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# On differential equations with state-dependent delay: The principles of linearized stability and instability revisited

*Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday*

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**Abstract.** This paper deals with a dynamical systems approach for studying nonlinear autonomous differential equations with bounded state-dependent delay. Starting with the semiflow generated by solutions of such an equation, we revisit the principles of linearized stability and instability enabling the local stability analysis of equilibria via linearization. In particular, we prove both principles in an elementary way by using only a quantitative version of continuous dependence of the semiflow on initial data together with basic properties of the discrete semi-dynamical system induced by iterations of some time-$t$-map.

**Keywords:** linearized stability, linearized instability, state-dependent delay, stability analysis, functional differential equation.

**2010 Mathematics Subject Classification:** 34K20, 34K21.

## 1 Introduction

Despite the fact that the first studies of differential equations with state-dependent delay may be dated back at least to the beginning of the 19th century, this type of equations became a subject of broader research activity in mathematics and other sciences only during the last sixty years. Particularly in about the past two decades there was a significantly increasing interest in differential equations with state-dependent delay, whereas in earlier times there were only a few studies of such equations as, for instance, carried out by Driver in his pioneering work [4–7], or some years later by Nussbaum in [15] or by Alt in [1,2].

In recent times more and more applications in numerous branches of science such as in mechanics (e.g., Insperger et al. [11]), population dynamics (e.g., Arino et al. [3]), infectious diseases (e.g., Qesmi et al. [16]), or in economy (e.g., [19]) were reported. Furthermore, at the beginning of this century the work [23] of Walther initiated the development of a general

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theory to study differential equations with (bounded) state-dependent delay from dynamical systems point of view. In [23] Walther shows the existence of a continuous semiflow with continuously differentiable time-$t$-maps for a class of abstract functional differential equations; and that is done under smoothness conditions which are typically satisfied when the right-hand side of the functional differential equation represents a differential equation with (bounded) state-dependent delay in a more abstract form.

In about last ten years, the semiflow from [23] and its properties were analyzed in various studies, and by now different concepts and methods from dynamical systems theory are well known. For instance, the survey paper [10] of Hartung et al. provides a detailed exposition of the linearization of the semiflow at equilibria and its spectral properties. Furthermore, [10] also contains a discourse on the existence of continuously differentiable local stable, center and unstable manifolds at an equilibrium. The existence of $C^1$-smooth local center-stable manifolds is discussed in Qesmi and Walther [17], whereas [18] shows the existence and [20] an attraction property of $C^1$-smooth local center-unstable manifolds. These two last-mentioned types of local invariant manifolds may also be used in order to give an alternative proof for the existence of local center manifolds as done in [22]. However, with respect to applications, the most significant results are certainly the so-called principles of linearized stability and instability.

Indeed, given a differential equation arising as a mathematical model in some application and describing the time evolution of the associated system, one of the first questions is the one about the existence of solutions; and the most simple kind of solutions are equilibria. Next, having determined the equilibria, it is natural to ask about their stability properties, and the most common technique here is to consider the linearized equation and its spectral properties. If all eigenvalues of the linearized equation have negative real part then the principle of linearized stability asserts that the equilibrium under consideration is locally asymptotically stable. On the other hand, if the linearized equation has some eigenvalues with positive real part then the principle of linearized instability asserts that the equilibrium under consideration is unstable.

Before the semiflow concept from Walther [23], it was unclear how to linearize a differential equation with state-dependent delay. And so, for a long time, heuristic or formal techniques were used to address the local stability analysis of differential equations with state-dependent delay by linearization. Here, the study [12] of Cooke and Huang or the studies [8, 9] of Hartung and Turi are indicative: after "freezing" the delay at some equilibrium, they linearize the resulting differential equation and then study the local stability of the equilibrium by means of the obtained formal linear equation with constant delay. However, as the discussion about the linearization at equilibria in Hartung et al. [10] substantiates this heuristic approach, the studies [8,9,12] may be considered as the first principles of linearized stability for general classes of differential equations with state-dependent delay. Furthermore, the works [8, 12] also contain the assertion (comp. [8, Remark 3.4] and [12, statement (ii) of Theorem 2.1]) that an equilibrium is unstable, provided the formal linear equation with constant delay has an eigenvalue with positive real part. But in both studies a detailed proof is omitted and only a short outline is sketched. In [13, comp. Section 5] Krisztin revisits the class of delay differential equations from [12] as an example and gives a proof for the corresponding principle of linearized instability.

In the context of the semiflow described above, the principle of linearized stability is stated and shown in Hartung et al. [10, Theorem 3.6.1 in Section 3.6]. It is a straightforward consequence of the discussion about the existence of local stable manifolds at equilibria. Indeed, if the linearization at some equilibrium does not have any center or unstable direction then all initial values sufficiently close to the equilibrium belong to some local stable manifold. Moreover, by shrinking the neighborhood about the equilibrium and using the continuous dependence on initial values together with the invariance and attraction property of local stable manifolds, it is possible to show that the associated solutions do not only stay close to the equilibrium for all $t \geq 0$ but also converge to the equilibrium as $t \to \infty$.

The principle of linearized instability for the semiflow from Walther [23] follows as well, more or less, from the discourse on local invariant manifolds in Hartung et al. [10], despite the fact that the survey work [10] does not contain a corresponding statement or a proof in detail. The point here is as follows: If the linearization at some equilibrium has an eigenvalue with positive real part then, as pointed out in [10], there exist local unstable manifolds of positive dimension at the equilibrium. In particular, such a local unstable manifold contains a solution which is different from the equilibrium, is defined for all $t \leq 0$ and which converges to the equilibrium as $t \to -\infty$. In other words, in this situation we find some neighborhood of the equilibrium with the property that any close vicinity of the equilibrium contains some initial value leading to a solution that leaves the neighborhood under consideration for some positive $t$. Compare here also the work [13] of Krisztin.

The main purpose of this study is now to revisit both, the principle of linearized stability – comp. Theorem 3.1 – and the principle of linearized instability – comp. Theorem 4.1 – for the semiflow from Walther [23], and to give more elementary proofs in detail. To be more precisely, we establish both principles by using only a result about continuous dependence on initial data and the dynamical behavior of the discrete semi-dynamical systems induced by the time-$t$-maps of the semiflow. Such an approach is natural for continuous (semi)-dynamical systems, and was, for instance, used by Diekmann et al. [14] to show analogous results for smooth semiflows generated by autonomous differential equations with constant delays. We adapt the technique from [14], and prove both principles without using the more advanced theory of local invariant manifolds as done in Hartung et al. [10].

It is worth to mention that the situation where the linearization at some equilibrium of the semiflow considered here does not have any eigenvalue with positive real part but at least one eigenvalue on the imaginary axis, and hence where an application of the principle of linearized stability or instability for the local stability analysis fails, was studied in the recent work [21]. In this case the equilibrium has the same stability behavior as the equilibrium of the ordinary differential equations obtained from a center manifold reduction.

The rest of this paper is organized as follows: The next section contains some basic facts about the mentioned semiflow approach from Walther [23] for studying differential equations with (bounded) state-dependent delay. In Section 3 we state and prove the principle of linearized stability whereas Section 4 presents the statement and the proof of the principle of linearized instability.

