

The Inverse Epsilon Distribution as an Alternative to Inverse Exponential Distribution with a Survival Times Data Example*

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Abstract

This paper is devoted to a new flexible two-parameter lower-truncated distribution, which is based on the inversion of the so-called epsilon distribution. It is called the inverse epsilon distribution. In some senses, it can be viewed as an alternative to the inverse exponential distribution, which has many applications in reliability theory and biology. Diverse properties of the new lower-truncated distribution are derived including relations with existing distributions, hazard and reliability functions, survival and reverse hazard rate functions, stochastic ordering, quantile function with related skewness and kurtosis measures, and moments. A demonstrative survival times data example is used to show the applicability of the new model.

Keywords: epsilon distribution, inverse exponential distribution, inverse epsilon distribution

1 Introduction

The exponential distribution and its generalizations play an important role in many areas of science, including physics, chemistry, medical sciences and reliability engineering (see e.g. [1, 2, 16, 19]). Dombi et al. [6] introduced the epsilon distribution,

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which may be treated as an alternative to the exponential distribution. Here, we will briefly review the epsilon distribution and its connection with the exponential distribution. Dombi et al. [6] defined the epsilon function as follows.

Definition 1. The epsilon function $\varepsilon_{\lambda,d}(x): (-d, d) \rightarrow (0, \infty)$ is given by

$$\varepsilon_{\lambda,d}(x) = \left(\frac{d+x}{d-x} \right)^{\lambda \frac{d}{2}},$$

where $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $d \in \mathbb{R}$, $d > 0$.

The following proposition concerns a key property of the epsilon function.

Proposition 1. For any $x \in (-d, +d)$, if $d \rightarrow \infty$, then

$$\varepsilon_{\lambda,d}(x) \rightarrow e^{\lambda x}.$$

Proof. See the proof of Theorem 1 in [6]. □

Utilizing the epsilon function given in Definition 1, the continuous random variable X said to have an epsilon distribution with the parameters $\lambda > 0$ and $d > 0$, if its cumulative distribution function (CDF) is given by

$$F_{\lambda,d}(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1 - \varepsilon_{-\lambda,d}(x), & \text{if } 0 < x < d \\ 1, & \text{if } x \geq d. \end{cases}$$

Notation 1. From now on, $X \sim \varepsilon(\lambda, d)$ will denote that the random variable X has an epsilon distribution with the parameters $\lambda > 0$ and $d > 0$.

Exploiting Proposition 1, we can state the following proposition.

Proposition 2. Let $X \sim \varepsilon(\lambda, d)$ and let $Y \sim \exp(\lambda)$, where $\lambda > 0, d > 0$. Then, for any $x \in \mathbb{R}$

$$\lim_{d \rightarrow \infty} P(X < x) = P(Y < x).$$

Proof. By making use of the definitions for the epsilon and the exponential distributions, the proposition immediately follows from Proposition 1. □

Based on Proposition 2, we may state that the asymptotic epsilon distribution is just the exponential distribution. It is worth mentioning that while the hazard function of an exponentially distributed random variable is constant, the hazard function of a random variable with an epsilon distribution can exhibit both constant and increasing shapes. That is, in reliability analyses, the epsilon distribution can be utilized to describe the distribution of the time to first failure random variable both in the second and in the third phases of the hazard function.

The reciprocal of a random variable with an exponential distribution is said to be a random variable with an inverse exponential distribution. The inverse exponential

distribution, like the exponential distribution, has a wide range of applications (see e.g. [17]). For example, if a random variable with an exponential distribution represents the time between failures of a system, then the reciprocal of this random variable, which has an inverse exponential distribution, describes the frequency of the system failures over time.

