 0,\quad V_i(0)=V_{i0}\geq 0,\qquad i=1,2,
\end{aligned}
\tag{2.2}
$$

where $E_{i0}(a)$, $R_{i0}(b)\in L^1_+(0,\infty)$, and $L^1_+(0,\infty)$ is the space of nonnegative and Lebesgue integrable functions on $(0,\infty)$. In model (2.1), Λ_h and Λ_m are the recruitment rates of human and mosquito population, respectively; $1/\mu_h$ and $1/\mu_m$ denote the life expectancy for human and the average lifespan of mosquito, respectively; β_i is the infectious rate from mosquito to human with strain i; γ_i is the recovery rate of human with strain i; d_i is the disease induced death rate in human with strain i and α_i is the infectious rate from human to mosquito with strain i, $i=1,\ 2$. All these parameters are assumed to be positive.

For model (2.1), the following hypotheses are reasonable.

(H1) $\varepsilon_i(\cdot),\theta_i(\cdot)\in L^1_+(0,\infty)$ are bounded with essential upper bound $\bar{\varepsilon}_i$, $\bar{\theta}_i$, and Lipschitz continuous on $\mathbb{R}_+$ with Lipschitz coefficients $M_{\varepsilon i}$, $M_{\theta i}$, $i=1,\ 2$, respectively. Besides, assuming that $\theta_i(\cdot)\in[0,1)$, if $\theta_i(\cdot)\in(0,1)$, then there exists cross-immunity between the two strains; if $\theta_i(\cdot)=0$, then individuals recovered from primary infection with one strain confer lifelong immunity to both strains.

(H2) $\bar{a}_i$ and $\bar{b}_i$ are the maximum ages of latency and cross immunity, the $\int_{\bar{a}_i}^{\infty}E_{i0}(a)\mathrm{d}a=0$ and $\int_{\bar{b}_i}^{\infty}R_{i0}(b)\mathrm{d}b=0,\ i=1,\ 2.$

The state space of model (2.1) is defined as follows, $\mathbb{X}=\mathbb{R}_+\times L^1_+(0,\infty)\times L^1_+(0,\infty)\times\mathbb{R}^2_+\times L^1_+(0,\infty)\times L^1_+(0,\infty)\times\mathbb{R}^5_+$. For any $X=(x_1,\phi_1,\phi_2,x_2,x_3,\psi_1,\psi_2,x_4,x_5,x_6,x_7,x_8)\in\mathbb{X}$ the norm is defined by

$$
\|X\|_{\mathbb{X}}=\sum_{i=1}^{8}|x_i|+\int_0^\infty|\varphi_1(a)|\mathrm{d}a+\int_0^\infty|\varphi_2(a)|\mathrm{d}a+\int_0^\infty|\psi_1(b)|\mathrm{d}b+\int_0^\infty|\psi_2(b)|\mathrm{d}b.
$$

For the convenience, we denote the solution of model (2.1) by $X(t)=(S(t),\ E_1(t,\cdot),\ E_2(t,\cdot),\ I_1(t),\ I_2(t),\ R_1(t,\cdot),\ R_2(t,\cdot),\ Y_1(t),\ Y_2(t),\ U(t),\ V_1(t),\ V_2(t))$. Let $X_0:=(S_0,\ E_{10}(\cdot),\ E_{20}(\cdot),\ I_{10},\ I_{20},\ R_{10}(\cdot),\ R_{20}(\cdot),\ Y_{10},\ Y_{20},\ U_0,\ V_{10},\ V_{20})$, then the initial condition (2.2) is rewritten as $X(0)=X_0$. Furthermore, we denote by $X(t,X_0)$ the solution of model (2.1) with the initial condition $X(0)=X_0$.

3 The well-posedness

Solving $E_i(t,a)$ and $R_i(t,b)$ in the second and fourth equations of model (2.1) along the characteristic line $t-a=const$ and $t-b=const$, respectively, we have

$$
\begin{aligned}
E_i(t,a)&=\begin{cases}\beta_i S(t-a)V_i(t-a)\eta_i(a), & 0\leq a<t,\\[4pt] E_{i0}(a-t)\dfrac{\eta_i(a)}{\eta_i(a-t)}, & 0\leq t\leq a,\end{cases}\\
R_i(t,b)&=\begin{cases}\gamma_i I_i(t-b)\Omega_j(t,b), & 0\leq b<t,\\[4pt] R_{i0}(b-t)\dfrac{\Omega_j(t,b)}{\Omega_j(t,b-t)}, & 0\leq t\leq b,\end{cases}
\end{aligned}
\tag{3.1}
$$

where $\eta_i(a)=e^{-\int_0^a(\mu_h+\varepsilon_i(s))\mathrm{d}s}$, $\Omega_i(t,b)=e^{-\int_0^b[\beta_i\theta_i(s)V_i(t-b+s)+\mu_h]\mathrm{d}s}$, $i,\ j=1,\ 2,\ i\neq j$.

On the existence and nonnegativity of solution for model (2.1), we have the following result.

Theorem 3.1.

(i) *For any* $X_0 \in \mathbb{X}$, *model* (2.1) *has a unique solution* $X(t)$ *with the initial condition* $X(0) = X_0$ *defined in maximal existence interval* $[0, t_0)$ *with* $t_0 > 0$.

(ii) $X(t)$ *is non-negative for all* $t \in [0, t_0)$.

(iii) *If* $S_0 > 0$, $E_{i0}(a) > 0$, $I_{i0} > 0$, $R_{i0}(b) > 0$, $Y_{i0} > 0$, $U_0 > 0$, $V_{i0} > 0$ $(i = 1,\ 2)$, *then* $X(t)$ *also is positive for all* $t \in [0, t_0)$.

Proof. From the Ref. [39], it is clear that conclusion (i) holds. From (3.1), we directly yield that $E_i(t, a) > 0$ and $R_i(t, b) > 0$ $(i = 1,\ 2)$ for all $t \in [0, t_0)$. We can obtain that the solution $X(t)$ of model (2.1) with positive initial value remains is positive by the method of Ref. [38]. From the continuous dependence of solutions with respect to initial value, we immediately obtain that $X(t)$ is non-negative for all $t \in [0, t_0)$. This completes the proof. $\square$

Denote

$$\mathbb{D} = \Big\{ X = (S, E_1(a), E_2(a), I_1, I_2, R_1(b), R_2(b), Y_1, Y_2, U, V_1, V_2) \in \mathbb{X} : S + \sum_{i=1}^{2} \Big(\|E_i(a)\|_{L^1} + I_i + \|R_i(b)\|_{L^1} + Y_i \Big) \le \frac{\Lambda_h}{\mu_h},\ U + V_1 + V_2 \le \frac{\Lambda_m}{\mu_m} \Big\}.$$

The following result is on the boundedness of solutions of model (2.1).

Theorem 3.2. *For any initial value* $X_0 \in \mathbb{X}$, *solution* $X(t, X_0)$ *of model* (2.1) *is defined for all* $t \ge 0$ *and is ultimately bounded. Further,* $\mathbb{D}$ *is positively invariant for model* (2.1), *i.e.,* $X(t, X_0) \in \mathbb{D}$ *for all* $t \ge 0$ *and* $X_0 \in \mathbb{D}$, *and* $\mathbb{D}$ *attracts all points in* $\mathbb{X}$.

Proof. From Theorem 3.1, it is obvious that $X(t, X_0) \ge 0$ for all $t \in [0, t_0)$. Define

$$N_h(t) = S(t) + \sum_{i=1}^{2} \left(\int_0^\infty E_i(t, a) \mathrm{d}a + I_i(t) + \int_0^\infty R_i(t, b) \mathrm{d}b + Y_i(t) \right)$$

and $N_m(t) = U(t) + V_1(t) + V_2(t)$, from model (2.1), we have

$$\frac{\mathrm{d}N_h(t)}{\mathrm{d}t} = \Lambda_h - \mu_h N_h(t) - d(Y_1(t) + Y_2(t)) \le \Lambda_h - \mu_h N_h(t), \quad \frac{\mathrm{d}N_m(t)}{\mathrm{d}t} = \Lambda_m - \mu_m N_m(t), \tag{3.2}$$

which implies that

$$N_h(t) \le \max\left\{ N_h^0, \frac{\Lambda_h}{\mu_h} \right\}, \qquad N_m(t) \le \max\left\{ N_m^0, \frac{\Lambda_m}{\mu_m} \right\}.$$

Hence, $N_h(t)$ and $N_m(t)$ are bounded on $[0, t_0)$, which implies that $X(t, X_0)$ is defined for any $t \ge 0$. Further, from (3.2), we have $\limsup_{t\to\infty} N_h(t) \le \Lambda_h/\mu_h$, $\limsup_{t\to\infty} N_m(t) \le \Lambda_m/\mu_m$. It follows that $X(t, X_0)$ is ultimately bounded. Furthermore, $\mathbb{D}$ is positively invariant for model (2.1), and D attracts each point in $\mathbb{X}$. The proof is complete. $\square$

From Theorems 3.1 and 3.2, we obtain that all nonnegative solutions $X(t, X_0)$ of model (2.1) with the initial condition $X(0) = X_0$ generate a continuous semi-flow $\Phi : \mathbb{R}_+ \times \mathbb{X} \to \mathbb{X}$ as $\Phi_t(X_0) = X(t, X_0),\ t \ge 0,\ X_0 \in \mathbb{X}$.

On the asymptotically smoothness of the semi-flow $\{\Phi_t\}_{t\ge 0}$, we have the following result.

Theorem 3.3. *The semi-flow* $\{\Phi_t\}_{t\ge0}$ *generated by model* (2.1) *is asymptotically smooth. Furthermore, model* (2.1) *has a compact global attractor* $\mathcal{A}$ *contained in* $\mathbb{X}$.

This theorem can be proved by using the standard argument, see [40] for detailed proof methods.

4 The existence and stability of boundary equilibria

Model (2.1) always has a disease-free equilibrium $P_0 = (S^*, 0, 0, 0, 0, 0, 0, 0, 0, 0, U^*, 0, 0)$, where $S^* = \Lambda_h/\mu_h$, $U^* = \Lambda_m/\mu_m$. For the convenience, denote $K_i = \int_0^\infty \varepsilon_i(a)\eta_i(a)\mathrm{d}a,\ i = 1,\ 2$. It is clear that $K_i \in (0,1),\ i = 1,\ 2$.

Denote the basic reproduction number $\mathcal{R}_0$ by

$$\mathcal{R}_0 = \max\{\mathcal{R}_1, \mathcal{R}_2\},\ \ \mathcal{R}_i = \frac{\Lambda_h\Lambda_m\alpha_i\beta_i K_i}{\mu_h\mu_m^2(\gamma_i+\mu_h)} = \frac{\Lambda_h}{\mu_h}\times\frac{\Lambda_m}{\mu_m}\times\frac{\beta_i}{\mu_m}\times\frac{\alpha_i}{\gamma_i+\mu_h}\times K_i,\ \ i = 1,\ 2. \tag{4.1}$$

Here, β_i/μ_m represents the number of secondary infections one infectious mosquito will produce in a completely susceptible human population, $\alpha_i/(\gamma_i+\mu_h)$ represents the number of effective contact human to mosquito during the infectious period of human and K_i represents the probability of an exposed individual becomes infective. Therefore, $\mathcal{R}_i$ can be considered as the basic reproduction number of strain i, which is defined as the average number of secondary infective of strain i, produced by a single infective of strain i in a completely susceptible population.

Let $E_2(t,a) = I_2(t) = R_2(t,b) = Y_2(t) = V_2(t) = 0$ in model (2.1), then we obtain the subsystem that only strain 1 exists as follows

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda_h - \beta_1 S(t)V_1(t) - \mu_h S(t), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right)E_1(t,a) = -(\mu_h+\varepsilon_1(a))E_1(t,a),\ E_1(t,0) = \beta_1 S(t)V_1(t), \\ \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = \displaystyle\int_0^\infty \varepsilon_1(a)E_1(t,a)\mathrm{d}a - (\gamma_1+\mu_h)I_1(t), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial b}\right)R_1(t,b) = -\mu_h R_1(t,b),\ R_1(t,0) = \gamma_1 I_1(t), \\ \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \Lambda_m(t) - \alpha_1 I_1(t)U(t) - \mu_m U(t), \\ \dfrac{\mathrm{d}V_1(t)}{\mathrm{d}t} = \alpha_1 I_1(t)U(t) - \mu_m V_1(t). \end{cases} \tag{4.2}$$

Clearly, model (4.2) always has a disease-free equilibrium $p_0 = (\Lambda_h/\mu_h, 0, 0, 0, \Lambda_m/\mu_m, 0)$. Let $p_1 = (S_1^*,\ E_1^*(a),\ I_1^*,\ R_1^*(b),\ U_1^*,\ V_1^*)$ be the positive equilibrium of model (4.2), then

$$\begin{aligned} &\Lambda_h - \beta_1 S_1^* V_1^* - \mu_h S_1^* = 0, && \Lambda_m - \alpha_1 I_1^* U_1^* - \mu_m U_1^* = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}a}E_1^*(a) = -(\mu_h+\varepsilon_1(a))E_1^*(a), && \frac{\mathrm{d}}{\mathrm{d}b}R_1^*(b) = -\mu_h R_1^*(b), \\ &\int_0^\infty \varepsilon_1(a)E_1^*(a)\mathrm{d}a - (\gamma_1+\mu_h)I_1^* = 0, && E_1^*(0) = \beta_1 S_1^* V_1^*, \quad R_1^*(0) = \gamma_1 I_1^*. \end{aligned} \tag{4.3}$$

From (4.3),

$$R_1^*(b) = R_1^*(0)e^{-\mu_h b} = \gamma_1 I_1^* e^{-\mu_h b}, \qquad U_1^* = \frac{\Lambda_m}{\alpha_1 I_1^*+\mu_m}, \qquad V_1^* = \frac{\alpha_1 I_1^*\Lambda_m}{\mu_m(\alpha_1 I_1^*+\mu_m)},$$

$$E_1^*(a) = E_1^*(0)\eta_1(a) = \beta_1 S_1^* V_1^* \eta_1(a), \qquad S_1^* = \frac{\mu_m(\alpha_1 I_1^*+\mu_m)(\gamma_1+\mu_h)}{\alpha_1\beta_1\Lambda_m K_1}.$$

Substituting the above formulas for V_1^*, $E_1^*(a)$ and S_1^* into the first equation of (4.3) yields

$$I_1^* = \frac{\alpha_1\beta_1\Lambda_h\Lambda_m K_1 - \mu_h\mu_m^2(\gamma_1+\mu_h)}{\alpha_1(\gamma_1+\mu_h)(\beta_1\Lambda_m+\mu_h\mu_m)} = \frac{\mu_h\mu_m^2(\mathcal{R}_1-1)}{\alpha_1(\beta_1\Lambda_m+\mu_m\mu_h)}.$$

Thus, from the expressions of S_1^*, $E_1^*(a)$, I_1^*, $R_1^*(b)$, U_1^* and V_1^*, it can be easily seen that model (4.2) has a unique positive equilibrium p_1 if and only if $\mathcal{R}_1 > 1$. Therefore, model (2.1) has a strain 1 dominant boundary equilibrium $P_1 = (S_1^*, E_1^*, 0, I_1^*, 0, R_1^*(b), 0, 0, 0, U_1^*, V_1^*, 0)$ when $\mathcal{R}_1 > 1$, where

$$S_1^* = \frac{\mu_m^2(\gamma_1+\mu_h)(\mathcal{R}_1\mu_h\mu_m+\beta_1\Lambda_m)}{\alpha_1\beta_1\Lambda_m(\beta_1\Lambda_m+\mu_m\mu_h)K_1}, \qquad E_1^*(a) = \frac{\mu_m^2\mu_h\eta_1(a)(\gamma_1+\mu_h)(\mathcal{R}_1-1)}{\alpha_1(\beta_1\Lambda_m+\mu_m\mu_h)K_1},$$
$$I_i^* = \frac{(\mathcal{R}_1-1)\mu_h\mu_m^2}{\alpha_1(\beta_1\Lambda_m+\mu_m\mu_h)}, \qquad R_1^*(b) = \frac{(\mathcal{R}_1-1)\gamma_1\mu_h\mu_m^2e^{-\mu_h b}}{\alpha_1(\beta_1\Lambda_m+\mu_m\mu_h)},$$
$$U_1^* = \frac{\Lambda_m(\beta_1\Lambda_m+\mu_m\mu_h)}{\mu_m(\mathcal{R}_1\mu_h\mu_m+\beta_1\Lambda_m)}, \qquad V_1^* = \frac{\Lambda_m\mu_h(\mathcal{R}_1-1)}{(\mathcal{R}_1\mu_h\mu_m+\beta_1\Lambda_m)}.$$

Similarly, model (2.1) has a strain-2 dominant boundary equilibrium $P_2 = (S_2^*, 0, E_2^*, 0, I_2^*, 0, R_2^*(b), 0, 0, U_2^*, 0, V_2^*)$ when $\mathcal{R}_2 > 1$, where

$$S_2^* = \frac{\mu_m^2(\gamma_2+\mu_h)(\mathcal{R}_2\mu_h\mu_m+\beta_2\Lambda_m)}{\alpha_2\beta_2\Lambda_m(\beta_2\Lambda_m+\mu_m\mu_h)K_2}, \qquad E_2^*(a) = \frac{\mu_m^2\mu_h\eta_2(a)(\gamma_2+\mu_h)(\mathcal{R}_2-1)}{\alpha_2(\beta_2\Lambda_m+\mu_m\mu_h)K_2},$$
$$I_2^* = \frac{(\mathcal{R}_2-1)\mu_h\mu_m^2}{\alpha_2(\beta_2\Lambda_m+\mu_m\mu_h)}, \qquad R_2^*(b) = \frac{(\mathcal{R}_2-1)\gamma_2\mu_h\mu_m^2e^{-\mu_h b}}{\alpha_2(\beta_2\Lambda_m+\mu_m\mu_h)},$$
$$U_2^* = \frac{\Lambda_m(\beta_2\Lambda_m+\mu_m\mu_h)}{\mu_m(\mathcal{R}_2\mu_h\mu_m+\beta_2\Lambda_m)}, \qquad V_2^* = \frac{\Lambda_m\mu_h(\mathcal{R}_2-1)}{(\mathcal{R}_2\mu_h\mu_m+\beta_2\Lambda_m)}.$$

Summarizing the discussions above, we have the following theorem.