## 2 Preliminaries

In the following we summarize without proofs the relevant material on studying differential equations with state-dependent delay in the context of dynamical systems theory. For the proofs we refer the reader to Hartung et al. [10] and the references therein.

Throughout this paper, let $h > 0$ and $n \in \mathbb{N}$ be fixed. Further, let $C$ denote the Banach space of all continuous functions $\varphi : [-h, 0] \to \mathbb{R}^n$, equipped with the norm $\|\varphi\|_C = \max_{-h \leq s \leq 0} \|\varphi(s)\|_{\mathbb{R}^n}$ of uniform convergence. Similarly, we write $C^1$ for the Banach space of all continuously differentiable functions $\varphi : [-h,0] \to \mathbb{R}^n$, provided with the norm $\|\varphi\|_{C^1} = \|\varphi\|_C + \|\varphi'\|_C$. If $x$ is any continuous function with values in $\mathbb{R}^n$ and defined on some domain containing the interval $[t-h,t]$, $t \in \mathbb{R}$, then by the *segment* $x_t$ we understand the element of $C$ given by the formula $x_t(s) := x(t+s)$, $s \in [-h,0]$.

**The equation under consideration.** From now on, we consider the functional differential equation

$$x'(t) = f(x_t) \tag{2.1}$$

defined by some function $f : U \to \mathbb{R}^n$ from an open subset $U \subset C^1$ into $\mathbb{R}^n$. In doing so, we assume that the closed subset

$$X_f := \{\psi \in U \mid \psi'(0) = f(\psi)\} \tag{2.2}$$

of $U$ is non-empty and that additionally $f$ satisfies the following standing smoothness conditions:

**(S1)** $f$ is continuously differentiable, and

**(S2)** at each $\varphi \in U$ the derivative $Df(\varphi) : C^1 \to \mathbb{R}^n$ of $f$ at $\varphi$ extends to a linear mapping $D_e f(\varphi) : C \to \mathbb{R}^n$ such that the map

$$U \times C \ni (\varphi, \chi) \mapsto D_e f(\varphi)\chi \in \mathbb{R}^n$$

is continuous.

In particular, the above smoothness assumptions are typically satisfied if Eq. (2.1) represents a differential equation with a (bounded) state-dependent delay. To make this point more clear, consider for simplicity the differential equation

$$x'(t) = g(x(t - r(x(t)))) \tag{2.3}$$

defined by some function $g : \mathbb{R}^n \to \mathbb{R}^n$ and a delay function $r : \mathbb{R}^n \to [0,h]$. Defining the map $\tilde{f} : C^1 \to \mathbb{R}^n$ by $\tilde{f}(\varphi) := g(\varphi(-r(\varphi(0))))$ for all $\varphi \in C^1$ and using the segment notation, we obtain

$$x'(t) = g(x(t - r(x(t)))) = g(x_t(-r(x_t(0))) = \tilde{f}(x_t);$$

that is, the differential equation (2.3) with state-dependent delay takes the more abstract form of Eq. (2.1). Moreover, under the hypothesis that both $g$ and $r$ are continuously differentiable it is not hard to see that $\tilde{f}$ satisfies the smoothness conditions (S1) and (S2). If we now additionally assume that $g(0) = 0$ then the associated set $X_{\tilde{f}}$ is clearly non-empty due to $0 \in X_{\tilde{f}}$. As a consequence, instead of studying Eq. (2.3), we may just as well study Eq. (2.1) under all assumptions imposed above and with $f$ replaced by $\tilde{f}$.

**The continuous semiflow.** A *solution* of Eq. (2.1) is either a globally defined continuously differentiable function $x : \mathbb{R} \to \mathbb{R}^n$ satisfying both $x_t \in U$ and Eq. (2.1) for all $t \in \mathbb{R}$, or a continuously differentiable function $x : [t_0 - h, t_e) \to \mathbb{R}^n$, $t_0 < t_e \leq \infty$, with $x_t \in U$ for all $t_0 \leq t < t_e$ and $x$ satisfies Eq. (2.1) as $t_0 < t < t_e$. Under the assumptions considered here, the question about the existence and uniqueness of solutions for Eq. (2.1) was firstly addressed by Walther in [23]: The set $X_f$ defined by Eq. (2.2) forms a continuously differentiable submanifold of $U$ with codimension $n$, and each $\varphi \in X_f$ uniquely defines some $t_+(\varphi) > 0$ and an in the forward $t$-direction non-continuable solution $x^{\varphi} : [-h, t_+(\varphi)) \to \mathbb{R}^n$ of Eq. (2.1) with initial value $x_0^{\varphi} = \varphi$. All the segments $x_t^{\varphi}$, $\varphi \in X_f$ and $0 \le t < t_+(\varphi)$, belong to the *solution manifold* $X_f$ and the relations

$$F(t,\varphi) := x_t^{\varphi}$$

induce a continuous semiflow $F : \Omega \to X_f$ with domain

$$\Omega := \{(t,\psi) \in [0,\infty) \times X_f \mid 0 \le t < t_+(\psi)\}$$

and with continuously differentiable time-$t$-maps

$$F_t : \{\psi \in X_f \mid 0 \le t < t_+(\psi)\} \ni \varphi \mapsto F(t,\varphi) \in X_f.$$

**Equilibria and their stability properties.** Now, suppose that $\varphi_0 \in X_f$ is an equilibrium of the semiflow $F$; that is, suppose that $F(t,\varphi_0) = \varphi_0$ for all $t \ge 0$. We call $\varphi_0$ *stable* if for each $\varepsilon > 0$ there exists some $\delta(\varepsilon) > 0$ such that for all $\varphi \in X_f$ with $\|\varphi - \varphi_0\|_{C^1} < \delta(\varepsilon)$ we have

$$\|F(t,\varphi) - F(t,\varphi_0)\|_{C^1} = \|F(t,\varphi) - \varphi_0\|_{C^1} < \varepsilon$$

as $0 \le t < t_+(\varphi)$. If the equilibrium $\varphi_0$ is not stable, then $\varphi_0$ is called *unstable*. In terms of neighborhoods of $\varphi_0$, this properties may clearly be characterized as follows: If $\varphi_0$ is stable then, given any neighborhood $V \subset X_f$ of $\varphi_0$, for each initial point $\varphi \in V$, which is sufficiently close to $\varphi_0$, the orbit $\gamma([0,t_+(\varphi))$ of the associated trajectory $\gamma : [0,t_+(\varphi)) \ni t \mapsto F(t,\varphi) \in X_f$ stays in $V$. On the other hand, if $\varphi_0$ is unstable then there exists some neighborhood $V$ of $\varphi_0$ in $X_f$ with the property that for any $\delta > 0$ we find some initial value $\varphi \in V$ with $\|\varphi - \varphi_0\|_{C^1} < \delta$ but $F(t,\varphi) \notin V$ for some $0 < t < t_+(\varphi)$.

