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Fisher–Kolmogorov type perturbations of the mean curvature operator in Minkowski space

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. We provide a complete description of the existence/non-existence and multiplicity of distinct pairs of nontrivial solutions to the problem with Minkowski operator

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda u(1-a|u|^q) \quad \text{in } \Omega, \;\; u|_{\partial\Omega} = 0, \quad (a \geq 0 < q),$$

when $\lambda \in (0,\infty)$, in terms of the spectrum of the classical Laplacian. Beforehand, we obtain multiplicity of solutions for parameterized and non-parameterized Dirichlet problems involving odd perturbations of this operator. The approach relies on critical point theory for convex, lower semicontinuous perturbations of C^1-functionals.

Keywords: Minkowski operator, Fisher–Kolmogorov nonlinearities, Krasnoselskii's genus, critical point.

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1 Introduction and preliminaries

In this paper we deal with the Dirichlet boundary value problem

$$\begin{cases} -\mathcal{M}(u) = \lambda g(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with boundary $\partial\Omega$ of class C^2, $\lambda > 0$ is a real parameter, $g : \mathbb{R} \to \mathbb{R}$ is an odd continuous function and $\mathcal{M}$ stands for the mean curvature operator in Minkowski space:

$$\mathcal{M}(u) = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right).$$

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Problems involving the operator $\mathcal{M}$ are originated in differential geometry and relativity. These are related to maximal and constant mean curvature spacelike hypersurfaces (spacelike submanifolds of codimension one in the flat Minkowski space $\mathbb{L}^{N+1} := \{(x,t) : x \in \mathbb{R}^N,\ t \in \mathbb{R}\}$ endowed with the Lorentzian metric $\sum_{j=1}^N (dx_j)^2 - (dt)^2$, where (x,t) are the canonical coordinates in $\mathbb{R}^{N+1}$) having the property that the trace of the extrinsic curvature is zero, respectively, constant. On the other hand, assuming that a spacelike hypersurface in $\mathbb{L}^{N+1}$ is the graph of a smooth function $u : \Omega \to \mathbb{R}$ with Ω a domain in $\{(x,t) : x \in \mathbb{R}^N, t = 0\} \simeq \mathbb{R}^N$, the (strictly) spacelike condition implies $|\nabla u| < 1$ and u satisfies an equation of type

$$\mathcal{M}(u) = H(x,u) \quad \text{in } \Omega,$$

where H is a prescribed mean curvature function. If H is continuous and bounded, it has been shown in [4] that the above equation subjected to a Dirichlet condition has at least one solution. More recently, the existence of additional solutions, such as of mountain pass type, was obtained in [5,6] and the existence of Filippov type solutions for discontinuous Dirichlet problems involving the operator $\mathcal{M}$ was established in [7]. For other recent developments of the subject, we refer the reader to [2,3,9–11,15,16] and the references therein.

As in [10], by a *solution* of (1.1) we mean a function $u \in C^{0,1}(\overline{\Omega})$, such that $\|\nabla u\|_\infty < 1$, which vanishes on $\partial\Omega$ and satisfies

$$\int_\Omega \frac{\nabla u \cdot \nabla w}{\sqrt{1 - |\nabla u|^2}}\, dx = \lambda \int_\Omega g(u) w\, dx, \tag{1.2}$$

for every $w \in W_0^{1,1}(\Omega)$. Here and below, $\|\cdot\|_\infty$ stands for the usual sup-norm on $L^\infty(\Omega)$. As shown in [10, Remark 2], if u is a solution of (1.1), in the sense of the previous definition, then $u \in W^{2,r}(\Omega)$ for all finite $r \geq 1$ and satisfies the equation a.e. in Ω. Reciprocally, since, for $p > N$, one has

$$W^{2,p}(\Omega) \subset C^1(\overline{\Omega}) \subset W^{1,\infty}(\Omega) = C^{0,1}(\overline{\Omega}),$$

it is straightforward to check that if a function $u \in W^{2,p}(\Omega)$ for some $p > N$, with $\|\nabla u\|_\infty < 1$ satisfies the equation a.e. in Ω and vanishes on $\partial\Omega$, then it is a solution of (1.1).

