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A saddle point type solution for a system of operator equations

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Abstract. Let $\Omega \subset \mathbb{R}^n$, $n > 1$ and let $p, q \geq 2$. We consider the system of nonlinear Dirichlet problems

$$\begin{cases} (Au)(x) = N'_u(x, u(x), v(x)), & x \in \Omega, \\ -(Bv)(x) = N'_v(x, u(x), v(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \\ v(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $N : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 and is partially convex–concave and $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, $B : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ are monotone and potential operators. The solvability of this system is reached via the Ky–Fan minimax theorem.

Keywords: Ky–Fan minimax theorem, Dirichlet problem, potential operators, monotone operators

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1 Introduction

Let Ω be any bounded domain in $\mathbb{R}^n$, where $n \in \mathbb{N}$ and let $p, q \geq 2$, $p, q \in \mathbb{R}$ be fixed. The aim of this work is to consider the system of two nonlinear Dirichlet boundary value problems whose solvability is reached via the Ky–Fan minimax theorem (consult [14] for details) which is a more general version of classical Sion's minimax theorem [10]. We also use some reasoning applied usually in the monotonicity approach. Namely we use direct method of Calculus of Variations, and the fact that monotone and potential operators are actually convex and l.s.c. To be precise we investigate the following problem. Let $N : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function, with some more requirement for its derivative with respect to second and third variables, and let $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, $\mathcal{B} : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ be some monotone and potential operators (pertaining to the classical negative p-Laplacian).

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Problem 1 (Main problem)**.** Find $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ such that

$$\begin{aligned} \langle \mathcal{A}(u); \bar{u} \rangle &= \int_\Omega N_u'(x,u(x),v(x))\bar{u}(x)\,\mathrm{d}x, \\ -\langle \mathcal{B}(v); \bar{v} \rangle &= \int_\Omega N_v'(x,u(x),v(x))\bar{u}(x)\,\mathrm{d}x. \end{aligned}$$

for all $\bar{u} \in \mathrm{W}_0^{1,p}(\Omega), \bar{v} \in \mathrm{W}_0^{1,q}(\Omega)$.

We see that the above is a system of mixed operator and integral type formulas, which under certain assumption appears to admit a solution of saddle point type. The existence of boundary value problems with the p-Laplacian is well covered in the literature, see for example [4,7–9,15]. Some results investigating the relation between the monotonicity and variational approaches are given in [5]. The case in which operators on LHS are both monotone is well studied, and existence result was proved by the critical point theory. In our situation one of the operators (namely $\mathcal{A}$) is monotone while the other (namely $-\mathcal{B}$) only becomes monotone in case it is multiplied by -1. This observation forces us to adapt the approach known for elliptic systems, see for example [6,11] to the case that could include also more non-linear equations. When compared with [11] we adapt their methods to the nonlinear setting and also simplify whenever possible their arguments by using direct links to the monotonicity theory. For an approach using the mixture of abstract formulation of the operator together with the explicitly written RHS we refer to [3] while underlying that these authors considered single equations.

2 Some preliminary results

The following properties are well known, but the full proofs are actually quite hard to be found. Some short proofs are indicated in [13], here we provide a full proof of a slightly modified result.

Lemma 2.1 (On properties of the pointwise maximum [13, Th. 3.3.3])**.** *Assume $f\colon U \times V \to \mathbb{R}$, where U, V are some vector spaces over $\mathbb{R}$ and let for any $u \in U$ there exists such $\hat{v} \in Y$ that $f(u,\hat{v}) = \max_v f(u,v)$. If $u \mapsto f(u,v)$ is convex for any $v \in V$, then $u \mapsto \max_v f(u,v)$, is convex. If $u \mapsto f(u,v)$ is l.s.c. for any $v \in V$ then $u \mapsto \max_v f(u,v)$, is also lower semicontinuous.*

Proof. Let $u, w \in U$ and let $\alpha \in (0,1)$. Lets denote $\hat{v}$ be such element of V that

$$f(\alpha u + (1-\alpha)w, \hat{v}) = \max_v f(\alpha u + (1-\alpha)w, v).$$

Then

$$\begin{aligned} f(\alpha u + (1-\alpha)w, \hat{v}) &\le \alpha f(u,\hat{v}) + (1-\alpha) f(w,\hat{v}) \\ &\le \alpha \max_v f(u,v) + (1-\alpha) f(w,\hat{v}) \\ &\le \alpha \max_v f(u,v) + (1-\alpha) \max_v f(w,v). \end{aligned}$$