The equilibrium $\varphi_0$ is *locally asymptotically stable* if $\varphi_0$ is stable and if in addition there exists some $\varepsilon > 0$ such that for all $\varphi \in X_f$ with $\|\varphi - \varphi_0\|_{C^1} < \varepsilon$ we have $t_+(\varphi) = \infty$ and

$$\|F(t,\varphi) - F(t,\varphi_0)\|_{C^1} = \|F(t,\varphi) - \varphi_0\|_{C^1} \to 0 \quad \text{as } t \to \infty.$$

So, in this case the orbit $\gamma([0,\infty))$ of a trajectory $\gamma : [0,\infty) \ni t \mapsto F(t,\varphi) \in X_f$ with $\varphi$ in close vicinity of $\varphi_0$ does not only stay in a small neighborhood of $\varphi_0$ but is also *attracted* by $\varphi_0$ as $t \to \infty$.

**Remark 2.1.** It is worth to point out that an equilibrium $\varphi_0 \in X_f$ is stable if and only if for each $\varepsilon > 0$ there is some $\delta(\varepsilon) > 0$ such that for all $\varphi \in X_f$ with $\|\varphi - \varphi_0\|_{C^1} < \delta(\varepsilon)$ we have both $t_+(\varphi) = \infty$ and $\|F(t,\varphi) - \varphi_0\|_{C^1} < \varepsilon$ as $0 \le t < \infty$. The one direction of this assertion is obvious, whereas the other immediately follows from Proposition 3.3 in [21] which shows that, provided a solution $x^{\varphi} : [-h, t_+(\varphi)) \to \mathbb{R}^n$, $\varphi \in X_f$, of Eq. (2.1) stays in close vicinity of $\varphi_0$, we necessarily have $t_+(\varphi) = \infty$.

**The linearization and spectrum at an equilibrium.** The tangent space $T_{\varphi_0}X_f$ of the solution manifold $X_f$ at the equilibrium $\varphi_0$ is given by

$$T_{\varphi_0}X_f := \left\{\chi \in C^1 \mid \chi'(0) = Df(\varphi_0)\chi\right\}$$

and it is a Banach space with the norm $\|\cdot\|_{C^1}$ of the greater Banach space $C^1$. The strongly continuous semigroup $\{T(t)\}_{t\ge 0}$ of bounded linear operators $T(t) := D_2F(t,\varphi_0)$, $t \ge 0$, on $T_{\varphi_0}X_f$ forms the linearization of the semiflow $F$ at $\varphi_0$. Given any $\chi \in T_{\varphi_0}X_f$, we have

$$T(t)\chi = D_2F(t,\varphi_0)\chi = v_t^{\chi}$$

with the uniquely determined solution $v^{\chi} : [-h, \infty) \to \mathbb{R}^n$ of the linear initial value problem

$$v'(t) = Df(\varphi_0)v_t, \qquad v(0) = \chi.$$

In particular, we have $0 \in T_{\varphi_0}X_f$ and $T(t)0 = 0$ for all $t \geq 0$; that is, $0 \in T_{\varphi_0}X_f$ forms an equilibrium of the linearization. The infinitesimal generator of the linearization $T$ of $F$ at $\varphi_0$ is defined by the linear operator $G : \mathcal{D}(G) \ni \chi \mapsto \chi' \in T_{\varphi_0}X_f$ with domain

$$\mathcal{D}(G) := \left\{\chi \in C^2 \mid \chi'(0) = Df(\varphi_0)\chi, \quad \chi''(0) = Df(\varphi_0)\chi'\right\}$$

where $C^2$ denotes the set of all twice continuously differentiable functions $\chi : [-h, 0] \to \mathbb{R}^n$.

Now, recall from the standing assumption (S2) on page 4 that the bounded linear operator $L := Df(\varphi_0) : C^1 \to \mathbb{R}^n$ extends to a bounded linear operator $L_e := D_e f(\varphi_0) : C \to \mathbb{R}^n$ on the greater Banach space $C$. In particular, $L_e$ defines the linear retarded functional differential equation

$$v'(t) = L_e v_t.$$

As, for instance, discussed in Diekmann et al. [14], for each $\chi \in C$ the associated initial value problem

$$v'(t) = L_e v_t, \qquad v_0 = \chi \tag{2.4}$$

has a uniquely determined solution; i.e., there exists a unique continuous $v^{\chi} : [-h, \infty) \to \mathbb{R}^n$ which is continuously differentiable on $(0, \infty)$, its segment $v_0^{\chi}$ at $t = 0$ coincides with initial value $\chi$, and which satisfies $(v^{\chi})'(t) = L_e v_t^{\chi}$ for all $t > 0$. Further, the segments $v_t^{\chi}$, $\chi \in C$ and $t \in [0, \infty)$, of all these solutions of initial value problem (2.4) induce a strongly continuous semigroup $T_e = \{T_e(t)\}_{t \geq 0}$ of bounded linear operators $T_e(t) : C \to C$ on the Banach space $C$ where the action is given by $T_e(t)\chi = v_t^{\chi}$. The linear operator $G_e : \mathcal{D}(G_e) \ni \chi \mapsto \chi' \in C$ defined on

$$\mathcal{D}(G_e) = \{\psi \in C^1 \mid \psi'(0) = L_e\psi\}$$

forms the associated infinitesimal generator of $T_e$. Clearly, we have $\mathcal{D}(G_e) = T_{\varphi_0}X_f$. Moreover, as can be found in Hartung et al. [10], $T(t)\varphi = T_e(t)\varphi$ for all $\varphi \in \mathcal{D}(G_e)$ and all $t \geq 0$, and the two spectra $\sigma(G), \sigma(G_e) \subset \mathbb{C}$ of the generators $G$, $G_e$, respectively, coincide.

The spectrum $\sigma(G_e)$, and so as well the spectrum $\sigma(G)$, is given by the roots of a familiar characteristic equation. In particular, it is discrete and consists only of eigenvalues with finite-dimensional generalized eigenspaces. In addition, to the right of any line parallel to the imaginary axis in the complex plane there are at most a finite number of eigenvalues of $G_e$.

**Exponential trichotomy.** Let $\sigma_u(G_e)$, $\sigma_c(G_e)$, and $\sigma_s(G_e)$ denote the subsets of the spectrum $\sigma(G_e)$ with positive, zero, and negative real part, respectively. Obviously, we have

$$\sigma(G_e) = \sigma_u(G_e) \cup \sigma_c(G_e) \cup \sigma_s(G_e)$$

and each of the spectral sets $\sigma_u(G_e)$ and $\sigma_c(G_e)$ is either empty or finite. Hence, the associated (realified) generalized eigenspaces $C_u$ and $C_c$, which are called the *unstable* and *center space* of $G_e$, respectively, are finite dimensional subspaces of $C$. In contrast to $C_u$ and $C_c$, the *stable space* $C_s \subseteq C$, i.e., the (realified) generalized eigenspace associated with the spectral part $\sigma_s(G_e)$, is infinite dimensional. All these subspaces of $C$ are closed, invariant under $T_e(t)$ for all $t \geq 0$, and provide the decomposition

$$C = C_u \oplus C_c \oplus C_s$$

of the Banach space $C$. Further, $T_e$ may be extended to a one-parameter group on $C_u$ as well as on $C_c$ since the restriction of $T_e$ to each of these finite dimensional subspaces has a bounded generator. For the action of $T_e$ on the closed subspaces $C_u$, $C_c$, and $C_s$ we have the following exponential estimates: There are reals $K \geq 1$, $c_s < 0 < c_u$, and $c_c > 0$ with $c_c < \min\{-c_s, c_u\}$ such that

$$\begin{aligned} \|T_e(t)\varphi\|_C &\leq Ke^{c_u t}\|\varphi\|_C, & t \leq 0, \varphi \in C_u, \\ \|T_e(t)\varphi\|_C &\leq Ke^{c_c|t|}\|\varphi\|_C, & t \in \mathbb{R}, \varphi \in C_c, \\ \|T_e(t)\varphi\|_C &\leq Ke^{c_s t}\|\varphi\|_C, & t \geq 0, \varphi \in C_s. \end{aligned} \tag{2.5}$$