This study is mainly motivated by the result obtained in [17] concerning the multiplicity of T-periodic solutions for the equation with relativistic operator:

$$-\left(\frac{u'}{\sqrt{1 - |u'|^2}}\right)' = \lambda g_1(u) \quad \text{in } [0,T]; \tag{1.3}$$

by g_a we denote the Fisher–Kolmogorov type nonlinearity $g_a(t) = t(1 - a|t|^q)$, $\forall\ t \in \mathbb{R}$ ($a \geq 0 < q$). This type of nonlinearities was originally motivated by models in biological population dynamics and led to the reaction-diffusion equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u(1 - u^2),$$

referred to as *the classical Fisher–Kolmogorov equation* [12,13,18]. Also, higher-order equations of type

$$u^{iv} - pu'' = u(q(t) - r(t)u^2), \qquad \text{(with } q, r \text{ positive functions)}$$

which corresponds, if $p > 0$, to *the extended Fisher–Kolmogorov equations* are models for phase transitions and other bistable phenomena (see e.g. [8,20–23,27]). So, in [17, Theorem 2.1] it is shown that if $\lambda > 4\pi^2 m^3/T^2$ for some $m \geq 2$, then equation (1.3) subjected to periodic boundary conditions has at least $m-1$ distinct pairs of non-constant solutions. By comparison, in the case of the Dirichlet problem for the parametrized equation

$$-\mathcal{M}(u) = \lambda g_a(u) \quad \text{in } \Omega,$$

we obtain (see Theorem 2.5) a complete description of the existence/non-existence and multiplicity of distinct pairs of nontrivial solutions when $\lambda \in (0, \infty)$, in terms of the eigenvalues of the classical $-\Delta$. It is worth to point out that the multiplicity part of the result relies on a Clark type theorem for the general problem (1.1) (see Theorem 2.2). Moreover, this theorem enables us to derive existence of finitely or infinitely many solutions to Dirichlet problems for non-parametrized equations having the form

$$-\mathcal{M}(u) = f(u) \quad \text{in } \Omega,$$

with odd continuous $f : \mathbb{R} \to \mathbb{R}$, by controlling the asymptotic behavior of the primitive of f near the origin (see Corollary 2.3).

We conclude this introductory part by briefly recalling some notions and results in the frame of Szulkin's critical point theory [26], which will be needed in the sequel. Let $(Y, \|\cdot\|)$ be a real Banach space and $\mathcal{I} : Y \to (-\infty, +\infty]$ be a functional of the type

$$\mathcal{I} = \mathcal{F} + \psi, \tag{1.4}$$

where $\mathcal{F} \in C^1(Y, \mathbb{R})$ and $\psi : Y \to (-\infty, +\infty]$ is convex, lower semicontinuous and proper (i.e., $D(\psi) := \{u \in Y : \psi(u) < +\infty\} \neq \emptyset$). A point $u \in Y$ is said to be *a critical point* of $\mathcal{I}$ if $u \in D(\psi)$ and if it satisfies the inequality

$$\langle \mathcal{F}'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0 \quad \forall\, v \in D(\psi).$$

It is straightforward to see that each local minimum of $\mathcal{I}$ is necessarily a critical point of $\mathcal{I}$ [26, Proposition 1.1]. A sequence $\{u_n\} \subset D(\psi)$ is called a (PS)-*sequence* if $\mathcal{I}(u_n) \to c \in \mathbb{R}$ and

$$\langle \mathcal{F}'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\| \quad \forall\, v \in D(\psi),$$

where $\varepsilon_n \to 0$. The functional $\mathcal{I}$ is said *to satisfy the* (PS) *condition* if any (PS)-sequence has a convergent subsequence in Y.

Let Σ be the collection of all symmetric subsets of $Y \setminus \{0\}$ which are closed in Y. The *genus* (Krasnoselskii) of a nonempty set $A \in \Sigma$ is defined as being the smallest integer k with the property that there exists an odd continuous mapping $h : A \to \mathbb{R}^k \setminus \{0\}$; in this case we write $\gamma(A) = k$. If such an integer does not exist, $\gamma(A) = +\infty$. Also, if $A \in \Sigma$ is homeomorphic to S^{k-1} ($k-1$ dimension unit sphere in the Euclidean space $\mathbb{R}^k$) by an odd homeomorphism, then $\gamma(A) = k$ (see e.g. [25, Corollary 5.5]). For properties and more details of the notion of genus we refer the reader to [24, 25]. Denoting by $\Gamma \subset 2^Y$ the collection of all nonempty compact symmetric subsets of Y, considered with the Hausdorff–Pompeiu distance, we set

$$\Gamma_j := \mathrm{cl}\{A \in \Gamma : \ 0 \notin A,\ \gamma(A) \geq j\}.$$

The following is an immediate consequence of [26, Theorem 4.3].