Theorem 4.1.

(i) Model (2.1) always has a disease-free equilibrium P_0.

(ii) If $\mathcal{R}_1 > 1$, then model (2.1) has a strain 1 dominant equilibrium P_1.

(iii) If $\mathcal{R}_2 > 1$, then model (2.1) has a strain 2 dominant equilibrium P_2.

On the stability of boundary equilibria of model (2.1), we first obtain the following results.

Theorem 4.2. *If $\mathcal{R}_0 < 1$, then the disease-free equilibrium P_0 of model (2.1) is locally asymptotically stable, and if $\mathcal{R}_0 > 1$, then P_0 is unstable.*

Proof. Let $S(t) = S^* + s(t)$, $E_i(t,a) = e_i(t,a)$, $I_i(t) = i_i(t)$, $R_i(t,a) = r_i(t,a)$, $Y_i(t) = y_i(t)$, $U(t) = U^* + u(t)$ and $V_i(t) = v_i(t)$, $i = 1,2$. Linearizing model (2.1) at equilibrium P_0, one has

$$\begin{cases} \dfrac{\mathrm{d}s(t)}{\mathrm{d}t} = -\beta_1(t)S^*v_1(t) - \beta_2(t)S^*v_2(t) - \mu_h s(t), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) e_i(t,a) = -(\mu_h + \varepsilon_i(a))e_i(t,a),\ e_i(t,0) = \beta_i S^* v_i(t), \\ \dfrac{\mathrm{d}i_i(t)}{\mathrm{d}t} = \displaystyle\int_0^\infty \varepsilon_i(a)e_i(t,a)\mathrm{d}a - (\gamma_i+\mu_h)i_i(t), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) r_i(t,b) = -\mu_h r_i(t,b),\ r_i(t,0) = \gamma_i i_i(t),\ i = 1,2. \end{cases} \tag{4.4}$$

and

$$\begin{cases} \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = -(\gamma_i + d_i + \mu_h)y_i(t), \\ \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = -\alpha_1(i_1(t)+y_1(t))U^* - \alpha_2(i_2(t)+y_2(t))U^* - \mu_m u(t), \\ \dfrac{\mathrm{d}v_i(t)}{\mathrm{d}t} = \alpha_i(i_i(t)+y_i(t))U^* - \mu_m v_i(t),\ i = 1,2. \end{cases} \tag{4.5}$$

It is easy to obtain that $\lim_{t\to\infty} y_i(t) = 0,\ i = 1,\ 2$ from the first equation of model (4.5). Thus, we only need to consider model (4.4) and the following limit system of model (4.5)

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = -\alpha_1 i_1(t)U^* - \alpha_2 i_2(t)U^* - \mu_m u(t), \\ \dfrac{\mathrm{d}v_i(t)}{\mathrm{d}t} = \alpha_i i_i(t)U^* - \mu_m v_i(t),\ i = 1,\ 2. \end{cases} \tag{4.6}$$

Let $s(t) = \bar{s}e^{\lambda t}$, $e_i(t,a) = \bar{e}_i(a)e^{\lambda t}$, $i_i(t) = \bar{i}_i e^{\lambda t}$, $r_i(t,b) = \bar{r}_i(b)e^{\lambda t}$, $u(t) = \bar{u}e^{\lambda t}$ and $v_i(t) = \bar{v}_i e^{\lambda t}$, where $\bar{s}$, $\bar{i}_i$, $\bar{y}_i$, $\bar{u}$ and $\bar{v}_i$ $(i = 1,\ 2)$ are positive constants, $\bar{e}_i(a)$ and $\bar{r}_i(b)$ are nonnegative functions, then we obtain the following eigenvalue problem

$$(\lambda + \mu_h)\bar{s} = -\beta_1 S^* \bar{v}_1 - \beta_2 S^* \bar{v}_2, \qquad (\lambda + \mu_m)\bar{u} = -\alpha_1 \bar{i}_1 U^* - \alpha_2 \bar{i}_2 U^* \tag{4.7}$$

and

$$\begin{cases} (\lambda + \gamma_i + \mu_h)\bar{i}_i = \displaystyle\int_0^\infty \varepsilon_i(a)\bar{e}_i(a)\mathrm{d}a, & (\lambda + \mu_m)\bar{v}_i = \alpha_i \bar{i}_i U^* \\ \dfrac{\mathrm{d}\bar{e}_i(a)}{\mathrm{d}a} = -(\mu_h + \varepsilon_i(a) + \lambda)\bar{e}_i(a), & \dfrac{\mathrm{d}\bar{r}_i(b)}{\mathrm{d}b} = -(\mu_h + \lambda)\bar{r}_i(b), \\ \bar{e}_i(0) = \beta_i S^* \bar{v}_i,\ \ \bar{r}_i(0) = \gamma_i \bar{i}_i,\ i = 1,2. \end{cases} \tag{4.8}$$

From (4.7), it follows that

$$\lambda_1 = -\frac{\beta_1 S^* \bar{v}_1 + \beta_2 S^* \bar{v}_2}{\bar{s}} - \mu_h < 0, \qquad \lambda_2 = -\frac{\alpha_1 \bar{i}_1 U^* + \alpha_2 \bar{i}_2 U^*}{\bar{u}} - \mu_m < 0.$$

Therefore, the stability of P_0 depends on the eigenvalues of (4.8). Directly calculating from the equations of $\bar{i}_i$, $\bar{e}_i(a)$ and $\bar{v}_i$ in problem (4.8) yields the following characteristic equation

$$\lambda + \gamma_i + \mu_h = \frac{\alpha_i \beta_i \Lambda_h \Lambda_m}{\mu_h \mu_m (\lambda + \mu_m)} \int_0^\infty \varepsilon_i(a) e^{-\int_0^a (\lambda + \mu_h + \varepsilon_i(s))\mathrm{d}s}\mathrm{d}a, \qquad i = 1,2. \tag{4.9}$$

Denote

$$LHS = \lambda + \gamma_i + \mu_h, \qquad RHS = \mathcal{G}(\lambda) = \frac{\alpha_i \beta_i \Lambda_h \Lambda_m}{\mu_h \mu_m (\lambda + \mu_m)} \int_0^\infty \varepsilon_i(a) e^{-\int_0^a (\lambda + \mu_h + \varepsilon_i(s))\mathrm{d}s}\mathrm{d}a.$$

It is easy to verify that for any eigenvalue λ, if $\mathrm{Re}(\lambda) \geq 0$, when $\mathcal{R}_0 < 1$, then

$$|LHS| \geq \gamma_i + \mu_h, \quad |RHS| \leq \mathcal{G}(\mathrm{Re}\,\lambda) \leq \mathcal{G}(0) = \mathcal{R}_i(\gamma_i + \mu_h) < |LHS|, \qquad i = 1,2.$$

This leads to a contradiction. Thus, all eigenvalues λ of problem (4.8) have negative real parts, which implies that $\lim_{t\to\infty} i_i(t) = 0$, $\lim_{t\to\infty} e_i(t,a) = 0$, $\lim_{t\to\infty} v_i(t) = 0$ and $\lim_{t\to\infty} r_i(t,b) = 0$. Therefore, P_0 is locally asymptotically stable when $\mathcal{R}_0 < 1$.

Now, assume that $\mathcal{R}_0 > 1$ and rewrite the characteristic equation (4.9) in the form

$$\mathcal{G}_{1i}(\lambda) = (\lambda + \gamma_i + \mu_h) - \frac{\alpha_i \beta_i \Lambda_h \Lambda_m}{\mu_h \mu_m (\lambda + \mu_m)} \int_0^\infty \varepsilon_i(a) e^{-\int_0^a (\lambda + \mu_h + \varepsilon_i(s))\mathrm{d}s}\mathrm{d}a = 0, \qquad i = 1,2.$$

Obviously,

$$\mathcal{G}_{1i}(0) = (\gamma_i + \mu_h) - \frac{\alpha_i \beta_i \Lambda_h \Lambda_m}{\mu_h \mu_m^2} \int_0^\infty \varepsilon_i(a) e^{-\int_0^a (\mu_h + \varepsilon_i(s))\mathrm{d}s}\mathrm{d}a = (\gamma_i + \mu_h)(1 - \mathcal{R}_i) < 0,$$

and $\lim_{\lambda\to\infty} \mathcal{G}_{1i}(\lambda) = +\infty$. Hence, the characteristic equation (4.9) at least has a positive real root. It implies that equilibrium P_0 is unstable. This completes the proof. □

Next, we discuss the global stability of equilibrium P_0. To do so, define

$$q_i(a) = \int_a^{\infty} \varepsilon_i(s) e^{-\int_a^s (\mu_h + \varepsilon_i(\xi))\mathrm{d}\xi} \mathrm{d}s, \qquad i = 1, 2.$$

It is easy to obtain that

$$\frac{\mathrm{d}q_i(a)}{\mathrm{d}a} = (\mu_h + \varepsilon_i(a))q_i(a) - \varepsilon_i(a), \quad q_i(0) = K_i, \ i = 1, 2.$$

Theorem 4.3. *If $\mathcal{R}_0 \le \min\{K_1, K_2\}$, then disease-free equilibrium P_0 of model (2.1) is globally asymptotically stable.*

Proof. Define a Lyapunov functional as follows

$$L(t) = \sum_{i=1}^{2} \left(\int_0^{\infty} q_i(a) E_i(t, a)\mathrm{d}a + I_i(t) + K_i Y_i(t) + \frac{\beta_i \Lambda_h}{\mu_m \mu_h} K_i V_i(t) \right).$$

Calculating the time derivative of $L(t)$ along the solution of model (2.1), it can be easily obtained that

$$\begin{aligned}
\frac{\mathrm{d}L(t)}{\mathrm{d}t} &= \sum_{i=1}^{2} \Big(\beta_i K_i S(t) V_i(t) - (\gamma_i + \mu_h) I_i(t) \Big) + \sum_{i=1}^{2} \Big(\frac{\alpha_i \beta_i \Lambda_h K_i}{\mu_m \mu_h} U(t)(I_i(t) + Y_i(t)) \\
&\quad - \frac{\beta_i \Lambda_h K_i}{\mu_h} V_i(t) \Big) + \sum_{i=1}^{2} \Big(K_i \beta_i V_i \int_0^{\infty} \theta_i(b) R_j(t, b)\mathrm{d}b - (\gamma_i + d_i + \mu_h) K_i Y_i(t) \Big) \\
&\le \sum_{i=1}^{2} \Big(\beta_i K_i S(t) V_i(t) - (\gamma_i + \mu_h) I_i(t) \Big) + \sum_{i=1}^{2} \Big(\frac{\alpha_i \beta_i \Lambda_h \Lambda_m K_i}{\mu_m^2 \mu_h} (I_i(t) + Y_i(t)) \\
&\quad - \frac{\beta_i \Lambda_h K_i}{\mu_h} V_i(t) \Big) + \sum_{i=1}^{2} \Big(K_i \beta_i V_i \int_0^{\infty} R_j(t, b)\mathrm{d}b - (\gamma_i + d_i + \mu_h) K_i Y_i(t) \Big) \\
&\le \sum_{i=1}^{2} \Big(\beta_i K_i V_i(t) \Big(S(t) + \int_0^{\infty} R_j(t, b)\mathrm{d}b - \frac{\Lambda_h}{\mu_h} \Big) \Big) + \sum_{i=1}^{2} \Big((\gamma_i + \mu_h)(\mathcal{R}_i - 1) I_i(t) \Big) \\
&\quad + \sum_{i=1}^{2} \Big((\gamma_i + \mu_h)(\mathcal{R}_i - K_i) Y_i(t) \Big) - d_1 Y_1(t) - d_2 Y_2(t).
\end{aligned}$$

Restricting to set $\mathbb{D}$, we have $S(t) + \int_0^{\infty} R_j(t, b)\mathrm{d}b - \Lambda_h/\mu_h \le 0$ for all $t \ge 0$. Hence, when $\mathcal{R}_i \le K_i$ $(i = 1,\ 2)$, we have $\mathrm{d}L(t)/\mathrm{d}t \le 0$, and the equality holds only if $I_i(t) = Y_i(t) = 0$ and

$$V_i(t) \left(S(t) + \int_0^{\infty} R_j(t, b)\mathrm{d}b - \frac{\Lambda_h}{\mu_h} \right) = 0.$$

When $I_i(t) = Y_i(t) = 0$, it follows that $\lim_{t\to\infty} V_i(t) = 0$ and $\lim_{t\to\infty} U(t) = U^*$ from the sixth and seventh equations model (2.1). Further, it is clearly that $\lim_{t\to\infty} S(t) = S^*$ from the first equation model (2.1). Then, from the second and fourth equations of model (2.1), we obtain that $\lim_{t\to\infty} E_i(t, a) = 0$ and $\lim_{t\to\infty} R_i(t, b) = 0$. Thus, $\{P_0\}$ is the largest invariant subset of set $\{X \in \mathbb{D} : \mathrm{d}L(t)/\mathrm{d}t = 0\}$. By the LaSalle's invariance principle, P_0 is globally asymptotically stable. The proof is complete. □

Remark 4.4. In the Section 6, by the numerical example, we verify the disease-free equilibrium P_0 is globally asymptotically stable when $\mathcal{R}_0 < 1$. However, our theoretical analysis can only obtain the global stability of P_0 when $\mathcal{R}_0 < \min\{K_1, K_2\}$. This is an open question, and we will continue to work on it in future studies.