The unstable and stable space of $G$ coincide with $C_u$ and $C_c$, respectively, whereas the stable space of $G$ is given by the intersection $C_s \cap \mathcal{D}(G_e)$. Consequently, we obtain the spectral decomposition

$$Y = C_u \oplus C_c \oplus Y_s \tag{2.6}$$

for the Banach space $Y := T_{\varphi_0}X_f$, where $Y_s := C_s \cap \mathcal{D}(G_e)$. All the spaces $C_u$, $C_c$, $Y_s$ are invariant under the semigroup $T$, and similarly to $T_e$, $T$ forms a one-parameter group on each of the both finite dimensional spaces $C_u$ and $C_c$. Using the exponential trichotomy (2.5), it is not hard to see the analogous estimates

$$\begin{aligned} \|T(t)\varphi\|_{C^1} &\leq Ke^{c_u t}\|\varphi\|_{C^1}, & t \leq 0, \varphi \in C_u, \\ \|T(t)\varphi\|_{C^1} &\leq Ke^{c_c|t|}\|\varphi\|_{C^1}, & t \in \mathbb{R}, \varphi \in C_c, \\ \|T(t)\varphi\|_{C^1} &\leq Ke^{c_s t}\|\varphi\|_{C^1}, & t \geq 0, \varphi \in Y_s, \end{aligned} \tag{2.7}$$

characterizing the action of $T$ on the decomposition of $Y$.

**Local coordinates for the semiflow $F$ in a neighborhood of $\varphi_0$.** Recall that the tangent space $Y = T_{\varphi_0}X_f$ of $X_f$ at the equilibrium $\varphi_0$ is a closed subspace of $C^1$ with codimension $n$. Therefore, we find a closed linear subspace $E \subset C^1$ of dimension $n$ which is complementary to $Y$ in $C^1$; that is, $C^1 = Y \oplus E$. In particular, the projection $P$ of $C^1$ along $E$ onto $Y$ is continuously differentiable, and the equation

$$N(\varphi) = P(\varphi - \varphi_0)$$

defines a manifold chart for $X_f$ on some open neighborhood $V \subset X_f$ of the equilibrium $\varphi_0$. Thereby, the image $Y_0 := N(V)$ of $V$ under $N$ forms an open neighborhood of $0 = N(\varphi_0)$ in the Banach space $Y$ equipped with norm $\|\cdot\|_{C^1}$. The inverse of $N$ is given by a continuously differentiable map $R : Y_0 \to C^1$, and the derivative $DN(\varphi_0)$ of $N$ at $\varphi_0$ as well as the derivative $DR(0)$ of $R$ at $0 \in Y_0$ is the identity operator on $Y$ in each case. Therefore we may assume that there is a constant $L_R > 0$ with

$$\|R(\chi_1) - R(\chi_2)\|_{C^1} \leq L_R\|\chi_1 - \chi_2\|_{C^1} \tag{2.8}$$

for all $\chi_1, \chi_2 \in Y_0$.

Let now $a > 0$ be given. By compactness of the interval $[0, a]$ together with the continuity of the map

$$(\mathbb{R} \times V) \cap \Omega \ni (t, \chi) \longmapsto F(t, R(\chi)) \in X_f,$$

we find an open neighborhood $Y_a$ of 0 in $Y_0$ such that $F(t, R(\chi))$ is well-defined for all $(t, \chi) \in [0, a] \times Y_a$ and that $F([0, a], R(Y_a)) \subset V$. As a consequence, we are able to represent the semiflow $F$ in local coordinates, namely by the map

$$H^a : [0, a] \times Y_a \ni (t, \chi) \longmapsto N(F(t, R(\chi))) \in Y. \tag{2.9}$$

Obviously, we have $H^a(t,0) = 0$ and $H^a(t,Y_a) \subset Y_0$ for all $0 \le t \le a$. The function $H^a$ is also continuous. Moreover, for each $0 \le t \le a$ the induced map

$$H_t^a : Y_a \ni \chi \longmapsto H^a(t,\chi) \in Y$$

is continuously differentiable with derivative given by

$$DH_t^a(0) = DN(\varphi_0) \circ D_2F(t,\varphi_0) \circ DR(0) = D_2F(t,\varphi_0) = T(t).$$

Suppose that we have $H^a(s,\chi) \in Y_a$ for a fixed $(s,\chi) \in [0,a] \times Y_a$. Then all values $H^a(t, H^a(s,\chi))$, $0 \le t \le a$, are well-defined. Accordingly, by setting

$$H^a(t+s,\chi) := H^a(t, H^a(s,\chi))$$

for $0 \le t \le a$ we may represent the positive semi-orbit of the semiflow $F$ through $R(\chi)$ in the local coordinates constructed above at least on the interval $[0, s+a]$. Additionally, in this case we see at once that the semigroup property of $F$ implies

$$H^a(t+s, R(\chi)) = H^a(t, H^a(s,\chi)) = N(F(t+s, R(\chi)))$$

for $0 \le t \le a$. Therefore, we may extend the domain for the local representation $H^a$ of the semiflow $F$ to the set

$$\Big\{(t,\chi) \in [0,\infty) \times Y_a \mid N(F(\lfloor t/a \rfloor a, R(\chi))) \in Y_a\Big\},$$

where $\lfloor t/a \rfloor$ denotes the integer part of the real $t/a$. For instance, $[0,\infty) \times \{0\}$ belongs to this extended domain of the map $H^a$ and the equilibrium $\varphi_0 \in X_f$ can obviously be represented by $0 \in Y_0$ for all $t \ge 0$.

## 3 The principle of linearized stability

After the preliminaries in the last section we are now in the position to state the announced principle of linearized stability for the semiflow $F$ induced by solutions of Eq. (2.1).