Theorem 1.1. *Let $\mathcal{I}$ be of type (1.4) with $\mathcal{F}$ and ψ even. Also, suppose that $\mathcal{I}$ is bounded from below, satisfies the (PS) condition and $\mathcal{I}(0) = 0$. If*

$$\inf_{A \in \Gamma_m} \sup_{v \in A} \mathcal{I}(v) < 0,$$

then the functional $\mathcal{I}$ has at least m distinct pairs of nontrivial critical points.

2 Main results

Using the ideas from [5], we introduce the variational formulation for problem (1.1). Accordingly, let

$$K_0 := \{u \in W^{1,\infty}(\Omega) : \|\nabla u\|_\infty \le 1,\ u|_{\partial\Omega} = 0\}.$$

The convex set K_0 is compact in $C(\overline{\Omega})$ [5, Lemma 2.2]. The functional $\Psi : C(\overline{\Omega}) \to (-\infty, +\infty]$ defined by

$$\Psi(u) = \begin{cases} \int_\Omega [1 - \sqrt{1 - |\nabla u|^2}]dx, & \text{for } u \in K_0, \\ +\infty, & \text{for } u \in C(\overline{\Omega}) \setminus K_0 \end{cases}$$

is convex and lower semicontinuous [5, Lemma 2.4]. Also, it is easy to see that

$$\Psi(u) \le \int_\Omega |\nabla u|^2, \quad \forall\, u \in K_0. \tag{2.1}$$

Let the C^1-functional $\mathcal{G}_\lambda : C(\overline{\Omega}) \to \mathbb{R}$ be given by

$$\mathcal{G}_\lambda(u) = -\lambda \int_\Omega G(u)dx,$$

where

$$G(t) = \int_0^t g(\tau)d\tau.$$

Then, the energy functional $I_\lambda : C(\overline{\Omega}) \to (-\infty, +\infty]$ associated to problem (1.1) is

$$I_\lambda = \Psi + \mathcal{G}_\lambda$$

and it has the structure required by Szulkin's critical point theory. Also, by the compactness of $K_0 \subset C(\overline{\Omega})$ it is easy to see that I_λ satisfies the (PS) condition.

From [5, Theorem 2.1], one has the following:

Proposition 2.1. *If a function $u_\lambda \in C(\overline{\Omega})$ is a critical point of I_λ, then it is a solution of problem (1.1). Moreover, I_λ is bounded from below and attains its infimum at some $u_\lambda \in K_0$, which is a critical point of I_λ and hence, a solution of (1.1).*

We briefly recall some classical spectral aspects of the operator $-\Delta$ in the Sobolev space $H_0^1(\Omega)$ - which is seen as being endowed with the usual scalar product

$$(u, v)_1 = \int_\Omega \nabla u \cdot \nabla v\, dx, \quad \text{for all } u, v \in H_0^1(\Omega).$$

A real number $\lambda^\Delta \in \mathbb{R}$ is called an *eigenvalue* of $-\Delta$ in $H_0^1(\Omega)$, if problem

$$\begin{cases} -\Delta u = \lambda^\Delta u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

has a nontrivial weak solution φ, i.e. there exists $\varphi \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla\varphi \cdot \nabla v\, dx = \lambda^\Delta \int_\Omega \varphi v\, dx, \quad \text{for all } v \in H_0^1(\Omega).$$

The solution φ is called *eigenfunction* corresponding to the eigenvalue λ^Δ. It is known that there exists a sequence of eigenvalues $0 < \lambda_1^\Delta < \lambda_2^\Delta \leq \cdots \leq \lambda_j^\Delta \leq \cdots$ (going to $+\infty$) and a sequence of corresponding eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$ defining an orthonormal basis of $H_0^1(\Omega)$. Also, since $\partial\Omega$ is of class C^2 one has that each eigenfunction φ_j belongs to $H^2(\Omega)$ and by a bootstrap argument combining a standard regularity result [14, Theorem 9.15] and the Sobolev embedding theorem [1, Theorem 4.12] we get that φ_j actually belongs to $W^{2,p}(\Omega)$ with some $p > N$. Therefore, φ_j belongs to $C^1(\overline{\Omega})$ and hence $|\nabla\varphi_j| \in C(\overline{\Omega})$ for all $j \in \mathbb{N}$.

