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On the solvability of some discontinuous functional impulsive problems

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

Feliz Minhós[1], **Milan Tvrdý**[2] and **Mirosława Zima**[✉3]

[1]Departamento de Matemática, Escola de Ciências e Tecnologia, Universidade de Évora, Rua Romão Ramalho, 59, PT 7000-671 Évora, Portugal
[2]Institute of Mathematics, Czech Academy of Sciences, Žitná 25, CZ 115 67 Praha 1, Czech Republic
[3]Institute of Mathematics, University of Rzeszów, Pigonia 1, PL 35-959 Rzeszów, Poland

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Abstract. This paper deals with impulsive problems consisting of second order differential equation with impulsive effects depending implicitly on the solution and with rather general nonlocal boundary conditions. The arguments are based on the lower and upper solutions method and a fixed point theorem.

Keywords: functional problems, generalized impulsive conditions, upper and lower solutions, fixed point theory, boundary value problem.

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1 Introduction

Impulsive problems have been object of growing and constant interest, mainly because they provide adequate mathematical tools to describe evolution processes with sudden changes, and to model real phenomena in science, as, for instance, population and biological dynamics, biotechnology and ecology, engineering and industrial robotic, etc. As a result, differential equations with impulses have been recently studied by many authors. They employed various methods and techniques, such as, bifurcation theory [16, 17], method of lower and upper solutions [9, 14, 23, 24], fixed point theorems and fixed point index in cones [11, 12, 32], critical point theory and variational methods [22, 30, 33]. For contributions to general and classical theory we refer to e.g. [1, 13, 25].

Problems with implicit impulse conditions depending both on values of the solution and its derivative at the points of the impulse action have been considered by several authors (see [3, 4, 15, 19, 20] and the references therein). In particular, we refer to [18] dealing with the

✉Corresponding author. Email: mzima@ur.edu.pl

problem

$$\begin{gathered} u''(t) = f(t, u(t), u'(t)) \quad \text{a.e. on } [0, \infty), \\ \Delta u(t_k) = I_{0k}(t_k, u(t_k), u'(t_k)), \quad \Delta u'(t_k) = I_{1k}(t_k, u(t_k), u'(t_k)) \quad \text{for } k \in \mathbb{N}, \\ u(0) = A, \quad u'(\infty) = B, \end{gathered}$$

where $\{t_k\}$ is a sequence of points in $(0, \infty)$ such that $t_k < t_{k+1}$ for $k \in \mathbb{N}$ and $\lim_{k \to +\infty} t_k = \infty$; $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is an L^1-Carathéodory function;

$$u'(\infty) := \lim_{t \to \infty} u'(t), \ \Delta u^{(i)}(t) := u^{(i)}(t+) - u^{(i)}(t-) \quad \text{for } t \in (0, \infty)$$

and $i \in \{0, 1\}$; $A, B \in \mathbb{R}$ and $I_{ik} : (0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous for $i \in \{0, 1\}$ and $k \in \mathbb{N}$. The arguments included Green's functions, Schauder's fixed point theorem and, to have the compactness of the representing operator, also the equiconvergence both at ∞, and at each impulse moment t_k.

Similarly, functional boundary conditions generalize local boundary data and encompass a broad spectrum of conditions where global information on the unknown function is given, including integral and nonlocal conditions, advanced or delay data, maximum or minimum arguments, among others. Existence, nonexistence and multiplicity results for general boundary conditions were studied, for example, in [2, 5, 8, 26–29], for scalar differential equations and, in [6], for coupled systems of differential equations.

Our idea in this paper is to combine both techniques, applied in the papers mentioned above, in the study of impulsive problems with impulse effects depending both on the unknown function and on its first derivative and with nonlocal boundary conditions. In particular, our aim is to get results on the existence of solutions to the boundary value problem

$$-u''(t) = f(t, u(t), u'(t)) \quad \text{a.e. on } [0, 1], \tag{1.1}$$

$$\Delta u(t) = I_{0k}(t, u(t), u'(t)), \quad \Delta u'(t) = I_{1k}(t, u(t), u'(t)) \quad \text{if } t = t_k \in D, \tag{1.2}$$

$$L_0(u(0), u(1), u'(0), u) = 0, \quad L_1(u(0), u(1), u'(1), u) = 0, \tag{1.3}$$

where $m \in \mathbb{N}$, $D = \{t_1, \dots, t_m\} \subset (0, 1)$, $t_1 < \cdots < t_m$, $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathédory function, $I_{0k}, I_{1k} : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ with $k \in \{1, \dots, m\}$ and $L_0, L_1 : \mathbb{R}^3 \times \mathcal{PC}_D \to \mathbb{R}$ satisfy conditions given below, $\mathcal{PC}_D$ is the space of piecewise continuous functions defined below and the symbol Δ has a usual meaning, i.e. $\Delta v(t) = v(t+) - v(t-)$ for any $t \in [0, 1]$ and any function $v : [0, 1] \to \mathbb{R}$ such that both limits in the above formula are defined and have finite values.

As far as we know, nonlocal boundary conditions together with impulsive effects of the types (1.3) and (1.2) are treated in this paper for the first time. This was enabled due to the implemented technique: lower and upper solutions method together with a proper truncation argument. Let us emphasize that, on the contrary to e.g. periodic problem, for (1.1)–(1.3) no a priori estimate of the derivative of the sought solution is available.

The paper is organized as follows: in Section 2 the general framework is established and the basic definitions are introduced. In Section 3 we present our main result: existence and localization theorem and its proof. Last section provides a nontrivial example illustrating the power of our main result.

2 Preliminaries

For a given function $v : [0,1] \to \mathbb{R}$ and points $t \in (0,1]$ and $s \in [0,1)$, the symbols $v(t-)$ and $v(s+)$ stand respectively for the corresponding one-sided limits

$$v(t-) := \lim_{\tau \to t-} v(\tau) \text{ and } v(s+) := \lim_{\tau \to s+} v(\tau)$$

whenever these limits exist and have finite values. In such a case, for $t \in (0,1)$, we write $\Delta v(t) = v(t+) - v(t-)$. Note that the functions such that $v(t-) \in \mathbb{R}$ for all $t \in (0,1]$ and $v(s+) \in \mathbb{R}$ for all $s \in [0,1)$ are usually called regulated functions. The space $\mathcal{G}$ of such functions is known to be a Banach space with respect to the supremum norm

$$\|v\| = \|v\|_\infty := \sup_{t\in[0,1]} |v(t)| \quad \text{for } v \in \mathcal{G}.$$

For basic properties of regulated functions, see e.g. [7], [10] or [21]. For our purposes, the following compactness criterion for subspaces of $\mathcal{G}$ will be essential (cf. [7] or [21, Lemma 4.3.4 and Corollary 4.3.7]).