Now, we show the local stability of equilibrium P_1 of model (2.1). Let $S(t) = s(t) + S_1^*$, $E_1(t,a) = E_1^*(a) + e_1(t,a)$, $I_1(t) = I_1^* + i_1(t)$, $R_1(t,b) = R_1^*(b) + r_1(t,b)$, $U(t) = U_1^* + u(t)$, $V_1(t) = V_1^* + v(t)$, $I_2(t) = i_2(t)$, $R_2(t,b) = r_2(t,b)$, $Y_i(t) = y_i(t)$ and $V_2(t) = v_2(t)$, $i = 1, 2$, then the linearized system of model (2.1) at equilibrium P_1 is as follows

$$\begin{cases}\dfrac{\mathrm{d}s(t)}{\mathrm{d}t} = -\beta_1 S_1^* v_1(t) - \beta_1 s(t) V_1^* - \beta_2 S_1^* v_2(t) - \mu_h s(t),\\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) e_1(t,a) = -(\mu_h + \varepsilon_1(a)) e_1(t,a),\ e_1(t,0) = \beta_1 S_1^* v_1(t) + \beta_1 s(t) V_1^*,\\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) e_2(t,a) = -(\mu_h + \varepsilon_2(a)) e_2(t,a),\ e_2(t,0) = \beta_2 S_1^* v_2(t),\\ \dfrac{\mathrm{d}i_i(t)}{\mathrm{d}t} = \displaystyle\int_0^\infty \varepsilon_i(a) e_i(t,a)\mathrm{d}a - (\gamma_i + \mu_h) i_i(t),\ i = 1, 2,\\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) r_1(t,b) = -\beta_2 \theta_2(b) v_2(t) R_1^*(b) - \mu_h r_1(t,b),\ r_1(t,0) = \gamma_1 i_1(t),\\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) r_2(t,b) = -\beta_1 \theta_1(b) V_1^* r_2(t,b) - \mu_h r_2(t,b),\ r_2(t,0) = \gamma_2 i_2(t),\\ \dfrac{\mathrm{d}y_1(t)}{\mathrm{d}t} = \beta_1 V_1^* \displaystyle\int_0^\infty \theta_1(b) r_2(t,b)\mathrm{d}b - (\gamma_1 + d_1 + \mu_h) y_1,\\ \dfrac{\mathrm{d}y_2(t)}{\mathrm{d}t} = \beta_2 v_2(t) \displaystyle\int_0^\infty \theta_2(b) R_1^*(b)\mathrm{d}b - (\gamma_2 + d_2 + \mu_h) y_2,\\ \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = -\alpha_1(i_1(t) + y_1(t))U_1^* - \alpha_1 u(t) I_1^* - \alpha_2(i_2(t) + y_2(t))U_1^* - \mu_m u(t),\\ \dfrac{\mathrm{d}v_1(t)}{\mathrm{d}t} = \alpha_1(i_1(t) + y_1(t))U_1^* + \alpha_1 u(t) I_1^* - \mu_m v_1(t),\\ \dfrac{\mathrm{d}v_2(t)}{\mathrm{d}t} = \alpha_2(i_2(t) + y_2(t))U_1^* - \mu_m v_2(t).\end{cases} \tag{4.10}$$

Firstly, we discuss the equations with strain 2 in model (4.10). Let $e_2(t,a) = \tilde{e}_2(a)e^{\lambda t}$, $i_2(t) = \tilde{i}_2 e^{\lambda t}$, $r_2(t,b) = \tilde{r}_2(b)e^{\lambda t}$ and $v_2(t) = \tilde{v}_2 e^{\lambda t}$, where $\tilde{i}_2$, $\tilde{y}_2$ and $\tilde{v}_2$ are positive constants, $\tilde{e}_2(a)$ and $\tilde{r}_2(b)$ are nonnegative functions, then we can get the following eigenvalue problem

$$\begin{cases}\dfrac{\mathrm{d}\tilde{e}_2(a)}{\mathrm{d}a} = -(\lambda + \mu_h + \varepsilon_2(a))\tilde{e}_2(a), \quad \tilde{e}_2(0) = \beta_2 S_1^* \tilde{v}_2,\\ (\lambda + \gamma_2 + \mu_h)\tilde{i}_2 = \displaystyle\int_0^\infty \varepsilon_2(a)\tilde{e}_2(a)\mathrm{d}a,\\ \dfrac{\mathrm{d}\tilde{r}_2(b)}{\mathrm{d}b} = -(\lambda + \beta_1\theta_1(b)V_1^* + \mu_h)\tilde{r}_2(b), \quad \tilde{r}_2(0) = \gamma_2 \tilde{i}_2,\\ (\lambda + \gamma_2 + d_2 + \mu_h)\tilde{y}_2 = \beta_2 \tilde{v}_2 \displaystyle\int_0^\infty \theta_2(b) R_1^*(b)\mathrm{d}b,\\ (\lambda + \mu_m)\tilde{v}_2 = \alpha_2(\tilde{i}_2 + \tilde{y}_2)U_1^*,\end{cases} \tag{4.11}$$

and characteristic equation

$$\begin{aligned}\mathcal{G}_2(\lambda) &= (\lambda + \mu_m)(\lambda + \gamma_2 + d_2 + \mu_h) - \Bigg\{\alpha_2\beta_2 U_1^* \int_0^\infty \theta_2(b) R_1^*(b)\mathrm{d}b\\ &\quad - \frac{\alpha_2\beta_2\mu_m(\gamma_1 + \mu_h)(\lambda + \gamma_2 + d_2 + \mu_h)}{\alpha_1\beta_1 K_1(\lambda + \gamma_2 + \mu_h)} \int_0^\infty \varepsilon_2(a) e^{-\int_0^a (\lambda + \mu_h + \varepsilon_2(s))\mathrm{d}s}\mathrm{d}a\Bigg\}\\ &= \mathcal{G}_3(\lambda) - \mathcal{G}_4(\lambda) = 0.\end{aligned} \tag{4.12}$$

Suppose that

$$\mathcal{R}_2 > \mathcal{R}_1^* = \mathcal{R}_1 \left(1 - \frac{\alpha_2\beta_2 U_1^* \int_0^\infty \theta_2(b) R_1^*(b) \mathrm{d}b}{\mu_m(\gamma_2 + d_2 + \mu_h)}\right),$$

then, $\mathcal{G}_2(0) = \mathcal{R}_1\mu_m(\gamma_2 + d_2 + \mu_h)(\mathcal{R}_1^* - \mathcal{R}_2) < 0$. Furthermore, it is easy to verify that $\mathcal{G}_2(\lambda)$ is increasing with λ, and $\lim_{\lambda\to+\infty} \mathcal{G}_2(\lambda) = +\infty$. Hence, the equation (4.12) at least has a positive real root, which implies that P_1 is unstable.

On the other hand, if $\mathcal{R}_2 < \mathcal{R}_1^*$, then

$$\begin{aligned}
&|\mathcal{G}_3(\lambda)| \geq \mu_m(\gamma_2 + d_2 + \mu_h), \\
&|\mathcal{G}_4(\lambda)| \leq \mathcal{G}_4(\mathrm{Re}\,\lambda) \leq \mathcal{G}_4(0) = \mu_m(\gamma_2 + d_2 + \mu_h)\frac{\mathcal{R}_2}{\mathcal{R}_1} + \alpha_2\beta_2 U_1^* \int_0^\infty \theta_2(b) R_1^*(b) \mathrm{d}b < |\mathcal{G}_3(\lambda)|,
\end{aligned}$$

for the eigenvalue λ with $\mathrm{Re}(\lambda) \geq 0$. This leads to a contradiction. Hence, all eigenvalues of equation (4.12) have negative real parts when $\mathcal{R}_2 < \mathcal{R}_1^*$. In this case, the stability of P_1 depends on the eigenvalues of the following problem,

$$\begin{cases}
\dfrac{\mathrm{d}s(t)}{\mathrm{d}t} = -\beta_1 S_1^* v_1(t) - \beta_1 s(t) V_1^* - \mu_h s(t), \\
\dfrac{\mathrm{d}i_1(t)}{\mathrm{d}t} = \displaystyle\int_0^\infty \varepsilon_1(a) e_1(t,a) \mathrm{d}a - (\gamma_1 + \mu_h) i_1(t), \\
\left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) e_1(t,a) = -(\mu_h + \varepsilon_1(a)) e_1(t,a), \quad e_1(t,0) = \beta_1 S_1^* v_1(t) + \beta_1 s(t) V_1^*, \\
\left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) r_1(t,b) = -\mu_h r_1(t,b), \quad r_1(t,0) = \gamma_1 i_1(t), \\
\dfrac{\mathrm{d}y_1(t)}{\mathrm{d}t} = \beta_1 V_1^* \displaystyle\int_0^\infty \theta_1(b) r_2(t,b) \mathrm{d}b - (\gamma_1 + d_1 + \mu_h) y_1, \\
\dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = -\alpha_1(i_1(t) + y_1(t)) U_1^* - \alpha_1 u(t) I_1^* - \mu_m u(t), \\
\dfrac{\mathrm{d}v_1(t)}{\mathrm{d}t} = \alpha_1(i_1(t) + y_1(t)) U_1^* + \alpha_1 u(t) I_1^* - \mu_m v_1(t).
\end{cases} \tag{4.13}$$

The corresponding characteristic equation of problem (4.13) is as follow

$$\begin{aligned}
&(\lambda + \gamma_1 + \mu_h)(\lambda + \beta_1 V_1^* + \mu_h)(\lambda + \alpha_1 I_1^* + \mu_m) \\
&\qquad = \alpha_1\beta_1 S_1^* U_1^* (\lambda + \mu_h) \int_0^\infty \varepsilon_1(a) e^{-\int_0^a (\lambda + \varepsilon_1(a) + \mu_h)\mathrm{d}s} \mathrm{d}a.
\end{aligned} \tag{4.14}$$

Dividing both sides of (4.14) by $(\lambda + \mu_h)(\lambda + \mu_m)$, we obtain

$$\frac{(\lambda + \gamma_1 + \mu_h)(\lambda + \beta_1 V_1^* + \mu_h)(\lambda + \alpha_1 I_1^* + \mu_m)}{(\lambda + \mu_h)(\lambda + \mu_m)} = \frac{\mu_m(\gamma_1 + \mu_h)}{(\lambda + \mu_m) K_1} \int_0^\infty \varepsilon_1(a) e^{-\int_0^a (\lambda + \varepsilon_1(a) + \mu_h)\mathrm{d}s} \mathrm{d}a,$$

where, we also use the expressions of S_1^* and U_1^*. Denote

$$\begin{aligned}
\mathcal{G}_5(\lambda) &= \frac{(\lambda + \gamma_1 + \mu_h)(\lambda + \beta_1 V_1^* + \mu_h)(\lambda + \alpha_1 I_1^* + \mu_m)}{(\lambda + \mu_h)(\lambda + \mu_m)}, \\
\mathcal{G}_6(\lambda) &= \frac{\mu_m(\gamma_1 + \mu_h)}{(\lambda + \mu_m) K_1} \int_0^\infty \varepsilon_1(a) e^{-\int_0^a (\lambda + \varepsilon_1(a) + \mu_h)\mathrm{d}s} \mathrm{d}a.
\end{aligned}$$

If λ is a root of equation (4.14) with $\mathrm{Re}\,\lambda \geq 0$, then one further have

$$|\mathcal{G}_5(\lambda)| > \gamma_1 + \mu_h, \quad |\mathcal{G}_6(\lambda)| \leq |\mathcal{G}_6(Re\lambda)| \leq \mathcal{G}_6(0) = \gamma_1 + \mu_h < |\mathcal{G}_5(\lambda)|,$$

which leads to a contradiction. Hence, equation (4.14) has no any root with nonnegative real part. This shows that characteristic equation corresponding to model (4.10) has only roots with negative real parts. Consequently, the boundary equilibrium P_1 is locally asymptotically stable if $\mathcal{R}_1 > 1$ and $\mathcal{R}_2 < \mathcal{R}_1^*$. To sum up, the following results are true.

Theorem 4.5. *Assume $\mathcal{R}_1 > 1$, the boundary equilibrium P_1 of model (2.1) is locally asymptotically stable when $\mathcal{R}_2 < \mathcal{R}_1^*$. Moreover, if the inequality is reversed, then P_1 is unstable.*

Remark 4.6. If $\theta_2(b) = 0$ for all $b \geq 0$, then there is perfect cross-immunity and primary infection with strain 1 prevents secondary infection with strain 2. In this case, from Theorem 4.5 we have that the boundary equilibrium P_1 is locally asymptotically stable when $\mathcal{R}_1 > 1$ and $\mathcal{R}_1 > \mathcal{R}_2$, i.e., strain 1 is dominant.

On the boundary equilibrium P_2, we can also obtain similar result as follows.

Theorem 4.7. *Assume $\mathcal{R}_2 > 1$, the boundary equilibrium P_2 of model (2.1) is locally asymptotically stable when*

$$\mathcal{R}_1 < \mathcal{R}_2^* = \mathcal{R}_2 \left(1 - \frac{\alpha_1 \beta_1 U_2^* \int_0^\infty \theta_1(b) R_2^*(b) \mathrm{d}b}{\mu_m(\gamma_1 + d_1 + \mu_h)}\right).$$

Moreover, if $\mathcal{R}_1 \geq \mathcal{R}_2^*$, then P_2 is unstable.

Remark 4.8. If $\theta_1(b) = 0$ for all $b \geq 0$, then there is perfect cross-immunity and primary infection with strain 2 prevents secondary infection with strain 1. In this case, from Theorem 4.7 we have that the boundary equilibrium P_2 is locally asymptotically stable when $\mathcal{R}_2 > 1$ and $\mathcal{R}_2 > \mathcal{R}_1$, i.e., strain 2 is dominant.

Remark 4.9. Based on Remark 4.6 and Remark 4.8, we can conclude the following results. That is, if $\theta_1(b) = \theta_2(b) = 0$ for all $b \geq 0$, then there is no secondary infection in model (2.1) and there is competitive exclusion between strain 1 and strain 2.

5 Uniform persistence

Define $\hat{\mathbb{X}} = L_+^1(0,\infty) \times L_+^1(0,\infty) \times \mathbb{R}_+^6$ and

$$\hat{\mathcal{Y}} = \left\{(E_1(\cdot), E_2(\cdot), I_1, I_2, Y_1, Y_2, V_1, V_2) \in \hat{\mathbb{X}} : \int_0^{\bar{a}_i} E_i(\cdot, a)\mathrm{d}a + I_i(\cdot) + Y_i(\cdot) + V_i(\cdot) > 0, i = 1,2\right\},$$

$$\mathcal{Y} = \mathbb{R}_+ \times \hat{\mathcal{Y}} \times L_+^1(0,\infty) \times L_+^1(0,\infty) \times \mathbb{R}_+.$$

Obviously, $\partial\mathcal{Y} = \mathbb{X} \setminus \mathcal{Y}$ and

$$\begin{aligned}\partial\hat{\mathcal{Y}} = \hat{\mathbb{X}} \setminus \hat{\mathcal{Y}} = \Big\{&(E_1(\cdot), E_2(\cdot), I_1, I_2, Y_1, Y_2, V_1, V_2) \in \hat{\mathbb{X}} : \int_0^{\bar{a}_1} E_1(\cdot, a)\mathrm{d}a + I_1(\cdot) + Y_1(\cdot) \\ &+ V_1(\cdot) = 0 \text{ or } \int_0^{\bar{a}_2} E_2(\cdot, a)\mathrm{d}a + I_2(\cdot) + Y_2(\cdot) + V_2(\cdot) = 0\Big\},\end{aligned}$$

$$\begin{aligned}\partial\hat{\mathcal{Y}}_0 = \Big\{&(E_1(\cdot), E_2(\cdot), I_1, I_2, Y_1, Y_2, V_1, V_2) \in \hat{\mathbb{X}} : \\ &\int_0^{\bar{a}_i} E_i(\cdot, a)\mathrm{d}a + I_i(\cdot) + Y_i(\cdot) + V_i(\cdot) = 0, i = 1,2\Big\},\end{aligned}$$

$$\begin{aligned}\partial\hat{\mathcal{Y}}_i = \Big\{&(E_1(\cdot), E_2(\cdot), I_1, I_2, Y_1, Y_2, V_1, V_2) \in \hat{\mathbb{X}} : \int_0^{\bar{a}_i} E_i(\cdot, a)\mathrm{d}a + I_i(\cdot) + Y_i(\cdot) \\ &+ V_i(\cdot) > 0, \int_0^{\bar{a}_j} E_j(\cdot, a)\mathrm{d}a + I_j(\cdot) + Y_j(\cdot) + V_j(\cdot) = 0\Big\}, \quad i,j = 1,2,\ i \neq j.\end{aligned}$$

It is clear that

$$\partial\mathcal{Y} = \partial\mathcal{Y}_0 \cup \partial\mathcal{Y}_1 \cup \partial\mathcal{Y}_2, \;\; \partial\mathcal{Y}_i = \mathbb{R}_+ \times \partial\hat{\mathcal{Y}}_i \times L^1_+(0,\infty) \times L^1_+(0,\infty) \times \mathbb{R}_+, \; i = 0,\ 1,\ 2.$$

Theorem 5.1. *If $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_2 > \mathcal{R}_1^*$ and $\mathcal{R}_1 > \mathcal{R}_2^*$, then the semi-flow $\{\Phi(t)\}_{t\geq 0}$ is uniformly persistent with respect to the pair $(\mathcal{Y}, \partial\mathcal{Y})$, i.e., the disease of model (2.1) is uniformly persistent.*

Proof. We prove, firstly, the following conclusions:

(i) The disease-free equilibrium P_0 is globally asymptotically stable for semi-flow $\{\Phi(t)\}_{t\geq 0}$ restricted to $\partial\mathcal{Y}_0$.