Theorem 2.2. *If $\lambda > 2\lambda_m^\Delta$ for some $m \in \mathbb{N}$ and*

$$\liminf_{t\to 0+} \frac{2G(t)}{t^2} \geq 1, \tag{2.2}$$

then problem (1.1) *has at least m distinct pairs of nontrivial solutions.*

Proof. We apply Theorem 1.1 with $Y = C(\overline{\Omega})$ and $\mathcal{I} = I_\lambda$. Set

$$c_1(m) := \left(\sum_{j=1}^m \|\nabla\varphi_j\|_\infty^2\right)^{\frac{1}{2}} \quad \text{and} \quad c_2(m) := \left(\sum_{j=1}^m \|\varphi_j\|_\infty^2\right)^{\frac{1}{2}}.$$

Since $\lambda > 2\lambda_m^\Delta$, we can choose $\varepsilon \in (0,1)$ so that $\lambda > 2\lambda_m^\Delta/(1-\varepsilon)$ and by virtue of (2.2), there exists $\delta > 0$ such that

$$2G(t) \geq (1-\varepsilon)t^2 \quad \text{as } |t| \leq \delta. \tag{2.3}$$

Consider the finite dimensional space

$$X_m := \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_m\},$$

equipped with the norm

$$\|\alpha_1\varphi_1 + \cdots + \alpha_m\varphi_m\|_{X_m} = \left(\alpha_1^2 + \cdots + \alpha_m^2\right)^{\frac{1}{2}}.$$

and let $A_m(\rho)$ be the subset of $C(\overline{\Omega})$ defined by

$$A_m(\rho) := \{v \in X_m : \ \|v\|_{X_m} = \rho\},$$

where ρ is a positive number $\leq \min\left\{\frac{1}{c_1(m)}, \frac{\delta}{c_2(m)}\right\}$. Then, it is easy to see that the odd mapping $H : A_m(\rho) \to S^{m-1}$ defined by

$$H\left(\sum_{k=1}^m \alpha_k\varphi_k\right) = \left(\frac{\alpha_1}{\rho}, \ldots, \frac{\alpha_m}{\rho}\right)$$

is a homeomorphism between $A_m(\rho)$ and S^{m-1} and so, $\gamma(A_m(\rho)) = m$. Hence, $A_m(\rho) \in \Gamma_m \subset 2^{C(\overline{\Omega})}$.

Let $v = \sum_{k=1}^m \alpha_k\varphi_k \in A_m(\rho)$. Clearly, $v|_{\partial\Omega} = 0$ and we have

$$|\nabla v| \leq \sum_{k=1}^m |\alpha_k||\nabla\varphi_k| \leq \left(\sum_{k=1}^m \alpha_k^2\right)^{1/2} \left(\sum_{k=1}^m |\nabla\varphi_k|^2\right)^{1/2} \leq \rho c_1(m).$$

Therefore, as ρ was chosen $\leq 1/c_1(m)$, one get $\|\nabla v\|_\infty \leq 1$, meaning that $v \in K_0$. On the other hand, using that $\{\varphi_j\}_{j\in\mathbb{N}}$ is orthonormal in $H_0^1(\Omega)$, one has

$$\int_\Omega v^2 dx \geq \frac{\rho^2}{\lambda_m^\Delta} \quad \text{and} \quad \int_\Omega |\nabla v|^2 dx = \rho^2. \tag{2.4}$$

Then, from

$$|v| \leq \left(\sum_{k=1}^m \alpha_k^2\right)^{1/2} \left(\sum_{k=1}^m |\varphi_k|^2\right)^{1/2} \leq \rho c_2(m) \leq \delta,$$

together with (2.1), (2.3) and (2.4), we estimate I_λ as follows

$$\begin{aligned} I_\lambda(v) = \Psi(v) + \mathcal{G}_\lambda(v) &\leq \int_\Omega |\nabla v|^2 dx - \frac{\lambda}{2}(1-\varepsilon)\int_\Omega v^2 dx \\ &\leq \rho^2\left(1 - \frac{\lambda(1-\varepsilon)}{2\lambda_m^\Delta}\right) = \rho^2 \frac{2\lambda_m^\Delta - \lambda(1-\varepsilon)}{2\lambda_m^\Delta} < 0. \end{aligned}$$

This yields

$$\inf_{A\in\Gamma_m} \sup_{v\in A} \mathcal{I}_\lambda(v) \leq \sup_{v\in A_m(\rho)} \mathcal{I}_\lambda(v) < 0$$

and, since I_λ is bounded from below, the proof is accomplished by Theorem 1.1 and Proposition 2.1. □

The above theorem can be applied to derive multiplicity of nontrivial solutions for autonomous non-parameterized Dirichlet problems having the form

$$\begin{cases} -\mathcal{M}(u) = f(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.5}$$

where the mapping $f : \mathbb{R} \to \mathbb{R}$ is odd and continuous. We set $F(t) = \int_0^t f(\tau)d\tau$ $(t \in \mathbb{R})$.

