

# Analytical estimations of limit cycle amplitude for delay-differential equations

*Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday*

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**Abstract.** The amplitude of limit cycles arising from Hopf bifurcation is estimated for nonlinear delay-differential equations by means of analytical formulas. An improved analytical estimation is introduced, which allows more accurate quantitative prediction of periodic solutions than the standard approach that formulates the amplitude as a square-root function of the bifurcation parameter. The improved estimation is based on special global properties of the system: the method can be applied if the limit cycle blows up and disappears at a certain value of the bifurcation parameter. As an illustrative example, the improved analytical formula is applied to the problem of stick balancing.

**Keywords:** delay-differential equation, Hopf bifurcation, limit cycle, center manifold reduction, normal form theory.

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## 1 Introduction

Delay-differential equations (DDEs) appear in various fields of science. Examples include neural networks [18], human balancing [13], epidemiology [6, 15], control theory [12, 17], wheel shimmy [22], and machine tool vibrations [1, 21], just to mention a few. Investigating the dynamics of systems with time delay is therefore an important field of research. The mathematical analysis of DDEs is complicated by the fact that their phase space is infinite dimensional. The infinite-dimensional nature often yields rich dynamics, including the possibility of periodic, quasi-periodic, and even chaotic solutions [2, 21].

In this paper, we focus on the computation of periodic solutions (limit cycles) of nonlinear DDEs occurring via Hopf bifurcation. An analytical approach is proposed to give an accurate

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estimation of the amplitude of limit cycles. The whole concept is based on the center manifold reduction technique [3, 8, 11], by which a two-dimensional center subsystem can be decomposed from the infinite-dimensional DDE. With the two-dimensional subsystem at hand, the normal form theory of ordinary differential equations (ODEs) can be applied to deduce a polar-form equation, which determines the amplitude of the periodic solutions arising from the Hopf bifurcation. We remark that this polar form can be obtained by other methods as well (see e.g. the method of multiple scales [16]). Based on the polar-form equation, the standard analytical estimation of the limit cycle amplitude is given by a square-root function of the bifurcation parameter [7]. Although the square-root function provides good approximation in the vicinity of the Hopf bifurcation, its accuracy may be insufficient if the bifurcation parameter is far from the bifurcation point. In this paper, a special hyperbolic function is proposed for the limit cycle amplitude by considering special global properties of the DDE. This way, a more accurate analytical prediction of large-amplitude periodic solutions of DDEs can be given.

We consider nonlinear autonomous DDEs of the form

$$\dot{\mathbf{y}}(t) = \int_{-\sigma}^0 d\boldsymbol{\eta}(\theta)\mathbf{y}(t+\theta)d\theta + \mathbf{f}(\mathbf{y}_t). \quad (1.1)$$

The evolution of the system is described by the shift  $\mathbf{y}_t \in \mathcal{H}: [-\sigma, 0] \rightarrow \mathbb{R}^n$ ,  $\mathbf{y}_t(\vartheta) = \mathbf{y}(t + \vartheta)$  defined in the Hilbert space  $\mathcal{H}$  of continuously differentiable vector-valued functions. The integral on the right-hand side of Eq. (1.1), which accounts for the linear terms, is a Stieltjes integral with  $\boldsymbol{\eta}: [-\sigma, 0] \rightarrow \mathbb{R}^{n \times n}$  being a matrix-valued function of bounded variation. The nonlinearities in the system are included in the continuous functional  $\mathbf{f}: \mathcal{H} \rightarrow \mathbb{R}^n$ . In what follows, we assume that a Hopf bifurcation is associated with Eq. (1.1), and analyze the arising limit cycle.

The rest of the paper is organized as follows. Section 2 gives an introduction to center manifold reduction and shows the polar-form equation determining the amplitude of the limit cycle. Section 3 proposes an analytical approach by which the amplitude can be approximated by a higher-order function of the bifurcation parameter. Section 4 demonstrates the results through an example: periodic solutions are computed for the single-degree-of-freedom model of an inverted pendulum subjected to a nonlinear feedback control. Section 5 summarizes the conclusions of the paper.