(ii) The boundary equilibrium P_i is globally asymptotically stable and P_0 is unstable for model (2.1) restricted to $\partial\mathcal{Y}_i$ when $\mathcal{R}_i > 1$, $i = 1,\ 2$.

For conclusion (i). If model (2.1) is restricted to $\partial\mathcal{Y}_0$, then it degenerates into

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda_h - \mu_h S(t), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) R_i(t,b) = -\mu_h R_i(t,b),\; R_i(t,0) = 0,\; i = 1,2, \\ \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \Lambda_m(t) - \mu_m U(t). \end{cases} \tag{5.1}$$

We can obtain that $\lim_{t\to+\infty} S(t) = \Lambda_h/\mu_h$, $\lim_{t\to+\infty} R_i(t,b) = 0$ and $\lim_{t\to+\infty} U(t) = \Lambda_m/\mu_m$. Therefore, P_0 is globally asymptotically stable for model (2.1) restricted to $\partial\mathcal{Y}_0$. Then the conclusion (i) is true.

For conclusion (ii). If model (2.1) restricted in $\partial\mathcal{Y}_1$, then it degenerates into

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda_h - \beta_1 S(t)V_1(t) - \mu_h S(t), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) E_1(t,a) = -(\mu_h + \varepsilon_1(a))E_1(t,a),\; E_1(t,0) = \beta_1 S(t)V_1(t), \\ \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = \displaystyle\int_0^\infty \varepsilon_1(a)E_1(t,a)\mathrm{d}a - (\gamma_1 + \mu_h)I_1(t), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) R_1(t,b) = -\mu_h R_1(t,b),\; R_1(t,0) = \gamma_1 I_1(t), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) R_2(t,b) = -(\beta_1\theta_1(b)V_1(t) + \mu_h)R_2(t,b),\; R_2(t,0) = 0, \\ \dfrac{\mathrm{d}Y_1(t)}{\mathrm{d}t} = \beta_1 V_1(t)\displaystyle\int_0^\infty \theta_1(b)R_2(t,b)\mathrm{d}b - (\gamma_1 + d_1 + \mu_h)Y_1(t), \\ \dfrac{\mathrm{d}Y_2(t)}{\mathrm{d}t} = -(\gamma_2 + d_2 + \mu_h)Y_2(t), \\ \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \Lambda_m(t) - \alpha_1(I_1(t) + Y_1(t))U(t) - \mu_m U(t), \\ \dfrac{\mathrm{d}V_1(t)}{\mathrm{d}t} = \alpha_1(I_1(t) + Y_1(t))U(t) - \mu_m V_1(t). \end{cases} \tag{5.2}$$

Obviously, $\lim_{t\to+\infty} R_2(t,b) = \lim_{t\to+\infty} Y_1(t) = \lim_{t\to+\infty} Y_2(t) = 0$. Since the equation of $R_1(t,b)$ is decoupled from the other equations in model (5.2), we can consider the following system

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda_h - \beta_1 S(t)V_1(t) - \mu_h S(t), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) E_1(t,a) = -(\mu_h + \varepsilon_1(a))E_1(t,a),\ E_1(t,0) = \beta_1 S(t)V_1(t), \\ \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = \displaystyle\int_0^\infty \varepsilon_1(a)E_1(t,a)\mathrm{d}a - (\gamma_1 + \mu_h)I_1(t), \\ \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \Lambda_m(t) - \alpha_1 I_1(t)U(t) - \mu_m U(t), \\ \dfrac{\mathrm{d}V_1(t)}{\mathrm{d}t} = \alpha_1 I_1(t)U(t) - \mu_m V_1(t). \end{cases} \tag{5.3}$$

Model (5.3) has the equilibrium $(S_1^*, E_1^*(a), I_1^*, U_1^*, V_1^*)$. Define Lyapunov functional

$$\mathcal{W}(t) = \mathcal{W}_1(t) + \mathcal{W}_2(t) + \mathcal{W}_3(t) + \mathcal{W}_4(t) + \mathcal{W}_5(t),$$

where

$$\mathcal{W}_1(t) = K_1 S_1^* \phi\left(\frac{S(t)}{S_1^*}\right), \quad \mathcal{W}_2(t) = \int_0^\infty q_1(a)E_1^*(a)\phi\left(\frac{E_1(t,a)}{E_1^*(a)}\right)\mathrm{d}a,$$
$$\mathcal{W}_3(t) = I_1^*\phi\left(\frac{I_1(t)}{I_1^*}\right), \quad \mathcal{W}_4(t) = \frac{K_1 E_1^*(0)}{\alpha_1 I_1^*}\phi\left(\frac{U(t)}{U_1^*}\right), \quad \mathcal{W}_5(t) = \frac{K_1 E_1^*(0)V_1^*}{\alpha_1 I_1^* U_1^*}\phi\left(\frac{V_1(t)}{V_1^*}\right),$$

with $\phi(x) = x - 1 - \ln x$. Then, it yields that

$$\begin{aligned} \frac{\mathrm{d}\mathcal{W}_1(t)}{\mathrm{d}t} &= K_1 S_1^* \left(\frac{1}{S_1^*} - \frac{1}{S(t)}\right)\left[\Lambda_h - \frac{\Lambda_h S(t)}{S_1^*} + \beta_1 S(t)V_1^* - \beta_1 S(t)V_1(t)\right] \\ &= -\frac{\Lambda_h (S(t) - S_1^*)^2 K_1}{S(t)S_1^*} + K_1\beta_1 S_1^* V_1^*\left(\frac{1}{S_1^*} - \frac{1}{S(t)}\right)\left(S(t) - \frac{S(t)V_1(t)}{V_1^*}\right) \\ &= -K_1 S_1^*(\beta_1 V_1^* + \mu_h)\left(\phi\left(\frac{S(t)}{S_1^*}\right) + \phi\left(\frac{S_1^*}{S(t)}\right)\right) \\ &\quad + K_1\beta_1 S_1^* V_1^*\left(\frac{S(t)}{S_1^*} - 1 - S(t)V_1(t)S_1^*V_1^* + V_1(t)V_1^*\right), \\ \frac{\mathrm{d}\mathcal{W}_2(t)}{\mathrm{d}t} &= \int_0^\infty q_1(a)E_1^*(a)\frac{\partial}{\partial t}\phi\left(\frac{E_1(t,a)}{E_1^*(a)}\right)\mathrm{d}a \\ &= -\int_0^\infty q_1(a)E_1^*(a)\left(\frac{E_1(t,a)}{E_1^*(a)} - 1\right)\left(\frac{E_{1a}(t,a)}{E_1(t,a)} + \mu_h + \varepsilon_1(a)\right)\mathrm{d}a, \end{aligned}$$

where $E_{1a}(t,a) = \mathrm{d}E_1(t,a)/\mathrm{d}a$. Since

$$\frac{\partial}{\partial a}\phi\left(\frac{E_1(t,a)}{E_1^*(a)}\right) = \left(\frac{E_1(t,a)}{E_1^*(a)} - 1\right)\left(\frac{E_{1a}(t,a)}{E_1(t,a)} + \mu_h + \varepsilon_1(a)\right),$$

then

$$\begin{aligned} \frac{\mathrm{d}\mathcal{W}_2(t)}{\mathrm{d}t} &= -\int_0^\infty q_1(a)E_1^*(a)\frac{\partial}{\partial a}\phi\left(\frac{E_1(t,a)}{E_1^*(a)}\right)\mathrm{d}a \\ &= -q_1(a)E_1^*(a)\phi\left(\frac{E_1(t,a)}{E_1^*(a)}\right)\Big|_{a=\infty} + q_1(0)E_1^*(0)\phi\left(\frac{E_1(t,0)}{E_1^*(0)}\right) \\ &\quad - \int_0^\infty \varepsilon_1(a)E_1^*(a)\phi\left(\frac{E_1(t,a)}{E_1^*(a)}\right)\mathrm{d}a \\ &= K_1 E_1(t,0) - K_1 E_1^*(0)\left(1 - \ln\left(\frac{E_1(t,0)}{E_1^*(0)}\right)\right) - \int_0^\infty \varepsilon_1(a)E_1^*(a)\phi\left(\frac{E_1(t,a)}{E_1^*(a)}\right)\mathrm{d}a. \end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{W}_3(t)}{\mathrm{d}t} &= \int_0^\infty \varepsilon_1(a)\left(1-\frac{I_1^*}{I_1(t)}\right)\left(E_1(t,a)-E_1^*(a)\frac{I_1(t)}{I_1^*}\right)\mathrm{d}a\\
&= \int_0^\infty \varepsilon_1(a)E_1^*(a)\left(\frac{E_1(t,a)}{E_1^*(a)}-\frac{I_1^*}{I_1(t)}-\frac{I_1^*E_1(t,a)}{I_1(t)E_1^*(a)}+1\right)\mathrm{d}a,\\
\frac{\mathrm{d}\mathcal{W}_4(t)}{\mathrm{d}t} &= \frac{K_1E_1^*(0)}{\alpha_1 I_1^*}\left(\frac{1}{U_1^*}-\frac{1}{U(t)}\right)(\mu_m(U_1^*-U(t))+\alpha_1U(t)I_1^*-\alpha_1U(t)I_1(t))\\
&= -\frac{K_1E_1^*(0)\mu_m(U(t)-U_1^*)^2}{U(t)I_1^*U_1^*}+K_1E_1^*(0)\left(1-\frac{U(t)}{U_1^*}+\frac{I_1(t)}{I_1^*}-\frac{U(t)I_1(t)}{U_1^*I_1^*}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{W}_5(t)}{\mathrm{d}t} &= \frac{K_1E_1^*(0)V_1^*}{\alpha_1I_1^*U_1^*}\left(\frac{1}{V_1^*}-\frac{1}{V_1(t)}\right)\left(\alpha_1U(t)I_1(t)-\alpha_1U_1^*I_1^*\frac{V_1(t)}{V_1^*}\right)\\
&= K_1E_1^*(0)\left(\frac{I_1(t)U(t)}{I_1^*U_1^*}-\frac{V_1(t)}{V_1^*}-\frac{U(t)I_1(t)V_1^*}{U_1^*I_1^*V_1(t)}+1\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{W}(t)}{\mathrm{d}t} =& -K_1S_1^*(\beta_1V_1^*+\mu_h)\left[\phi\left(\frac{S(t)}{S_1^*}\right)+\phi\left(\frac{S_1^*}{S(t)}\right)\right]+K_1\beta_1S_1^*V_1^*\left(\frac{S(t)}{S_1^*}+\frac{V_1(t)}{V_1^*}\right.\\
&\left.-\ln\frac{S(t)V_1(t)}{S_1^*V_1^*}-2\right)+\int_0^\infty\varepsilon_1(a)E_1^*(a)\left(2-\frac{I_1(t)}{I_1^*}-\frac{I_1^*E_1(t,a)}{I_1(t)E_1^*(a)}+\ln\frac{E_1(t,a)}{E_1^*(a)}\right)\mathrm{d}a\\
&-\frac{K_1E_1^*(0)\mu_m(U(t)-U_1^*)^2}{U(t)I_1^*U_1^*}+K_1E_1^*(0)\left(2-\frac{U(t)}{U_1^*}+\frac{I_1(t)}{I_1^*}-\frac{V_1(t)}{V_1^*}-\frac{U(t)I_1(t)V_1^*}{U_1^*I_1^*V_1(t)}\right)\\
=& -K_1S_1^*(\beta_1V_1^*+\mu_h)\left[\phi\left(\frac{S(t)}{S_1^*}\right)+\phi\left(\frac{S_1^*}{S(t)}\right)\right]+K_1S_1^*\beta_1V_1^*\left[\phi\left(\frac{S(t)}{S_1^*}\right)+\phi\left(\frac{V_1^*}{V_1(t)}\right)\right]\\
&-\int_0^\infty\varepsilon_1(a)E_1^*(a)\left[\phi\left(\frac{E_1(t,a)I_1^*}{E_1^*(a)I_1(t)}\right)+\phi\left(\frac{I_1(t)}{I_1^*}\right)\right]\mathrm{d}a-\frac{K_1E_1^*(0)\mu_m(U(t)-U_1^*)^2}{U(t)I_1^*U_1^*}\\
&-K_1E_1^*(0)\left[\phi\left(\frac{U_1^*}{U(t)}\right)+\phi\left(\frac{V_1(t)}{V_1^*}\right)+\phi\left(\frac{U(t)I_1(t)V_1^*}{U_1^*I_1^*V_1(t)}\right)\right]+K_1E_1^*(0)\phi\left(\frac{I_1(t)}{I_1^*}\right)\\
=& -K_1S_1^*\beta_1V_1^*\phi\left(\frac{S_1^*}{S(t)}\right)-K_1S_1^*\mu_h\left[\phi\left(\frac{S(t)}{S_1^*}\right)+\phi\left(\frac{S_1^*}{S(t)}\right)\right]\\
&-\int_0^\infty\varepsilon_1(a)E_1^*(a)\phi\left(\frac{E_1(t,a)I_1^*}{E_1^*(a)I_1(t)}\right)\mathrm{d}a-\frac{K_1E_1^*(0)\mu_m(U(t)-U_1^*)^2}{U(t)I_1^*U_1^*}\\
&-K_1E_1^*(0)\left[\phi\left(\frac{U_1^*}{U(t)}\right)+\phi\left(\frac{V_1(t)}{V_1^*}\right)+\phi\left(\frac{U(t)I_1(t)V_1^*}{U_1^*I_1^*V_1(t)}\right)\right]\le 0.
\end{aligned}$$

It is clear that $\mathrm{d}\mathcal{W}(t)/\mathrm{d}t \le 0$ for any $S(t)>0$, $E_1(t,a)>0$, $I_1(t)>0$, $U(t)>0$ and $V_1(t)>0$, and $\mathrm{d}\mathcal{W}(t)/\mathrm{d}t=0$ implies that $(S(t),E_1(t,a),I_1(t),U(t),V_1(t))\equiv(S_1^*,E_1^*(a),I_1^*,U_1^*,V_1^*)$ for all $t>0$. Thus, by LaSalle's invariance principle, equilibrium $(S_1^*,E_1^*(a),I_1^*,U_1^*,V_1^*)$ is globally asymptotically stable for model (5.3) when $\mathcal{R}_1>1$. From model (5.2), we easily obtain that $\lim_{t\to\infty}R_1(t,b)=R_1^*(b)$. This shows that equilibrium P_1 is globally asymptotically stable for model (2.1) restricted to $\partial\mathcal{Y}_1$ when $\mathcal{R}_1>1$. Moreover, from Theorem 4.2, we can obtain that P_0 is unstable for model (2.1) restricted to $\partial\mathcal{Y}_1$ when $\mathcal{R}_1>1$. That is, conclusion (ii) is hold.

Similarly, we can show that P_2 is globally asymptotically stable and P_0 is unstable for model (2.1) restricted to $\partial\mathcal{Y}_2$ when $\mathcal{R}_2>1$.

Next, we claim that $W^s(P_0)\cap\mathcal{Y}=\emptyset$, $W^s(P_i)\cap\mathcal{Y}=\emptyset$, $i=1,2$, where $W^s(P_0)=\{X_0\in\mathcal{Y}:\lim_{t\to\infty}X(t,X_0)=P_0\}$ and $W^s(P_i)=\{X_0\in\mathcal{Y}:\lim_{t\to\infty}X(t,X_0)=P_i\}$, $i=1,2$.