Thus it follows that $u \mapsto \max_v f(u,v)$, is convex. For the second part we assume $u_0 \in U$ and $\bar{v} \in V$ to be an arbitrary element. Then

$$\liminf_{u\to u_0} \max_v f(u,v) \ge \liminf_{u \to u_0} f(u,\bar{v}) \ge f(u_0,\bar{v}).$$

As we apply maximum over $\bar{v}$ we gets

$$\liminf_{u \to u_0} \max_v f(u,v) \geq \max_v f(u_0,v).$$

Since $u_0 \in X$ was arbitrary, thus $u \mapsto \max_v f(u,v)$, is also lower semicontinuous. $\square$

Corollary 2.2 (On properties of the pointwise minimum)**.** *Assume $f\colon U \times V \to \mathbb{R}$, where U, V are some vector spaces and let for any u there exists such $\hat{v} \in Y$ that $f(u,\hat{v}) = \min_v f(u,v)$. Let $u \mapsto f(u,v)$ be concave for any $v \in V$, then $u \mapsto \min_v f(u,v)$, is concave. If $u \mapsto f(u,v)$ be u.s.c. for any $v \in V$ then $u \mapsto \min_v f(u,v)$, is also upper semicontinuous.*

E will stand for a real and reflexive Banach space in this section. Since we shall use monotone operator approach lets recall its definition. We refer to [2] and [16] for some background.

Definition 2.3 (Properties of operators)**.** Let $\mathcal{A} : E \to E^*$. Then

- $\mathcal{A}$ is called monotone iff
$$\langle \mathcal{A}(u) - \mathcal{A}(v); u - v \rangle \geq 0,$$
for all $u, v \in E$;
- $\mathcal{A}$ is called coercive iff
$$\lim_{\|u\|_E \to \infty} \frac{\langle \mathcal{A}(u); u \rangle}{\|u\|_E} = +\infty.$$
- $\mathcal{A}$ is called anticoercive iff operator $-\mathcal{A}$ is coercive.
- $\mathcal{A}$ is said to be demicontinuous iff $u_n \to u$ as $n \to \infty$ implies that
$$\mathcal{A}u_n \rightharpoonup \mathcal{A}u,$$
as $n \to \infty$.
- $\mathcal{A}$ is potential if there exists a functional $f : E \to \mathbb{R}$ differentiable in the sense of Gâteaux and such that
$$f' = \mathcal{A},$$
Then f is called potential of $\mathcal{A}$.

Lemma 2.4. *Assume $\mathcal{A} : E \to E^*$ is potential and monotone. Then its potential is convex and weakly lower semicontinuous (w.l.s.c. for short). Also $\mathcal{A}$ is demicontinuous.*

Lemma 2.5. *Assume $\mathcal{A} : E \to E^*$ is potential and demicontinuous. Then*

$$v \mapsto \int_0^1 \langle \mathcal{A}(tv); v \rangle \, \mathrm{d}t, v \in E,$$

is a potential of $\mathcal{A}$.

Sufficient conditions for existence of solution may describe in terms of some constant provided by the following Sobolev embedding theorem.

Theorem 2.6 (Sobolev imbedding theorem [1, Th. 4.12])**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. Then*

- *if $p \geq n$ then*
$$\mathrm{W}_0^{1,p}(\Omega) \to \mathrm{L}^q(\Omega),$$
for $1 \leq q \leq \infty$, and
- *if $p < n$ then*
$$\mathrm{W}_0^{1,p}(\Omega) \to \mathrm{L}^q(\Omega),$$
for $1 \leq q \leq \frac{np}{n-p}$.

We shall require following two constants. Let $\lambda_{1,p} > 0$ be such that for all $u \in \mathrm{W}_0^{1,p}(\Omega)$:
$$\lambda_{1,p} \|u\|^p_{\mathrm{L}^p(\Omega)} \leq \|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)}.$$
Also let $\lambda_{1,q} > 0$ satisfy similar condition for q and $v \in \mathrm{W}_0^{1,q}(\Omega)$.

Definition 2.7 (L^s-Carathéodory function [5]). Assume $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $s \geq 1$ holds. We shall say that f is L^s-Carathéodory, if

- for all $(u, v) \in \mathbb{R} \times \mathbb{R}$ function $x \mapsto f(x, u, v)$ is measurable;
- for a. e. $x \in \Omega$ function $(u, v) \mapsto f(x, u, v)$ is continuous;
- for each $d > 0$ there exists a function $f_d \in \mathrm{L}^s(\Omega)$ such that for a. e. $x \in \Omega$
$$\max_{(u,v)\in[-d,d]\times[-d,d]} |f(x, u, v)| \leq f_d(x);$$

3 Variational framework and the existence of a solution

(A) Operator $\mathcal{A}$ is potential and monotone.