## 2 Flow on center manifold

In this section, we shortly revise a possible method to derive the polar-form equation that determines the amplitude of limit cycles arising from the Hopf bifurcation associated with Eq. (1.1). The analysis is based on the center manifold reduction technique discussed in [3, 8, 11], which allows us to characterize the long-term dynamics of infinite-dimensional time-delay systems undergoing Hopf bifurcation. The center manifold reduction uses an abstract representation of system (1.1) given by the operator differential equation (OpDE) form

$$\dot{\mathbf{y}}_t = \mathcal{A}\mathbf{y}_t + \mathcal{F}(\mathbf{y}_t), \quad (2.1)$$

where  $\mathcal{A}, \mathcal{F}: \mathcal{H} \rightarrow \mathcal{H}$  are the linear and the nonlinear operators, respectively, defined by

$$\mathcal{A}\mathbf{u} = \begin{cases} \frac{d}{d\vartheta}\mathbf{u}(\vartheta) & \text{if } \vartheta \in [-\sigma, 0), \\ \int_{-\sigma}^0 d\boldsymbol{\eta}(\theta)\mathbf{u}(\theta)d\theta & \text{if } \vartheta = 0, \end{cases} \quad (2.2)$$

$$\mathcal{F}(\mathbf{u}) = \begin{cases} \mathbf{0} & \text{if } \vartheta \in [-\sigma, 0), \\ \mathbf{f}(\mathbf{u}) & \text{if } \vartheta = 0. \end{cases} \quad (2.3)$$

We assume that system (2.1) has a trivial equilibrium  $\mathbf{y}(t) \equiv \mathbf{0}$ . Then, the associated linear system is described by operator  $\mathcal{A}$ , and the eigenvalues of  $\mathcal{A}$  (called as the characteristic exponents) determine the stability and bifurcations of the equilibrium. Let  $p$  denote the bifurcation parameter and assume that a Hopf bifurcation takes place at  $p = p_H$ . When the equilibrium of system (2.1) loses stability via Hopf bifurcation, a complex conjugate pair  $\lambda = \pm i\omega$  of eigenvalues lies on the imaginary axis ( $i^2 = -1, \omega > 0$ ), whereas all the other infinitely many eigenvalues of operator  $\mathcal{A}$  are located in the left half of the complex plane. Accordingly, a two-dimensional center manifold embedded in the infinite-dimensional phase space attracts the solutions of the differential equation. The long-term dynamics of the system is therefore determined by the flow on the two-dimensional center manifold. The flow on this manifold can be analyzed by decomposing the two-dimensional center subsystem. This procedure is referred to as center manifold reduction, and it can be done using the decomposition theorem given by Eqs. (3.10) and (3.11) in Chapter 7 of [8].

Here, we do not present all the details of the decomposition, we just remark that the center manifold reduction technique uses the operator  $\mathcal{A}^*$ , which is formally adjoint to operator  $\mathcal{A}$  relative to a certain bilinear form. We rather focus on the analysis of the center subsystem. The center manifold reduction allows us to decouple the two-dimensional center subsystem from the infinite-dimensional time-delay system to obtain the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{\mathbf{y}}_{tn} \end{bmatrix} = \begin{bmatrix} 0 & \omega & \mathcal{O} \\ -\omega & 0 & \mathcal{O} \\ \mathbf{o} & \mathbf{o} & \mathcal{A} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \mathbf{y}_{tn} \end{bmatrix} + \begin{bmatrix} g_1(z_1, z_2, \mathbf{y}_{tn}) \\ g_2(z_1, z_2, \mathbf{y}_{tn}) \\ \mathcal{G}(z_1, z_2, \mathbf{y}_{tn}) \end{bmatrix}, \quad (2.4)$$

where  $\mathbf{o}: \mathbb{R} \rightarrow \mathcal{H}$  is a zero operator,  $\mathcal{O}: \mathcal{H} \rightarrow \mathbb{R}$  is a zero functional. Here,  $z_1$  and  $z_2$  denote the two local coordinates on the center manifold, whereas  $\mathbf{y}_{tn}$  represents the remaining infinite-dimensional component of  $\mathbf{y}_t$  transverse to the center subspace. The first term on the right-hand side of Eq. (2.4) is linear, while  $g_1, g_2: \mathbb{R} \times \mathbb{R} \times \mathcal{H} \rightarrow \mathbb{R}$  and  $\mathcal{G}: \mathbb{R} \times \mathbb{R} \times \mathcal{H} \rightarrow \mathcal{H}$  contain all nonlinear terms. Parameter  $\omega$  gives the approximate angular frequency of the arising periodic solutions.