For $W^s(P_0) \cap \mathcal{Y} = \emptyset$. Suppose that there exists a $X_0 \in \mathcal{Y}$ such that $\lim_{t\to\infty} X(t, X_0) = P_0$. Then, for any constant $\epsilon > 0$, there exists a $T_0 > 0$ such that

$$\begin{aligned}
&S^* - \epsilon < S(t) < S^* + \epsilon,\ 0 < E_i(t,a) < \epsilon,\ 0 < I_i(t) < \epsilon,\ 0 < R_i(t,b) < \epsilon,\\
&0 < Y_i(t) < \epsilon,\ U^* - \epsilon < U(t) < U^* + \epsilon,\ 0 < V_i(t) < \epsilon,\ i = 1,\ 2,
\end{aligned}$$

for all $t > T_0$. From the third and eighth equations of model (2.1), it follows that

$$\begin{aligned}
\frac{\mathrm{d}I_i(t)}{\mathrm{d}t} &= \int_0^\infty \varepsilon_i(a)E_i(t,a)\mathrm{d}a - (\gamma_i + \mu_h)I_i(t)\\
&\geq \beta_i(S^* - \epsilon_1)\int_0^t \varepsilon_i(a)V_i(t-a)\eta_i(a)\mathrm{d}a - (\gamma_i + \mu_h)I_i(t),\\
\frac{\mathrm{d}V_i(t)}{\mathrm{d}t} &\geq \alpha_i I_i(t)U(t) - \mu_m V_i(t) \geq \alpha_i(U^* - \epsilon)I_i(t) - \mu_m V_i(t).
\end{aligned}$$

Let us take the Laplace transform of both sides of above inequalities. Since all functions above are bounded, the Laplace transform of the functions exist for $\lambda > 0$. Denote the Laplace transform of the function $f(t)$ by $\mathcal{L}[f(t)]$. Using the convolution property of the Laplace transform, we obtain the following inequalities for $\mathcal{L}[I_i(t)]$ and $\mathcal{L}[V_i(t)]$,

$$\begin{cases}
\lambda\mathcal{L}[I_i(t)] - I_i(0) \geq \beta_i(S^* - \epsilon)\displaystyle\int_0^\infty \varepsilon_i(a)\eta_i(a)e^{-\lambda a}\mathrm{d}a\mathcal{L}[V_i(t)] - (\gamma_i + \mu_h)\mathcal{L}[I_i(t)],\\
\lambda\mathcal{L}[V_i(t)] - V_i(0) \geq \alpha_i(U^* - \epsilon)\mathcal{L}[I_i(t)] - \mu_m\mathcal{L}[V_i(t)].
\end{cases}$$

Eliminating $\mathcal{L}[V_i(t)]$ yields

$$\mathcal{L}[I_i(t)] \geq \frac{\alpha_i\beta_i(S^* - \epsilon)(U^* - \epsilon)\int_0^\infty \varepsilon_i(a)\eta_i(a)e^{-\lambda a}\mathrm{d}a}{(\lambda + \mu_m)(\lambda + \gamma_i + \mu_h)}\mathcal{L}[I_i(t)] + \frac{I_i(0)}{\lambda + \gamma_i + \mu_h}. \tag{5.4}$$

Since $\mathcal{R}_i > 1,\ i = 1, 2$, we can choose λ and ϵ small enough such that

$$\frac{\alpha_i\beta_i(S^* - \epsilon)(U^* - \epsilon)\int_0^\infty \varepsilon_i(a)\eta_i(a)e^{-\lambda a}\mathrm{d}a}{(\lambda + \mu_m)(\lambda + \gamma_i + \mu_h)} > 1, \quad i = 1,\ 2.$$

Therefore, inequality (5.4) does not hold. This implies that $W^s(P_0) \cap \mathcal{Y} = \emptyset$.

For $W^s(P_1) \cap \mathcal{Y} = \emptyset$. Suppose that there exists a $X_1 \in \mathcal{Y}$ such that $\lim_{t\to\infty} X(t, X_1) = P_1$. Then, for any constant $\epsilon > 0$ there exists a $T_1 > 0$ such that for all $t > T_1$ one have

$$\begin{aligned}
&S_1^* - \epsilon_1 < S(t) < S_1^* + \epsilon_1, \quad E_1^* - \epsilon_1 < E_1(t,a) < E_1^* + \epsilon_1, \quad 0 < E_2(t,a) < \epsilon_1,\\
&0 < I_2(t) < \epsilon_1, \qquad I_1^* - \epsilon_1 < I_1(t) < I_1^* + \epsilon_1, \quad R_1^*(b) - \epsilon_1 < R_1(t,b) < R_1^*(b) + \epsilon_1,\\
&0 < R_2(t,b) < \epsilon_1, \quad 0 < Y_i(t) < \epsilon_1, \quad U_1^* - \epsilon_1 < U(t) < U_1^* + \epsilon_1,\\
&V_1^* - \epsilon_1 < V_1(t) < V_1^* + \epsilon_1, \quad 0 < V_2(t) < \epsilon_1, \quad i = 1,\ 2.
\end{aligned}$$

From model (2.1), we can obtain

$$\begin{cases}
\dfrac{\mathrm{d}I_2(t)}{\mathrm{d}t} = \displaystyle\int_0^\infty \varepsilon_2(a)E_2(t,a)\mathrm{d}a - (\gamma_2 + \mu_h)I_2(t)\\
\qquad\quad \geq \beta_2(S_1^* - \epsilon_1)\displaystyle\int_0^t \varepsilon_2(a)V_2(t-a)\eta_2(a)\mathrm{d}a - (\gamma_2 + \mu_h)I_2(t),\\
\dfrac{\mathrm{d}Y_2(t)}{\mathrm{d}t} = \beta_2 V_2(t)\displaystyle\int_0^\infty \theta_2(b)R_1(t,b)\mathrm{d}b - (\gamma_2 + d_2 + \mu_h)Y_2(t)\\
\qquad\quad \geq \beta_2 V_2(t)\displaystyle\int_0^\infty \theta_2(b)(R_1^*(b) - \epsilon_1)\mathrm{d}b - (\gamma_2 + d_2 + \mu_h)Y_2(t),\\
\dfrac{\mathrm{d}V_2(t)}{\mathrm{d}t} \geq \alpha_2(U_1^* - \epsilon_1)(I_2(t) + Y_2(t)) - \mu_m V_2(t).
\end{cases} \tag{5.5}$$

Take the Laplace transform of both sides of inequalities (5.5). Since all functions above are bounded, the Laplace transform of the functions exist for $\lambda > 0$. Then, we can get the following inequalities for $\mathcal{L}[I_2(t)]$, $\mathcal{L}[Y_2(t)]$ and $\mathcal{L}[V_2(t)]$,

$$\begin{cases} \lambda\mathcal{L}[I_2(t)] - I_2(0) \geq \beta_2(S_1^* - \epsilon_1)\displaystyle\int_0^\infty \varepsilon_2(a)\eta_2(a)e^{-\lambda a}\mathrm{d}a\mathcal{L}[V_2(t)] - (\gamma_2+\mu_h)\mathcal{L}[I_2(t)], \\ \lambda\mathcal{L}[Y_2(t)] - Y_2(0) \geq \beta_2\displaystyle\int_0^\infty \theta_2(b)(R_1^*(b) - \epsilon_1)\mathrm{d}b\mathcal{L}[V_2(t)] - (\gamma_2+d_2+\mu_h)\mathcal{L}[Y_2(t)], \\ \lambda\mathcal{L}[V_2(t)] - V_2(0) \geq \alpha_2(U_1^* - \epsilon_1)(\mathcal{L}[I_2(t)] + \mathcal{L}[Y_2(t)]) - \mu_m\mathcal{L}[V_2(t)]. \end{cases}$$

Eliminating $\mathcal{L}[I_2(t)]$ and $\mathcal{L}[Y_2(t)]$ yields

$$\begin{aligned} \mathcal{L}[V_2(t)] \geq \alpha_2\beta_2(U_1^* - \epsilon_1)\mathcal{L}[V_2(t)]\bigg\{ & \frac{(S_1^* - \epsilon_1)\int_0^\infty \varepsilon_2(a)\eta_2(a)\eta_2(a)e^{-\lambda a}\mathrm{d}a}{(\lambda+\mu_m)(\lambda+\gamma_2+\mu_h)} \\ & + \frac{\int_0^\infty \theta_2(b)(R_1^*(b) - \epsilon_1)\mathrm{d}b}{(\lambda+\mu_m)(\lambda+\gamma_2+d_2+\mu_h)}\bigg\} + \frac{V_2(0)}{\lambda+\mu_m} \\ & + \frac{\alpha_2(U_1^* - \epsilon_1)}{\lambda+\mu_m}\bigg\{\frac{I_2(0)}{\lambda+\gamma_2+\mu_h} + \frac{Y_2(0)}{\lambda+\gamma_2+d_2+\mu_h}\bigg\}. \end{aligned}$$

This is impossible when $\mathcal{R}_2 > \mathcal{R}_1^*$. By calculation, we have $S_1^*U_1^* = \mu_m(\gamma_1+\mu_h)/\alpha_1\beta_1K_1$, and

$$\frac{\alpha_2\beta_2S_1^*U_1^*\int_0^\infty \varepsilon_2\eta_2(a)\mathrm{d}a}{\mu_m(\gamma_2+\mu_h)} + \frac{\alpha_2\beta_2U_1^*\int_0^\infty \theta_2(b)R_1^*(b)\mathrm{d}b}{\mu_m(\gamma_2+d_2+\mu_h)} = \frac{\mathcal{R}_2}{\mathcal{R}_1} + \frac{\alpha_2\beta_2U_1^*\int_0^\infty \theta_2(b)R_1^*(b)\mathrm{d}b}{\mu_m(\gamma_2+d_2+\mu_h)} > 1.$$

Therefore, we can choose λ and ϵ_1 small enough such that

$$\alpha_2\beta_2(U_1^* - \epsilon_1)\bigg\{\frac{(S_1^* - \epsilon_1)\int_0^\infty \varepsilon_2(a)\eta_2(a)\eta_2(a)e^{-\lambda a}\mathrm{d}a}{(\lambda+\mu_m)(\lambda+\gamma_2+\mu_h)} + \frac{\int_0^\infty \theta_2(b)(R_1^*(b) - \epsilon_1)\mathrm{d}b}{(\lambda+\mu_m)(\lambda+\gamma_2+d_2+\mu_h)}\bigg\} > 1.$$

This contradiction implies that $W^s(P_1) \cap \mathcal{Y} = \emptyset$.

Similarly, we can verify $W^s(P_2) \cap \mathcal{Y} = \emptyset$, when $\mathcal{R}_1 > \mathcal{R}_2^*$. Thus, Theorem 4.2 in Hale and Waltman [18] implies the semi-flow $\{\Phi(t)\}_{t\geq 0}$ is uniformly persistent with respect to the pair $(\mathcal{Y}, \partial\mathcal{Y})$ if $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_1 > \mathcal{R}_2^*$ and $\mathcal{R}_2 > \mathcal{R}_1^*$. This completes the proof. □

As a consequence of Theorem 5.1, we have the following Corollary 5.2.

Corollary 5.2. *If $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_1 > \mathcal{R}_2^*$ and $\mathcal{R}_2 > \mathcal{R}_1^*$, then model (2.1) has at least a coexistence equilibrium denoted by $P_3 = (\widetilde{S}^*, \widetilde{E}_1^*(a), \widetilde{E}_2^*(a), \widetilde{I}_1^*, \widetilde{I}_2^*, \widetilde{R}_1^*(b), \widetilde{R}_2^*(b), \widetilde{Y}_1^*, \widetilde{Y}_2^*, \widetilde{U}^*, \widetilde{V}_1^*, \widetilde{V}_2^*)$.*

Based on the discussion in Section 4 and Section 5, we can conclude the existence and stability of the equilibria of model (2.1), as shown in Table 5.1. Here, LAS and GAS denote locally asymptotically stable and globally asymptotically stable, respectively.

Remark 5.3. It should be pointed out that the numerical simulations show that if the coexistence equilibrium of model (2.1) is existence, then it is stable. In fact, we have also obtained the sufficient conditions for the stability of the coexistence equilibrium by constructing the Lyapunov function. Due to additional technical conditions, we put this result in the appendix.

Case	Existence or stability	Case	Existence or stability
$\mathcal{R}_0 < 1$	P_0 is LAS	$\mathcal{R}_0 < \min\{K_1, K_2\}$	P_0 is GAS
$\mathcal{R}_1 > 1$	P_1 exists	$\mathcal{R}_2 > 1$	P_2 exists
$\mathcal{R}_1 > 1$, $\mathcal{R}_2 < \mathcal{R}_1^*$	P_1 is LAS	$\mathcal{R}_2 > 1$, $\mathcal{R}_1 < \mathcal{R}_2^*$	P_2 is LAS
$\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_2 > \mathcal{R}_1^*$, $\mathcal{R}_1 > \mathcal{R}_2^*$	coexistence equilibrium exists		

Table 5.1: Summarizing the different scenarios depending on the threshold parameters.

6 Numerical simulation and discussions

In this section, some numerical simulations are conducted to illustrate our theoretical analysis results. Since the longer one stay in the latency stage, the more one is likely to exposed to the disease, and the risk of infection will increase, we assume that the age-dependent removal rate $\varepsilon_i(a)$ in model (2.1) takes the form $\varepsilon_i(a) = x_i a^2 \exp(-y_i a)$, where $x_i,\ y_i > 0,\ i = 1, 2$, see [21]. Similarly, in order to describe the primary recovery period and the level about losing cross vaccine protection, we choose cross immunity waning rate function as $\theta_i(b) = u_i(1 + 5\exp(-v_i b))^{-1}$, where $u_i,\ v_i > 0,\ i = 1, 2$. Furthermore, the values of other parameters of the model (2.1) are based on Refs. [6,36,42] and the references cited therein.

Example 6.1. The global asymptotic stability of the disease-free equilibrium of model (2.1).

We choose model parameters as follows: $\Lambda_h = 25$, $\beta_1 = 2.38 \times 10^{-6}$, $\beta_2 = 2.25 \times 10^{-6}$, $\mu_h = 0.004$, $\gamma_1 = \gamma_2 = 0.14$, $d_1 = d_2 = 0.0001$, $\Lambda_m = 21000$, $\alpha_1 = 3.75 \times 10^{-6}$, $\alpha_2 = 3.95 \times 10^{-6}$, $\mu_m = 0.09$, $\varepsilon_1(a) = 0.01a^2\exp(-0.2a)$, $\varepsilon_2(a) = 0.01\exp(-0.18a)a^2$, $\theta_1(b) = 0.45(1+5\exp(-0.026b))^{-1}$ and $\theta_2(b) = 0.48(1 + 5\exp(-0.026b))^{-1}$ in model (2.1). By numerical calculations, we obtain $K_1 \approx 0.882$, $K_2 \approx 0.931$, and basic reproduction number $\mathcal{R}_1 \approx 0.8687 < K_1$ and $\mathcal{R}_2 \approx 0.913 < K_2$. Then, by Theorem 4.3, the disease-free equilibrium P_0 of model (2.1) is globally asymptotically stable. The plots in Figures 6.1(a)-(c) show this theoretical result.

Further, we only adjust the values of transmission rates β_1 and β_2 and let $\beta_1 = 2.48 \times 10^{-6}$ and $\beta_2 = 2.32 \times 10^{-6}$ in model (2.1), then by numerical calculations it is obtained that the basic reproduction numbers $\mathcal{R}_1 \approx 0.9052$ and $\mathcal{R}_2 \approx 0.9414$. The values of K_1 and K_2 remain the same as above, then $\mathcal{R}_1 > K_1$ and $\mathcal{R}_2 > K_2$. In this case, numerical simulations show that the disease-free equilibrium P_0 is globally asymptotically stable, as shown in Figure 6.1(d). However, numerical simulations show the disease-free equilibrium is globally asymptotically stable if $\mathcal{R}_0 < 1$ without additional conditions. Therefore, we put forward an interesting open question: If $\mathcal{R}_0 < 1$, then the disease-free equilibrium is globally asymptotically stable.

Example 6.2. The existence and stability of strain i $(i = 1,\ 2)$ dominant equilibrium of (2.1).