In this study, we will present the inverse epsilon distribution and, by the means of an illustrative data example, show that it may be viewed as an alternative to the inverse exponential distribution. The key features of the inverse epsilon distribution and the main motivations of our study can be summarized as follows:

- (a) It is a new, flexible, lower-truncated power-polynomial distribution.
- (b) The famous inverse exponential distribution is just the limit of the inverse epsilon distribution.
- (c) The literature lacks of a flexible inverted lower-truncated distributions.
- (d) The hazard function of the inverse exponential distribution has a first, increasing part and a second, slowly decreasing part (see [18]). This explains why in the course of the study of mortality associated with some diseases, the inverse exponential distribution may be utilized as a life distribution model (see [12, 5]). Taking into account the asymptotic property of the inverse epsilon distribution, this latter one can also be utilized in mortality studies.

This paper is structured as follows. In Section 2, we will introduce the inverse epsilon distribution and describe its key properties including the hazard function, survival and reverse hazard rate functions, stochastic ordering, quantile function and moments. Next, in Section 3, we will present a demonstrative example of the application of the new distribution on survival times data. Lastly, in Section 4, our main findings are summarized.

2 Theoretical aspects

2.1 Basics on the inverse epsilon distribution

Here, we will present the inverse epsilon distribution and show that it may be viewed as an alternative to the inverse exponential distribution.

Now, let the random variable X have an epsilon distribution with the parameters $\lambda, d > 0$; that is, $X \sim \varepsilon(\lambda, d)$. Next, let $Y = 1/X$, where $X > 0$, and let $G_{\lambda,d}: \mathbb{R}^+ \rightarrow (0, 1)$ be the CDF of Y . Then, noting that X and Y are continuous random variables, after direct calculation, we have

$$G_{\lambda,d}(y) = P(Y < y) = P\left(X > \frac{1}{y}\right) = 1 - P\left(X \leq \frac{1}{y}\right) = 1 - F_{\lambda,d}\left(\frac{1}{y}\right).$$

Therefore,

$$G_{\lambda,d}(y) = \begin{cases} 0, & \text{if } 0 < y \leq \frac{1}{d} \\ \left(\frac{d+y^{-1}}{d-y^{-1}}\right)^{-\lambda \frac{d}{2}}, & \text{if } y > \frac{1}{d}. \end{cases}$$

By taking the derivative of function $G_{\lambda,d}(x)$, we get the probability density function (PDF) $g_{\lambda,d}(x) = G'_{\lambda,d}(x)$ of the random variable Y :

$$g_{\lambda,d}(y) = \begin{cases} 0, & \text{if } 0 < y \leq \frac{1}{d} \\ \lambda \frac{d^2}{d^2 y^2 - 1} \left(\frac{d+y^{-1}}{d-y^{-1}} \right)^{-\lambda \frac{d}{2}}, & \text{if } y > \frac{1}{d}. \end{cases}$$

Following this line of thinking, we define the inverse epsilon distribution as follows.

Definition 2. *The continuous random variable $X > 0$ has an inverse epsilon distribution with the parameters $\lambda > 0$ and $d > 0$, if the PDF $f_{\lambda,d}$ of X is given by*

$$f_{\lambda,d}(x) = \begin{cases} 0, & \text{if } 0 < x \leq \frac{1}{d} \\ \lambda \frac{d^2}{d^2 x^2 - 1} \left(\frac{d+x^{-1}}{d-x^{-1}} \right)^{-\lambda \frac{d}{2}}, & \text{if } x > \frac{1}{d}. \end{cases} \quad (1)$$

Note that the CDF of the inverse epsilon distribution given in Definition 2 is

$$F_{\lambda,d}(x) = \begin{cases} 0, & \text{if } 0 < x \leq \frac{1}{d} \\ \left(\frac{d+x^{-1}}{d-x^{-1}} \right)^{-\lambda \frac{d}{2}}, & \text{if } x > \frac{1}{d}. \end{cases} \quad (2)$$

Notation 2. *From now on, $X \sim \bar{\varepsilon}(\lambda, d)$ will denote that the random variable $X > 0$ has an inverse epsilon distribution with the parameters $\lambda > 0$ and $d > 0$.*

It is a familiar fact that a continuous random variable $X > 0$ has an inverse exponential distribution with the parameter $\lambda > 0$, if the PDF $f_{\lambda}(x)$ and the CDF $F_{\lambda}(x)$ of X are given by

$$f_{\lambda}(x) = \frac{\lambda}{x^2} e^{-\lambda \frac{1}{x}}, \quad F_{\lambda}(x) = e^{-\lambda \frac{1}{x}}, \quad (3)$$

respectively.