Note that the two-dimensional center subsystem described by  $z_1$  and  $z_2$  is decoupled only on the linear level from the remaining infinite-dimensional stable subsystem described by  $\mathbf{y}_{tn}$ . There is still a coupling through the nonlinear terms  $g_1$  and  $g_2$ . In order to fully decouple the two-dimensional center subsystem in the first two rows of Eq. (2.4), the dynamics must be restricted to the center manifold. This manifold is embedded in the infinite-dimensional phase space and can be given in the form  $\mathbf{y}_{tn} = \mathbf{y}_{tn}^{\text{CM}}(z_1, z_2)$ . The expansion of the center manifold  $\mathbf{y}_{tn}^{\text{CM}}(z_1, z_2)$  into Taylor series in terms of  $z_1$  and  $z_2$  allows us to construct a third-order approximation of the decoupled center subsystem in the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \sum_{j+k=2,3} a_{jk} z_1^j z_2^k \\ \sum_{j+k=2,3} b_{jk} z_1^j z_2^k \end{bmatrix}. \quad (2.5)$$

That is, the nonlinearity is approximated only by quadratic and cubic terms, which is necessary and suitable for bifurcation analysis. Here, we do not investigate the effect of higher-order nonlinearities.

Finally, using near-identity transformation, the center subsystem (2.5) with quadratic and cubic nonlinearity can be transformed into a simple polar form. In the vicinity of the Hopf bifurcation taking place at  $p = p_H$ , the polar-form system reads

$$\dot{r} = r(\sigma(p) + \delta(p)r^2), \quad (2.6)$$

$$\dot{\varphi} = \alpha(p) + \beta(p)r^2, \quad (2.7)$$

where  $r$  and  $\varphi$  are the amplitude and the phase variable, respectively. The coefficients  $\sigma(p)$ ,  $\alpha(p)$ ,  $\delta(p)$ , and  $\beta(p)$  are functions of the bifurcation parameter  $p$ . Actually,  $\sigma(p)$  is the real and  $\alpha(p)$  is the imaginary part of the eigenvalues  $\lambda = \sigma(p) \pm i\alpha(p)$  that cross the imaginary axis during the Hopf bifurcation. Consequently, at the Hopf bifurcation, i.e., at  $p = p_H$ , the real part is zero:  $\sigma_H := \sigma(p_H) = 0$ , and the imaginary part is equal to parameter  $\omega$ :  $\alpha(p_H) = \omega$ . Besides, parameter  $\delta(p)$  is related to the so-called Poincaré-Lyapunov constant (PLC):  $\delta_H := \delta(p_H)$ . The criticality of the Hopf bifurcation is determined by the sign of the PLC: the bifurcation is subcritical for  $\delta_H > 0$  and supercritical for  $\delta_H < 0$ . A limit cycle arising from the Hopf bifurcation is associated with the nontrivial equilibrium of Eq. (2.6):

$$r^2(p) = -\frac{\sigma(p)}{\delta(p)}. \quad (2.8)$$

The limit cycle is stable (attractive) if the bifurcation is supercritical and unstable (repelling) if it is subcritical.

### 3 Analysis of limit cycles

As indicated in Eq. (2.8), the limit cycle amplitude  $r$  is a function of the bifurcation parameter  $p$ . In this section, we propose methods to accurately estimate this function. The standard approach [7] is to expand the parameter  $\sigma(p)$  into Taylor series in terms of  $p$  up to first order and approximate the parameter  $\delta(p)$  by constant (by the PLC). According to Eq. (2.8), this yields a linear function for  $r^2(p)$  and hence a square-root function for the amplitude  $r(p)$ :

$$r^{\text{stan}}(p) = \sqrt{-\frac{\sigma'_H(p - p_H)}{\delta_H}}, \quad (3.1)$$

where prime indicates differentiation with respect to  $p$ , subscript H refers to the substitution  $p = p_H$ , and we used  $\sigma_H = 0$ . The calculation of  $\sigma'_H$  (and also the higher derivatives of  $\sigma$ ) is possible via the implicit differentiation of the characteristic equation  $\text{Ker}(\mathcal{A} - \lambda\mathcal{I}) \neq \{0\} \Leftrightarrow D(\lambda) = 0$  defining the characteristic exponents, where  $\mathcal{I}$  is the unit operator and  $D(\lambda)$  is the characteristic function. After the implicit differentiation of  $D(\lambda) = 0$  with respect to  $p$ , and after the substitution of  $p = p_H$  and  $\lambda = \pm i\omega$ , the derivative  $\sigma'_H = \Re(\lambda'_H)$  can be expressed analytically. The PLC  $\delta_H$  can also be calculated by a closed-form formula, see [9].

