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Implicit elliptic equations via Krasnoselskii–Schaefer type theorems

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. Existence of solutions to the Dirichlet problem for implicit elliptic equations is established by using Krasnoselskii–Schaefer type theorems owed to Burton–Kirk and Gao–Li–Zhang. The nonlinearity of the equations splits into two terms: one term depending on the state, its gradient and the elliptic principal part is Lipschitz continuous, and the other one only depending on the state and its gradient has a superlinear growth and satisfies a sign condition. Correspondingly, the associated operator is a sum of a contraction with a completely continuous mapping. The solutions are found in a ball of a Lebesgue space of a sufficiently large radius established by the method of *a priori* bounds.

Keywords: implicit elliptic equation, fixed point, Krasnoselskii theorem for the sum of two operators.

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1 Introduction

Krasnoselskii's fixed point theorem for the sum of two operators [12] – a typical hybrid fixed point result – has been used to prove the existence of solutions for many classes of problems when the associated operators do not comply to a pure fixed point principle. Its hybrid character is given by a combination of the Banach and Schauder fixed point theorems.

Theorem 1.1 (Krasnoselskii)**.** *Let D be a bounded closed convex nonempty subset of a Banach space $(X, |\cdot|)$ and let A, B be two operators such that*

(i) *$A : D \to X$ is a contraction;*

(ii) *$B : D \to X$ is continuous with $B(D)$ relatively compact;*

(iii) *$A(x) + B(y) \in D$ for every $x, y \in D$.*

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Then the operator $A+B$ has at least one fixed point, i.e., there exists $x \in D$ such that $x = A(x) + B(x)$.

There are many extensions of Krasnoselskii's theorem in several directions, for single and multi-valued mappings, self and non-self mappings, for generalized contractions and generalized compact-type operators, see for example [2,5,6,10,14,18].

The strong invariance condition (iii) is required by the similar condition from Schauder's fixed point theorem. The last one is removed and replaced with the Leray–Schauder boundary condition by Schaefer's fixed point theorem [17].

Theorem 1.2 (Schaefer). *Let D_R be the closed ball centered at the origin and of radius R of a Banach space X, and let $N : D_R \to X$ be continuous with $N(D_R)$ relatively compact. If*

$$\lambda N(x) \neq x \quad \text{for all } x \in \partial D_R,\ \lambda \in (0,1), \tag{1.1}$$

then N has at least one fixed point.

There are known hybrid theorems of Krasnoselskii type that combine Banach's contraction principle with Schaefer's fixed point theorem. Such a result is owed to Burton and Kirk [6].

Theorem 1.3 (Burton–Kirk). *Let D_R be the closed ball centered at the origin and of radius R of a Banach space X, and let A, B be operators such that*

(j) *$A : X \to X$ is a contraction;*

(jj) *$B : D_R \to X$ is continuous with $B(D_R)$ relatively compact;*

(jjj) *$x \neq \lambda A\left(\frac{1}{\lambda}x\right) + \lambda B(x)$ for all $x \in \partial D_R$ and $\lambda \in (0,1)$.*

Then the operator $A+B$ has at least one fixed point, i.e., there exists $x \in D_R$ such that $x = A(x) + B(x)$.

A similar result is owed to Gao, Li and Zhang [11].

Theorem 1.4 (Gao–Li–Zhang). *Let D_R be the closed ball centered at the origin and of radius R of a Banach space X, and let A, B be operators such that*

(h) *$A : X \to X$ is a contraction;*

(hh) *$B : D_R \to X$ is continuous with $B(D_R)$ relatively compact;*

(hhh) *$x \neq A(x) + \lambda B(x)$ for all $x \in \partial D_R$ and $\lambda \in (0,1)$.*

Then the operator $A+B$ has at least one fixed point, i.e., there exists $x \in D_R$ such that $x = A(x) + B(x)$.

In proof, the difference between Theorem 1.3 and Theorem 1.4 consists in the homotopy that is considered. In the first case, the homotopy is $\lambda(I-A)^{-1}B$, while in the second case, it is $(I-A)^{-1}\lambda B$.