Notation 3. *Hereafter, $X \sim \text{invexp}(\lambda)$ will denote that the random variable $X > 0$ has an inverse exponential distribution with the parameter $\lambda > 0$.*

The following proposition concerns the connection between the inverse exponential and inverse epsilon distributions.

Proposition 3. *Let $X \sim \bar{\varepsilon}(\lambda, d)$ and let $Y \sim \text{invexp}(\lambda)$, where $\lambda > 0, d > 0$ and $X, Y > 0$. Then, for any $x > 0$*

$$\lim_{d \rightarrow \infty} P(X < x) = P(Y < x).$$

Proof. Noting the CDFs of X and Y given in Eq. (2) and Eq. (3), respectively, and applying Proposition 1, for any $x > 0$, we can write

$$\lim_{d \rightarrow \infty} P(X < x) = \lim_{d \rightarrow \infty} \left(\frac{d+x^{-1}}{d-x^{-1}} \right)^{-\lambda \frac{d}{2}} = e^{-\lambda \frac{1}{x}} = P(Y < x).$$

□

Based on Proposition 3, the inverse exponential distribution may be viewed as the asymptotic inverse epsilon distribution.

Using the results above, we may state that the interests in the inverse epsilon distribution is based on the following facts:

- (a) It is a new, flexible, lower-truncated power-polynomial distribution.
- (b) The famous inverse exponential distribution is just the limit of the inverse epsilon distribution.
- (c) The literature lacks of a flexible inverted lower-truncated distributions.

Some example plots of the CDFs of the inverse epsilon distribution are shown in Figure 1.

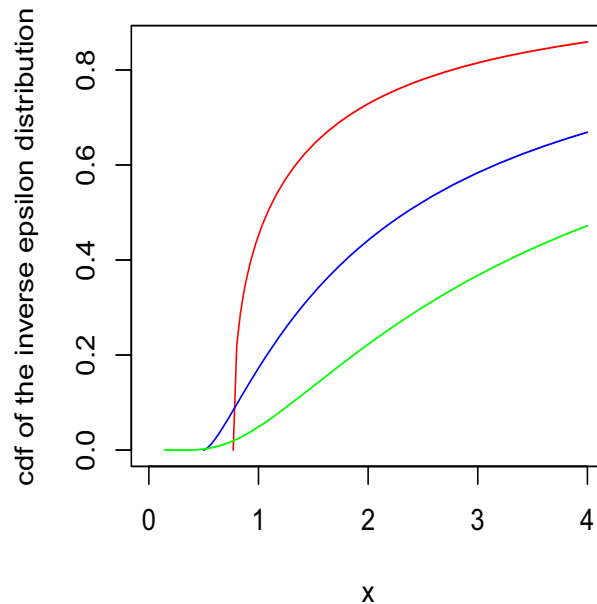


Figure 1: CDF of the inverse epsilon distribution with three sets of parameters for (λ, d) : $(0.6, 1.3)$, $(1.6, 2)$ and $(3, 7)$.

We observe that the CDF can be more or less concave (for the considered values).

2.2 Hazard function

By making use of the PDF and CDF of the inverse exponential distribution, we get that its hazard function $h_\lambda: (0, \infty) \rightarrow (0, \infty)$ is

$$h_\lambda(x) = \frac{f_\lambda(x)}{1 - F_\lambda(x)} = \frac{\frac{\lambda}{x^2} e^{-\lambda \frac{1}{x}}}{1 - e^{-\lambda \frac{1}{x}}},$$

where $\lambda > 0$.