Corollary 2.3.

(i) If

$$\liminf_{t\to 0+} \frac{F(t)}{t^2} > \lambda_m^\Delta \tag{2.6}$$

for some $m \in \mathbb{N}$, then problem (2.5) has at least m distinct pairs of nontrivial solutions.

(ii) If

$$\lim_{t\to 0+} \frac{F(t)}{t^2} = +\infty, \tag{2.7}$$

then problem (2.5) has infinitely many distinct pairs of nontrivial solutions.

Proof. (i) By (2.6), there exists $\overline{\lambda}$ such that

$$\liminf_{t\to 0+} \frac{2F(t)}{t^2} \geq \overline{\lambda} > 2\lambda_m^\Delta$$

and the result follows from Theorem 2.2 with $g(t) = f(t)/\overline{\lambda}$.

(ii) This is immediate from (i) and (2.7). □

Example 2.4.

(i) For any $m \in \mathbb{N}$ and $\varepsilon > 0$, problem

$$\begin{cases} -\mathcal{M}(u) = 2(\lambda_m^\Delta + \varepsilon)\sin u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

has at least m distinct pairs of nontrivial solutions.

(ii) If $\alpha \in (0,1)$, then problem

$$\begin{cases} -\mathcal{M}(u) = |u|^{\alpha-1}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

has infinitely many distinct pairs of nontrivial solutions.

Now, we study existence/non-existence and multiplicity of nontrivial solutions for Dirichlet problems involving Fisher-Kolmogorov nonlinearities:

$$\begin{cases} -\mathcal{M}(u) = \lambda u(1 - a|u|^q) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.8}$$

where $a \geq 0$ and $q > 0$ are constants. Notice, in this case one has

$$G(t) = \frac{t^2}{2} - a\frac{|t|^{q+2}}{q+2}, \quad \forall\, t \in \mathbb{R} \tag{2.9}$$

and

$$I_\lambda(u) = \Psi(u) - \lambda \int_\Omega \left[\frac{u^2}{2} - a\frac{|u|^{q+2}}{q+2}\right] dx, \quad u \in C(\overline{\Omega}). \tag{2.10}$$

The next theorem will invoke the constant

$$a_\Omega := \frac{\text{diam}(\Omega)}{2},$$

where $\text{diam}(\Omega)$ stands for the diameter of Ω. Using the mean value theorem, it is straightforward to check that any solution u of a problem of type (1.1) satisfies

$$\|u\|_\infty < a_\Omega. \tag{2.11}$$

Theorem 2.5.

(i) If $\lambda > 2\lambda_m^\Delta$, for some $m \geq 2$, then problem (2.8) has at least m distinct pairs of nontrivial solutions.

(ii) If $\lambda > \lambda_1^\Delta$, then problem (2.8) has at least one pair of nontrivial solutions $(u_\lambda, -u_\lambda)$, with u_λ a minimizer of the corresponding I_λ. In addition, if $a \in [0, a_\Omega^{-q})$, one may suppose that $u_\lambda > 0$ on Ω.

(iii) If $\lambda \in (0, \lambda_1^\Delta]$, the only solution of (2.8) is the trivial one.

Proof. (i) This follows from Theorem 2.2 and (2.9).

(ii) Let $\varphi_1 > 0$ be an eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ corresponding to the first eigenvalue λ_1^Δ and set

$$\psi_1 := \frac{\varphi_1}{\|\nabla\varphi_1\|_\infty}.$$

As $\varphi_1 \in C^1(\overline{\Omega})$, it is clear that $\psi_1 \in K_0 \setminus \{0\}$. Since

$$\lambda_1^\Delta = \frac{\displaystyle\int_\Omega |\nabla\psi_1|^2\,dx}{\displaystyle\int_\Omega \psi_1^2\,dx},$$

we have (as observed in [19]):

$$\lim_{t\to 0^+} \frac{\displaystyle\int_\Omega \left[1-\sqrt{1-|t\nabla\psi_1|^2}\right]dx}{\displaystyle\frac{1}{2}\int_\Omega (t\psi_1)^2 dx} = \lim_{t\to 0^+} \frac{\displaystyle\int_\Omega \frac{t|\nabla\psi_1|^2}{\sqrt{1-|t\nabla\psi_1|^2}}\,dx}{\displaystyle t\int_\Omega \psi_1^2\,dx} = \lambda_1^\Delta. \tag{2.12}$$