Theorem 2.1 (Fraňková)**.** *A given subset B of the space $\mathcal{G}$ of regulated functions is relatively compact if and only if*

- *B is the set of equi-regulated functions, i.e. for every $\varepsilon > 0$ there is a division $\{\alpha_0 < \ldots < \alpha_n\}$ of the interval $[0,1]$ such that for every $v \in B$, $j \in \{1,\ldots,n\}$ and $t,s \in (\alpha_{j-1},\alpha_j)$ we have $|v(t) - v(s)| < \varepsilon$*

and

- *the set $\{v(t) : v \in B\} \subset \mathbb{R}$ is bounded for each $t \in [0,1]$.*

In what follows, the symbol D stands for the fixed set $D = \{t_1,\ldots,t_m\}$ of points of impulses in the open interval $(0,1)$ ordered in such a way that $0 < t_1 < \cdots < t_m < 1$. It will be helpful to denote also $t_0 = 0$ and $t_{m+1} = 1$. The symbols $\mathcal{PC}_D$ and $\mathcal{PC}^1_D$ then denote respectively the corresponding sets of functions piecewise continuous on $[0,1]$ or with a derivative piecewise continuous on $[0,1]$. More precisely, $\mathcal{PC}_D$ is the set of all functions $u : [0,1] \to \mathbb{R}$ continuous at every $t \in [0,1] \setminus D$, continuous from the left at every $t \in D$ and having, in addition, finite right limits $u(s+)$ for all $s \in D$. Obviously, when equipped with usual algebraic operations, the space $\mathcal{PC}_D$ is a closed subspace of the Banach space $\mathcal{G}$ of regulated functions. Therefore, it is also a Banach space (with respect to the supremum norm). Analogously, $\mathcal{PC}^1_D$ is the set of all functions $u \in \mathcal{PC}_D$ having a finite derivative $u'(t)$ at each $t \in [0,1] \setminus D$, while u' is continuous at each $t \in [0,1] \setminus D$ and, in addition, it has finite limits $u'(t-)$ and $u'(s+)$ for all $t,s \in D$. For a given $u \in \mathcal{PC}^1_D$, by u' we always mean a function which coincides with the derivative of u on $(0,1) \setminus D$ and is extended to the whole interval $[0,1]$ by the prescriptions

$$u'(0) = u'(0+), \qquad u'(1) = u'(1-) \quad \text{and} \quad u'(t) = u'(t-) \quad \text{if } t \in D.$$

Of course, for such an extension of the derivative we have $u' \in \mathcal{PC}_D$ whenever $u \in \mathcal{PC}^1_D$. It is easy to verify that both the mappings

$$u \in \mathcal{PC}^1_D \to (u', u(0), \Delta u(t_1), \ldots, \Delta u(t_m)) \in \mathcal{PC}_D \times \mathbb{R}^{m+1},$$

and

$$(v, d_0, d_1, \dots, d_m) \in \mathcal{PC}_D \times \mathbb{R}^{m+1}$$
$$\to u(t) = d_0 + \int_0^t v(s)\,\mathrm{d}s + \sum_{k=1}^m d_k\, \chi_{(t_k,t]}(t) \in \mathcal{PC}_D^1,$$

where $\chi_M(t) = 1$ if $t \in M$ and $\chi_M(t) = 0$ if $t \notin M$, are continuous with respect to the norms

$$\|u\|_{\mathcal{PC}^1} := \|u\|_\infty + \|u'\|_\infty \text{ on } \mathcal{PC}_D^1$$

and

$$\|(v, d_0, d_1, \dots, d_m)\| = \|v\|_\infty + \sum_{k=0}^m |d_k| \text{ on } \mathcal{PC}_D \times \mathbb{R}^{m+1}$$

and provide a one-to-one correspondence between the spaces $\mathcal{PC}_D^1$ and $\mathcal{PC}_D \times \mathbb{R}^{m+1}$. As a consequence, $\mathcal{PC}_D^1$ is a Banach space when equipped with the norm $\|\cdot\|_{\mathcal{PC}^1}$. This together with Theorem 2.1 leads directly to the following compactness criterion for subsets of $\mathcal{PC}_D^1$.

Corollary 2.2. *A subset B of the space $\mathcal{PC}_D^1$ is relatively compact if and only if*

- *the set $\{u' : u \in B\}$ is equi-regulated*

and

- *for a given $t \in [0,1]$, the set $\{u(t) : u \in B\} \subset \mathbb{R}$ is bounded.*

Solutions to our problem (1.1)–(1.3) will be understood in the Carathéodory sense as functions with piecewise absolutely continuous derivatives. More precisely, the symbol $\mathcal{AC}_D^1$ stands for the set of functions $u \in \mathcal{PC}_D^1$ having first derivatives absolutely continuous on each subinterval (t_{k-1}, t_k) with $k \in \{1, \dots, m+1\}$ and solutions to (1.1)–(1.3) are defined as follows.

Definition 2.3. By a solution u of problem (1.1)–(1.3) we understand a function $u \in \mathcal{AC}_D^1$ satisfying equation (1.1) a.e. on $[0,1]$ together with conditions (1.2) and (1.3).

Throughout the paper we consider the following assumptions:

(A) $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ of (1.1) satisfies the Carathéodory conditions, i.e.

- $f(\cdot, x, y)$ is Lebesgue integrable for all $(x, y) \in \mathbb{R}^2$,
- $f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^2$ for a.e. $t \in [0,1]$,
- for each $\rho > 0$ there is a function μ_ρ Lebesgue integrable on $[0,1]$ and such that $|f(t, x, y)| \le \mu_\rho(t)$ for a.e. $t \in [0,1]$ and all $x, y \in \mathbb{R}$ such that $|x| \le \rho$ and $|y| \le \rho$.