Obviously, if A is identically zero, then both results by Burton–Kirk and Gao–Li–Zhang reduce to Schaefer's theorem.

Remark 1.5 (Method of *a priori* bounds)**.** In applications, usually both operators A, B are defined on the whole space X and a ball D_R as required by condition (jjj) of Theorem 1.3 and (hhh) of Theorem 1.4 exists if the set of all solutions for $\lambda \in (0,1)$ of the equations

$$x = \lambda A\left(\frac{1}{\lambda}x\right) + \lambda B(x)$$

and

$$x = A(x) + \lambda B(x),$$

respectively, is bounded in X.

The aim of this paper is to give an application of the previous Krasnoselskii–Schaefer type theorems to the Dirichlet problem for implicit elliptic equations. Such equations have been intensively studied in the literature, see for example [7, 9]. Our result extends and complements previous contributions in this direction such as those in [4, 13, 15, 16].

We conclude the Introduction by some basic notions and results from the linear theory of partial differential equations [3, 16].

We shall work in the Sobolev space $H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is open bounded, endowed with the energetic norm

$$|u|_{H_0^1} = |\nabla u|_{L^2} = \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}.$$

Its dual space is $H^{-1}(\Omega)$ and the pairing of a functional $v \in H^{-1}(\Omega)$ and a function $u \in H_0^1(\Omega)$ is denoted by (v, u). We identify $L^2(\Omega)$ to its dual and thus we have $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. Then, in particular, for $v \in L^2(\Omega)$, one has

$$(v, u) = (v, u)_{L^2} = \int_\Omega uv, \quad u \in H_0^1(\Omega).$$

Recall that the operator $(-\Delta)^{-1}$ is an isometry between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, so

$$|v|_{H^{-1}} = \left|(-\Delta)^{-1} v\right|_{H_0^1}, \quad v \in H^{-1}(\Omega).$$

Also, the embedding $H_0^1(\Omega) \subset L^p(\Omega)$ holds and is continuous for $1 \leq p \leq 2^* = 2n/(n-2)$, and the same happens for the embedding $L^q(\Omega) \subset H^{-1}(\Omega)$ if $q \geq (2^*)' = 2n/(n+2)$. These embeddings are compact for $p < 2^*$ and $q > (2^*)'$, respectively.

2 Application

We discuss here the Dirichlet problem for implicit nonlinear elliptic equations,

$$\begin{cases} -\Delta u = f(x, u, \nabla u, \Delta u) + g(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is open bounded $(n \geq 3)$; $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy the Carathéodory conditions.

To give sense to the composition $f(x, u, \nabla u, \Delta u)$, we need to look for solutions $u \in H_0^1(\Omega)$ such that Δu is a function. More exactly we shall require that $\Delta u \in L^q(\Omega)$ for a given number $q \geq (2^*)'$.

If we let $v := -\Delta u$, then the equation becomes

$$v = f\Big(x, (-\Delta)^{-1}v, \nabla(-\Delta)^{-1}v, -v\Big) + g\Big(x, (-\Delta)^{-1}v, \nabla(-\Delta)^{-1}v\Big).$$

As noted above, this equation will be solved in a Lebesgue space $L^q(\Omega)$ with $q \geq (2^*)'$. We assume in addition that $q \leq 2$, which implies $L^2(\Omega) \subset L^q(\Omega)$.

Let $A, B : L^q(\Omega) \to L^q(\Omega)$ be given by

$$A(v) = f\Big(\cdot, (-\Delta)^{-1}v, \nabla(-\Delta)^{-1}v, -v\Big)$$
$$B(v) = g\Big(\cdot, (-\Delta)^{-1}v, \nabla(-\Delta)^{-1}v\Big).$$

Clearly we need some additional conditions on f and g to guarantee that the two operators are well-defined from $L^q(\Omega)$ to itself, and then, wishing to apply Theorem 1.3 or Theorem 1.4 we have to guarantee that A is a contraction, and B is completely continuous.