(B) Operator $\mathcal{B}$ is potential and monotone.

(C) Operator $\mathcal{A}$ fulfils that there exists $\hat{\alpha_1} > 0$,
$$\langle \mathcal{A}(u); u\rangle \geq \hat{\alpha_1} \|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)},$$
for all $u \in \mathrm{W}_0^{1,p}(\Omega)$.

(D) Operator $\mathcal{B}$ fulfils that there exists $\hat{\alpha_2} > 0$,
$$\langle \mathcal{B}(v); v\rangle \geq \hat{\alpha_2} \|v\|^q_{\mathrm{W}_0^{1,q}(\Omega)},$$
for all $v \in \mathrm{W}_0^{1,q}(\Omega)$.

(E) Function $N : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is L^1-Carathéodory. Moreover, derivatives N'_u, N'_v exists and $N'_u : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $\mathrm{L}^{p'}$-Carathéodory, and $N'_v : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $\mathrm{L}^{q'}$-Carathéodory.

(F) for each $v \in \mathrm{W}_0^{1,q}(\Omega)$ there exists functions $\beta_1 \in \mathrm{L}^2(\Omega)$, $\gamma_1 \in \mathrm{L}^1(\Omega)$ and $0 < \alpha_1 < \lambda_{1,p}\frac{\hat{\alpha_1}}{p}$ that
$$N(x, u, v(x)) \geq -\alpha_1|u|^p + \beta_1(x) \cdot u + \gamma_1(x),$$
for almost every $x \in \Omega$ and all $u \in \mathbb{R}$.

(G) for each $u \in W_0^{1,p}(\Omega)$ there exists functions $\beta_2 \in L^2(\Omega)$, $\gamma_2 \in L^1(\Omega)$ and $0 < \alpha_2 < \lambda_{1,q}\frac{\hat{\alpha_2}}{q}$ that

$$N(x, u(x), v) \leq \alpha_2 |v|^q + \beta_2(x) \cdot v + \gamma_2(x),$$

for almost every $x \in \Omega$ and all $v \in \mathbb{R}$.

(H) For any fixed $v \in W_0^{1,q}(\Omega)$ functional

$$u \mapsto \int_\Omega N(x, u(x), v(x))\, dx$$

is convex.

(I) For any fixed $u \in W_0^{1,p}(\Omega)$ functional

$$v \mapsto \int_\Omega N(x, u(x), v(x))\, dx$$

is concave.

Let A be a potential to $\mathcal{A}$, and B to $\mathcal{B}$. Also by N be shall denote the Nemyckij's operator to N.

In order to obtain the existence result, we consider the following reformulation of Problem 1 to a critical point-type problem:

Problem 2 (Variational form of the main problem)**.** Consider the following functional

$$J\colon W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \to \mathbb{R}$$

given by the formula

$$J(u, v) = \int_0^1 \langle \mathcal{A}(tu); u \rangle \, dt - \int_0^1 \langle \mathcal{B}(tv); v \rangle \, dt + \int_\Omega N(x, u(x), v(x))\, dx.$$

Find such $\hat{u}, \hat{v}$ that

$$\sup_{v \in W_0^{1,q}(\Omega)} \inf_{u \in W_0^{1,p}(\Omega)} J(u, v) = \inf_{u \in W_0^{1,p}(\Omega)} \sup_{v \in W_0^{1,q}(\Omega)} J(u, v) = J(\hat{u}, \hat{v}).$$

We can easily observe that if conditions (A), (B), (E), (F), (G) holds then any solution to problem 2 is a solution to Problem 1.