(B) $L_0, L_1 : \mathbb{R}^3 \times \mathcal{PC}_D \to \mathbb{R}$ and $I_{0k}, I_{1k} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous for all $k \in \{1, \dots, m\}$.

Important tools for our proofs will be associated lower and upper solutions given by the following definition.

Definition 2.4. A function $\alpha \in \mathcal{AC}^1_D$ is a lower solution of (1.1)–(1.3) if

$$\begin{cases} -\alpha''(t) \le f(t,\alpha(t),\alpha'(t)) & \text{a.e. on } [0,1], \\ \Delta\alpha(t_k) \le I_{0k}(t_k,\alpha(t_k),\alpha'(t_k)) & \text{for } k \in \{1,\dots,m\}, \\ \Delta\alpha'(t_k) > I_{1k}(t_k,\alpha(t_k),\alpha'(t_k)) & \text{for } k \in \{1,\dots,m\}, \\ L_0(\alpha(0),\alpha(1),\alpha'(0),\alpha) \ge 0, \\ L_1(\alpha(0),\alpha(1),\alpha'(1),\alpha) \ge 0, \end{cases} \tag{2.1}$$

while a function $\beta \in \mathcal{AC}^1_D$ is an upper solution of (1.1)–(1.3) if

$$\begin{cases} -\beta''(t) \ge f(t,\beta(t),\beta'(t)) & \text{a.e. on } t \in [0,1], \\ \Delta\beta(t_k) \ge I_{0k}(t_k,\beta(t_k),\beta'(t_k)) & \text{for } k \in \{1,\dots,m\}, \\ \Delta\beta'(t_k) < I_{1k}(t_k,\beta(t_k),\beta'(t_k)) & \text{for } k \in \{1,\dots,m\}, \\ L_0(\beta(0),\beta(1),\beta'(0),\beta) \le 0, \\ L_1(\beta(0),\beta(1),\beta'(1),\beta) \le 0. \end{cases} \tag{2.2}$$

The following lemma enables us to construct the operator representation of our problem. Its proof is obvious and can be left to readers.

Lemma 2.5. *Linear problem*

$$\begin{gathered} -u''(t) = h(t) \quad \textit{for a.e. } t \in [0,1], \\ \Delta u(t_k) = C_k,\ \Delta u'(t_k) = D_k \quad \textit{for } k \in \{1,\dots,m\}, \\ u(0) = A, \quad u'(1) = B \end{gathered}$$

has a unique solution for any h Lebesgue integrable on $[0,1]$, $A, B \in \mathbb{R}$, $C_k, D_k \in \mathbb{R}$ $(k \in \{1,\dots,m\})$. This solution is given by

$$u(t) = A + B\,t + \int_0^1 G(t,s)\,h(s)\,ds + \sum_{k=1}^m C_k\,\chi_{(t_k,1]}(t) + \sum_{k=1}^m D_k\,(t-t_k)\,\chi_{(t_k,1]}(t) - t\sum_{k=1}^m D_k\,,$$

where

$$G(t,s) = \begin{cases} s & \textit{if } 0 \le s \le t \le 1, \\ t & \textit{if } 0 \le t \le s \le 1, \end{cases}$$

is the Green function associated to the homogeneous problem

$$-u''(t) = 0, \quad u(0) = 0,\ u'(1) = 0.$$

Remark 2.6. In what follows, the following evident estimate

$$\max\left\{ \sup_{t,s\in[0,1]} |G(t,s)|,\ \sup_{t,s\in[0,1]} \left|\frac{\partial G}{\partial t}(t,s)\right| \right\} = 1 \tag{2.3}$$

will be useful.

Our main existence tool will be the Schauder fixed point theorem (see e.g. [31, Theorem 2.A]).

Theorem 2.7 (Schauder). *Let B be a nonempty, closed, bounded and convex subset of a Banach space X and let $T : B \to X$ be a compact operator mapping B into B. Then T has at least one fixed point in B.*

3 Main result

First, we will construct a proper auxiliary problem and its operator representation. To this aim, the existence of associated lower and upper solutions will be needed. Thus, we will make use of the following assumption.

(C) Problem (1.1)–(1.3) possesses a pair α, β of a lower and an upper solutions such that

$$\alpha(t) \le \beta(t) \ \text{ and } \ \alpha'(t) \le \beta'(t) \ \text{ for } t \in [0,1]. \tag{3.1}$$

Then, for $t \in [0,1]$ and $x, y, w \in \mathbb{R}$, define

$$\delta_0(t,w) = \begin{cases} \alpha(t) & \text{if } w < \alpha(t), \\ w & \text{if } w \in [\alpha(t), \beta(t)], \\ \beta(t) & \text{if } w > \beta(t), \end{cases} \tag{3.2}$$

$$\delta_1(t,w) = \begin{cases} \alpha'(t) & \text{if } w < \alpha'(t), \\ w & \text{if } w \in [\alpha'(t), \beta'(t)], \\ \beta'(t) & \text{if } w > \beta'(t), \end{cases} \tag{3.3}$$

and

$$\widetilde{f}(t,x,y) = f(t, \delta_0(t,x), \delta_1(t,y)) + \frac{\delta_1(t,y) - y}{1 + |y - \delta_1(t,y)|}, \tag{3.4}$$

and consider the following auxiliary problem

$$\begin{cases} -u''(t) = \widetilde{f}(t, u(t), u'(t)) \quad \text{for } t \in [0,1] \setminus D, \\ \Delta u(t_k) = I_{0k}\big(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k))\big), \\ \Delta u'(t_k) = I_{1k}\big(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k))\big), \\ u(0) = \delta_0\big(0, u(0) + L_0(u(0), u(1), u'(0), u)\big), \\ u'(1) = \delta_1\big(1, u'(1) + L_1(u(0), u(1), u'(1), u)\big). \end{cases} \tag{3.5}$$

Finally, we define

$$
\begin{cases}
(Tu)(t) = \delta_0\big(0, u(0) + L_0(u(0), u(1), u'(0), u)\big) \\
\quad +\delta_1\big(1, u'(1) + L_1(u(0), u(1), u'(1), u)\big)\, t \\
\quad + \sum_{k=1}^{m} \big[I_{0k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)))\big]\chi_{(t_k,1]}(t) \\
\quad + \sum_{k=1}^{m} \big[I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)))\,(t - t_k)\big]\chi_{(t_k,1]}(t) \\
\quad -t \sum_{k=1}^{m} I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k))) \\
\quad + \int_0^1 G(t,s)\, \widetilde{f}(s, u(s), u'(s))\, \mathrm{d}s \quad \text{for } u \in \mathcal{PC}^1_D \ \text{ and } t \in [0,1].
\end{cases}
\tag{3.6}
$$

The relationship between the operator T and the auxiliary problem (3.5) is described by the following assertion.