We begin by a technical lemma concerning the embedding constants. By an *embedding constant* for a continuous embedding $X \subset Y$ of two Banach spaces $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$, we mean a number $c > 0$ such that

$$|x|_Y \leq c|x|_X \quad \text{for every} \quad x \in X.$$

Note that if c is an embedding constant for the inclusion $X \subset Y$, then c is also an embedding constant for the dual inclusion $Y' \subset X'$. Indeed, for any $u \in Y'$, one has

$$|u|_{X'} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|(u,x)|}{|x|_X} \leq \sup_{\substack{x \in X \\ x \neq 0}} \frac{|(u,x)|}{c^{-1}|x|_Y} \leq c \sup_{\substack{x \in Y \\ x \neq 0}} \frac{|(u,x)|}{|x|_Y} = c|u|_{Y'}.$$

Recall that, according to the Poincaré inequality, the best (smallest) embedding constant for the inclusions $H_0^1(\Omega) \subset L^2(\Omega)$ and $L^2(\Omega) \subset H^{-1}(\Omega)$ is $1/\sqrt{\lambda_1}$, where λ_1 is the first eigenvalue of the Dirichlet problem for the operator $-\Delta$.

Lemma 2.1. *Let* $(2^*)' \leq q \leq 2$ *and let* c_1, c_2, c_3 *be embedding constants for the inclusions*

$$H_0^1(\Omega) \subset L^q(\Omega), \quad L^2(\Omega) \subset L^q(\Omega), \quad L^q(\Omega) \subset H^{-1}(\Omega). \tag{2.2}$$

Then one may consider

$$c_2 = c_1\sqrt{\lambda_1}, \quad c_3 = \frac{1}{c_1\lambda_1}.$$

Proof. From $H_0^1(\Omega) \subset L^2(\Omega) \subset L^q(\Omega)$, if $u \in H_0^1(\Omega)$, one has

$$|u|_{L^q} \leq c_2|u|_{L^2} \leq \frac{c_2}{\sqrt{\lambda_1}}|u|_{H_0^1},$$

hence $c_1 = c_2/\sqrt{\lambda_1}$, or $c_2 = c_1\sqrt{\lambda_1}$. To prove the second equality, let $u \in H_0^1(\Omega)$. On the one hand, using twice Poincaré's inequality, we have

$$|u|_{H^{-1}} \leq \frac{1}{\sqrt{\lambda_1}}|u|_{L^2} \leq \frac{1}{\lambda_1}|u|_{H_0^1},$$

and on the other hand,

$$|u|_{H^{-1}} \leq c_3|u|_{L^q} \leq c_1c_3|u|_{H_0^1}.$$

Hence $c_1c_3 = 1/\lambda_1$. □

The next lemma guarantees that the operator A is a contraction.

Lemma 2.2. *Assume that there exist constants $a,b,c \geq 0$ such that*

$$|f(x,y,z,w) - f(x,\overline{y},\overline{z},\overline{w})| \leq a|y-\overline{y}| + b|z-\overline{z}| + c|w-\overline{w}|$$

for all $y,\overline{y},w,\overline{w} \in \mathbb{R}$; $z,\overline{z} \in \mathbb{R}^n$ and a.a. $x \in \Omega$. Also assume that $f(\cdot,0,0,0) \in L^2(\Omega)$. If

$$l := \frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} + c < 1,$$

then A is a contraction on the space $L^q(\Omega)$ for any $q \in [1,2]$.

Proof. From the basic result about Nemytskii's operator (see, a.e., [16]), we have that A maps $L^q(\Omega)$ to itself. Let $v,w \in L^q(\Omega)$. Then using the embedding constants for the inclusions (2.2) and the relationships between them given by Lemma 2.1, we have

$$\begin{aligned}
|A(v)-A(w)|_{L^q} &\leq a\left|(-\Delta)^{-1}(v-w)\right|_{L^q} + b\left|\nabla(-\Delta)^{-1}(v-w)\right|_{L^q} + c|v-w|_{L^q} \\
&\leq ac_1\left|(-\Delta)^{-1}(v-w)\right|_{H_0^1} + bc_2\left|\nabla(-\Delta)^{-1}(v-w)\right|_{L^2} + c|v-w|_{L^q} \\
&= ac_1|v-w|_{H^{-1}} + bc_2\left|(-\Delta)^{-1}(v-w)\right|_{H_0^1} + c|v-w|_{L^q} \\
&= (ac_1+bc_2)|v-w|_{H^{-1}} + c|v-w|_{L^q} \\
&\leq ((ac_1+bc_2)c_3+c)|v-w|_{L^q} \\
&= \left(\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} + c\right)|v-w|_{L^q}.
\end{aligned}$$

□

Furthermore, we have the following result about the complete continuity of the operator B on the space $L^q(\Omega)$.