Now, let $\lambda > \lambda_1^\Delta$ and let us fix some $\varepsilon > 0$ with $\lambda_1^\Delta < \lambda - \varepsilon$. On account of (2.12), there exists $t_{\lambda,\varepsilon} \in (0,1)$ such that

$$\frac{\displaystyle\int_\Omega \left[1-\sqrt{1-|t\nabla\psi_1|^2}\right]dx}{\displaystyle\frac{1}{2}\int_\Omega (t\psi_1)^2 dx} < \lambda - \varepsilon, \quad \forall\, t \in (0, t_{\lambda,\varepsilon}). \tag{2.13}$$

Next, from (2.13) and taking $t^*_{\lambda,\varepsilon} \in (0, t_{\lambda,\varepsilon})$ with

$$\lambda a \frac{(t^*_{\lambda,\varepsilon}\psi_1(x))^q}{q+2} < \frac{\varepsilon}{2}, \quad \forall\, x \in \overline{\Omega},$$

we estimate I_λ in (2.10) as follows

$$\begin{aligned}
I_\lambda(t^*_{\lambda,\varepsilon}\psi_1) &= \Psi(t^*_{\lambda,\varepsilon}\psi_1) - \lambda \int_\Omega \left[\frac{(t^*_{\lambda,\varepsilon}\psi_1)^2}{2} - a\frac{(t^*_{\lambda,\varepsilon}\psi_1)^{q+2}}{q+2}\right]dx \\
&= \int_\Omega \left[1-\sqrt{1-|\nabla(t^*_{\lambda,\varepsilon}\psi_1)|^2}\right]dx - \lambda\int_\Omega \left[\frac{(t^*_{\lambda,\varepsilon}\psi_1)^2}{2} - a\frac{(t^*_{\lambda,\varepsilon}\psi_1)^{q+2}}{q+2}\right]dx \\
&< \frac{\lambda-\varepsilon}{2}\int_\Omega (t^*_{\lambda,\varepsilon}\psi_1)^2 dx - \frac{\lambda}{2}\int_\Omega (t^*_{\lambda,\varepsilon}\psi_1)^2 dx + \lambda\int_\Omega a\frac{(t^*_{\lambda,\varepsilon}\psi_1)^{q+2}}{q+2}\,dx \\
&= \int_\Omega (t^*_{\lambda,\varepsilon}\psi_1)^2 \left[\lambda a\frac{(t^*_{\lambda,\varepsilon}\psi_1)^q}{q+2} - \frac{\varepsilon}{2}\right]dx < 0 = I_\lambda(0).
\end{aligned}$$

From Proposition 2.1 we infer that, if $\lambda > \lambda_1^\Delta$, the even functional I_λ attains its infimum at some $u_\lambda \in K_0 \setminus \{0\}$, hence problem (2.8) has a pair of nontrivial solutions $(u_\lambda, -u_\lambda)$. Since $|u_\lambda|$ is still a minimizer of I_λ, it also solves (2.8) and, taking into account (2.11), we obtain

$$-\mathcal{M}(|u_\lambda|) = \lambda|u_\lambda|(1-a|u_\lambda|^q) \geq \lambda|u_\lambda|\left(1-a\,a_\Omega^q\right).$$

Then, since $|u_\lambda| > 0$ in a subset of Ω having positive measure, from [11, Lemma 2.6] it follows that actually $|u_\lambda| > 0$ in the whole Ω.

(iii) Assume, by contradiction, that for such a λ, a function u is a nontrivial solution of (2.8). On account of (1.2), one gets

$$\lambda \int_\Omega u^2(1 - a|u|^q)\,dx = \int_\Omega \frac{|\nabla u|^2}{\sqrt{1-|\nabla u|^2}}\,dx \geq \int_\Omega |\nabla u|^2\,dx \geq \lambda_1^\Delta \int_\Omega u^2\,dx. \tag{2.14}$$

If $a > 0$, we have

$$0 > -\lambda a \int_\Omega |u|^{q+2}\,dx \geq (\lambda_1^\Delta - \lambda) \int_\Omega u^2\,dx \geq 0,$$

i.e. a contradiction. In the case $a = 0$, if $\lambda < \lambda_1^\Delta$, as above we obtain the contradiction

$$0 \geq (\lambda_1^\Delta - \lambda) \int_\Omega u^2\,dx > 0.$$

Also, if $\lambda = \lambda_1^\Delta$, from (2.14) (with $a = 0$) we have that

$$\int_\Omega |\nabla u|^2 \left(\frac{1}{\sqrt{1-|\nabla u|^2}} - 1 \right)\,dx = 0,$$

or,

$$\int_\Omega \frac{|\nabla u|^4}{\left(1+\sqrt{1-|\nabla u|^2}\right)\sqrt{1-|\nabla u|^2}}\,dx = 0$$

which, since $u \in C^1(\overline{\Omega})$, implies $|\nabla u| = 0$ on $\overline{\Omega}$. It follows that u is constant and then, as $u \in K_0$, we infer that $u \equiv 0$ – a contradiction. Hence, (2.8) has only the trivial solution provided that $\lambda \in (0, \lambda_1^\Delta]$ and the proof is now complete. □