From this point on, we refer to Eq. (3.1) as *standard analytical estimation* as indicated by the superscript of  $r$ . The standard estimation is accurate only for  $p \approx p_H$ , and it may become inaccurate if  $p$  lies farther from  $p_H$ . The standard estimation can be improved if the following special global properties hold for the system.

1. Without loss of generality, let us assume that the point where the special properties hold is at  $p = 0$ . Note that  $p = 0$  does not have to lie in the vicinity of  $p = p_H$ .

2. Assume that  $r^{\text{stan}}(0)$  exists and is real, but the time-delay system (1.1) in fact does not have a periodic solution for  $p = 0$ . That is, the periodic solution vanishes when changing  $p$  from  $p_H$  to 0, which is not described by the standard analytical estimation (3.1).
3. Furthermore, assume that the Hopf bifurcation is unique in the sense that no other Hopf bifurcation takes place between 0 and  $p_H$ .
4. Finally, assume that the actual amplitude  $r(p)$  is a bijective function of the bifurcation parameter.

These assumptions imply that the periodic solution can only vanish at  $p = 0$  by fulfilling

$$\lim_{p \rightarrow 0} r(p) = \infty. \quad (3.2)$$

Now we propose two methods by which the standard analytical estimation (3.1) can be improved if the above assumptions hold.

The first method to improve the standard estimation is by means of the expansion of  $\sigma(p)$  and  $\delta(p)$  into higher-order Taylor polynomials. It requires the calculation of the derivatives of  $\sigma$  and  $\delta$  at  $p = p_H$ . The higher derivatives of  $\sigma$  can be calculated by implicit differentiation in a similar manner to  $\sigma'_H$ , however, the derivatives of  $\delta$  cannot easily be obtained. Still, we can use this approach by taking advantage of property (3.2). The simplest way to improve the analytical estimation is the approximation of both  $\sigma(p)$  and  $\delta(p)$  by a linear Taylor polynomial:

$$r(p) \approx \sqrt{-\frac{\sigma'_H(p - p_H)}{\delta_H + \delta'_H(p - p_H)}}. \quad (3.3)$$

The unknown coefficient  $\delta'_H$  can be determined so that property (3.2) is fulfilled. Equivalently, the denominator on the right-hand side of Eq. (3.3) must be zero for  $p = 0$ . This yields the coefficient  $\delta'_H = \delta_H / p_H$  and the *improved analytical estimation*

$$r^{\text{impr}}(p) = \sqrt{-\frac{\sigma'_H(p - p_H)}{\frac{\delta_H}{p_H} p}}. \quad (3.4)$$

The same result can be obtained by a slightly different approach. This second method also incorporates the global property (3.2), and fits a bifurcation curve to the limit cycle amplitude by considering the behavior away from the bifurcation (at  $r \rightarrow \infty$ ). However, now we approximate  $r^2(p)$  by a hyperbola as

$$r^2(p) \approx a_0 + \frac{a_1}{p} \quad (3.5)$$

rather than expanding  $\sigma(p)$  and  $\delta(p)$  into polynomials. The core idea is that Eq. (3.5) is the simplest function, which ensures the automatic fulfillment of Eq. (3.2). The coefficients  $a_0$  and  $a_1$  can be calculated based on Eq. (2.8) and using  $\sigma_H = 0$ , which give

$$r^2(p_H) = 0, \quad (3.6)$$

$$\frac{dr^2}{dp}(p_H) = -\frac{\sigma'_H}{\delta_H}. \quad (3.7)$$

This way, Eqs. (3.5), (3.6), and (3.7) imply

$$a_0 = -\frac{\sigma'_H p_H}{\delta_H}, \quad (3.8)$$

$$a_1 = \frac{\sigma'_H p_H^2}{\delta_H}, \quad (3.9)$$

which yields the improved analytical estimation

$$r^{\text{impr}}(p) = \sqrt{-\frac{\sigma'_H p_H}{\delta_H} \left(1 - \frac{p_H}{p}\right)}. \quad (3.10)$$

Note that estimation (3.10) is exactly the same as that in Eq. (3.4).