Lemma 2.3. *Assume that there exist constants $a_0,b_0 \geq 0$; $\alpha \in [1, 2^*/(2^*)')$, $\beta \in [1, 2/(2^*)')$; and function $h \in L^2(\Omega)$ such that*

$$|g(x,y,z)| \leq a_0|y|^\alpha + b_0|z|^\beta + h(x) \tag{2.3}$$

for all $y \in \mathbb{R}$, $z \in \mathbb{R}^n$ and a.a. $x \in \Omega$. Then the operator $B : L^q(\Omega) \to L^q(\Omega)$ is well-defined and completely continuous for

$$q = \min\left\{\frac{2^*}{\alpha}, \frac{2}{\beta}\right\}. \tag{2.4}$$

Proof. First note that the restrictions on α and β imply that q given by (2.4) satisfies $(2^*)' < q \leq 2$.

Now the operator B is the composition NPJ of three operators

$$\begin{aligned}
&J : L^q(\Omega) \to H^{-1}(\Omega), && J(v) = v \\
&P : H^{-1}(\Omega) \to L^{2^*}(\Omega) \times L^2(\Omega;\mathbb{R}^n), && P(v) = \left((-\Delta)^{-1}v,\ \nabla(-\Delta)^{-1}v\right) \\
&N : L^{2^*}(\Omega) \times L^2(\Omega;\mathbb{R}^n) \to L^q(\Omega), && N(u,v) = g(\cdot,u,v).
\end{aligned}$$

Here J is completely continuous since the embedding $L^q(\Omega) \subset H^{-1}(\Omega)$ is compact $\left(q > (2^*)'\right)$, and obviously, the linear operator P is bounded. Next we show that N is well-defined, continuous and bounded (maps bounded sets into bounded sets). According to the basic result about Nemytskii's operator, this happens if we have a growth condition on g of the form

$$|g(x,w_1,w_2)| \le a_0|w_1|^{\frac{2^*}{q}} + b_0|w_2|^{\frac{2}{q}} + h_0(x) \quad (w_1 \in \mathbb{R},\ w_2 \in \mathbb{R}^n,\ \text{a.a. } x \in \Omega) \tag{2.5}$$

with $a_0, b_0 \in \mathbb{R}_+$ and $h_0 \in L^q(\Omega)$. From (2.4), we have

$$1 \le \alpha \le \frac{2^*}{q}, \quad 1 \le \beta \le \frac{2}{q}.$$

Then the exponents α, β in (2.3) can be replaced by the larger ones $2^*/q$ and $2/\beta$ and thus the growth condition (2.3) implies (2.5), with a suitable function h_0 that incorporates h. Hence N has the desired properties.

The above properties of the operators J, P and N imply that B is well-defined and completely continuous from $L^q(\Omega)$ to itself. □

It remains to find *a priori* bounds of the solutions as required by Remark 1.5.

Lemma 2.4. *Under the assumptions of Lemmas 2.2 and 2.3, if in addition g satisfies the sign condition*

$$yg(x,y,z) \le 0 \tag{2.6}$$

for all $y \in \mathbb{R}$, $z \in \mathbb{R}^n$ and a.a. $x \in \Omega$, then the sets of solutions of the equations

$$v = \lambda A\left(\frac{1}{\lambda}v\right) + \lambda B(v) \quad (\lambda \in (0,1)) \tag{2.7}$$

and of the equations

$$v = A(v) + \lambda B(v) \quad (\lambda \in (0,1)) \tag{2.8}$$

are bounded in $L^q(\Omega)$.