Lemma 3.1 (Growth estimate on A and B)**.** *Under (C) for any $u \in W_0^{1,p}(\Omega)$ the following holds:*

$$A(u) = \int_0^1 \langle \mathcal{A}(tu); u \rangle \, dt \geq \frac{\hat{\alpha_1}}{p} \|u\|^p_{W_0^{1,p}(\Omega)}.$$

Similarly under (D) for any $v \in W_0^{1,q}(\Omega)$ the following holds:

$$B(v) = \int_0^1 \langle \mathcal{B}(tv); v \rangle \, dt \geq \frac{\hat{\alpha_2}}{q} \|v\|^q_{W_0^{1,q}(\Omega)}.$$

Proof of Lemma 3.1. Let $u \in \mathrm{W}_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\int_0^1 \langle \mathcal{A}(tu); u\rangle \,\mathrm{d}t &= \int_0^1 \frac{1}{t}\langle \mathcal{A}(tu); tu\rangle \,\mathrm{d}t\\ &\geq \int_0^1 \frac{1}{t}\|tu\|^p_{\mathrm{W}_0^{1,p}(\Omega)}\hat{\alpha_1}\,\mathrm{d}t\\ &= \|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)}\hat{\alpha_1}\int_0^1 t^{p-1}\,\mathrm{d}t = \|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)}\frac{\hat{\alpha_1}}{p}.\end{aligned}$$

Similarly we prove the second part. □

We also need a following auxiliary result used in order to prove the main theorem.

Lemma 3.2 (Properties of $\mathrm{F_v}$)**.** *Assume (E), (F), (A), (C), (H). Let $v \in \mathrm{W}_0^{1,q}(\Omega)$ be fixed. The functional $\mathrm{F_v}\colon \mathrm{W}_0^{1,p}(\Omega) \to \mathbb{R}$, given by formula*

$$\mathrm{F_v} := u \mapsto \mathrm{A}(u) + \mathrm{N}(u,v),$$

has a minimizer, is convex and w.l.s.c.

Proof. Let $v \in \mathrm{W}_0^{1,q}(\Omega)$ be fixed. Potential A is convex and l.s.c. Also N is convex and l.s.c. Thus functional $\mathrm{F_v}$ is convex and weakly l.s.c. In order to show that $\mathrm{F_v}$ has a minimizer it suffices to estimate it from below by some coercive functional.

Let $u \in \mathrm{W}_0^{1,p}(\Omega)$, $\hat{\beta}_1^v$ denotes $\|\beta_1^v\|_{\mathrm{L}^2(\Omega)}$ multiplied by a constant from embedding of $\mathrm{W}_0^{1,p}(\Omega) \to \mathrm{L}^2(\Omega)$. By (F) we have

$$\begin{aligned}\mathrm{N}(u,v) &= \int_\Omega N(x,u(x),v(x))\,\mathrm{d}x\\ &\geq \int_\Omega -\alpha_1|u(x)|^p + \beta_1^v(x)u(x) + \gamma_1^v(x)\,\mathrm{d}x\\ &\geq -\alpha_1\|u\|^p_{\mathrm{L}^p(\Omega)} - \|\beta_1^v\|_{\mathrm{L}^2(\Omega)}\|u\|_{\mathrm{L}^2(\Omega)} - \|\gamma_1\|_{\mathrm{L}^1(\Omega)}\\ &\geq -\frac{\alpha_1}{\lambda_{1,p}}\|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)} - \hat{\beta}_1^v\|u\|_{\mathrm{W}_0^{1,p}(\Omega)} - \|\gamma_1\|_{\mathrm{L}^1(\Omega)}.\end{aligned}$$

By (C) and Lemma 3.1 we have

$$\begin{aligned}\mathrm{F_v}(u) &= \mathrm{A}(u) + \mathrm{N}(u,v)\\ &\geq \left(\frac{\hat{\alpha}_1}{p} - \frac{\alpha_1}{\lambda_{1,p}}\right)\|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)} - \hat{\beta}_1^v\|u\|_{\mathrm{W}_0^{1,p}(\Omega)} - \|\gamma_1\|_{\mathrm{L}^1(\Omega)}.\end{aligned}$$

Since $\left(\frac{\hat{\alpha}_1}{p} - \frac{\alpha_1}{\lambda_{1,p}}\right) > 0$ we know that $\mathrm{F_v}$ is bounded from below by a coercive functional. Thus since it is also w.l.s.c. functional, it must have a minimizer, however not necessarily unique. □

Lemma 3.3 (Properties of $\mathrm{G_u}$)**.** *Assume (E), (G), (B), (D), (I). Let $u \in \mathrm{W}_0^{1,p}(\Omega)$ be fixed. The functional $\mathrm{G_u}\colon \mathrm{W}_0^{1,q}(\Omega) \to \mathbb{R}$ given by formula*

$$\mathrm{G_u} := v \mapsto -\mathrm{B}(v) + \mathrm{N}(u,v).$$

has a maximizer (not necessarily unique), is concave and weakly upper semicontinuous (w.u.s.c. for short).