Proposition 3.1. *A function $u \in \mathcal{PC}^1_D$ is a solution to* (3.5) *if and only if it is a fixed point of the operator T given by* (3.6)*.*

Proof. From the construction of the operator T it is clear that any fixed point of T has piecewise absolutely continuous derivative, more precisely it belongs to the set $\mathcal{AC}^1_D$. Moreover, having in mind Lemma 2.5 we easily verify that u solves problem (3.5) if and only if it is a fixed point of T. □

Remark 3.2. If for a given $\rho > 0$ the function μ_ρ has a meaning from (A), then having in mind definitions (3.2)–(3.4), we can see that the following estimate of $\widetilde{f}$ is true:

$$|\widetilde{f}(t,x,y)| \le \mu_{r_0}(t) + 1 \quad \text{for a.e. } t \in [0,1] \text{ and all } x, y \in \mathbb{R},$$

where

$$r_0 = \max\big\{\|\alpha\|_\infty, \|\beta\|_\infty, \|\alpha'\|_\infty, \|\beta'\|_\infty\big\}. \tag{3.7}$$

As a result, we may put

$$\mu_\rho(t) = \mu_{r_0}(t) \quad \text{for all } \rho \ge r_0 \text{ and a.e. } t \in [0,1]. \tag{3.8}$$

Next, we will find conditions ensuring the solvability of problem (3.5).

Proposition 3.3. *Let assumptions* (A)–(C) *hold. Then problem* (3.5) *has at least one solution $\bar{u} \in \mathcal{PC}^1_D$.*

Proof. We will prove that the operator T satisfies the assumptions of the Schauder fixed point theorem (Theorem 2.7).

For better transparency, this proof is divided into several steps.

Step 1. *We will show that the operator T maps $\mathcal{PC}^1_D$ into $\mathcal{PC}^1_D$.*

Clearly, $Tu \in \mathcal{PC}_D$ for every $u \in \mathcal{PC}^1_D$. Furthermore, differentiating the relation (3.6), we get

$$\begin{cases} (Tu)'(t) = \delta_1\big(1, u'(1) + L_1(u(0), u(1), u'(1), u)\big) \\ \quad + \displaystyle\sum_{k=1}^{m} \big[I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)))\big]\chi_{(t_k,1]}(t) \\ \quad - \displaystyle\sum_{k=1}^{m} I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k))) + \int_0^1 \frac{\partial G}{\partial t}(t,s)\, \widetilde{f}(s, u(s), u'(s))\, ds \end{cases} \tag{3.9}$$

for $u \in \mathcal{PC}^1_D$ and $t \in [0,1] \setminus D$, wherefrom, taking into account the properties of the Green function G, we deduce immediately that $Tu \in \mathcal{PC}^1_D$ for each $u \in \mathcal{PC}^1_D$.

Step 2. *Let $t \in [0,1]$ and a bounded subset B of $\mathcal{PC}^1_D$ be given. We will show that the set $(TB)(t) = \big\{(Tu)(t) : u \in B\big\}$ is then bounded subset of $\mathbb{R}$.*

Choose an arbitrary $t \in [0,1]$ and let $\|u\|_{\mathcal{PC}^1} = \|u\|_\infty + \|u'\|_\infty \le \rho < \infty$ for every $u \in B$. Our aim is to find a uniform estimate for elements of $(TB)(t)$.

First, by (3.2) and (3.3) we have

$$\big|\delta_0\big(0, u(0) + L_0(u(0), u(1), u'(0), u)\big)\big| \le \max\big\{|\alpha(0)|, |\beta(0)|\big\}$$

and

$$\big|\delta_1\big(0, u'(1) + L_1(u(0), u(1), u'(1), u)\big)\, t\big| \le \max\big\{|\alpha'(1)|, |\beta'(1)|\big\}.$$

Further, due to continuity of I_{0k} and I_{1k}, for an arbitrary $k \in \{1 \ldots, m\}$ we get

$$\big|I_{0k}(t_k, \delta_0(t_k, u(t_k), u'(t_k)))\, \chi_{(t_k,1]}(t)\big| \le M_{0k} := \max_{(x,y)\in Q_k} |I_{0k}(t_k, x, y)| < \infty$$

and

$$\big|I_{1k}(t_k, \delta_0(t_k, u(t_k), u'(t_k)))\, (t - t_k)\, \chi_{(t_k,1]}(t)\big| \le M_{1k} := \max_{(x,y)\in Q_k} |I_{1k}(t_k, x, y)| < \infty,$$

where $Q_k = [\alpha(t_k), \beta(t_k)] \times [\alpha'(t_k), \beta'(t_k)]$.

Finally, by (2.3) we have $|G(t,s)| \le 1$ for $t, s \in [0,1]$ and consequently by the third point of (A) and by the definition (3.4) of $\widetilde{f}$ we have

$$\left|\int_0^1 G(t,s)\, \widetilde{f}(s, u(s), u'(s))\, ds\right| \le \int_0^1 (\mu_\rho(s) + 1)\, ds.$$

To summarize, the relation

$$\begin{cases} |(Tu)(t)| \le \max\big\{|\alpha(0)|, |\beta(0)|\big\} + \max\big\{|\alpha'(1)|, |\beta'(1)|\big\} \\ \qquad\qquad + M_0 + 2\, M_1 + \displaystyle\int_0^1 (\mu_\rho(s) + 1)\, ds < \infty, \end{cases}$$

where

$$M_0 = \sum_{k=1}^{m} M_{0k} \quad \text{and} \quad M_1 = \sum_{k=1}^{m} M_{1k},$$

holds for any $u \in B$. This proves our claim.