Proof. We shall prove the statement for the family of equations (2.7). The proof is similar for (2.8).

Step 1: We first prove the boundedness of the solutions in $H^{-1}(\Omega)$. Let $v \in L^q(\Omega)$ be any solution of (2.7). Since $v \in H^{-1}(\Omega)$, we may write

$$\left(v, (-\Delta)^{-1}v\right) = \lambda\left(A\left(\frac{1}{\lambda}v\right), (-\Delta)^{-1}v\right) + \lambda\left(B(v), (-\Delta)^{-1}v\right). \tag{2.9}$$

On the left side we have $\left|(-\Delta)^{-1}v\right|^2_{H^1_0}$ which is equal to $|v|^2_{H^{-1}}$. Also, from (2.6) we have

$$\left(B(v), (-\Delta)^{-1}v\right) = \int_\Omega g\left(x, (-\Delta)^{-1}v, \nabla(-\Delta)^{-1}v\right)(-\Delta)^{-1}v \le 0.$$

Next, using the Lipschitz property of f, and denoting $\gamma_0 := |f(\cdot,0,0,0)|_{L^2}$ we obtain

$$
\begin{aligned}
&\lambda\left(A\left(\frac{1}{\lambda}v\right),(-\Delta)^{-1}v\right)\\
&\quad= \lambda\int_\Omega f\left(x,\frac{1}{\lambda}(-\Delta)^{-1}v,\frac{1}{\lambda}\nabla(-\Delta)^{-1}v,-\frac{1}{\lambda}v\right)(-\Delta)^{-1}v\\
&\quad\le \int_\Omega\left(a\left|(-\Delta)^{-1}v\right|+b\left|\nabla(-\Delta)^{-1}v\right|+c|v|+|f(x,0,0,0)|\right)\left|(-\Delta)^{-1}v\right|\\
&\quad\le a\left|(-\Delta)^{-1}v\right|_{L^2}^2+b\left|\nabla(-\Delta)^{-1}v\right|_{L^2}\left|(-\Delta)^{-1}v\right|_{L^2}\\
&\qquad+c\int_\Omega|v|\left|(-\Delta)^{-1}v\right|+\gamma_0\left|(-\Delta)^{-1}v\right|_{L^2}\\
&\quad\le \frac{a}{\lambda_1}\left|(-\Delta)^{-1}v\right|_{H_0^1}^2+\frac{b}{\sqrt{\lambda_1}}\left|(-\Delta)^{-1}v\right|_{H_0^1}^2\\
&\qquad+c\int_\Omega|v|\left|(-\Delta)^{-1}v\right|+\frac{1}{\sqrt{\lambda_1}}\gamma_0\left|(-\Delta)^{-1}v\right|_{H_0^1}\\
&\quad= \frac{a}{\lambda_1}|v|_{H^{-1}}^2+\frac{b}{\sqrt{\lambda_1}}|v|_{H^{-1}}^2+c\int_\Omega|v|\left|(-\Delta)^{-1}v\right|+\frac{1}{\sqrt{\lambda_1}}\gamma_0|v|_{H^{-1}}.
\end{aligned}
$$

Since

$$\int_\Omega|v|\left|(-\Delta)^{-1}v\right|=\left(v,s(-\Delta)^{-1}v\right),$$

where function s has only two values ±1 giving the sign of $v(-\Delta)^{-1}v$, we then have

$$\int_\Omega|v|\left|(-\Delta)^{-1}v\right|\le|v|_{H^{-1}}\left|s(-\Delta)^{-1}v\right|_{H_0^1}=|v|_{H^{-1}}\left|(-\Delta)^{-1}v\right|_{H_0^1}=|v|_{H^{-1}}^2.$$

It follows that

$$\lambda\left(A\left(\frac{1}{\lambda}v\right),(-\Delta)^{-1}v\right)\le\left(\frac{a}{\lambda_1}+\frac{b}{\sqrt{\lambda_1}}+c\right)|v|_{H^{-1}}^2+d|v|_{H^{-1}},$$

where $d=\gamma_0/\sqrt{\lambda_1}$. Thus (2.9) gives

$$|v|_{H^{-1}}^2\le l|v|_{H^{-1}}^2+d|v|_{H^{-1}}$$

which based on $l<1$ implies that

$$|v|_{H^{-1}}\le C_1, \tag{2.10}$$

where $C_1=d/(1-l)$ does not depend on λ.