Using the PDF and the CDF of the inverse epsilon distribution with the parameters $\lambda, d > 0$, the hazard function $h_{\lambda,d}: (0, \infty) \rightarrow [0, \infty)$ of this distribution is

$$h_{\lambda,d}(x) = \frac{f_{\lambda,d}(x)}{1 - F_{\lambda,d}(x)} = \begin{cases} 0, & \text{if } 0 < x \leq \frac{1}{d} \\ \lambda \frac{\frac{d^2}{d^2 x^2 - 1} \left(\frac{d+x-1}{d-x-1}\right)^{-\lambda \frac{d}{2}}}{1 - \left(\frac{d+x-1}{d-x-1}\right)^{-\lambda \frac{d}{2}}}, & \text{if } x > \frac{1}{d}. \end{cases}$$

Proposition 4. *Let $X \sim \bar{\varepsilon}(\lambda, d)$ and let $Y \sim \text{inveexp}(\lambda)$, where $\lambda > 0, d > 0$ and $X, Y > 0$. Furthermore let $h_{\lambda,d}: (0, \infty) \rightarrow [0, \infty)$ and $h_{\lambda}: (0, \infty) \rightarrow (0, \infty)$ be the hazard functions of X and Y , respectively. Then, for any $x > 0$*

$$\lim_{d \rightarrow \infty} h_{\lambda,d}(x) = h_{\lambda}(x).$$

Proof. This proposition immediately follows from Proposition 1. □

The first derivative of the hazard function $h_{\lambda,d}(x)$ is

$$\frac{dh_{\lambda,d}(x)}{dx} = - \frac{\lambda d^4 \left((2x - \lambda) \left(\frac{dx+1}{dx-1} \right)^{\frac{\lambda d}{2}} - 2x \right)}{(dx-1)^2 (dx+1)^2 \left(\left(\frac{dx+1}{dx-1} \right)^{\frac{\lambda d}{2}} - 1 \right)^2}.$$

Using the first derivative of $h_{\lambda,d}(x)$, one can see that

- if $0 < \lambda d \leq 2$, then $h_{\lambda,d}(x)$ is strictly decreasing in the interval $(\frac{1}{d}, \infty)$
- if $\lambda d > 2$, then in the interval $(\frac{1}{d}, \infty)$, $h_{\lambda,d}(x)$ is first increasing, and then decreasing; that is, $h_{\lambda,d}(x)$ has a local maxima.

Figure 2 shows example plots of hazard functions of the inverse exponential distribution and the inverse epsilon distribution.

It is an acknowledged fact that in the course of the study of mortality associated with some diseases, the hazard function has a first, increasing part and a second, slowly decreasing part [18]. We can see that the hazard function of the inverse exponential distribution (see the left upper plot in Figure 2) exhibits such a shape. This is why the inverse exponential distribution may be utilized as a life distribution model (see [12, 5]).

Now, by taking into account the above mentioned characteristics of the hazard function of the inverse epsilon distribution, we can draw the following practical conclusions.

- The hazard function of the inverse epsilon distribution with the parameters $\lambda, d > 0$ may be viewed as an alternative to the hazard function of the inverse exponential distribution, if the value of parameter d is sufficiently large. If $d \rightarrow \infty$, then the two hazard functions coincide.