**Theorem 3.1** (The principle of linearized stability)**.** *Suppose the function $f : U \longrightarrow \mathbb{R}^n$, $U \subset C^1$ open, satisfies (S1) and (S2), and $\varphi_0 \in X_f$ is an equilibrium of the semiflow $F$. If $\Re(\lambda) < 0$ for all eigenvalues $\lambda \in \sigma(G_e)$, then $\varphi_0$ is locally asymptotically stable as an equilibrium of $F$.*

*Moreover, under the conditions stated above, the (local) attraction rate of $\varphi_0$ is exponential; that is, there exist reals $\varepsilon > 0$, $\gamma > 0$ and $\kappa \ge 0$ such that for each $\varphi \in X_f$ with $\|\varphi - \varphi_0\|_{C^1} < \varepsilon$ we have $t_+(\varphi) = \infty$ and*

$$\|F(t,\varphi) - \varphi_0\|_{C^1} \le \kappa e^{-\gamma t} \quad \textit{for all } t \ge 0.$$

As mentioned in the introduction, we will prove Theorem 3.1 in an elementary way by reducing the question about the stability of $\varphi_0$ for the continuous dynamical system given by the semiflow $F$ to the one for the discrete dynamical system given by some (appropriate) time-$t$ map $F(t,\cdot)$. But in doing so, we may not ignore all the "pieces in between" of a trajectory. Here, we will need the next result from Hartung et al. [10] about a quantitative version of continuous dependence of semiflow $F$ on initial data. For the sake of completeness, we also repeat the proof of the statement.

**Proposition 3.2** (Proposition 3.5.3 in [10]). *Let $f : U \to \mathbb{R}^n$, $U \subset C^1$ open, with (S1) and (S2) be given, and let $\varphi_0$ be an equilibrium of the semiflow $F$. Then for each $a > 0$ there exist an open neighborhood $X_{f,a}$ of $\varphi_0$ in $X_f$ and some constant $c_a \geq 0$ such that $[0, a] \times X_{f,a} \subset \Omega$ and*

$$\|F(t, \varphi) - \varphi_0\|_{C^1} \leq c_a \|\varphi - \varphi_0\|_{C^1} \tag{3.1}$$

*for all $(t, \varphi) \in [0, a] \times X_{f,a}$.*

*Proof.* 1. First, observe that, by using smoothness condition (S2) of function $f$, it is not hard to see that $f$ has the following local Lipschitz property: there exist a neighborhood $V_L$ of $\varphi_0$ in $U$ and a real $L_V \geq 0$ with

$$\|f(\psi) - f(\chi)\|_{\mathbb{R}^n} \leq L_V \|\psi - \chi\|_C$$

for all $\psi, \chi \in V_L$ (see, for instance, Corollary 1 in Walther [24]).

2. Let now $a > 0$ be given. Then, the continuity of the semiflow $F : \Omega \to X_f$ and the compactness of the interval $[0, a]$ implies the existence of some neighborhood $X_{f,a}$ of $\varphi_0$ in $X_f$ such that $[0, a] \times X_{f,a} \subset \Omega$ and $F([0, a] \times X_{f,a}) \subset V_L$ with $V_L$ from the last part.

3. Set $\xi := \varphi_0(0)$, and let $\varphi \in X_{f,a}$ be given. Then, in view of $f(\varphi_0) = 0$ and the first part, it follows that for all $0 \leq t \leq a$ we have

$$\begin{aligned} \|x^\varphi(t) - \xi\|_{\mathbb{R}^n} &= \left\| x^\varphi(0) - \xi + \int_0^t (x^\varphi)'(s) ds \right\|_{\mathbb{R}^n} \\ &= \left\| x^\varphi(0) - \xi + \int_0^t f(x_s^\varphi) ds \right\|_{\mathbb{R}^n} \\ &\leq \|\varphi - \varphi_0\|_C + L_V \int_0^t \|x_s^\varphi - \varphi_0\|_C \, ds. \end{aligned}$$

Given any $t \in [0, a]$, observe that there is some $t_0 \in [t - h, t]$ satisfying

$$\|x^\varphi(t_0) - \xi\|_{\mathbb{R}^n} = \|x_t^\varphi - \varphi_0\|_C.$$

In case $t_0 < 0$, we have

$$\|x_t^\varphi - \varphi_0\|_C = \|\varphi - \varphi_0\|_C,$$

whereas in the other case $t_0 \geq 0$ we obtain

$$\begin{aligned} \|x_t^\varphi - \varphi_0\|_C &\leq \|x_0^\varphi - \varphi_0\|_C + L_V \int_0^{t_0} \|x_s^\varphi - \varphi_0\|_C \, ds \\ &\leq \|\varphi - \varphi_0\|_C + L_V \int_0^t \|x_s^\varphi - \varphi_0\|_C \, ds. \end{aligned}$$

However, in any case we have

$$\|x_t^\varphi - \varphi_0\|_C \leq \|\varphi - \varphi_0\|_C + L_V \int_0^t \|x_s^\varphi - \varphi_0\|_C \, ds$$

such that Gronwall's lemma shows

$$\|F(t, \varphi) - \varphi_0\|_C = \|x_t^\varphi - \varphi_0\|_C \leq \|\varphi - \varphi_0\|_C \, e^{L_V t}$$

for all $t \in [0, a]$. Furthermore, by combining the last estimate with the first part, we also see that

$$\|(x^\varphi)'(t)\|_{\mathbb{R}^n} \leq \|f(x_t^\varphi) - f(\varphi_0)\|_{\mathbb{R}^n} \leq L_V \, e^{L_V t} \|\varphi - \varphi_0\|_C$$

as $t \in [0, a]$. Consequently, it follows that

$$\|(x_t^\varphi)' - (\varphi_0)'\|_C \leq L_V\, e^{L_V t} \|\varphi - \varphi_0\|_C$$

and finally

$$\|F(t, \varphi) - \varphi_0\|_{C^1} = \|x_t^\varphi - \varphi_0\|_{C^1} \leq (1 + L_V) e^{L_V t} \|\varphi - \varphi_0\|_C \leq (1 + L_V) e^{L_V t} \|\varphi - \varphi_0\|_{C^1}$$

on $[0, a]$. Setting $c_a := (1 + L_V) e^{L_V a}$ completes the proof. □

As the last preparatory step towards a proof of the principle of linearized stability, we show that, under the assumptions of Theorem 3.1 and for $a > 0$ sufficiently large, the associated time-$t$-map $H_a^a = H^a(a, \cdot) : Y_a \to Y$ in local coordinates is a contractive self-map on a neighborhood of $0 \in Y$. To be more precisely, we prove the following result.

**Proposition 3.3.** *Let the hypothesis of Theorem 3.1 hold. Then for each sufficiently large $a > 0$ there exist an open neighborhood $Y_{c,a} \subset Y_a$ of $0 \in Y$ such that the restriction $H := H_a^a|_{Y_{c,a}}$ satisfies $H(Y_{c,a}) \subset Y_{c,a}$ and*

$$\|H(\chi_1) - H(\chi_2)\|_{C^1} \leq \frac{1}{2} \|\chi_1 - \chi_2\|_{C^1} \tag{3.2}$$

*for all $\chi_1, \chi_2 \in Y_{c,a}$.*

*Proof.* 1. Under given assumptions, we clearly have $C_u = C_c = \{0\}$ and $Y_s = Y$. Consequently, (2.7) implies

$$\|T(t)\| \leq K e^{c_s t}$$

for all $t \geq 0$. Fix any $a > 0$ with $K e^{c_s a} < 1/4$, which is possible due to the fact $c_s < 0$, and let $H : Y_a \to Y$ denote the corresponding time-$a$-map $H^a(a, \cdot)$ in the following.