Step 2. $|B(v)|_{L^q}\le C_2$ for some constant C_2. Indeed, one has

$$|B(v)|_{L^q}\le a_0\left|\left|(-\Delta)^{-1}v\right|^\alpha\right|_{L^q}+b_0\left|\left|\nabla(-\Delta)^{-1}v\right|^\beta\right|_{L^q}+|h|_{L^q} \tag{2.11}$$

Furthermore, since $\alpha q\le 2^*$, we have the continuous embedding $H_0^1(\Omega)\subset L^{\alpha q}(\Omega)$, and so for some constant $\bar{c}$, we have

$$\left|\left|(-\Delta)^{-1}v\right|^\alpha\right|_{L^q}=\left|(-\Delta)^{-1}v\right|_{L^{\alpha q}}^\alpha\le\bar{c}\left|(-\Delta)^{-1}v\right|_{H_0^1}^\alpha=\bar{c}|v|_{H^{-1}}^\alpha. \tag{2.12}$$

Similarly, since $\beta q \leq 2$, we have

$$\begin{aligned}\left|\left|\nabla(-\Delta)^{-1}v\right|^{\beta}\right|_{L^q} &= \left|\nabla(-\Delta)^{-1}v\right|^{\beta}_{L^{\beta q}} \leq \bar{\bar{c}}\left|\nabla(-\Delta)^{-1}v\right|^{\beta}_{L^2} \\ &= \bar{\bar{c}}\left|(-\Delta)^{-1}v\right|^{\beta}_{H^1_0} = \bar{\bar{c}}|v|^{\beta}_{H^{-1}}.\end{aligned} \tag{2.13}$$

Now (2.10)–(2.13) lead to the conclusion at Step 2.

Step 3. $|v|_{L^q} \leq C$ for some constant C. Indeed, if $\gamma = |f(\cdot,0,0,0)|_{L^q}$, then one has

$$|v|_{L^q} \leq \lambda\left|A\left(\frac{1}{\lambda}v\right)\right|_{L^q} + \lambda|B(v)|_{L^q} \leq l|v|_{L^q} + \gamma + |B(v)|_{L^q}.$$

Hence

$$|v|_{L^q} \leq \frac{1}{1-l}(|B(v)|_{L^q} + \gamma),$$

which together with the result at Step 2 gives the conclusion with $C = (C_2 + \gamma)\,/(1-l)$. □

The above lemmas together with Theorem 1.3 (or alternatively, Theorem 1.4) and Remark 1.5 allow us to state the following existence result.

Theorem 2.5. *If f and g satisfy the conditions in Lemmas 2.2–2.4, then problem (2.1) has at least one solution $u \in H^1_0(\Omega)$ with $\Delta u \in L^q(\Omega)$, where $q = \min\{2^*/\alpha,\ 2/\beta\}$.*

Remark 2.6. The sign condition (2.6) can be replaced by

$$yg(x,y,z) \leq \sigma y^2$$

for all $y \in \mathbb{R}$, $z \in \mathbb{R}^n$ and a.a. $x \in \Omega$, for some $\sigma < (1-l)\lambda_1$.

Remark 2.7. If $g(x,y,z)$ has a linear growth in $y,\ z$ with constants a_0 and b_0, and

$$\frac{a+a_0}{\lambda_1} + \frac{b+b_0}{\sqrt{\lambda_1}} + c < 1,$$

then the conclusion of Theorem 2.5 can be obtain using Krasnoselskii's theorem, without a sign condition on g. This happens, since in this case, it is possible to find a ball of $L^q(\Omega)$ of a sufficiently large radius such that the strong invariance condition of Krasnoselskii's theorem is fulfilled.

Finally we would like to mention that the result can be adapted to a general elliptic operator replacing the Laplacian, and the technique is possible to be used for treating other classes of implicit differential equations.

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