Let $\Lambda_h = 25$, $\beta_1 = 9.85 \times 10^{-6}$, $\beta_2 = 6.85 \times 10^{-6}$, $\mu_h = 0.004$, $\gamma_1 = 0.07$, $\gamma_2 = 0.14$, $d_1 = d_2 = 0.0001$, $\Lambda_m = 21000$, $\alpha_1 = 1.75 \times 10^{-6}$, $\alpha_2 = 3.75 \times 10^{-6}$, $\mu_m = 0.07$, $\varepsilon_1(a) = \varepsilon_2(a) = 0.01a^2\exp(-0.28a)$, $\theta_1(b) = 0.45(1+5\exp(-0.026b))^{-1}$ and $\theta_2(b) = 0.48(1+5\exp(-0.028b))^{-1}$ in model (2.1). It is easy to calculate that parameter values satisfy all conditions of Theorem 4.5, that is, $\mathcal{R}_1 \approx 3.528 > 1$ and $\mathcal{R}_2 \approx 2.7 < \mathcal{R}_1^* \approx 3.5055$. By Theorem 4.5, the strain 1 dominant equilibrium P_1 is locally asymptotically stable which is consistent with the simulation results as shown in Figures 6.2(a)-(d). As we can see, in Figure 6.2(e), solution curves of $I_1(t)$, $Y_1(t)$ and $S(t)$ from different initial values all tend to a point in the first quadrant various, and the number of $I_2(t)$, $Y_1(t) + Y_2(t)$ and $V_2(t)$ all tend to zero, which is shown Figure 6.2(f). Therefore,

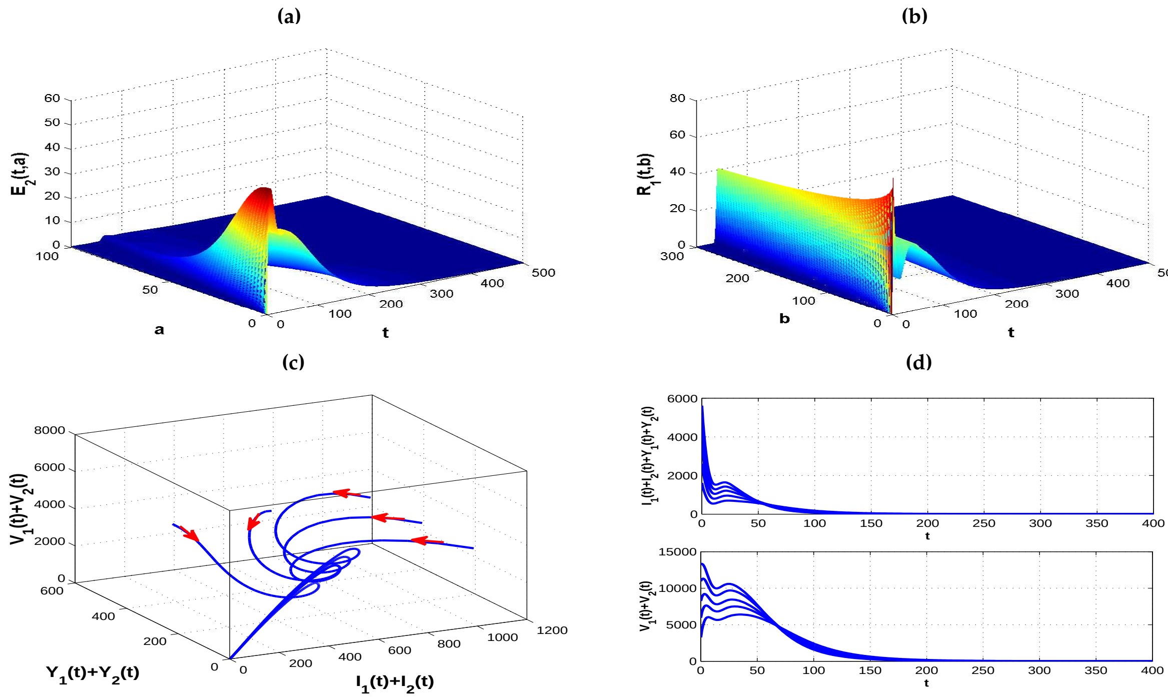

Figure 6.1: The global asymptotical stability of disease-free equilibrium of model (2.1) with the basic reproduction number $\mathcal{R}_0 < 1$, which implies that the disease dies out.

numerical simulations imply that the strain 1 dominant equilibrium P_1 of model (2.1) is globally asymptotically stable. In addition, the numerical simulation for the existence and stability of P_2 is similar to that of P_1, hence we omit it here.

Example 6.3. The persistence of disease, the existence and stability of coexistence equilibrium for model (2.1).

We choose $\Lambda_h = 100$, $\beta_1 = 2.85 \times 10^{-5}$, $\beta_2 = 4.25 \times 10^{-5}$, $\mu_h = 0.004$, $\gamma_1 = \gamma_2 = 0.07$, $d_1 = d_2 = 0.0001$, $\Lambda_m = 5500$, $\alpha_1 = 8.75 \times 10^{-6}$, $\alpha_2 = 8.45 \times 10^{-6}$, $\mu_m = 0.05$, $\varepsilon_1(a) = 0.01\exp(-0.25a)a^2$, $\varepsilon_2(a) = 0.01\exp(-0.31a)a^2$ and $\theta_1(b) = \theta_2(b) = 0.4(1+50\exp(-0.05b))^{-1}$ in model (2.1). Numerical calculation follows that $\mathcal{R}_1 \approx 48.22$, $\mathcal{R}_2 \approx 47.23$, $\mathcal{R}_1^* \approx 41.3200$ and $\mathcal{R}_2^* \approx 40.8999$, which satisfy the conditions of Theorem 5.1 and Corollary 5.2. Therefore, the disease is uniformly persistent and model (2.1) exists coexistence equilibrium which is consistent with the simulation results as shown in Figures 6.3(a)–(d). Particularly, as we can see, in Figures 6.3(c) and (d), solution curves of $I_1(t)$ and $V_2(t)$ from different initial values all tend to a positive constants rather than zero. This implies that model (2.1) exists a globally asymptotically stable coexistence equilibrium. Of course, we also verify the globally asymptotically stability of coexistence equilibrium by constructing a Lyapunov functional in the Appendix with some strong constraint conditions, but these conditions are difficult to verify. This may be related to our research methods and the selection of Lyapunov functional. This encourages us to propose new research methods or construct more suitable Lyapunov functional to solve this problem in the future research.

In additional, we fixed parameter values of model (2.1) as above, and only adjust the value of cross immunity wane rate $\theta_2(b)$ to be $0.2(1+50\exp(-0.05b))^{-1}$, $0.3(1+50\exp(-0.05b))^{-1}$, $0.4(1+50\exp(-0.05b))^{-1}$ and $0.6(1+50\exp(-0.05b))^{-1}$, respectively, we obtain the Fig.6.3(e).

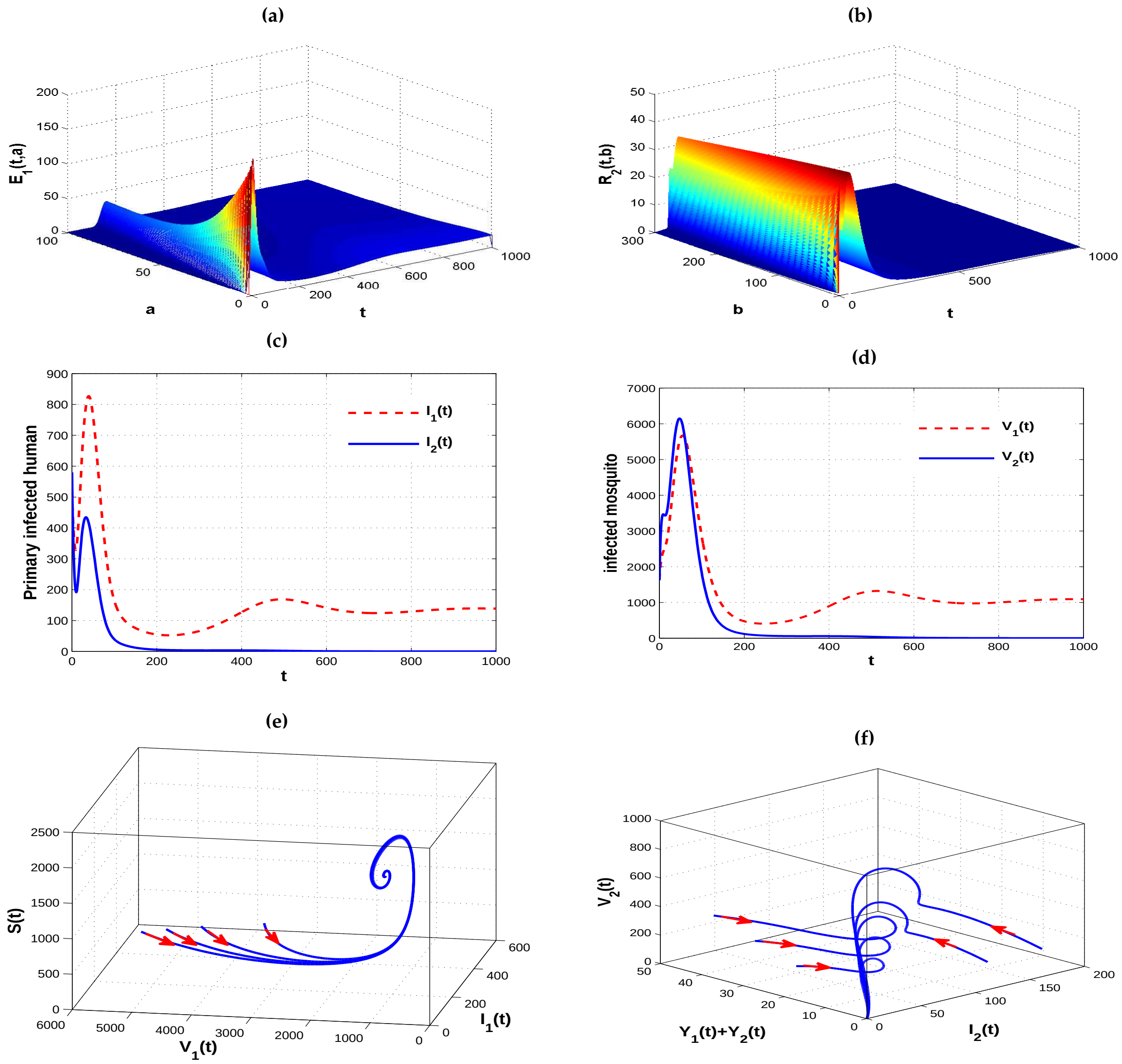

Figure 6.2: The numerical simulation of the stability of strain 1 dominant equilibrium for model (2.1) with the basic reproduction number are$\mathcal{R}_1 \approx 3.528$ and $\mathcal{R}_2 \approx 2.7$.

It is easily to see that $\theta_i(b)$ does not appear in the expression of the basic reproduction number (i.e., the value of $\theta_i(b)$ does not affect the dynamic behavior of the model) from equation (4.1). However, the plot in Figure 6.3(e) show that the peak of secondary infection individuals number with strain 2 increases remarkably with $\theta_2(b)$ increases when the model persistent. This illustrates that the value of cross immunity wane rate still play very important role in the transmission of dengue fever.

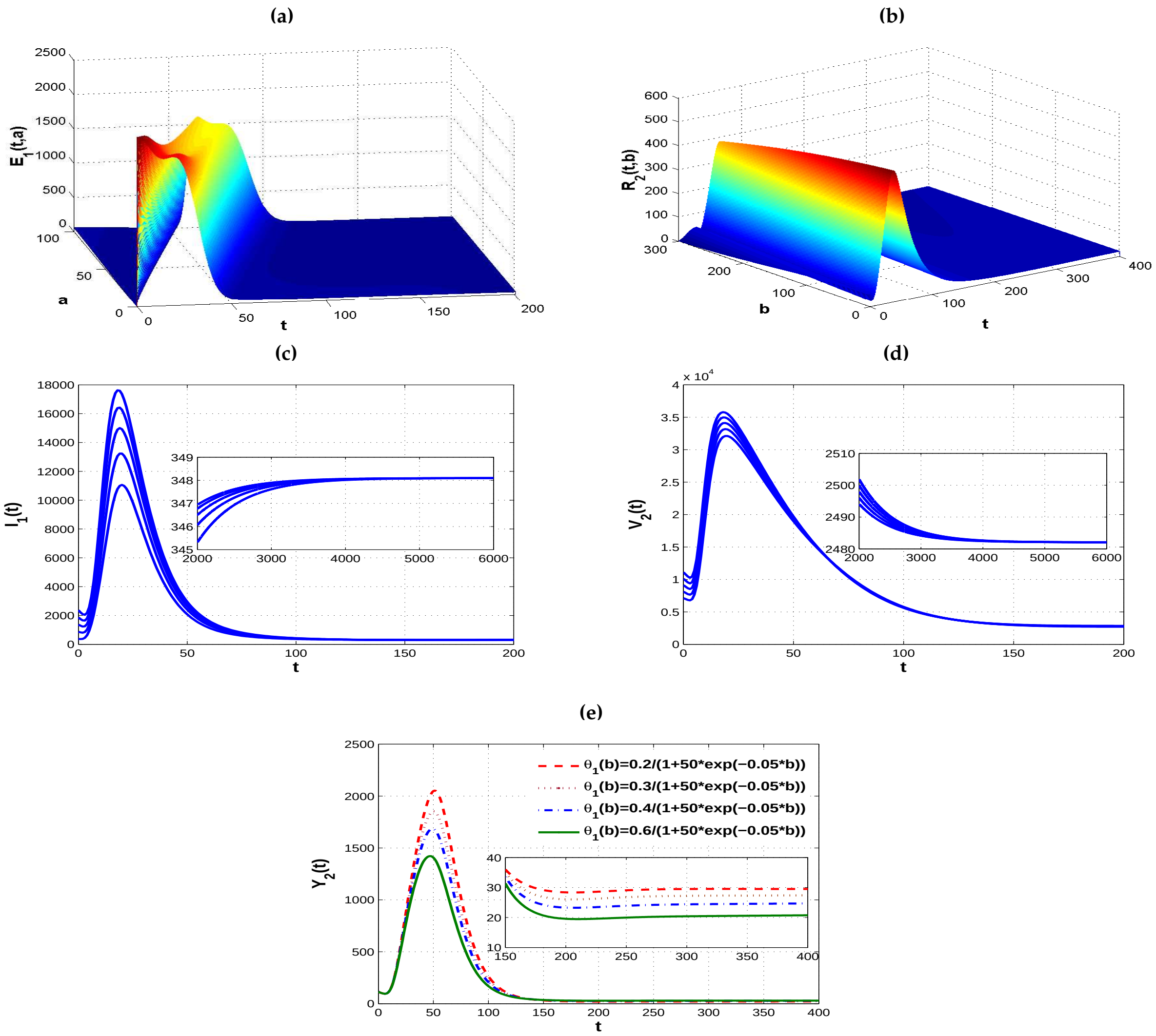

Figure 6.3: Numerical simulations of the persistence, the existence and stability of coexistence equilibrium of model (2.1) with $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_1 > \mathcal{R}_2^*$ and $\mathcal{R}_1 > \mathcal{R}_1^*$.

7 Conclusion

In recent years, many scholars have established lots of multi-strain dengue fever transmission models, studied the existence and stability of the disease-free equilibria, endemic equilibria, stain dominant equilibria, and competitive exclusion, and discussed the effects of the and mutual immune of strains on on the spread and control of dengue fever [9, 11, 17, 19, 27, 29, 32, 34, 41, 42]. However, most of which are described by ordinary differential equations (ODEs). In this paper, based on the two-strain dengue fever model proposed in Ref. [42], we propose a two-strain dengue fever transmission model with age structure to investigate the effects of latency age and cross immunity on the transmission dynamics of dengue virus. This extends the existing single-strain age structure models [4, 5, 8, 36], which is a highlight of our paper.

By using these methods proposed in Refs. [18, 33, 38–40], we first obtain the non-negativity, boundedness and asymptotic smoothness for solutions of our model. Further, the basic reproduction number $\mathcal{R}_0 = \max\{\mathcal{R}_1, \mathcal{R}_2\}$ are defined, which plays a sharp threshold role in the process of this disease outbreaks. That is, if $\mathcal{R}_0 < 1$, then the disease-free equilibrium P_0 is locally stable, and P_0 is unstable for $\mathcal{R}_0 > 1$. Further, we also obtained sufficient conditions for the global asymptotic stability of P_0. To be specific, if $\mathcal{R}_0 < \min\{K_1, K_2\}$, then P_0 is globally asymptotically stable. Of course, our numerical simulations suggest that P_0 is also globally asymptotically stable when $\mathcal{R}_0 < 1$ (see Figure 6.1(d)). In addition, if $\mathcal{R}_i > 1$, this model has a strain-i dominant equilibrium P_i which is locally stable for $\mathcal{R}_j < \mathcal{R}_i^*$ $(i,\ j = 1,\ 2,\ i \neq j)$. This condition is similar to the threshold condition for the stability of strain-i dominant equilibria of these multi strain ordinary differential equations [11,42]. And we have given sufficient conditions for the uniform persistence of disease and the coexistence of the two strains. Finally, the numerical simulation implies that the strain-i dominant equilibrium is global asymptotic stability for $\mathcal{R}_i > 1$ and $\mathcal{R}_j < \mathcal{R}_i^*$ (see Figures 6.3(c)–(d)). However, due to the limitations of these research methods, the global attractivity of coexistence equilibrium obtained by us is subject to certain technical conditions. Therefore, this issue needs further research.