Remark 2.6. (i) It is worth noticing that in the particular case $a = 0$, Theorem 2.5 recovers and improves the main result of paper [19], which states that problem

$$\begin{cases} -\mathcal{M}(u) = \lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

has a nontrivial solution iff $\lambda > \lambda_1^\Delta$ and for such a λ, a nontrivial solution can be chosen to be nonnegative on Ω and to minimize the corresponding I_λ.

(ii) In Theorem 2.5 it is assumed: if $m = 1$, $\lambda > \lambda_m^\Delta$, and if $m > 1$, $\lambda > 2\lambda_m^\Delta$, instead of $\lambda > \lambda_m^\Delta$. This comes from the fact that in Theorem 2.2 we were not able to prove that $\lambda > 2\lambda_m$ can be replaced by the weaker condition $\lambda > \lambda_m^\Delta$. Actually, at the moment it is not clear that this can be done under assumption (2.2) – this remains an open problem. Nevertheless, it is worth to point out that Theorem 2.2 yields the following: problem (1.1) has at least m ($\in \mathbb{N}$) distinct pairs of nontrivial solutions if $\lambda > \lambda_m^\Delta$ and

$$\liminf_{t \to 0+} \frac{G(t)}{t^2} \geq 1. \tag{2.15}$$

To see this, rewrite the equation in (1.1) as

$$-\mathcal{M}(u) = 2\lambda \tilde{g}(u) \quad \text{in } \Omega,$$

with $\tilde{g}(u) = g(u)/2$ and apply Theorem 2.2. In this form this seems to allow in Theorem 2.5 the more natural assumption $\lambda > \lambda_m^\Delta$, instead of $\lambda > 2\lambda_m^\Delta$, for $m > 1$. However, this cannot be applied to problem (2.8) since G defined in (2.9) does not satisfy (2.15).

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References

[1] R. Adams, J. Fournier, *Sobolev Spaces*, Second edition, Pure and applied Mathematics, Volume 140, Academic Press, 2003. MR2424078

[2] A. Azzollini, Ground state solution for a problem with mean curvature operator in Minkowski space, *J. Funct. Anal.* **26**(2014), 2086–2095. https://doi.org/10.1016/j.jfa.2013.10.002; MR3150152

[3] A. Azzollini, On a prescribed mean curvature equation in Lorentz–Minkowski space, *J. Math. Pures Appl.* **106**(2016), 1122–1140. https://doi.org/10.1016/j.matpur.2016.04.003; MR3565417

[4] R. Bartnik, L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, *Comm. Math. Phys.* **87**(1982-83), 131–152. https://doi.org/10.1007/BF01211061; MR0680653

[5] C. Bereanu, P. Jebelean, J. Mawhin, The Dirichlet problem with mean curvature operator in Minkowski space – a variational approach, *Adv. Nonlinear Stud.* **14**(2014), 315–326. https://doi.org/10.1515/ans-2014-0204; MR3194356

[6] C. Bereanu, P. Jebelean, J. Mawhin, Corrigendum to: "The Dirichlet problem with mean curvature operator in Minkowski space – a variational approach, *Adv. Nonlinear Stud.* **14**(2014), 315–326", *Adv. Nonlinear Stud.* **16**(2016), 173–174. https://doi.org/10.1515/ans-2015-5030; MR3456755

[7] C. Bereanu, P. Jebelean, C. Șerban, The Dirichlet problem for discontinuous perturbations of the mean curvature operator in Minkowski space, *Electron. J. Qual. Theory Differ. Equ.* **2015**, No. 35, 1–7. https://doi.org/10.14232/ejqtde.2015.1.35; MR3371438

[8] J. Chaparova, L. A. Peletier, S. Tersian, Existence and nonexistence of nontrivial solutions of semilinear fourth and sixth order differential equations, *Appl. Math. Lett.* **17**(2004), 1207–1212. https://doi.org/10.1016/j.aml.2003.05.014; MR2016681