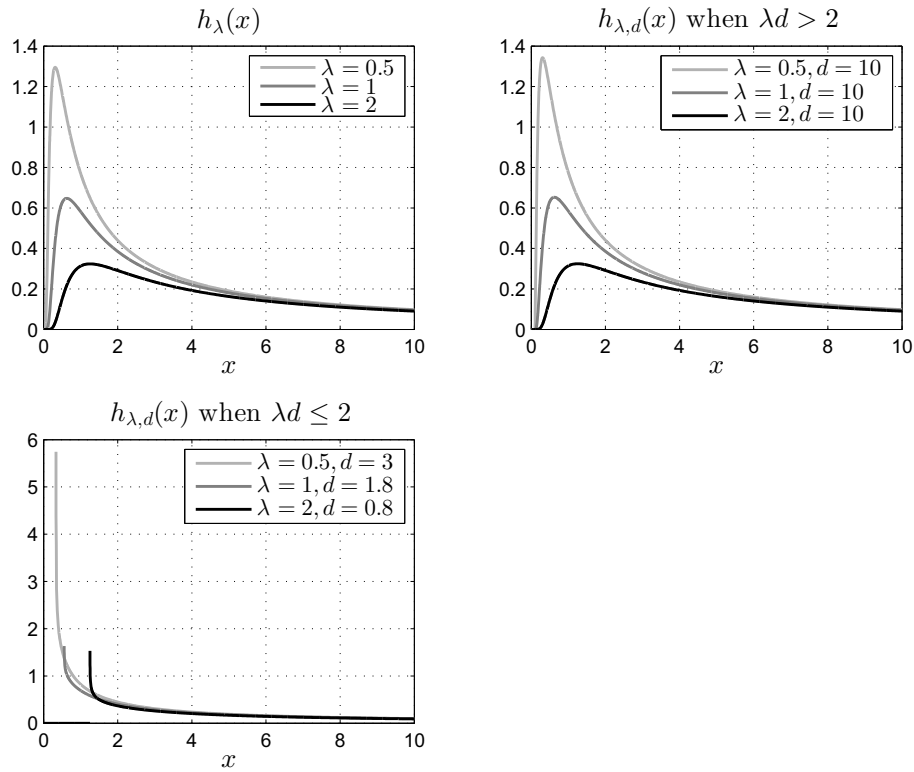


Figure 2: Example plots of hazard functions of the inverse exponential distribution and the inverse epsilon distribution.

- If $\lambda d > 2$, then the shape of the hazard function of the inverse epsilon distribution is very similar to that of the hazard function of the inverse exponential distribution. In this case, the hazard function is first increasing and then it is slowly decreasing (see the upper plots in Figure 2).
- If $0 < \lambda d \leq 2$, then the hazard function of the inverse epsilon distribution is strictly decreasing in the interval $(\frac{1}{d}, \infty)$ (see the lower plot in Figure 2).

Therefore, the inverse epsilon distribution can be used to model life time data that have either first monotonically increasing and then decreasing hazard rates, or monotonically decreasing hazard rates.

2.3 Survival and reverse hazard rate functions

The following functions are of interest, mainly in hazard and reliability analysis. The survival function of the inverse epsilon distribution is obtained as

$$S_{\lambda,d}(x) = 1 - F_{\lambda,d}(x) = \begin{cases} 1, & \text{if } 0 < x \leq \frac{1}{d} \\ 1 - \left(\frac{d+x^{-1}}{d-x^{-1}}\right)^{-\lambda\frac{d}{2}}, & \text{if } x > \frac{1}{d}. \end{cases}$$

The reversed hazard rate function of the inverse epsilon distribution is given by

$$r_{\lambda,d}(x) = \frac{f_{\lambda,d}(x)}{F_{\lambda,d}(x)} = \begin{cases} 0, & \text{if } 0 < x \leq \frac{1}{d} \\ \lambda \frac{d^2}{d^2 x^2 - 1}, & \text{if } x > \frac{1}{d}. \end{cases}$$

The cumulative hazard rate function is expressed as

$$H_{\lambda,d}(x) = -\ln(S_{\lambda,d}(x)) = \begin{cases} 0, & \text{if } 0 < x \leq \frac{1}{d} \\ -\ln \left[1 - \left(\frac{d+x^{-1}}{d-x^{-1}}\right)^{-\lambda\frac{d}{2}} \right], & \text{if } x > \frac{1}{d}, \end{cases}$$

where the logarithmic term can be decomposed as

$$\begin{aligned} -\ln \left[1 - \left(\frac{d+x^{-1}}{d-x^{-1}}\right)^{-\lambda\frac{d}{2}} \right] &= -\ln \left[(d-x^{-1})^{-\lambda\frac{d}{2}} - (d+x^{-1})^{-\lambda\frac{d}{2}} \right] \\ &\quad - \lambda\frac{d}{2} \ln(d-x^{-1}). \end{aligned}$$

Further details on these functions can be found in [11].