4 Main result – the existence of a saddle point

Theorem 4.1 (Existence of saddle point)**.** *Assume (A)–(I). There exists a solution to Problem 2.*

Lets recall the main abstract result we use:

Theorem 4.2 (Ky–Fan minimax theorem [14, Th. 5.2.2.])**.** *Let X and Y be Hausdorff topological vector spaces, $A \subset X$ and $B \subset Y$ be convex sets and $f\colon A \times B \to \mathbb{R}$ be a function which satisfies the following conditions*

(i) for each $z_2 \in B$ the function $z_1 \mapsto f(z_1, z_2)$ is convex and lower semicontinuous on A;

(ii) for each $z_1 \in A$ the function $z_2 \mapsto f(z_1, z_2)$ is concave and upper semicontinuous on B;

(iii) for some $\hat{z_1} \in A$ and some

$$\delta_0 < \inf_{z_1 \in A} \sup_{z_2 \in B} f(z_1, z_2),$$

the set $\{z_2 \in B : f(\hat{z_1}, z_2) \geq \delta_0\}$ is compact. Then

$$\sup_{z_2} \inf_{z_1} f(z_1, z_2) = \inf_{z_1} \sup_{z_2} f(z_1, z_2).$$

It is almost immediate to have (i) and (ii) fulfilled for our problem. But the hardest part is to obtain the last technical condition.

Proof of Theorem 4.1. First we start by proving (i) and (ii). Lets recall that for all $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$:

$$J(u, v) = A(u) - B(v) + N(u, v).$$

Let us begin with (i). Let $v \in W_0^{1,q}(\Omega)$ be fixed. Since (A) holds by Lemma 2.4, A is convex and w.l.s.c. By (H) and since N is L^1-Carathéodory $u \mapsto N(u, v)$ is convex and w.l.s.c. $B(v)$ is a constant - thus (i) holds. Similarly (ii) holds.

Actually we shall not use Ky–Fan theorem directly for J but for $J|_{A\times B}$ where A, B are some closed balls respectively in $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$. Since J fulfils (i) and (ii), those properties will remain unchanged for $J|_{A\times B}$.

We shall proceed as follows:

1. We shall define two more auxiliary functionals J^+, J^-, and bound each of them by yet another functional.

2. We prove that both
$$\sup_v \inf_u J(u, v) \quad \text{and} \quad \inf_u \sup_v J(u, v)$$
are attained.

3. We prove that each minimax argument must lie within balls of certain radius.

4. We deduce a suitable constant δ and show the compactness of the required set.

We consider the following functional $J^-\colon W_0^{1,q}(\Omega) \to \mathbb{R}$ given by the formula

$$J^- := v \mapsto \min_{u \in W_0^{1,p}(\Omega)} J(u, v).$$

We shall prove that that this functional is: well defined, concave and w.u.s.c. and anticoercive.

Let start with fixing $v \in W_0^{1,q}(\Omega)$. Then we see that $u \mapsto J(u,v)$ differs from F_v by only a constant element $-B(v)$. Then by Lemma 3.2 a minimum must be attained. Since $v \in W_0^{1,q}(\Omega)$ was arbitrary thus J^- is well defined.

Let fix $u \in W_0^{1,p}(\Omega)$. Then we see that $v \mapsto J(u,v)$ differs from G_u by only a constant element $A(u)$. Then by Lemma 3.3 each of such functionals must be u.s.c. and concave. Then by Corollaries 2.2 and 2.2 its is clear that J^- is u.s.c. and concave.

Let $v \in W_0^{1,q}(\Omega)$. By Assumption (G) and (D) we have

$$\begin{aligned} J^-(v) &\leq J(0,v) \\ &= -\int_0^1 \langle \mathcal{B}(tv); v\rangle \, dt + \int_\Omega N(x,0,v(x)) \, dx \\ &\leq \left(\frac{\alpha_2}{\lambda_{1,q}} - \frac{\hat{\alpha_2}}{q}\right) \|v\|^q_{W_0^{1,q}(\Omega)} + \hat{\beta}_2^0 \|v\|_{W_0^{1,q}(\Omega)} + \|\gamma_2^0\|_{L^1(\Omega)} . \end{aligned}$$

Since $\left(\frac{\alpha_2}{\lambda_{1,q}} - \frac{\hat{\alpha_2}}{q}\right) < 0$, J^-, it follows that is anticoercive.