Step 3. *Let B be a bounded subset of $\mathcal{PC}^1_D$. We will show that the set $\big\{(Tu)' : u \in B\big\}$ is equi-regulated.*

Let $B \subset \mathcal{PC}^1_D$ be bounded and let $\rho > 0$ be such that $B \subset B_\rho = \{u \in \mathcal{PC}^1_D : \|u\|_{\mathcal{PC}^1} \le \rho\}$. Further, let $\varepsilon > 0$ be given and let $[s,t] \subset (t_{\ell-1}, t_\ell)$ for some $\ell \in \{1,\dots,m+1\}$. Then

$$\begin{aligned}
&\sum_{k=1}^{m} \big[I_{1k}(t_k, \delta_0(t_k,u(t_k)), \delta_1(t_k,u'(t_k)))\big]\chi_{(t_k,1]}(t) \\
&\qquad = \sum_{k=\ell-1}^{m} \big[I_{1k}(t_k, \delta_0(t_k,u(t_k)), \delta_1(t_k,u'(t_k)))\big]\chi_{(t_k,1]}(t) \\
&\qquad = \sum_{k=\ell-1}^{m} \big[I_{1k}(t_k, \delta_0(t_k,u(t_k)), \delta_1(t_k,u'(t_k)))\big]\chi_{(t_k,1]}(s) \\
&\qquad = \sum_{k=1}^{m} \big[I_{1k}(t_k, \delta_0(t_k,u(t_k)), \delta_1(t_k,u'(t_k)))\big]\chi_{(t_k,1]}(s)
\end{aligned}$$

and

$$(Tu)'(t) - (Tu)'(s) = \int_0^1 \left(\frac{\partial G}{\partial t}(t,\tau) - \frac{\partial G}{\partial t}(s,\tau)\right) f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau$$

for all $u \in B$. Further, since

$$\frac{\partial G}{\partial t}(t,\tau) - \frac{\partial G}{\partial t}(s,\tau) = \begin{cases} 0, & \text{if } 0 \le \tau < s < 1 \text{ or } 0 < t < \tau \le 1, \\ 1, & \text{if } 0 < s < \tau < t < 1, \end{cases}$$

it follows that

$$\begin{aligned}
\big|(Tu)'(t) - (Tu)'(s)\big| &\le \int_s^t \left|\frac{\partial G}{\partial t}(t,\tau) - \frac{\partial G}{\partial t}(s,\tau)\right| (\mu_\rho(\tau)+1)\,\mathrm{d}\tau \\
&\le \int_s^t (\mu_\rho(\tau)+1)\,\mathrm{d}\tau \quad \text{for all } u \in B
\end{aligned}$$

and hence

$$\big|(Tu)'(t) - (Tu)'(s)\big| < \varepsilon \quad \text{for all } u \in B \text{ whenever } \quad \int_s^t (\mu_\rho(\tau)+1)\,\mathrm{d}\tau < \varepsilon.$$

Therefore, any refinement $\{\alpha_0,\dots,\alpha_n\}$ of $\{t_0,\dots,t_{m+1}\}$ which is such that

$$\int_{\alpha_{j-1}}^{\alpha_j} (\mu_\rho(\tau)+1)\,\mathrm{d}\tau < \varepsilon \quad \text{for all } j \in \{1,\dots,n\}$$

satisfies the requirements from the definition of equi-regulatedness contained in Theorem 2.1. Consequently, the set $\big\{(Tu)' : u \in B\big\}$ is equi-regulated and this completes the proof of our claim.

Step 4. *We will construct a nonempty, closed, bounded and convex subset B of $\mathcal{PC}^1_D$ such that $T\,B \subset B$.*

Let $B \subset \mathcal{PC}^1_D$ be bounded and let $\rho > 0$ be such that $B \subset B_\rho = \{u \in \mathcal{PC}^1_D : \|u\|_{\mathcal{PC}^1} \le \rho\}$. Recall that by Step 2 we have

$$\begin{aligned}
\|Tu\|_\infty \le{}& \max\big\{|\alpha(0)|, |\beta(0)|\big\} + \max\big\{|\alpha'(1)|, |\beta'(1)|\big\} \\
&+ M_0 + 2\,M_1 + \int_0^1 (\mu_\rho(s)+1)\,\mathrm{d}s \quad \text{for all } u \in B_\rho.
\end{aligned}$$

Similarly, from (3.9) we deduce that the inequality

$$|(Tu)'(t)| \le \max\{|\alpha'(1)|, |\beta'(1)|\} + 2\,M_1 + \int_0^1 (\mu_\rho(s)+1)\,\mathrm{d}s$$

holds for any $u \in B$ and any $t \in [0,1]$, i.e.

$$\|(Tu)'\|_\infty \le \max\{|\alpha'(1)|, |\beta'(1)|\} + 2\,M_1 + \int_0^1 (\mu_\rho(s)+1)\,\mathrm{d}s$$

for all $u \in B_\rho$. Hence, with respect to (3.8), we conclude that

$$\|Tu\|_{\mathcal{PC}^1} = \|Tu\|_\infty + \|(Tu)'\|_\infty \le \varkappa(r_0) \quad \text{for all } u \in B_\rho \text{ and } \rho \ge r_0$$

where

$$\left\{\begin{aligned} \varkappa(\rho) := {} & \max\{|\alpha(0)|, |\beta(0)|\} + 2\max\{|\alpha'(1)|, |\beta'(1)|\} \\ & + M_0 + 4\,M_1 + 2\int_0^1 (\mu_\rho(s)+1)\,\mathrm{d}s \quad \text{for } \rho > 0. \end{aligned}\right. \tag{3.10}$$

Now, if we put $R = \max\{r_0, \varkappa(r_0)\}$ and $B = B_R$, then the inequality $\|Tu\|_{\mathcal{PC}^1} \le R$ will be true for all $u \in B$. This proves our claim.

To summarize, by Steps 1–3 and Corollary 2.2, the operator T is compact in $\mathcal{PC}^1_D$ and, by Step 4, it maps the nonempty, closed, bounded and convex set $B = B_R$ into itself. By Theorem 2.7 it follows that T has a fixed point $\bar{u} \in B$ which is a solution of (3.5) according to Proposition 3.1. □

Now we can formulate our main result. It provides sufficient conditions for the existence of at least one solution of problem (1.1)–(1.3), as well as its localization.