2.4 Stochastic ordering

The following stochastic ordering result on the inverse epsilon distribution holds.

Proposition 5. *Let $F_{\lambda,d}$ be the CDF of the inverse epsilon distribution as defined by (2). Then, for $d_2 \geq d_1 > 0$ and any $x > 0$, we have*

$$F_{\lambda,d_2}(x) \geq F_{\lambda,d_1}(x);$$

and for any $\lambda_2 \geq \lambda_1 > 0$ and any $x > 0$, we have

$$F_{\lambda_1,d}(x) \geq F_{\lambda_2,d}(x).$$

Proof. For $x < 1/d_2$, the CDFs are equal to 0. For $x \in [1/d_2, 1/d_1]$, since $F_{\lambda,d_1}(x) = 0$, the inequality is clear too. Now, for $x > 1/d_1 > 1/d_2$, by using the following inequality:

$$\frac{1}{2} \ln \left(\frac{1+x}{x-1} \right) < \frac{x}{x^2-1}$$

for $x > 1$, we get

$$\frac{\partial}{\partial d} F_{\lambda,d}(x) = \lambda \left(\frac{d+x^{-1}}{d-x^{-1}} \right)^{-\lambda \frac{d}{2}} \left[\frac{dx}{d^2 x^2 - 1} - \frac{1}{2} \ln \left(\frac{1+dx}{dx-1} \right) \right] > 0,$$

implying that $F_{\lambda,d}(x)$ is increasing with respect to d .

For $\lambda_2 \geq \lambda_1$, we have

$$F_{\lambda_1,d}(x) \geq F_{\lambda_2,d}(x).$$

Indeed, the function $F_{\lambda,d}(x)$ is decreasing with respect to λ : we have

$$\frac{\partial}{\partial \lambda} F_{\lambda,d}(x) = -\frac{d}{2} \left(\frac{d+x^{-1}}{d-x^{-1}} \right)^{-\lambda \frac{d}{2}} \ln \left(\frac{d+x^{-1}}{d-x^{-1}} \right) < 0.$$

□

Under the conditions of Proposition 5, we see that $X_1 \sim \bar{\varepsilon}(\lambda_1, d)$ first order stochastically dominates $X_2 \sim \bar{\varepsilon}(\lambda_2, d)$.

2.5 Quantile function

The quantile function of the inverse epsilon distribution is obtained by inverting $F_{\lambda,d}(x)$. After some developments, we arrive at

$$Q_{\lambda,d}(u) = \frac{1}{d} \left(\frac{u^{-\frac{2}{d\lambda}} + 1}{u^{-\frac{2}{d\lambda}} - 1} \right) = \frac{1}{d} \left(\frac{1 + u^{\frac{2}{d\lambda}}}{1 - u^{\frac{2}{d\lambda}}} \right), \quad u \in (0, 1).$$

This function is of importance because it allows us to define the main quartiles of the inverse epsilon distribution, as the first quartile: $Q_{\lambda,d}(1/4)$, the median: $Q_{\lambda,d}(1/2)$ and the third quartile: $Q_{\lambda,d}(3/4)$. Also, it can be served to generate values from the inverse epsilon distribution.

We should also add that we can use $Q_{\lambda,d}(u)$ to define measures of skewness and kurtosis as the Bowley skewness and Moors kurtosis are given by

$$B_{\lambda,d} = \frac{Q_{\lambda,d}(1/4) - 2Q_{\lambda,d}(1/2) + Q_{\lambda,d}(3/4)}{Q_{\lambda,d}(3/4) - Q_{\lambda,d}(1/4)}$$

and

$$M_{\lambda,d} = \frac{Q_{\lambda,d}(7/8) - Q_{\lambda,d}(5/8) + Q_{\lambda,d}(3/8) - Q_{\lambda,d}(1/8)}{Q_{\lambda,d}(6/8) - Q_{\lambda,d}(2/8)}.$$

These measures provide alternative definitions to the skewness and kurtosis measures defined with moments. For more details on these alternative definitions see [8] and [13].