Since J^- is concave, u.s.c. (weakly) and anticoercive it must attain a maximum. Thus there must exist a pair $(\hat{u}, \hat{v})$ such that

$$\sup_v \inf_u J(u,v) = J(\hat{u},\hat{v}) .$$

Lets use the previous estimate to define a functional $j^-\colon W_0^{1,q}(\Omega) \to \mathbb{R}$:

$$J^-(v) \leq \left(\frac{\alpha_2}{\lambda_{1,q}} - \frac{\hat{\alpha_2}}{q}\right) \|v\|^q_{W_0^{1,q}(\Omega)} + \hat{\beta}_2^0 \|v\|_{W_0^{1,q}(\Omega)} + \|\gamma_2^0\|_{L^1(\Omega)} =: j^-(v) .$$

It is obviously a concave, continuous and anticoercive functional. Similarly we can define the following functional $J^+\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ given by the formula

$$J^+(u) = \max_{v \in W_0^{1,q}(\Omega)} J(u,v).$$

By using the same argument we prove that that this functional is well defined, convex, w.l.s.c., coercive.

Let us fix $u \in W_0^{1,p}(\Omega)$. Then we see that $v \mapsto J(u,v)$ differs from G_u by only a constant element $A(u)$. Then by Lemma 3.3 a maximum must be attained. So J^+ is well defined since u was set arbitrary.

Let set $v \in W_0^{1,q}(\Omega)$. Then we see that $u \mapsto J(u,v)$ differs from F_v by only a constant element $-B(v)$. Then by Lemma 3.2 each of such functionals must be w.l.s.c. and convex. Then by Lemmas 2.1 and 2.1 its is clear that J^+ is w.l.s.c. and convex.

Let $u \in W_0^{1,p}(\Omega)$. By Assumption (F) and (C) we have

$$\begin{aligned} J^+(u) &\geq J(u,0) \\ &= A(u) + N(u,0) \\ &\geq \left(\frac{\hat{\alpha_1}}{p} - \frac{\alpha_1}{\lambda_{1,p}}\right) \|u\|^p_{W_0^{1,p}(\Omega)} - \hat{\beta}_1^0 \|u\|_{W_0^{1,p}(\Omega)} + \hat{\gamma}_1^0. \end{aligned}$$

Where $\hat{\beta_1^0}$ and $\hat{\gamma_1^0}$ are some nonnegative constants. Since $\left(\frac{\hat{\alpha_1}}{p} - \frac{\alpha_1}{\lambda_{1,p}}\right) > 0$, it follows J^+ is coercive.

Since J^+ is convex, l.s.c. (weakly) and coercive it must attain a minimum. Thus there must exist a pair $(\hat{u}, \hat{v})$ which satisfies that

$$\inf_u \sup_v \mathrm{J}\left(u, v\right) = \mathrm{J}\left(\hat{u}, \hat{v}\right).$$

Let us use the previous estimate to define a functional $j^+ \colon \mathrm{W}_0^{1,p}\left(\Omega\right) \to \mathbb{R}$

$$\mathrm{J}^+\left(u\right) \geq \left(\frac{\alpha_1}{\lambda_{1,p}} - \frac{\hat{\alpha_1}}{p}\right) \|u\|^q_{\mathrm{W}_0^{1,p}(\Omega)} + \hat{\beta_1^0} \|u\|_{\mathrm{W}_0^{1,p}(\Omega)} + \left\|\gamma_1^0\right\|_{\mathrm{L}^1(\Omega)} =: j^+\left(u\right).$$

It is a continuous, coercive and convex functional.

Now we shall focus on the balls which contain all the minimax points. Assume that $(\bar{u}, \bar{v}) \in \mathrm{W}_0^{1,p}\left(\Omega\right) \times \mathrm{W}_0^{1,q}\left(\Omega\right)$ is a pair such that

$$\mathrm{J}(\bar{u}, \bar{v}) = \max_v \min_u \mathrm{J}(u, v).$$

Then

$$\begin{aligned} \mathrm{J}^-(\bar{v}) &\geq \mathrm{J}^-(0) = \min_u \mathrm{J}(u, 0) \\ &\geq \min_u j^+(u). \end{aligned}$$

Minimum of a coercive functional is in this case obviously a finite number which we shall denote as δ_2.