Theorem 3.4. *Let the assumptions of Proposition 3.3 be satisfied. Furthermore, suppose:*

$$\left\{\begin{aligned} & f(t,\alpha(t),\alpha'(t))) \le f(t,x,\alpha'(t)) \\ & \quad \text{for a.e. } t \in [0,1] \text{ and } x \in [\alpha(t),\beta(t)], \\ & f(t,x,\beta'(t)) \le f(t,\beta(t),\beta'(t))) \\ & \quad \text{for a.e. } t \in [0,1] \text{ and } x \in [\alpha(t),\beta(t)], \end{aligned}\right. \tag{3.11}$$

$$\left\{\begin{aligned} & I_{0k}(t_k,\alpha(t_k),\alpha'(t_k)) \le I_{0k}(t_k,x,y) \le I_{0k}(t_k,\beta(t_k),\beta'(t_k)) \\ & \quad \text{for } (x,y) \in [\alpha(t_k),\beta(t_k)]\times[\alpha'(t_k),\beta'(t_k)] \text{ and } k \in \{1,\dots,m\}, \end{aligned}\right. \tag{3.12}$$

$$\left\{\begin{aligned} & I_{1k}(t_k,x,\alpha'(t_k)) \le I_{1k}(t_k,\alpha(t_k),\alpha'(t_k)) \\ & \quad \text{for } x \ge \alpha(t_k) \text{ and } k \in \{1,\dots,m\}, \\ & I_{1k}(t_k,x,\beta'(t_k)) \ge I_{1k}(t_k,\beta(t_k),\beta'(t_k)) \\ & \quad \text{for } x \le \beta(t_k) \text{ and } k \in \{1,\dots,m\}, \end{aligned}\right. \tag{3.13}$$

$$\left\{\begin{aligned} & L_0(\beta(0),y,z,u) \le L_0(\beta(0),\beta(1),\beta'(1),\beta) \\ & \quad \text{for } y \le \beta(1),\ z \le \beta'(1) \text{ and } u \in \mathcal{PC}_D \text{ such that } u \le \beta \text{ on } [0,1], \\ & L_0(\alpha(0),y,z,u) \ge L_0(\alpha(0),\alpha(1),\alpha'(1),\alpha) \\ & \quad \text{for } y \ge \alpha(1),\ z \ge \alpha'(1) \text{ and } u \in \mathcal{PC}_D \text{ such that } u \ge \alpha \text{ on } [0,1] \end{aligned}\right. \tag{3.14}$$

and

$$\begin{cases} L_1(x,y,\beta'(1),u) \le L_1(\beta(0),\beta(1),\beta'(1),\beta) \\ \quad \textit{for } x \le \beta(0),\ y \le \beta(1) \textit{ and } u \in \mathcal{PC}_D \textit{ such that } u \le \beta \textit{ on } [0,1], \\ L_1(x,y,\alpha'(1),u) \ge L_0(\alpha(0),\alpha(1),\alpha'(1),\alpha) \\ \quad \textit{for } x \ge \alpha(0),\ y \ge \alpha(1) \textit{ and } u \in \mathcal{PC}_D \textit{ such that } u \ge \alpha \textit{ on } [0,1]. \end{cases} \tag{3.15}$$

Then problem (1.1)–(1.3) *has at least one solution u such that*

$$\alpha(t) \le u(t) \le \beta(t) \quad \textit{and} \quad \alpha'(t) \le u'(t) \le \beta'(t) \quad \textit{for } t \in [0,1].$$

Proof. By Proposition 3.3 the auxiliary problem (3.5) has a solution $\bar{u}$ such that $\|\bar{u}\|_{\mathcal{PC}^1} \le R$, where $R = \max\{r_0, \varkappa(r_0)\} > 0$ is given by (3.7) and (3.10). Thus, it remains to show that $\bar{u}$ satisfies the following set of inequalities

$$\alpha(t) \le \bar{u}(t) \le \beta(t) \quad \text{for } t \in [0,1], \tag{3.16}$$

$$\alpha'(t) \le \bar{u}'(t) \le \beta'(t) \quad \text{for } t \in [0,1] \tag{3.17}$$

and

$$\alpha(0) \le \bar{u}(0) + L_0(\bar{u}(0),\bar{u}(1),\bar{u}'(0),\bar{u}) \le \beta(0), \tag{3.18}$$

$$\alpha'(1) \le \bar{u}'(1) + L_1(\bar{u}(0),\bar{u}(1),\bar{u}'(1),\bar{u}) \le \beta'(1). \tag{3.19}$$

Indeed, in such a case, in view of (3.2) and (3.3), the relations

$$\begin{aligned} \widetilde{f}(t,\bar{u}(t),\bar{u}'(t)) &= f(t,\bar{u}(t),\bar{u}'(t)), \\ \delta_0\big(0,\bar{u}(0)+L_0(\bar{u}(0),\bar{u}(1),\bar{u}'(0),\bar{u})\big) &= \bar{u}(0)+L_0(\bar{u}(0),\bar{u}(1),\bar{u}'(0),\bar{u}), \\ \delta_1\big(1,\bar{u}'(1)+L_1(\bar{u}(0),\bar{u}(1),\bar{u}'(1),\bar{u})\big) &= \bar{u}'(1)+L_1(\bar{u}(0),\bar{u}(1),\bar{u}'(1),\bar{u}), \\ I_{0k}\big(t_k,\delta_0(t_k,\bar{u}(t_k)),\delta_1(t_k,\bar{u}'(t_k))\big) &= I_{0k}\big(t_k,\bar{u}(t_k)),\bar{u}'(t_k))\big) \end{aligned}$$

and

$$I_{1k}\big(t_k,\delta_0(t_k,\bar{u}(t_k)),\delta_1(t_k,\bar{u}'(t_k))\big) = I_{1k}\big(t_k,\bar{u}(t_k)),\bar{u}'(t_k))\big)$$

are true for all $t \in [0,1]$ and $k \in \{1,\dots,m\}$. Therefore, it follows immediately that then $\bar{u}$ is the desired solution of the given problem (1.1)–(1.3).