Figure 3 shows the plots of Bowley skewness and Moors kurtosis as functions of the parameters λ and d . The graphics for $B_{\lambda,d}$ and $M_{\lambda,d}$ are useful to determine

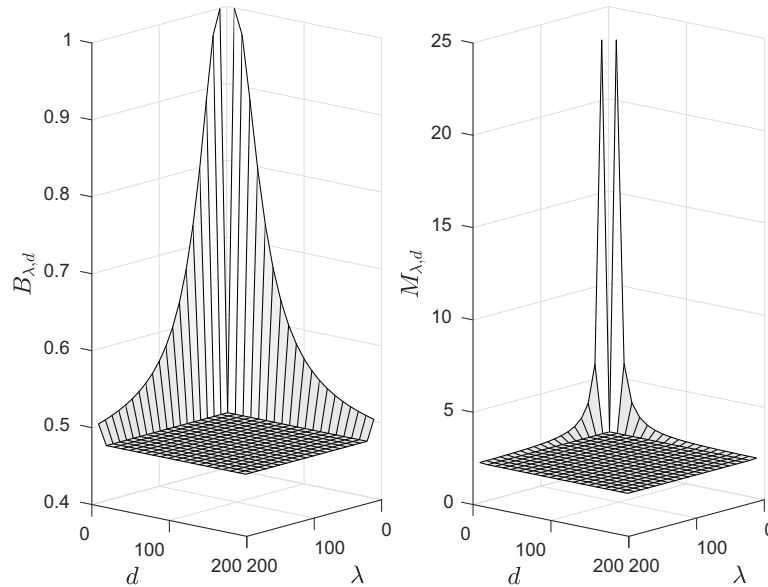


Figure 3: Plots of Bowley skewness and Moors kurtosis

the ability of the inverse epsilon distribution in skewness and kurtosis. This is very interesting for the inverse epsilon distribution because it does not admit mean (and obviously raw moments of superior order). This aspect is developed in the next section.

Also, upon differentiation of $Q_{\lambda,d}(u)$ according to u , the quantile density function is defined by

$$q_{\lambda,d}(u) = \frac{4}{\lambda d^2} \frac{u^{\frac{2}{d\lambda}-1}}{\left(1 - u^{\frac{2}{d\lambda}}\right)^2}, \quad u \in (0, 1).$$

This function is of interest since it appears in several statistical tools. For further details see [10].

2.6 Moments

Let us now investigate the moments of $X \sim \bar{\varepsilon}(\lambda, d)$. Then, assuming that it exists, the mean of X^r is defined by

$$\mu'_r = E(X^r) = \int_{1/d}^{+\infty} x^r f_{\lambda,d}(x) dx = \int_{1/d}^{+\infty} x^r \lambda \frac{d^2}{d^2 x^2 - 1} \left(\frac{d+x^{-1}}{d-x^{-1}} \right)^{-\lambda \frac{d}{2}} dx.$$

Proposition 6. *The mean of X^r exists if and only if $r \in (-\lambda d/2, 1)$, and it is given as*

$$\mu'_r = \frac{\lambda}{d^r} B\left(1 - r, \frac{\lambda d}{2} + 1\right) {}_2F_1\left(\frac{\lambda d}{2} + 1, 1 - r; 1 - r + \frac{\lambda d}{2}; -1\right),$$

where $B(a, b)$ and ${}_2F_1(a, b; c; x)$ are the classical beta and Gauss hypergeometric functions, respectively.