## 4 Applications

The idea of the improved analytical estimation (3.4) for the amplitude of limit cycles originates in the problem of regenerative machine tool vibrations in metal cutting [4, 20], where Hopf bifurcation occurs due to the nonlinearity of the cutting force characteristics and gives rise to an unstable limit cycle in the vicinity of the linearly stable equilibrium. Correspondingly, an unsafe zone exists in the plane of the technological parameters, where the equilibrium is only locally (but not globally) stable. From practical point of view, it is important to avoid these unsafe parameters, therefore in [14] we used the above approach for the special case of machine tool vibrations to accurately estimate large-amplitude periodic solutions and to accurately compute the unsafe zone. The present paper generalizes the results of [14] for a class of DDEs and provides the theoretical background, as the approach is not restricted to the problem of machine tool vibrations. In the metal cutting example, the difference of the delayed and the actual variable appears in the governing equation. Similar expressions can be found for example in the Pyragas control strategy [17] and in the Duffing oscillator with delayed feedback [23]. The method in Section 3 may help in the bifurcation analysis of these systems. As an additional example, now we demonstrate the application of formula (3.4) on a model of stick balancing: an inverted pendulum subjected to a nonlinear proportional-derivative (PD) controller is investigated.

The equation of motion related to the single-degree-of-freedom model of an inverted pendulum with delayed PD control can be written in the form [10]

$$\ddot{x}(t) - ax(t) = -p(x(t-\tau) + x^3(t-\tau)) - d\dot{x}(t-\tau). \quad (4.1)$$

Accordingly, the inverted pendulum is represented by an unstable second-order system ( $a > 0$ ) on the left-hand side. The expression on the right-hand side is originated in the control force exerted by the PD controller in order to maintain the stick at its upright position (at the equilibrium  $x(t) \equiv 0$ ). Parameters  $p$  and  $d$  are the proportional and the derivative control gains, respectively, whereas  $\tau$  represents the delay in the control loop. The proportional term on the right-hand side has a cubic nonlinearity, which can be considered as a simplified smooth model of sensory dead zone [13].

Stability and bifurcation analysis of Eq. (4.1) was carried out for  $\tau = 1$ ,  $a = 0.1$ , and  $d = 1$ , using  $p$  as bifurcation parameter. The equilibrium  $x(t) \equiv 0$  is linearly stable for  $a < p < p_H$ . At  $p = a = 0.1$ , a fold bifurcation occurs, whereas at  $p = p_H = 0.5983$ , a subcritical Hopf

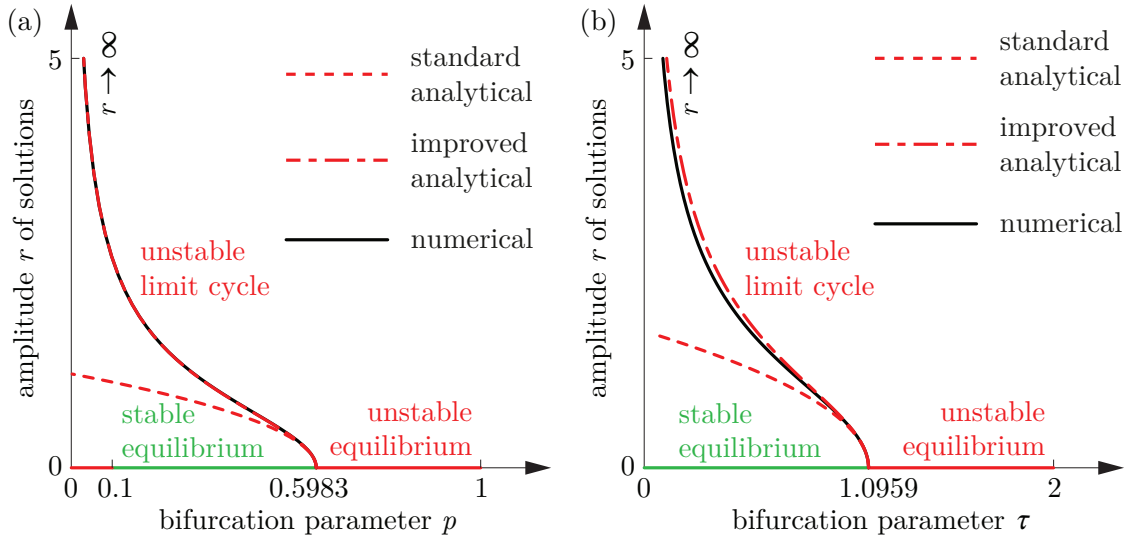


Figure 4.1: Bifurcation diagrams showing the amplitude of limit cycles.