From the expression of $\mathcal{R}_i$, it is easy to observe that their value depends on $\varepsilon_i(a)$. Numerical simulations also shows that if the period of cross-immunity between the two strains increased (i.e., the rate of cross immunity waning decreased), the number of individuals with secondary infection decreased, and then the number of severe dengue cases decreased (see Figure 6.3(d)). This means that the latent age and cross immunity age play a important role in the transmission of dengue fever. Additionally, other model parameters also have an impact on the value of $\mathcal{R}_i$, such as the rates of transmission (α_i and β_i), the death rate and the recruitment rate of mosquito (μ_m and Λ_m), and so on. Therefore, control or prevent the transmission of dengue fever is mainly to reduce the number of mosquito and to increase personal protect awareness.

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Competing interests

The authors declare that they have no competing interests.

Appendix

According to the Corollary 5.2 and the Figure 6.3, the coexistence equilibrium P_3 is globally asymptotically stable. Hence, we attempt to construct a Lyapunov functional to obtain the theoretical analysis.

Theorem A.1. *If the condition of Corollary 5.2 and the following inequalities hold*

$$\widetilde{S}^* + \int_0^\infty \theta_1(b)\widetilde{R}_2^*(b)\mathrm{d}b < \frac{\mu_m}{\beta_1}, \qquad \widetilde{S}^* + \int_0^\infty \theta_2(b)\widetilde{R}_1^*(b)\mathrm{d}b < \frac{\mu_m}{\beta_2},$$

$$\widetilde{U}^* < \min\left\{\frac{\mu_h + \gamma_1(1-K_1)}{\alpha_1 K_1}, \frac{1}{\alpha_1}(\gamma_1 + d_1 + \mu_h), \frac{\mu_h + \gamma_2(1-K_2)}{\alpha_2 K_2}, \frac{1}{\alpha_2}(\gamma_2 + d_2 + \mu_h)\right\},$$

then model (2.1) *has a unique coexistence equilibrium* P_3 *which is globally attractive.*

Proof. Consider the Lyapunov functional as follows

$$\mathcal{L}(t) = \mathcal{L}_1(t) + \mathcal{L}_6(t) + \sum_{i=1}^{2} \left\{ \mathcal{L}_{2i}(t) + \mathcal{L}_{3i}(t) + \mathcal{L}_{4i}(t) + \mathcal{L}_{5i}(t) + \mathcal{L}_{7i}(t) \right\},$$

where

$$\mathcal{L}_1(t) = \widetilde{S}^* \phi\left(\frac{S(t)}{\widetilde{S}^*}\right), \qquad \mathcal{L}_{2i}(t) = \frac{1}{K_i}\int_0^\infty q_i(a)\widetilde{E}_i^*(a)\phi\left(\frac{E_i(t,a)}{\widetilde{E}_i^*(a)}\right)\mathrm{d}a,$$

$$\mathcal{L}_{3i}(t) = \frac{1}{K_i}\widetilde{I}_i^* \phi\left(\frac{I_i(t)}{\widetilde{I}_i^*}\right), \qquad \mathcal{L}_{4i}(t) = \int_0^\infty \widetilde{R}_i^*(b)\phi\left(\frac{R_i(t,b)}{\widetilde{R}_i^*(b)}\right)\mathrm{d}a,$$

$$\mathcal{L}_{5i}(t) = \widetilde{Y}_i^* \phi\left(\frac{Y_i(t)}{\widetilde{Y}_i^*}\right), \qquad \mathcal{L}_6(t) = \widetilde{U}^* \phi\left(\frac{U(t)}{\widetilde{U}^*}\right), \qquad \mathcal{L}_{7i}(t) = \widetilde{V}_i^* \phi\left(\frac{V_i(t)}{\widetilde{V}_i^*}\right).$$

Because of the complexity of the expressions, we make the derive of each component of the Lyapunov functional separately.

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_1(t)}{\mathrm{d}t} &= \widetilde{S}^*\left(\frac{1}{\widetilde{S}^*} - \frac{1}{S(t)}\right)\left(\Lambda_h - \frac{\Lambda_h S(t)}{\widetilde{S}^*} + \beta_1 S(t)\widetilde{V}_1^* + \beta_1 S(t)\widetilde{V}_2^* - \beta_1 S(t)V_1(t) - \beta_2 S(t)V_2(t)\right) \\
&= \Lambda_h\left(2 - \frac{\widetilde{S}^*}{S(t)} - \frac{S(t)}{\widetilde{S}^*}\right) + \beta_1\widetilde{S}^*\widetilde{V}_1^*\left(1 - \frac{\widetilde{S}^*}{S(t)}\right)\left(\frac{S(t)}{\widetilde{S}^*} - \frac{S(t)V_1(t)}{\widetilde{S}^*\widetilde{V}_1^*}\right) \\
&\quad + \beta_2\widetilde{S}^*\widetilde{V}_2^*\left(1 - \frac{\widetilde{S}^*}{S(t)}\right)\left(\frac{S(t)}{\widetilde{S}^*} - \frac{S(t)V_2(t)}{\widetilde{S}^*\widetilde{V}_2^*}\right) \\
&= -\Lambda_h\left[\phi\left(\frac{S(t)}{\widetilde{S}^*}\right) + \phi\left(\frac{\widetilde{S}^*}{S(t)}\right)\right] + \beta_1\widetilde{S}^*\widetilde{V}_1^*\left(\frac{S(t)}{\widetilde{S}^*} - \frac{S(t)V_1(t)}{\widetilde{S}^*\widetilde{V}_1^*} - 1 + \frac{V_1(t)}{\widetilde{V}_1^*}\right) \\
&\quad + \beta_2\widetilde{S}^*\widetilde{V}_2^*\left(\frac{S(t)}{\widetilde{S}^*} - \frac{S(t)V_2(t)}{\widetilde{S}^*\widetilde{V}_2^*} - 1 + \frac{V_2(t)}{\widetilde{V}_2^*}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_{2i}(t)}{\mathrm{d}t} &= \frac{1}{K_i}\int_0^\infty q_i(a)\widetilde{E}_i^*(a)\frac{\partial}{\partial t}\phi\left(\frac{E_i(t,a)}{\widetilde{E}_i^*(a)}\right)\mathrm{d}a \\
&= -\frac{1}{K_i}\int_0^\infty q_i(a)\widetilde{E}_i^*(a)\left(\frac{E_i(t,a)}{\widetilde{E}_i^*(a)} - 1\right)\left(\frac{E_{ia}(t,a)}{E_i(t,a)} + \mu_h + \varepsilon_i(a)\right)\mathrm{d}a,
\end{aligned}$$

where $E_{ia}(t,a) = \mathrm{d}E_i(t,a)/\mathrm{d}a$. Since

$$\frac{\partial}{\partial a}\phi\left(\frac{E_i(t,a)}{\widetilde{E}_i^*(a)}\right) = \left(\frac{E_i(t,a)}{\widetilde{E}_i^*(a)} - 1\right)\left(\frac{E_{ia}(t,a)}{E_i(t,a)} + \mu_h + \varepsilon_i(a)\right),$$

it can be easily shown that

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_{2i}(t)}{\mathrm{d}t} &= -\frac{1}{K_i}\int_0^\infty q_i(a)\widetilde{E}_i^*(a)\frac{\partial}{\partial a}\phi\left(\frac{E_i(t,a)}{\widetilde{E}_i^*(a)}\right)\mathrm{d}a \\
&= -\frac{1}{K_i}q_i(a)\widetilde{E}_i^*(a)\phi\left(\frac{E_i(t,a)}{\widetilde{E}_i^*(a)}\right)\Bigg|_0^\infty - \frac{1}{K_i}\int_0^\infty \varepsilon_i(a)\widetilde{E}_i^*(a)\phi\left(\frac{E_i(t,a)}{\widetilde{E}_i^*(a)}\right)\mathrm{d}a \\
&= \beta_i\widetilde{S}^*\widetilde{V}_i^*\phi\left(\frac{S(t)V_i(t)}{\widetilde{S}^*\widetilde{V}_i^*}\right) - \frac{1}{K_i}\int_0^\infty \varepsilon_i(a)\widetilde{E}_i^*(a)\phi\left(\frac{E_i(t,a)}{\widetilde{E}_i^*(a)}\right)\mathrm{d}a.
\end{aligned}$$

By directly calculating, we have

$$\begin{aligned}\frac{\mathrm{d}\mathcal{L}_{3i}(t)}{\mathrm{d}t} &= \frac{1}{K_i}\int_0^\infty \varepsilon_i(a)\left(1-\frac{\widetilde{I}_i^*}{I_i(t)}\right)\left(E_i(t,a)-\widetilde{E}_i^*(a)\frac{I_i(t)}{\widetilde{I}_i^*}\right)\mathrm{d}a\\ &= \frac{1}{K_i}\int_0^\infty \varepsilon_i(a)\widetilde{E}_i^*(a)\left(\frac{E_i(t,a)}{\widetilde{E}_i^*(a)}-\frac{\widetilde{I}_i^*}{I_i(t)}-\frac{\widetilde{I}_i^*E_i(t,a)}{I_i(t)\widetilde{E}_i^*(a)}+1\right)\mathrm{d}a\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}\mathcal{L}_{4i}(t)}{\mathrm{d}t} &= \int_0^\infty \widetilde{R}_i^*(b)\frac{\partial}{\partial b}\phi\left(\frac{R_i(t,b)}{\widetilde{R}_i^*(b)}\right)\mathrm{d}b\\ &= -\int_0^\infty \widetilde{R}_i^*(b)\left(\frac{1}{\widetilde{R}_i^*(b)}-\frac{1}{R_i(t,b)}\right)\left(\frac{\partial}{\partial b}R_i(t,b)+\beta_j\theta_j(b)V_j(t)R_i(t,b)+\mu_hR_i(t,b)\right)\mathrm{d}b\\ &= -\int_0^\infty \widetilde{R}_i^*(b)\left(\frac{R_i(t,b)}{\widetilde{R}_i^*(b)}-1\right)\\ &\quad\times\left[\left(\frac{R_{ib}(t,b)}{R_i(t,b)}+\beta_j\theta_j(b)\widetilde{V}_j^*+\mu_h\right)+\beta_j\theta_j(b)\widetilde{V}_j^*\left(\frac{V_j(t)}{\widetilde{V}_j^*}-1\right)\right]\mathrm{d}b,\end{aligned}$$

where $R_{ib}(t,b)=\mathrm{d}R_i(t,b)/\mathrm{d}b,\ i,\ j=1,\ 2,\ i\neq j$. Since

$$\frac{\partial}{\partial b}\phi\left(\frac{R_i(t,b)}{\widetilde{R}_i^*(b)}\right)=\left(\frac{R_i(t,b)}{\widetilde{R}_i^*(b)}-1\right)\left(\frac{R_{ib}(t,b)}{R_i(t,b)}+\beta_j\theta_j(b)\widetilde{V}_j^*+\mu_h\right),$$

this gives

$$\begin{aligned}\frac{\mathrm{d}\mathcal{L}_{4i}(t)}{\mathrm{d}t} &= -\int_0^\infty\left[\widetilde{R}_i^*(b)\frac{\partial}{\partial b}\phi\left(\frac{R_i(t,b)}{\widetilde{R}_i^*(b)}\right)+\beta_j\theta_j(b)\widetilde{V}_j^*\widetilde{R}_i^*(b)\left(\frac{R_i(t,b)}{\widetilde{R}_i^*(b)}-1\right)\left(\frac{V_j(t)}{\widetilde{V}_j^*}-1\right)\right]\mathrm{d}b\\ &= -\widetilde{R}_i^*(b)\phi\left(\frac{R_i(t,b)}{\widetilde{R}_i^*(b)}\right)\Bigg|_0^\infty-\int_0^\infty\phi\left(\frac{R_i(t,b)}{\widetilde{R}_i^*(b)}\right)\left(\beta_j\theta_j(b)\widetilde{V}_j^*\widetilde{R}_i^*(b)+\mu_h\widetilde{R}_i^*(b)\right)\mathrm{d}b\\ &= \gamma_i\widetilde{I}_i^*\phi\left(\frac{I_i(t)}{\widetilde{I}_i^*}\right)-\mu_h\int_0^\infty\widetilde{R}_i^*(b)\phi\left(\frac{R_i(t,b)}{\widetilde{R}_i^*(b)}\right)\mathrm{d}b\\ &\quad+\int_0^\infty\beta_j\theta_j(b)\widetilde{R}_i^*(b)\widetilde{V}_j^*\left(\frac{R_i(t,b)V_j(t)}{\widetilde{R}_i^*(b)\widetilde{V}_j^*}-\frac{V_j(t)}{\widetilde{V}_j^*}-\ln\frac{R_i(t,b)}{\widetilde{R}_i^*(b)}\right)\mathrm{d}b.\end{aligned}$$

Furthermore, it can be easily calculated that

$$\begin{aligned}\frac{\mathrm{d}\mathcal{L}_{5i}(t)}{\mathrm{d}t} &= \beta_i\int_0^\infty\theta_i(b)\left(1-\frac{\widetilde{Y}_i^*}{Y_i(t)}\right)\left(V_i(t)R_j(t,b)-\frac{Y_i(t)}{\widetilde{Y}_i^*}\widetilde{V}_i^*\widetilde{R}_j^*(b)\right)\mathrm{d}b\\ &= \beta_i\int_0^\infty\theta_i(b)\widetilde{V}_i^*\widetilde{R}_j^*(b)\left(\frac{V_i(t)R_j(t,b)}{\widetilde{V}_i^*\widetilde{R}_j^*(b)}-\frac{Y_i(t)}{\widetilde{Y}_i^*}-\frac{\widetilde{Y}_i^*V_i(t)R_j(t,b)}{Y_i(t)\widetilde{V}_i^*\widetilde{R}_j^*(b)}+1\right)\mathrm{d}b.\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_6(t)}{\mathrm{d}t} =& \Lambda_m\left(1-\frac{\widetilde{U}^*}{U(t)}\right)\left(1-\frac{U(t)}{\widetilde{U}^*}\right)+\left(1-\frac{\widetilde{U}^*}{U(t)}\right)\left(\alpha_1(\widetilde{I}_1^*+\widetilde{Y}_1^*)U(t)-\alpha_1(I_1(t)\right.\\
&\left.+Y_1(t))U(t)\right)+\left(1-\frac{\widetilde{U}^*}{U(t)}\right)\Big(\alpha_2(\widetilde{I}_2^*+\widetilde{Y}_2^*)U(t)-\alpha_2(I_2(t)+Y_2(t))U(t)\Big)\\
=& -\Lambda_m\left[\phi\left(\frac{U(t)}{\widetilde{U}^*}\right)+\phi\left(\frac{\widetilde{U}^*}{U(t)}\right)\right]+\alpha_1\widetilde{I}_1^*\widetilde{U}^*\left(\frac{U(t)}{\widetilde{U}^*}-1-\frac{U(t)I_1(t)}{\widetilde{U}^*\widetilde{I}_1^*}+\frac{I_1(t)}{\widetilde{I}_1^*}\right)\\
&+\alpha_1\widetilde{Y}_1^*\widetilde{U}^*\left(\frac{U(t)}{\widetilde{U}^*}-1-\frac{U(t)Y_1(t)}{\widetilde{U}^*\widetilde{Y}_1^*}+\frac{Y_1(t)}{\widetilde{Y}_1^*}\right)+\alpha_2\widetilde{I}_2^*\widetilde{U}^*\left(\frac{U(t)}{\widetilde{U}^*}-1-\frac{U(t)I_2(t)}{\widetilde{U}^*\widetilde{I}_2^*}\right.\\
&\left.+\frac{I_2(t)}{\widetilde{I}_2^*}\right)+\alpha_2\widetilde{Y}_2^*\widetilde{U}^*\left(\frac{U(t)}{\widetilde{U}^*}-1-\frac{U(t)Y_2(t)}{\widetilde{U}^*\widetilde{Y}_2^*}+\frac{Y_2(t)}{\widetilde{Y}_2^*}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_{7i}(t)}{\mathrm{d}t} &= \alpha_i\left(1-\frac{V_i(t)}{\widetilde{V}_i^*}\right)\left(I_i(t)U(t)-\frac{V_i(t)}{\widetilde{V}_i^*}\widetilde{I}_i^*\widetilde{U}^*+Y_i(t)U(t)-\frac{V_i(t)}{\widetilde{V}_i^*}\widetilde{Y}_i^*\widetilde{U}^*\right)\\
&= \alpha_i\widetilde{I}_i^*\widetilde{U}^*\left(\frac{I_i(t)U(t)}{\widetilde{I}_i^*\widetilde{U}^*}-\frac{V_i(t)}{\widetilde{V}_i^*}-\frac{I_i(t)U(t)\widetilde{V}_i^*}{\widetilde{I}_i^*\widetilde{U}^*V_i(t)}+1\right)\\
&\quad+\alpha_i\widetilde{Y}_i^*\widetilde{U}^*\left(\frac{Y_i(t)U(t)}{\widetilde{Y}_i^*\widetilde{U}^*}-\frac{V_i(t)}{\widetilde{V}_i^*}-\frac{Y_i(t)U(t)\widetilde{V}_i^*}{\widetilde{Y}_i^*\widetilde{U}^*V_i(t)}+1\right).
\end{aligned}
$$