• **ad** (3.17): Suppose that there is a $\bar{t} \in [0,1]$ such that

$$\bar{u}'(\bar{t}) - \beta'(\bar{t}) = \max_{t\in[0,1]} \big(\bar{u}'(t) - \beta'(t)\big) > 0. \tag{3.20}$$

As, by the definition (3.3) of δ_1 and by the last relation in (3.5) we have

$$\bar{u}'(1) = \delta_1(1,\bar{u}'(1)+L_1(\bar{u}(0),\bar{u}(1),\bar{u}'(1),\bar{u})) \le \beta'(1),$$

it follows that $\bar{t} < 1$.

Assume that $\bar{t} \in [0,1)\backslash D$. Then either $\bar{t} \in [0,t_1)$ or $\bar{t} \in (t_{k-1},t_k)$ for some $k \in \{2,\dots,m+1\}$. In both cases there is a $\Delta > 0$ such that $\bar{t}+\Delta < t_k$ and $\bar{u}'(s) - \beta'(s) > 0$ for all $s \in [\bar{t},\bar{t}+\Delta]$.

In particular, $\delta_1(s,\bar{u}(s)) = \beta'(s)$ for $s \in [\bar{t}, \bar{t}+\Delta]$. Now, using (3.11) and the first inequality in (2.2), we will deduce for $t \in [\bar{t}, \bar{t}+\Delta]$

$$\begin{aligned}
0 &\geq \big(\bar{u}'(t) - \beta'(t)\big) - \big(\bar{u}'(\bar{t}) - \beta'(\bar{t})\big) = \int_{\bar{t}}^{t} \big(\bar{u}''(s) - \beta''(s)\big)\,\mathrm{d}s \\
&= \int_{\bar{t}}^{t} \Big(-f(s,\delta_0(s,\bar{u}(s)),\delta_1(s,\bar{u}'(s))) - \frac{\delta_1(s,\bar{u}'(s)) - \bar{u}'(s)}{|\bar{u}'(s) - \delta_1(s,\bar{u}'(s))| + 1} - \beta''(s) \Big)\,\mathrm{d}s \\
&= \int_{\bar{t}}^{t} \Big(-f(s,\delta_0(s,\bar{u}(s)),\beta'(s))) + \frac{\bar{u}'(s) - \beta'(s)}{\bar{u}'(s) - \beta'(s) + 1} - \beta''(s) \Big)\,\mathrm{d}s \\
&> \int_{\bar{t}}^{t} \Big(-f(s,\delta_0(s,\bar{u}(s)),\beta'(s))) - \beta''(s) \Big)\,\mathrm{d}s \\
&\geq \int_{\bar{t}}^{t} \Big(-f(s,\beta(s),\beta'(s)) - \beta''(s) \Big)\,\mathrm{d}s \geq 0,
\end{aligned}$$

a contradiction, of course.

It remains to consider the possibility that there exists $k \in \{1,2,...,m\}$ such that either (3.20) with $\bar{t} = t_k$ or

$$\bar{u}'(t_k+) - \beta'(t_k+) = \sup_{t\in[0,1]} \big(\bar{u}'(t) - \beta'(t)\big) > 0$$

holds. The latter case leads to a contradiction by arguments analogous to those used above. So, let (3.20) with $\bar{t} = t_k$ for some $k \in \{1,2,...,m\}$ be the case. In particular, we have

$$\bar{u}'(t_k+) - \beta'(t_k+) \leq \bar{u}'(t_k) - \beta'(t_k), \;\; \delta_1(t_k,\bar{u}'(t_k)) = \beta'(t_k)$$

and $\Delta\bar{u}'(t_k) \leq \Delta\beta'(t_k)$, i.e.

$$\begin{aligned}
0 &\geq \Delta\bar{u}'(t_k) - \Delta\beta'(t_k) = I_{1k}(t_k,\delta_0(t_k,\bar{u}(t_k)),\delta_1(t_k,\bar{u}'(t_k))) - \Delta\beta'(t_k) \\
&= I_{1k}(t_k,\delta_0(t_k,\bar{u}(t_k)),\beta'(t_k)) - \Delta\beta'(t_k).
\end{aligned}$$

Thanks to (3.13), Definition 2.4 (cf. the third line in (2.2)) and the third line in (3.5) this leads to a contradiction

$$\begin{aligned}
0 &\geq I_{1k}(t_k,\delta_0(t_k,\bar{u}(t_k)),\delta_1(t_k,\bar{u}'(t_k))) - \Delta\beta'(t_k) \\
&\geq I_{1k}(t_k,\delta_0(t_k,\bar{u}(t_k)),\beta'(t_k)) - \Delta\beta'(t_k). \geq I_{1k}(t_k,\beta(t_k),\beta'(t_k)) - \Delta\beta'(t_k) > 0.
\end{aligned}$$

This means that $\bar{u}'(t) \leq \beta'(t)$ holds for $t \in [0,1]$.

Similarly, we can prove that also $\alpha'(t) \leq \bar{u}'(t)$ holds for $t \in [0,1]$. This completes the proof of (3.17).

• **ad** (3.16): Integrating the inequality $\alpha'(t) \leq \bar{u}'(t)$ over $[0,t]$ for $t \in (0,t_1]$, we get

$$\alpha(t) - \alpha(0) \leq \bar{u}(t) - \bar{u}(0) \quad \text{for } t \in [0,t_1]. \tag{3.21}$$

Further, as $\bar{u}(0) = \delta_0(0,\bar{u}(0) + L_0(\bar{u}(0),\bar{u}(1),\bar{u}'(0),\bar{u})) \geq \alpha(0)$, it follows that

$$\alpha(t) \leq \bar{u}(t) + \alpha(0) - \bar{u}(0) \leq \bar{u}(t) \quad \text{for} \;\; t \in [0,t_1].$$