Similarly assume $(\bar{u}, \bar{v}) \in \mathrm{W}_0^{1,p}\left(\Omega\right) \times \mathrm{W}_0^{1,q}\left(\Omega\right)$ be a point such that

$$\mathrm{J}(\bar{u}, \bar{v}) = \min_u \max_v \mathrm{J}(u, v).$$

Then

$$\begin{aligned} \mathrm{J}^+(\bar{u}) &\leq \mathrm{J}^+(0) = \max_v \mathrm{J}(0, v) \\ &\leq \max_v j^-(v). \end{aligned}$$

Maximum of an anticoercive functional is in this case obviously a finite number which we shall denote as δ_1.

Assume that $(\bar{u}, \bar{v}) \in \mathrm{W}_0^{1,p}\left(\Omega\right) \times \mathrm{W}_0^{1,q}\left(\Omega\right)$ is a pair such that

$$\mathrm{J}(\bar{u}, \bar{v}) = \max_v \min_u \mathrm{J}(u, v) = \min_u \max_v \mathrm{J}(u, v).$$

Then

$$\begin{aligned} \bar{v} &\in \left\{v \in \mathrm{W}_0^{1,q}\left(\Omega\right) : \mathrm{J}^-(v) \geq \delta_2\right\} \\ &\subset \left\{v \in \mathrm{W}_0^{1,q}\left(\Omega\right) : j^-(v) \geq \delta_2\right\}. \end{aligned}$$

The set $\left\{v \in \mathrm{W}_0^{1,q}\left(\Omega\right) : j^-(v) \geq \delta_2\right\}$, since j^- is anticoercive, must be a bounded one. Thus one could choose such a radius r_2 that zero-centred ball $B(r_2) \supset \left\{v \in \mathrm{W}_0^{1,q}\left(\Omega\right) : j^-(v) \geq \delta_2\right\} \ni \bar{v}$. Also

$$\begin{aligned} \bar{u} &\in \left\{u \in \mathrm{W}_0^{1,p}\left(\Omega\right) : \mathrm{J}^+(u) \leq \delta_1\right\} \\ &\subset \left\{u \in \mathrm{W}_0^{1,p}\left(\Omega\right) : j^+(u) \leq \delta_1\right\}. \end{aligned}$$

Then again, the set $\{u \in W_0^{1,p}(\Omega) : j^+(u) \le \delta_1\}$, since j^+ is coercive, must be bounded. Thus one could choose such a radius r_1 that zero-centred ball

$$B(r_1) \supset \left\{u \in W_0^{1,p}(\Omega) : J^+(u) \le \delta_1\right\} \ni \bar{u}.$$

It follows that $(\bar{u}, \bar{v}) \in B(r_1) \times B(r_2)$. So if we restrict the domain of J to $B(r_1) \times B(r_2)$ we will not exclude any solution to Problem 2.

Finally we deduce (iii). Take $\hat{z_1} = 0$, $A = B(r_1)$, $B = B(r_2)$ and $\delta_0 < \delta_2$. Then

- $\hat{z_1} \in A$ obviously holds.
- It follows that:

$$\min_u \max_v \mathrm{J}(u,v) \ge \min_u \mathrm{J}(u,0) \ge \min_u j^+(u) = \delta_2 > \delta_0.$$

- And finally

$$\{v \in B : \mathrm{J}(0,v) \ge \delta_0\} \subset \left\{v \in B : j^-(v) \ge \delta_0\right\},$$

  is bounded since j^- is anticoercive and weakly closed (since $J(0,v)$ is concave and w.u.s.c). Thus by Banach–Alouglu theorem, and since a closed subset of compact set is compact - it is a weakly compact set. All the requirements of the Ky–Fan minimax theorem are fulfilled, so there exists $\hat{u} \in W_0^{1,p}(\Omega)$, and $\hat{v} \in W_0^{1,q}(\Omega)$ that

$$\max_v \min_u \mathrm{J}(u,v) = \min_u \max_v \mathrm{J}(u,v) = \mathrm{J}(\hat{u},\hat{v}).$$

  This concludes the proof. □

The following corollary follows instantly from the prove above.

Corollary 4.3. *Assume that we replace convexity with strict convexity in (H) and concavity with strict concavity in (I). If we also assume conditions (A)–(G) from Theorem 4.1 then Problem 2 has exactly 1 solution.*

Assumptions (F), (G) can obviously have a stronger form, but without the upper bound requirement on constants $\hat{\alpha_1}, \hat{\alpha_2}$.