2. Recall that $H$ is continuously differentiable and that $DH(0) = DH_a^a = T(a)$. For this reason, we find some open ball $B_\varepsilon(0) = \{\chi \in Y \mid \|\chi\|_{C^1} < \varepsilon\}$ of radius $\varepsilon > 0$ about 0 in $Y$ with $B_\varepsilon(0) \subset Y_a$ and

$$\|DH(\chi) - DH(0)\| < \frac{1}{4}$$

for all $\chi \in B_\varepsilon(0)$. Combining that with the first part gives

$$\|DH(\chi)\| \leq \|DH(0)\| + \frac{1}{4} = \|T(a)\| + \frac{1}{4} \leq K e^{c_s a} + \frac{1}{4} < \frac{1}{2}$$

as $\chi \in B_\varepsilon(0)$. Hence, given any $\chi_1, \chi_2 \in B_\varepsilon(0)$,

$$\begin{aligned}
\|H(\chi_1) - H(\chi_2)\|_{C^1} &\leq \int_0^1 \|DH(\chi_2 + s(\chi_1 - \chi_2))(\chi_1 - \chi_2)\|_{C^1} ds \\
&\leq \max_{s \in [0,1]} \|DH(\chi_2 + s(\chi_1 - \chi_2))\| \|\chi_1 - \chi_2\|_{C^1} \\
&\leq \sup_{\chi \in B_\varepsilon(0)} \|DH(\chi)\| \|\chi_1 - \chi_2\|_{C^1} \\
&\leq \frac{1}{2} \|\chi_1 - \chi_2\|_{C^1}.
\end{aligned}$$

3. It remains to prove that $H$ is a self-map of $B_\varepsilon(0)$. But this point is immediate in consideration of $H(0) = 0$ and the contraction property from the part above. Indeed, we have

$$\|H(\chi)\|_{C^1} = \|H(\chi) - H(0)\|_{C^1} \leq \frac{1}{2} \|\chi - 0\|_{C^1} = \|\chi\|_{C^1} < \varepsilon$$

for all $\chi \in B_\varepsilon(0)$. □

**Remark 3.4.** Note that the last result proves that the fixed point $\chi_0 = 0$ of the discrete semi-dynamical system induced by iterations of $H : Y_{c,a} \to Y_{c,a}$ is asymptotically stable. Indeed, for each $\chi \in Y_{c,a}$ and all $k \in \mathbb{N}_0$ we have

$$\|H^k(\chi)\|_{C^1} \leq \frac{1}{2}\|H^{(k-1)}(\chi)\|_{C^1} \leq \left(\frac{1}{2}\right)^k \|\chi\|_{C^1}$$

and so $H^k(\chi) \to 0$ as $k \to \infty$.

Now, we are able to prove the principle of linearized stability.

*Proof of Theorem 3.1.* 1. To begin with, let constant $a > 0$, open neighborhood $Y_{c,a} \subset Y_a$ of $0 \in Y$, and the map $H$ as in Proposition 3.3 be given. Set $V_N := R(Y_{c,a})$. Further, fix constant $c_a \geq 0$ and open neighborhood $X_{f,a} \subset X_f$ of equilibrium $\varphi_0 \in X_f$ as in Proposition 3.2. Observe that, in view of the proof of Proposition 3.3, there is no loss of generality in assuming $V_N \subset X_{f,a}$. Indeed, otherwise we could start with a smaller neighborhood $Y_{c,a} \subset Y_a$ of $0 \in Y$.

2. For each $\varphi \in V_N$ we have $F(a, \varphi) \in V_N$. In fact, by Proposition 3.3, $H(\chi) \in Y_{c,a}$ for $\chi := N(\varphi) \in N(V_N) = N(R(Y_{c,a})) = Y_{c,a}$ and so

$$F(a, \varphi) = R(N(F(a, R(\chi)))) = R(H(\chi)) \in R(Y_{c,a}) = V_N.$$

As $V_N \subset X_{f,a}$ it clearly follows that $[0, \infty) \times V_N \subset \Omega$. Moreover, any point $\psi \in V_N$ defines a trajectory $\{\psi_j\}_{j \in \mathbb{N}_0}$ with $\psi_0 := \psi$ of the time-$a$-map $F(a, \cdot)$ in $V_N$, and the associated points $\chi_j := N(\psi_j)$ a trajectory of $H$ in $Y_{c,a}$ as

$$\chi_{j+1} = N(\varphi_{j+1}) = N(F(a, \psi_j)) = N(F(a, R(\chi_j))) = H(\chi_j).$$

Using the Lipschitz continuity (2.8) of $R$ and contraction property (3.2) of $H$, we obtain

$$\begin{aligned}
\|\psi_j - \varphi_0\|_{C^1} &= \|R(\chi_j) - R(0)\|_{C^1} \\
&\leq L_R \|\chi_j - 0\|_{C^1} \\
&= L_R \|H(\chi_{j-1})\|_{C^1} \\
&\leq L_R \frac{1}{2} \|\chi_{j-1}\|_{C^1} \\
&\leq L_R \left(\frac{1}{2}\right)^j \|\chi_0\|_{C^1} \\
&= L_R \left(\frac{1}{2}\right)^j \|N(\psi_0)\|_{C^1} \\
&= L_R \left(\frac{1}{2}\right)^j \|P(\psi_0 - \varphi_0)\|_{C^1} \\
&\leq L_R \|P\| \left(\frac{1}{2}\right)^j \|\psi_0 - \varphi_0\|_{C^1}
\end{aligned}$$

for all $j \geq 0$ and all trajectories $\{\psi_j\}_{j \in \mathbb{N}_0}$ with $\psi_0 \in V_N$.

3. Set $\gamma := -\frac{\log(1/2)}{a} > 0$, and let $\psi \in V_N$ and $t \geq 0$ be given. Fix $j \in \mathbb{N}_0$ satisfying

$ja \leq t < (j+1)a$. Then, in view of Proposition 3.2 and the last part,

$$\begin{aligned}
\|F(t,\psi) - \varphi_0\|_{C^1} &= \|F(t-ja, F(ja,\psi)) - \varphi_0\|_{C^1} \\
&\leq c_a \|F(ja,\psi) - \varphi_0\|_{C^1} \\
&\leq c_a L_R \|P\| \left(\frac{1}{2}\right)^j \|\psi - \varphi_0\|_{C^1} \\
&= c_a L_R \|P\| e^{j\log(1/2)} \|\psi - \varphi_0\|_{C^1} \\
&= c_a L_R \|P\| e^{(t\log(1/2))/a} e^{\left(j-\frac{t}{a}\right)\log(1/2)} \|\psi - \varphi_0\|_{C^1} \\
&\leq c_a L_R \|P\| e^{-\gamma t} e^{-\log(1/2)} \|\psi - \varphi_0\|_{C^1} \\
&= \kappa e^{-\gamma t} \|\psi - \varphi_0\|_{C^1}
\end{aligned}$$

with $\kappa := 2c_a L_R \|P\| \geq 0$.