Now, let $k \in \{1,\ldots,m\}$ be such that

$$\alpha(t) \leq \bar{u}(t) \quad \text{for} \;\; t \in [0,t_k]. \tag{3.22}$$

Analogously to (3.21) we derive

$$\alpha(t) - \alpha(t_k+) \le \bar{u}(t) - \bar{u}(t_k+) \quad \text{for } t \in (t_k, t_{k+1}]$$

and, with respect to the second line in (3.5), we get

$$\begin{aligned}\alpha(t) &\le \bar{u}(t) + \alpha(t_k+) - \bar{u}(t_k+)\\ &= \bar{u}(t) + \alpha(t_k+) - I_{0k}(t_k, \delta_0(t_k, \bar{u}(t_k)), \delta_1(t_k, \bar{u}'(t_k))) - \bar{u}(t_k)\end{aligned}$$

for $t \in (t_k, t_{k+1}]$. Furthermore, having in mind that

$$\alpha(t_k) \le \delta_0(t_k, \bar{u}(t_k)) \le \beta(t_k) \quad \text{and} \quad \alpha'(t_k) \le \delta_1(t_k, \bar{u}'(t_k)) \le \beta'(t_k)$$

due to (3.2) and (3.3), we can use (3.12), the second line in (2.1) and hypothesis (3.22) to deduce that

$$\begin{aligned}\alpha(t) &= \bar{u}(t) + \alpha(t_k+) - I_{0k}(t_k, \delta_0(t_k, \bar{u}(t_k)), \delta_1(t_k, \bar{u}'(t_k))) - \bar{u}(t_k)\\ &\le \bar{u}(t) + \alpha(t_k+) - I_{0k}(t_k, \alpha(t_k), \alpha'(t_k)) - \bar{u}(t_k)\\ &\le \bar{u}(t) + \alpha(t_k) - \bar{u}(t_k) \le \bar{u}(t) \quad \text{for } t \in (t_k, t_{k+1}].\end{aligned}$$

By induction principle, we can conclude that $\alpha(t) \le \bar{u}(t)$ holds on the whole interval $[0,1]$. Similarly we can prove that $\bar{u}(t) \le \beta(t)$ on $[0,1]$. This completes the proof of (3.16).

• **ad** (3.18): Suppose that

$$\bar{u}(0) + L_0(\bar{u}(0), \bar{u}(1), \bar{u}'(0), \bar{u}) > \beta(0).$$

Then by (3.5) and (3.2)

$$\bar{u}(0) = \delta_0(0, \bar{u}(0) + L_0(\bar{u}(0), \bar{u}(1), \bar{u}'(0), \bar{u}) = \beta(0)$$

and using the monotonicity type condition (3.14) we obtain

$$\begin{aligned}0 &< \bar{u}(0) + L_0(\bar{u}(0), \bar{u}(1), \bar{u}'(0), \bar{u}) - \beta(0)\\ &= L_0(\beta(0), \bar{u}(1), \bar{u}'(0), \bar{u}) \le L_0(\beta(0), \beta(1), \beta'(0), \beta) \le 0,\end{aligned}$$

a contradiction. Hence, it must be

$$\bar{u}(0) + L_0(\bar{u}(0), \bar{u}(1), \bar{u}'(0), \bar{u}) \le \beta(0).$$

Similarly, we would show that

$$\alpha(0) \le \bar{u}(0) + L_0(\bar{u}(0), \bar{u}(1), \bar{u}'(0), \bar{u}),$$

is true, as well. Thus, the relations (3.18) are true.

• **ad** (3.19): Suppose that

$$\bar{u}'(1) + L_1(\bar{u}(0), \bar{u}(1), \bar{u}'(1), \bar{u}) > \beta'(1).$$

Then by (3.5) and (3.3)

$$\bar{u}'(1) = \delta_1(1, \bar{u}'(1) + L_1(\bar{u}(0), \bar{u}(1), \bar{u}'(1), \bar{u}) = \beta'(1).$$

Furthermore, the monotonicity type condition (3.15) together with (2.2) yield the following contradiction:

$$\begin{aligned} 0 &< \bar{u}'(1) + L_1(\bar{u}(0), \bar{u}(1), \bar{u}'(1), \bar{u}) - \beta'(1) \\ &= L_1(\bar{u}(0), \bar{u}(1), \beta'(1), \bar{u}) \le L_1(\beta(0), \beta(1), \beta'(1), \beta) \le 0. \end{aligned}$$

Consequently, it has to be

$$\bar{u}'(1) + L_1(\bar{u}(0), \bar{u}(1), \bar{u}'(1), \bar{u}) \le \beta'(1).$$

Similarly it can be shown that

$$\alpha'(1) \le \bar{u}'(1) + L_1(\bar{u}(0), \bar{u}(1), \bar{u}'(1), \bar{u}).$$

This completes the proof of (3.19).

To summarize, all the relations (3.16)–(3.19) are true and hence the fixed point $\bar{u}$ of T is a solution of the given problem (1.1)–(1.3). $\square$

4 Example

To illustrate the range of applications of our main result, let us consider problem (1.1)–(1.3), with $m = 1$, $D = \{t_1\} = \{\frac{1}{2}\}$,

$$f(t,x,y) = \begin{cases} 0.001\,[(t-2)\,y^3 + x] & \text{if } \ 0 \le t \le \frac{1}{2}, \\ 0.001\,[(t-6)\,y^3 + x] & \text{if } \ \frac{1}{2} < t \le 1, \end{cases}$$

$$I_{01}(\tfrac{1}{2}, x, y) = 0.1\left[\tfrac{3}{2} + \tfrac{1}{3}\,x + y^3\right], \quad I_{11}(\tfrac{1}{2}, x, y) = 0.1\left[\tfrac{1}{2} - \tfrac{1}{4}\,x + y^3\right],$$

$$L_0(x,y,z,u) = -\,x + \tfrac{1}{6}\Big(z + \sup_{t\in[0,1]} \,u(t)\Big), \; L_0(x,y,z,u) = -\,2y - z + \int_0^1 u(t)\,\mathrm{d}t.$$

It is easy to verify that (A), (B) are satisfied. Furthermore, the functions

$$\alpha(t) = -(t+1) \ \text{ for } t \in [0,1] \quad \text{and} \quad \beta(t) = \begin{cases} t+1 & \text{if } 0 \le t \le \frac{1}{2}, \\ t+4 & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

are lower and upper solutions of the given problem and conditions (3.1), (3.11), (3.12), (3.13), (3.14) and (3.15) hold. Therefore, our Theorem 3.4 ensures the existence of its solution $u_* \in \mathcal{PC}^1_D$ such that

$$\alpha(t) \le u_*(t) \le \beta(t) \ \text{ and } \ -1 \le u_*'(t) \le 1.$$

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