(F1) for each $v \in W_0^{1,q}(\Omega)$ there exists functions $\beta_1 \in L^2(\Omega)$, $\gamma_1 \in L^1(\Omega)$, $1 < \hat{p} < p$ and $\alpha_1 \in \mathbb{R}^+$ that

$$N(x,u,v(x)) \ge -\alpha_1|u|^{\hat{p}} + \beta_1(x) \cdot u(x) + \gamma_1(x),$$

for almost every $x \in \Omega$ and all $u \in \mathbb{R}$.

(G1) for each $u \in W_0^{1,p}(\Omega)$ there exists functions $\beta_2 \in L^2(\Omega)$, $\gamma_2 \in L^1(\Omega)$ $1 < \hat{q} < q$ and $\alpha_2 \in \mathbb{R}^+$ that

$$N(x,u(x),v) \le \alpha_2|v|^{\hat{q}} + \beta_2(x) \cdot v(x) + \gamma_2(x),$$

for almost every $x \in \Omega$ and all $v \in \mathbb{R}$.

Corollary 4.4. *Assume (A), (B), (C), (D), (E), (F1), (G1), (H), (I). Then Problem 2 has a solution.*

It is easy to check that each step of proof to Theorem 4.1 can be used with the above setting.

5 Example

Lets consider a constants [12] λ_p and λ_q which are the first nonlinear eigenvalues of $-\Delta_p$ and $-\Delta_q$ respectively, namely

$$\lambda_p = \min_{u \in W_0^{1,p}([0,1]), u \neq 0} \frac{\int_0^1 |u'(x)|^p \, dx}{\int_0^1 |u(x)|^p \, dx}, \quad \lambda_q = \min_{u \in W_0^{1,q}([0,1]), u \neq 0} \frac{\int_0^1 |u'(x)|^q \, dx}{\int_0^1 |u(x)|^q \, dx}.$$

Example 5.1. Lets $p = 6, q = 4$, and $\Omega = [0,1]$. We consider the system of the following form

$$\begin{aligned} \int_0^1 |u'(t)|^{p-2} u'(t) \bar{u}'(t) \, dt &= -\frac{\lambda}{2p} \int_0^1 \left(|u(t)|^{p-2} u(t) + v(t) \right) \bar{u}(t) \, dt, \\ \int_0^1 |v'(t)|^{q-2} v'(t) \bar{v}'(t) \, dt &= \frac{\lambda}{2p} \int_0^1 \left(|v(t)|^{q-2} v(t) - u(t) \right) \bar{v}(t) \, dt \end{aligned} \tag{Ex1}$$

for all $\bar{u} \in W_0^{1,p}([0,1]), \bar{v} \in W_0^{1,q}([0,1])$. We consider a functional which critical points corresponds to solution Problem (Ex1). Such a functional has a form:

$$J(u,v) = \frac{1}{p} \int_0^1 |u'(t)|^p \, dt - \frac{1}{q} \int_0^1 |v'(t)|^q \, dt + \int_0^1 \frac{\lambda}{2p} \left(\frac{1}{p} u(t)^p - \frac{1}{q} v(t)^q + u(t)v(t) \right) dt.$$

We shall apply Theorem 4.1 to prove the existence of a critical point (saddle point) to this functional. Lets check all the required assumptions

(A), (B) Negative p-Laplace operator $(-\Delta_p)$ is know to be potential and monotone.

(C), (D) the conditions are fulfilled with $\hat{\alpha}_1 = \frac{1}{p}$ and $\hat{\alpha}_2 = \frac{1}{q}$.

(E) is obviously fulfilled.

(F), (G) If $v \in W_0^{1,q}([0,1])$ then it must be bounded a.e. as a continuous function by a positive constant. Lets check the condition on α_1. It is easy to observe that

$$\alpha_1 := \frac{\lambda}{p^2} \leq \lambda_p \frac{1}{p^2} = \lambda_p \frac{\hat{\alpha}_1}{p} = \lambda_{1,p} \frac{\hat{\alpha}_1}{p}.$$

Thus the condition holds. (G) follows in a similar manner.

(H) With v fixed functional $u \mapsto N(u,v)$ has a plot similar to a function

$$u \mapsto \frac{1}{2} u^p + cu + C.$$

Since its second derivative is nonnegative $(p = 6)$ – it is a convex function.

(I) With u fixed functional $v \mapsto N(u,v)$ has a plot similar to a function

$$v \mapsto -\frac{1}{2} v^q + cv + C.$$

Since its second derivative is nonpositive $(q = 4)$ – it is a concave function.

Thus from Theorem 4.1 it follows that Problem (Ex1) admits a solution.

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