[9] C. Corsato, F. Obersnel, P. Omari, The Dirichlet problem for gradient dependent prescribed mean curvature equations in the Lorenz–Minkowski space, *Georgian Math. J.* **24**(2017), 113–134. https://doi.org/10.1515/gmj-2016-0078; MR3607245

[10] C. Corsato, F. Obersnel, P. Omari, S. Rivetti, On the lower and upper solution method for the prescribed mean curvature equation in Minkowski space, *Discrete Contin. Dyn. Syst.*, Dynamical systems, differential equations and applications. 9th AIMS Conference. Suppl. (2013), 159–169. https://doi.org/10.3934/proc.2013.2013.159; MR3462363

[11] C. Corsato, F. Obersnel, P. Omari, S. Rivetti, Positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space, *J. Math. Anal. Appl.* **405**(2013), 227–239. https://doi.org/10.1016/j.jmaa.2013.04.003; MR3053503

[12] P. C. Fife, J. B. McLeod, The approach of solutions of nonlinear diffusion equations to traveling wave solutions, *Arch. Rational. Mech. Anal.* **65**(1977), 335–362. https://doi.org/10.1007/BF00250432; MR0442480

[13] R. A. Fisher, The wave of advance of advantageous genes, *Ann. Eugen.* **7**(1937), 335–369. https://doi.org/10.1111/j.1469-1809.1937.tb02153.x

[14] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001. https://doi.org/10.1007/978-3-642-61798-0; MR1814364

[15] D. Gurban, P. Jebelean, Positive radial solutions for multiparameter Dirichlet systems with mean curvature operator in Minkowski space and Lane–Emden type nonlinearities, *J. Differential Equations* **266**(2019), 5377–5396. https://doi.org/10.1016/j.jde.2018.10.030; MR3912753

[16] D. Gurban, P. Jebelean, C. Şerban, Non-potential and non-radial Dirichlet systems with mean curvature operator in Minkowski space, *Discrete Contin. Dyn. Syst.* **40**(2020), 133–151. https://doi.org/10.3934/dcds.2020006; MR4026955

[17] P. Jebelean, C. Şerban, Fisher-Kolmogorov type perturbations of the relativistic operator: differential vs. difference, *Proc. Amer. Math. Soc.* **144**(2018), 2005–2014. https://doi.org/10.1090/proc/13978; MR3767352

[18] A. Kolmogorov, I. Petrovski, N. Piscounov, Étude de l'équation de la diffusion avec croissance de la quantité de matière at son application à un problème biologique, *Bull. Univ. d'État à Moscou, Sér. Int., Sect. A* **1**(1937), 1–25.

[19] M. Mihăilescu, The spectrum of the mean curvature operator, *Proc. Roy. Soc. Edinburgh Sect. A Mathematics* **2020**, 1–13. https://doi.org/10.1017/prm.2020.25

[20] L. A. Peletier, W. C. Troy, A topological shooting method and the existence of kinks of the extended Fisher–Kolmogorov equation, *Topol. Methods Nonlinear Anal.* **6**(1995), 331–355. MR1399544

[21] L. A. Peletier, W. C. Troy, Spatial patterns described by the extended Fisher–Kolmogorov equation: periodic solutions, *SIAM J. Math. Anal.* **28**(1997), 1317–1353. https://doi.org/10.1137/S0036141095280955; MR1474217

[22] L. A. Peletier, W. C. Troy, *Spatial patterns: higher order models in physics and mechanics*, Birkäuser, Boston, 2001. https://doi.org/10.1007/978-1-4612-0135-9

[23] L. A. Peletier, W. C. Troy, R. Van der Vorst, Stationary solutions of a fourth-order nonlinear diffusion equation, *Differential Equations* **31**(1995), No. 2, 301–314. MR1373793

[24] P. H. Rabinowitz, Variational methods for nonlinear eigenvalue problems, chapter in: *Eigenvalues of Non-Linear Problems*, G. Prodi (ed.), Vol. 67, *C.I.M.E. Summer Schools*, 1974, pp. 139–195.

[25] P. H. Rabinowitz, *Some aspects of critical point theory*, MRC Tech. Rep. #2465, Madison, Wisconsin, 1983.

[26] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3**(1986), 77–109. MR0837231

[27] S. Tersian, J. Chaparova, Periodic and homoclinic solutions of extended Fisher-Kolmogorov equations, *J. Math. Anal. Appl.* **260**(2001), No. 2, 490–506. https://doi.org/10.1006/jmaa.2001.7470; MR1845566