4. Let now $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $\kappa\delta < \varepsilon$ and such that for all $\varphi \in X_f$ with $\|\varphi - \varphi_0\|_{C^1} < \delta$ we have $\varphi \in V_N$ with $V_N$ defined above. Then, given $\varphi \in X_f$ with $\|\varphi - \varphi_0\|_{C^1} < \delta$, we have $t_+(\varphi) = +\infty$ and the last part clearly implies both

$$\|F(t,\varphi) - \varphi_0\|_{C^1} < \kappa\delta < \varepsilon$$

for all $t \geq 0$ as well as $F(t,\varphi) \to \varphi_0$ exponentially as $t \to \infty$. This shows the assertion. □

## 4 The principle of linearized instability

This final section is devoted to prove the *principle of linearized instability* which allows to infer the instability of an equilibrium $\varphi_0 \in X_f$ of the semiflow $F$ from the instability of the trivial equilibrium of the associated linearization $T$. More precisely, we will establish the following result.

**Theorem 4.1** (The principle of linearized instability)**.** *Suppose the function $f : U \longrightarrow \mathbb{R}^n$, $U \subset C^1$ open, satisfies (S1) and (S2), and $\varphi_0 \in X_f$ is an equilibrium of the semiflow $F$. If $\Re(\lambda) > 0$ for some eigenvalue $\lambda \in \sigma(G_e)$, then $\varphi_0$ is unstable for the semiflow $F$.*

Similarly to the last section, we begin with a statement concerning the dynamics induced by iterations of some time-$t$-map of $F$ in local coordinates before proving the principle of linearized instability.

**Proposition 4.2.** *Let the hypothesis of Theorem 3.1 hold. Then there is some $a > 0$ such that $\chi_0 = 0 \in Y$ is unstable as a fixed point of the discrete semi-dynamical system generated by iterations of the map $H := H_a^a : Y_a \to Y$. In fact, there exists an open neighborhood $Y_{c,a} \subset Y_a$ of $\chi_0 = 0 \in Y$ such that for each $\varepsilon > 0$ there is some $\chi \in Y_{c,a}$ with $\|\chi\|_{C^1} < \varepsilon$ but $H^k(\chi) \notin Y_{c,a}$ for some $k \in \mathbb{N}$.*

*Proof.* 1. Consider the decomposition (2.6) of $Y$ and the associated trichotomy given by (2.7). Defining $Y_{cs} := C_c \oplus Y_s$ and $c_{cs} := c_c$, we obtain the decomposition

$$Y = C_u \oplus Y_{cs}$$

with the exponential estimates

$$\begin{aligned}
\|T(t)\chi\|_{C^1} &\leq K e^{c_u t} \|\varphi\|_{C^1}, & t \leq 0,\ \chi \in C_u, \\
\|T(t)\chi\|_{C^1} &\leq K e^{c_{cs} t} \|\varphi\|_{C^1}, & t \geq 0,\ \chi \in Y_{cs}.
\end{aligned} \tag{4.1}$$

Fix $0 < q < 1$ with $1/q > K \geq 1$. Then for all $t \geq 0$ and $\chi \in C_u$ we have

$$\|\chi\|_{C^1} = \|T(-t)\,T(t)\,\chi\|_{C^1} \leq Ke^{-c_u t}\|T(t)\,\chi\|_{C^1} \leq \frac{1}{q}\,e^{-c_u t}\|T(t)\,\chi\|_{C^1},$$

that is,

$$q\,e^{c_u t}\|\chi\|_{C^1} \leq \|T(t)\,\chi\|_{C^1}. \tag{4.2}$$

Since $c_{cs} < c_u$ it follows that $\vartheta_1 := q\,e^{c_u a} - K\,e^{c_{cs} a} > 0$ for all sufficiently large $a > 0$. Fix such a constant $a > 0$. Then the estimates (4.1) and (4.2) for the linear operator $L := T(a)$ imply the two inequalities

$$\begin{aligned} &\|L\,\chi\|_{C^1} \geq (\vartheta_1 + \vartheta_2)\|\chi\|_{C^1}, && \chi \in C_u,\\ &\|L\,\chi\|_{C^1} \leq \vartheta_2\|\chi\|_{C^1}, && \chi \in Y_{cs}, \end{aligned} \tag{4.3}$$

where $\vartheta_2 := K\,e^{c_{cs} a} > 1$.

2. Let $\widehat{P}_u : Y \longrightarrow Y$ denote the projection of $Y$ along $Y_{cs}$ onto the unstable space $C_u$ of the operator $G$. Using the continuity of $\widehat{P}_u$, it is easily seen that

$$\|\varphi\|_u := \|\widehat{P}_u\,\varphi\|_{C^1} + \|(id - \widehat{P}_u)\,\varphi\|_{C^1},$$

where $id$ denotes the identity operator, defines a norm on $Y$. In particular, the norm $\|\cdot\|_u$ is equivalent to $\|\cdot\|_{C^1}$ on $Y$. Consider now the time-$a$-map $H := H^a(a,\cdot) : Y_a \longrightarrow Y$ of the semiflow $F$ in local coordinates. Since $Y_a$ is an open neighborhood of the origin in $Y$, and $L$ is the derivative of $H$ at $\chi = 0$, we find a sufficiently small $\varepsilon_1 > 0$ such that for all $\chi \in Y$ with $\|\chi\|_u < \varepsilon_1$ we have $\chi \in Y_a$ and

$$\|H(\chi) - L\,\chi\|_u \leq \frac{1}{4}\,\vartheta_1\,\|\chi\|_u. \tag{4.4}$$

Suppose for $\chi \in Y$ with $\|\chi\|_u < \varepsilon_1$ there holds $\|(id - \widehat{P}_u)\,\chi\|_{C^1} \leq \|\widehat{P}_u\,\chi\|_{C^1}$. Then we claim that the value $H(\chi)$ satisfies the same cone condition as $\chi$; that is,

$$\|(id - \widehat{P}_u)(H(\chi))\|_{C^1} \leq \|\widehat{P}_u(H(\chi))\|_{C^1}.$$

To see this, note first that the above assumptions on $\chi$ immediately imply the inequality $\|\chi\|_u \leq 2\|\widehat{P}_u\,\chi\|_{C^1}$. Therefore the invariance of the spaces $C_u$, $Y_{cs}$ for $L$ and the estimates (4.3), (4.4) yield

$$\begin{aligned} \|\widehat{P}_u\,H(\chi)\|_{C^1} &\geq \|\widehat{P}_u\,L\,\chi\|_{C^1} - \|\widehat{P}_u(H(\chi) - L\,\chi)\|_{C^1}\\ &= \|L\,\widehat{P}_u\,\chi\|_{C^1} - \|\widehat{P}_u(H(\chi) - L\,\chi)\|_{C^1}\\ &\geq (\vartheta_1 + \vartheta_2)\|\widehat{P}_u\,\chi\|_{C^1} - \|H(\chi) - L\,\chi\|_u\\ &\geq (\vartheta_1 + \vartheta_2)\|\widehat{P}_u\,\chi\|_{C^1} - \frac{1}{4}\,\vartheta_1\|\chi\|_u\\ &\geq (\vartheta_1 + \vartheta_2)\|\widehat{P}_u\,\chi\|_{C^1} - \frac{1}{2}\,\vartheta_1\|\widehat{P}_u\,\chi\|_{C^1}\\ &\geq \left(\vartheta_2 + \frac{1}{2}\,\vartheta_1\right)\|\widehat{P}_u\,\chi\|_{C^1} \end{aligned}$$