bifurcation takes place. Due to the subcritical Hopf bifurcation, an unstable limit cycle exists for  $0 < p \leq p_H$ . The bifurcation scenario is shown in Fig. 4.1 (a). Solid line indicates the numerical result  $r^{\text{num}}(p)$  for the limit cycle amplitude obtained by the continuation software DDE-BIFTOOL [5]. Here,  $r^{\text{num}}$  stands for the half of the peak-to-peak amplitude of the non-harmonic periodic solution. According to numerical continuation, the limit cycle blows up at  $p = 0$ , which implies that property (3.2) holds and, along with this, the bijective property of  $r(p)$  and the uniqueness of the Hopf bifurcation for  $0 < p \leq p_H$  is fulfilled. Therefore, the improved analytical estimation  $r^{\text{impr}}(p)$  given by Eq. (3.4) can be computed as shown by the dash-dot line in the figure. The parameters  $\sigma'_H = 0.6976$  and  $\delta_H = 0.3173$  were determined numerically by DDE-BIFTOOL, although they can also be derived analytically. Dashed line shows the standard analytical estimation  $r^{\text{stan}}(p)$  given by Eq. (3.1).

As shown in Fig. 4.1 (a), the standard analytical estimation is accurate and agrees well with the numerical results only in the vicinity of the Hopf bifurcation at  $p = p_H$ . For zero proportional gain ( $p = 0$ ), the nonlinearity on the right-hand side of Eq. (4.1) vanishes. Since the resulting linear system does not have a periodic solution, the limit cycle vanishes (blows up) at  $p = 0$ . The standard analytical estimation fails to capture this phenomenon as  $r^{\text{stan}}(p)$  exists and is real at  $p = 0$ . In contrast, the improved analytical estimation captures the blow-up phenomenon at  $p = 0$ , and is everywhere in a very good agreement with the numerical results: the dash-dot curve overlaps with the solid line.

Similarly, Fig. 4.1 (b) shows the bifurcation diagram with the same color scheme for  $a = 0.1$ ,  $p = 0.5$ ,  $d = 1$ , and with bifurcation parameter  $\tau$ . The corresponding parameters are  $\tau_H = 1.0959$ ,  $\sigma'_H = 0.6776$ , and  $\delta_H = 0.2668$ . Again, the periodic solution blows up at  $\tau = 0$ , since the corresponding delay-free system does not have a limit cycle. The standard analytical estimation again fails to capture this phenomenon and the dashed curve deviates from the numerical bifurcation curve, while the improved analytical estimation agrees well with numerics.

Consequently, we can use the improved analytical estimation (3.4) to give an accurate analytical prediction of the unstable oscillations, even at large amplitudes. The unstable large-amplitude oscillations are important, because they determine the basin of attraction of the stable equilibrium and affect global stability.

## 5 Conclusions

In this paper, the analytical estimation of the amplitude of limit cycles arising from Hopf bifurcation was addressed for nonlinear DDEs. The standard analytical estimation (3.1) known from the literature describes the amplitude as a square-root function of the bifurcation parameter. Here, an improved analytical estimation (3.4) was proposed to estimate the amplitude more accurately for large-amplitude vibrations outside the vicinity of the bifurcation. The method uses the concept of center manifold reduction and the polar-form equation of the two-dimensional center subsystem. From the polar-form equation, the limit cycle amplitude is expressed as a special hyperbolic function of the bifurcation parameter, whose coefficients can be determined using some special global properties of the system. Namely, the limit cycle must disappear at a certain value of the bifurcation parameter, where the amplitude according to the standard analytical estimation exists and is real. The application of the method was demonstrated through the example of the inverted pendulum subjected to a nonlinear feedback control.

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