Thus, to sum up, we can get

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t} =& \frac{\mathrm{d}\mathcal{L}_1(t)}{\mathrm{d}t}+\frac{\mathrm{d}\mathcal{L}_6(t)}{\mathrm{d}t}+\sum_{i=1}^{2}\left\{\frac{\mathrm{d}\mathcal{L}_{2i}(t)}{\mathrm{d}t}+\frac{\mathrm{d}\mathcal{L}_{3i}(t)}{\mathrm{d}t}+\frac{\mathrm{d}\mathcal{L}_{4i}(t)}{\mathrm{d}t}+\frac{\mathrm{d}\mathcal{L}_{5i}(t)}{\mathrm{d}t}+\frac{\mathrm{d}\mathcal{L}_{7i}(t)}{\mathrm{d}t}\right\}\\
=& -\Lambda_h\left[\phi\left(\frac{S(t)}{\widetilde{S}^*}\right)+\phi\left(\frac{\widetilde{S}^*}{S(t)}\right)\right]-\Lambda_m\left[\phi\left(\frac{U(t)}{\widetilde{U}^*}\right)+\phi\left(\frac{\widetilde{U}^*}{U(t)}\right)\right]+\gamma_1\widetilde{I}_1^*\phi\left(\frac{I_1(t)}{\widetilde{I}_1^*}\right)\\
&-\mu_h\int_0^\infty\widetilde{R}_1^*(b)\phi\left(\frac{R_1(t,b)}{\widetilde{R}_1^*(b)}\right)\mathrm{d}b+\gamma_2\widetilde{I}_2^*\phi\left(\frac{I_2(t)}{\widetilde{I}_2^*}\right)-\mu_h\int_0^\infty\widetilde{R}_2^*(b)\phi\left(\frac{R_2(t,b)}{\widetilde{R}_2^*(b)}\right)\mathrm{d}b\\
&+\sum_{i=1}^{2}\Big(\mathcal{H}_{1i}(t)+\mathcal{H}_{2i}(t)+\mathcal{H}_{3i}(t)+\mathcal{H}_{4i}(t)+\mathcal{H}_{5i}(t)\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}_{1i}(t) :=& \beta_i\widetilde{S}^*\widetilde{V}_i^*\left[\frac{S(t)}{\widetilde{S}^*}-\frac{S(t)V_i(t)}{\widetilde{S}^*\widetilde{V}_i^*}-1+\frac{V_i(t)}{\widetilde{V}_i^*}+\phi\left(\frac{S(t)V_i(t)}{\widetilde{S}^*\widetilde{V}_i^*}\right)\right]\\
=& \beta_i\widetilde{S}^*\widetilde{V}_i^*\left[\phi\left(\frac{S(t)}{\widetilde{S}^*}\right)+\phi\left(\frac{\widetilde{V}_i^*}{V_i(t)}\right)\right],\\
\mathcal{H}_{2i}(t) :=& \frac{1}{K_i}\int_0^\infty\varepsilon_i(a)\widetilde{E}_i^*(a)\left[\frac{E_i(t,a)}{\widetilde{E}_i^*(a)}-\frac{\widetilde{I}_i^*}{I_i(t)}-\frac{\widetilde{I}_i^*E_i(t,a)}{I_i(t)\widetilde{E}_i^*(a)}+1-\phi\left(\frac{E_i(t,a)}{\widetilde{E}_i^*(a)}\right)\right]\mathrm{d}a\\
=& \frac{1}{K_i}\int_0^\infty\varepsilon_i(a)\widetilde{E}_i^*(a)\left[\phi\left(\frac{E_i(t,a)\widetilde{I}_i^*}{\widetilde{E}_i^*(a)I_i(t)}\right)+\phi\left(\frac{I_i(t)}{\widetilde{I}_i^*}\right)\right]\mathrm{d}a,\\
\mathcal{H}_{3i}(t) :=& \int_0^\infty\beta_i\theta_i(b)\widetilde{R}_j^*(b)\widetilde{V}_i^*\left(1+\frac{V_i(t)}{\widetilde{V}_i^*}+\ln\frac{R_j(t,b)}{\widetilde{R}_j^*(b)}-\frac{Y_i(t)}{\widetilde{Y}_i^*}-\frac{\widetilde{Y}_i^*V_i(t)R_j(t,b)}{Y_i(t)\widetilde{V}_i^*\widetilde{R}_j^*(b)}\right)\mathrm{d}b
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty \beta_i\theta_i(b)\widetilde{R}_j^*(b)\widetilde{V}_i^* \left[\phi\left(\frac{V_i(t)}{\widetilde{V}_i^*}\right) - \phi\left(\frac{\widetilde{Y}_i^* V_i(t) R_j(t,b)}{Y_i(t)\widetilde{V}_i^*\widetilde{R}_j^*(b)}\right) - \phi\left(\frac{Y_i(t)}{\widetilde{Y}_i^*}\right)\right] \mathrm{d}b, \\
\mathcal{H}_{4i}(t) &:= \alpha_i \widetilde{I}_i^*\widetilde{U}^* \left(\frac{U(t)}{\widetilde{U}^*} + \frac{I_i(t)}{\widetilde{I}_i^*} - \frac{V_i(t)}{\widetilde{V}_i^*} - \frac{\widetilde{V}_i^* U(t) I_i(t)}{V_i(t)\widetilde{U}^*\widetilde{I}_i^*}\right) \\
&= \alpha_i \widetilde{I}_i^*\widetilde{U}^* \left[\phi\left(\frac{U(t)}{\widetilde{U}^*}\right) + \phi\left(\frac{I_i(t)}{\widetilde{I}_i^*}\right) - \phi\left(\frac{V_i(t)}{\widetilde{V}_i^*}\right) - \phi\left(\frac{\widetilde{V}_i^* U(t) I_i(t)}{V_i(t)\widetilde{U}^*\widetilde{I}_i^*}\right)\right], \\
\mathcal{H}_{5i}(t) &:= \alpha_i \widetilde{Y}_i^*\widetilde{U}^* \left(\frac{U(t)}{\widetilde{U}^*} + \frac{Y_i(t)}{\widetilde{Y}_i^*} - \frac{V_i(t)}{\widetilde{V}_i^*} - \frac{\widetilde{V}_i^* U(t) Y_i(t)}{V_i(t)\widetilde{U}^*\widetilde{Y}_i^*}\right) \\
&= \alpha_i \widetilde{Y}_i^*\widetilde{U}^* \left[\phi\left(\frac{U(t)}{\widetilde{U}^*}\right) + \phi\left(\frac{Y_i(t)}{\widetilde{Y}_i^*}\right) - \phi\left(\frac{V_i(t)}{\widetilde{V}_i^*}\right) - \phi\left(\frac{\widetilde{V}_i^* U(t) Y_i(t)}{V_i(t)\widetilde{U}^*\widetilde{Y}_i^*}\right)\right].
\end{aligned}
$$

Note that equilibrium P_3 satisfies

$$
\begin{aligned}
&\Lambda_h = \beta_1\widetilde{S}^*\widetilde{V}_1^* + \beta_2\widetilde{S}^*\widetilde{V}_2^* + \mu_h\widetilde{S}^*, \quad \int_0^\infty \varepsilon_i(a)\widetilde{E}_i^*(a)\mathrm{d}a = (\gamma_i + \mu_h)\widetilde{I}_i^*, \\
&\mu_m\widetilde{V}_i^* = \alpha_i(\widetilde{I}_i^* + \widetilde{Y}_i^*)\widetilde{U}^*, \qquad \beta_i\int_0^\infty \theta_i(b)\widetilde{V}_i^*\widetilde{R}_j^*(b)\mathrm{d}b = (\gamma_i + d_i + \mu_h)Y_i, \\
&\Lambda_m = \alpha_1(\widetilde{I}_1^* + \widetilde{Y}_1^*)\widetilde{U}^* + \alpha_2(\widetilde{I}_2^* + \widetilde{Y}_2^*)\widetilde{U}^* + \mu_m\widetilde{U}^*, \qquad i,j = 1,2,\ i \neq j.
\end{aligned}
$$

Therefore, we finally obtain

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t} = &-\mu_h\widetilde{S}^*\phi\left(\frac{S(t)}{\widetilde{S}^*}\right) - \Lambda_h\phi\left(\frac{\widetilde{S}^*}{S(t)}\right) - \frac{1}{K_1}\int_0^\infty \varepsilon_1(a)\widetilde{E}_1^*(a)\phi\left(\frac{E_1(t,a)\widetilde{I}_1^*}{\widetilde{E}_1^*(a)I_1(t)}\right)\mathrm{d}a \\
&-\mu_h\int_0^\infty \widetilde{R}_1^*(b)\phi\left(\frac{R_1(t,b)}{\widetilde{R}_1^*(b)}\right)\mathrm{d}b - \mu_h\int_0^\infty \widetilde{R}_2^*(b)\phi\left(\frac{R_2(t,b)}{\widetilde{R}_2^*(b)}\right)\mathrm{d}b \\
&-\frac{1}{K_2}\int_0^\infty \varepsilon_2(a)\widetilde{E}_2^*(a)\phi\left(\frac{E_2(t,a)\widetilde{I}_2^*}{\widetilde{E}_2^*(a)I_2(t)}\right)\mathrm{d}a - \mu_m\widetilde{U}^*\phi\left(\frac{U(t)}{\widetilde{U}^*}\right) - \Lambda_m\phi\left(\frac{\widetilde{U}^*}{U(t)}\right) \\
&-\int_0^\infty \beta_1\theta_1(b)\widetilde{R}_2^*(b)\widetilde{V}_1^*\phi\left(\frac{\widetilde{Y}_1^*V_1(t)R_2(t,b)}{Y_1(t)\widetilde{V}_1^*\widetilde{R}_2^*(b)}\right)\mathrm{d}b - \alpha_1\widetilde{I}_1^*\widetilde{U}^*\phi\left(\frac{\widetilde{V}_1^*U(t)I_1(t)}{V_1(t)\widetilde{U}^*\widetilde{I}_1^*}\right) \\
&-\int_0^\infty \beta_2\theta_2(b)\widetilde{R}_1^*(b)\widetilde{V}_2^*\phi\left(\frac{\widetilde{Y}_2^*V_2(t)R_1(t,b)}{Y_2(t)\widetilde{V}_2^*\widetilde{R}_1^*(b)}\right)\mathrm{d}b - \alpha_2\widetilde{I}_2^*\widetilde{U}^*\phi\left(\frac{\widetilde{V}_2^*U(t)I_2(t)}{V_2(t)\widetilde{U}^*\widetilde{I}_2^*}\right) \\
&+\widetilde{I}_1^*\phi\left(\frac{I_1(t)}{\widetilde{I}_1^*}\right)\left[\alpha_1\widetilde{U}^* + \gamma_1\left(1 - \frac{1}{K_1}\right) - \frac{\mu_h}{K_1}\right] + (\alpha_1\widetilde{U}^* - (\gamma_1 + d_1 + \mu_h))\widetilde{Y}_1^* \\
&\times\phi\left(\frac{Y_1(t)}{\widetilde{Y}_1^*}\right) + \widetilde{I}_1^*\phi\left(\frac{I_1(t)}{\widetilde{I}_1^*}\right)\left[\alpha_1\widetilde{U}^* + \gamma_1\left(1 - \frac{1}{K_1}\right) - \frac{\mu_h}{K_1}\right] \\
&+(\alpha_1\widetilde{U}^* - (\gamma_1 + d_1 + \mu_h))\widetilde{Y}_1^*\phi\left(\frac{Y_1(t)}{\widetilde{Y}_1^*}\right) - \alpha_1\widetilde{Y}_1^*\widetilde{U}^*\phi\left(\frac{\widetilde{V}_1^*U(t)Y_1(t)}{V_1(t)\widetilde{U}^*\widetilde{Y}_1^*}\right) \\
&-\alpha_2\widetilde{Y}_2^*\widetilde{U}^*\phi\left(\frac{\widetilde{V}_2^*U(t)Y_2(t)}{V_2(t)\widetilde{U}^*\widetilde{Y}_2^*}\right) + \left(\beta_1\widetilde{S}^* + \beta_1\int_0^\infty \theta_1(b)\widetilde{R}_2^*(b)\mathrm{d}b - \mu_m\right) \\
&\times\widetilde{V}_1^*\phi\left(\frac{V_1(t)}{\widetilde{V}_1^*}\right) + \left(\beta_2\widetilde{S}^* + \beta_2\int_0^\infty \theta_2(b)\widetilde{R}_1^*(b)\mathrm{d}b - \mu_m\right)\widetilde{V}_2^*\phi\left(\frac{V_2(t)}{\widetilde{V}_2^*}\right).
\end{aligned}
$$

It is easy to see that the sufficient condition for $\mathrm{d}\mathcal{L}(t)/\mathrm{d}t < 0$ are

$$\begin{aligned}
&\alpha_1\widetilde{U}^* + \gamma_1\left(1 - \frac{1}{K_1}\right) - \frac{\mu_h}{K_1} < 0, \quad \alpha_1\widetilde{U}^* - (\gamma_1 + d_1 + \mu_h) < 0,\\
&\alpha_2\widetilde{U}^* + \gamma_2\left(1 - \frac{1}{K_2}\right) - \frac{\mu_h}{K_2} < 0, \qquad \alpha_2\widetilde{U}^* - (\gamma_2 + d_2 + \mu_h) < 0,\\
&\beta_1\widetilde{S}^* + \beta_1\int_0^\infty \theta_1(b)\widetilde{R}_2^*(b)\mathrm{d}b - \mu_m < 0, \quad \beta_2\widetilde{S}^* + \beta_2\int_0^\infty \theta_2(b)\widetilde{R}_1^*(b)\mathrm{d}b - \mu_m < 0.
\end{aligned}$$

That is,

$$\begin{aligned}
&\widetilde{S}^* + \int_0^\infty \theta_1(b)\widetilde{R}_2^*(b)\mathrm{d}b < \frac{\mu_m}{\beta_1}, \qquad \widetilde{S}^* + \int_0^\infty \theta_2(b)\widetilde{R}_1^*(b)\mathrm{d}b < \frac{\mu_m}{\beta_2},\\
&\widetilde{U}^* < \min\left\{\frac{\mu_h + \gamma_1(1-K_1)}{\alpha_1 K_1}, \frac{1}{\alpha_1}(\gamma_1 + d_1 + \mu_h), \frac{\mu_h + \gamma_2(1-K_2)}{\alpha_2 K_2}, \frac{1}{\alpha_2}(\gamma_2 + d_2 + \mu_h)\right\}.
\end{aligned}$$

This shows that equilibrium P_3 is globally attractive. This completes the proof. □

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