and

$$
\begin{aligned}
\|(id-\widehat{P}_u)\,H(\chi)\|_{C^1} &\leq \|(id-\widehat{P}_u)\,L\,\chi\|_{C^1} + \|(id-\widehat{P}_u)(H(\chi)-L\,\chi)\|_{C^1} \\
&= \|L\,(id-\widehat{P}_u)\,\chi\|_{C^1} + \|(id-\widehat{P}_u)(H(\chi)-L\,\chi)\|_{C^1} \\
&\leq \vartheta_2\|(id-\widehat{P}_u)\,\chi\|_{C^1} + \|H(\chi)-L\,\chi\|_u \\
&\leq \vartheta_2\|(id-\widehat{P}_u)\,\chi\|_{C^1} + \frac{1}{4}\,\vartheta_1\|\chi\|_u \\
&\leq \vartheta_2\|(id-\widehat{P}_u)\,\chi\|_{C^1} + \frac{1}{2}\,\vartheta_1\|\widehat{P}_u\,\chi\|_{C^1} \\
&\leq \left(\vartheta_2 + \frac{1}{2}\,\vartheta_1\right)\|\widehat{P}_u\,\chi\|_{C^1},
\end{aligned}
$$

which proves the claim. Thus for all $\chi \in Y$ with $\|\chi\|_u < \varepsilon_1$ we have the implication

$$\|(id-\widehat{P}_u)\,\chi\|_{C^1} \leq \|\widehat{P}_u\,\chi\|_{C^1} \quad \Longrightarrow \quad \|(id-\widehat{P}_u)(H(\chi))\|_{C^1} \leq \|\widehat{P}_u(H(\chi))\|_{C^1}.$$

3. Consider now any $0 < \varepsilon_2 < \varepsilon_1/(\|\widehat{P}_u\| + \|id-\widehat{P}_u\|)$, and assume that for every sufficiently small $\chi \in Y$ with $\|\chi\|_u < \varepsilon_1$ and $\|(id-\widehat{P}_u)\,\chi\|_{C^1} \leq \|\widehat{P}_u\,\chi\|_{C^1}$ there holds

$$\|H^k(\chi)\|_{C^1} < \varepsilon_2$$

for all $k \in \mathbb{N}$. Then we would have

$$
\begin{aligned}
\|H^k(\chi)\|_u &= \|\widehat{P}_u(H^k(\chi))\|_{C^1} + \|(id-\widehat{P}_u)(H^k(\chi))\|_{C^1} \\
&\leq \|\widehat{P}_u\|\,\|H^k(\chi)\|_{C^1} + \|id-\widehat{P}_u\|\,\|H^k(\chi)\|_{C^1} \\
&\leq (\|\widehat{P}_u\| + \|id-\widehat{P}_u\|)\,\varepsilon_2 \\
&< \varepsilon_1,
\end{aligned}
$$

and hence by the part above

$$\|\widehat{P}_u(H^k(\chi))\|_{C^1} \geq \left(\vartheta_2 + \frac{1}{2}\,\vartheta_1\right)^k \|\widehat{P}_u\,\chi\|_{C^1}$$

for all $k \in \mathbb{N}$. Subsequently, in consideration of $\vartheta_1 > 0$ and $\vartheta_2 > 1$, this would imply

$$\|\widehat{P}_u(H^k(\chi))\|_{C^1} \to \infty$$

for $k \to \infty$ whenever $\chi \neq 0$. But, as by hypothesis of the proposition $\dim C_u \geq 1$, we see at once the existence of any desired small $\chi_u \in Y\backslash\{0\}$ satisfying

$$\|(id-\widehat{P}_u)\,\chi_u\|_{C^1} \leq \|\widehat{P}_u\,\chi_u\|_{C^1} \leq \|\chi_u\|_u < \varepsilon_1.$$

This leads to a contradiction to our assumption on boundedness for the iterations of $H$. Thus, setting $Y_{c,a} := \{\chi \in Y_a \mid \|\chi\|_{C^1} < \varepsilon_2\}$ shows the assertion. □

**Remark 4.3.** Note that the statement of the last result may be sharpened. For instance, our proof shows that the assertion is true for all time-$t$-maps $H_t^t : Y_t \to Y$ with $t \geq a$. However, for our purpose, namely, a proof of Theorem 4.1, it is sufficient to have only a single time-$t$-map with the stated property.

Now, we return to the proof of the principle of linearized instability. Observe that, contrary to the principle of linearized stability, the "pieces in between" of a trajectory may be ignored such that the instability assertion carries over, more or less, immediately from the discrete dynamical system to the continuous dynamical system.

*Proof of Theorem 4.1.* Contrary to the statement, suppose that the equilibrium $\varphi_0 \in X_f$ is stable for the semiflow $F$. Choose $a > 0$ and open neighborhood $Y_{c,a}$ of $\chi_0 = 0 \in Y$ according Proposition 4.2, and let $P : C^1 \to C^1$ denote the projection operator along $Y$ onto $E$ involved in our construction of local coordinates for $X_f$. Then there clearly is some sufficiently small $\varepsilon > 0$ such that the open ball $\{\chi \in Y \mid \|\chi\|_{C^1} < \|P\|\varepsilon\}$ is contained in $Y_{c,a}$. Next, by assumption and Remark 2.1, we find a constant $0 < \delta < \varepsilon$ such that for all $\varphi \in X_f$ with $\|\varphi - \varphi_0\|_{C^1} < \delta$ and all $t \geq 0$

$$\|F(t,\varphi) - \varphi_0\|_{C^1} < \varepsilon$$

holds. Now, consider any $\chi \in Y_{c,a}$ satisfying $\|\chi\|_{C^1} < \delta / L_R$ where $L_R > 0$ is the constant from the Lipschitz condition (2.8) for the map $R$. As

$$\|R(\chi) - \varphi_0\|_{C^1} = \|R(\chi) - R(0)\|_{C^1} \leq L_R \|\chi\|_{C^1} < \delta$$

it follows that

$$\begin{aligned} \|H^k(\chi)\|_{C^1} &= \|H^a(k\,a,\chi)\|_{C^1} \\ &= \|N(F(k\,a,R(\chi)))\|_{C^1} \\ &= \|P(F(k\,a,R(\chi)) - \varphi_0)\|_{C^1} \\ &\leq \|P\|\,\|F(k\,a,R(\chi)) - \varphi_0\|_{C^1} \\ &< \|P\|\varepsilon \end{aligned}$$

for all $k \in \mathbb{N}$. For this reason, if $\varphi_0$ would be stable then for all sufficiently small $\chi \in Y_{c,a}$ we would have $H^k(\chi) \in Y_{c,a}$ as $k \in \mathbb{N}$. But this is clearly impossible due to Proposition 3.3. Thus $\varphi_0$ is unstable, which proves the theorem. □

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