Proof. When $x \rightarrow 1/d$, we have $f_{\lambda, d}(x) \sim \lambda d^2 2^{-\lambda \frac{d}{2} - 1} (xd - 1)^{\lambda \frac{d}{2} - 1}$ so, by the Riemann integrability, the integral converge in 0 if and only if $1 - \lambda d/2 - r < 1$, hence $r > -\lambda d/2$. Also, when $x \rightarrow +\infty$, we have $f_{\lambda, d}(x) \sim \lambda x^{r-2}$ so, by the Riemann integrability, the integral converge in $+\infty$ if and only if $2 - r > 1$, hence $r < 1$. That is, μ'_r exists if and only if $r \in (-\lambda d/2, 1)$. Following the lines of [15] with the use of $Y \sim \bar{\varepsilon}(\lambda, d)$ and the change of variable $y = x/d$, we have

$$\begin{aligned} \mu'_r &= E(Y^{-r}) = \lambda d^2 \int_0^d \frac{x^{-r}}{d^2 - x^2} \left(\frac{d+x}{d-x}\right)^{-\lambda \frac{d}{2}} dx \\ &= \frac{\lambda}{d^r} \int_0^1 y^{-r} (1-y)^{\lambda \frac{d}{2} - 1} (1+y)^{-\lambda \frac{d}{2} - 1} dy. \end{aligned}$$

The desired result involving the beta and Gauss hypergeometric functions is an immediate application of Eq. 3.197.3 of [9]. □

In particular, from Proposition 6, we see that the mean of X doesn't exist. Some inverse raw moments of X exists, depending on the large values for λ and d .

Remark 1. Alternatively, by applying the change of variable $x = Q_{\lambda, d}(u)$ and use he general binomial theorem, one can also express μ'_r as

$$\begin{aligned} \mu'_r &= \int_0^1 [Q_{\lambda, d}(u)]^r du = \frac{1}{d^r} \int_0^1 \left(\frac{1 + u \frac{2}{d\lambda}}{1 - u \frac{2}{d\lambda}}\right)^r du \\ &= \frac{1}{d^r} \int_0^1 \left[\sum_{k=0}^{+\infty} \binom{r}{k} u^{k \frac{2}{d\lambda}} \right] \left[\sum_{\ell=0}^{+\infty} \binom{-r}{\ell} (-1)^\ell u^{\ell \frac{2}{d\lambda}} \right] du \\ &= \frac{1}{d^r} \sum_{k, \ell=0}^{+\infty} \binom{r}{k} \binom{-r}{\ell} (-1)^\ell \frac{1}{2(k + \ell)/(d\lambda) + 1}, \end{aligned}$$

from which an acceptable approximation can be given by substituting $+\infty$ by any large integer. If λ or d are sufficiently large, the negative moments of X can be investigated for moments analysis.

The incomplete moments of X exists when $r \geq 0$; the r th incomplete moment of X^r at $t > 1/d$ is given by

$$\mu'_r(t) = E(X^r 1_{\{X \leq t\}}) = \int_{1/d}^t x^r f_{\lambda, d}(x) dx = \int_{1/d}^t x^r \lambda \frac{d^2}{d^2 x^2 - 1} \left(\frac{d+x^{-1}}{d-x^{-1}}\right)^{-\lambda \frac{d}{2}} dx$$

or, equivalently,

$$\mu'_r(t) = \int_0^{F_{\lambda,d}(t)} [Q_{\lambda,d}(u)]^r du = \frac{1}{d^r} \int_0^{\left(\frac{d+t-1}{d-t-1}\right)^{-\lambda \frac{d}{2}}} \left(\frac{1+u \frac{2}{d\lambda}}{1-u \frac{2}{d\lambda}}\right)^r du.$$

To our knowledge, there is no close form $\mu'_r(t)$. For known parameters (including t), we can have a numerical value of it. As a complementary approach, a series expansion of $\mu'_r(t)$ is possible through the application of the generalized binomial series expansion. Following this approach, we get

$$\mu'_r(t) = \frac{1}{d^r} \sum_{k,\ell=0}^{+\infty} \binom{r}{k} \binom{-r}{\ell} (-1)^\ell \frac{1}{2(k+\ell)/(d\lambda) + 1} \left(\frac{d+t-1}{d-t-1}\right)^{-(k+\ell)-\lambda \frac{d}{2}}.$$

From the incomplete moments, one can define applied curves, functions or indexes of interest, such as the Lorenz curves, Gini index and mean residual life or others. See, for instance, [3].

3 A demonstrative survival times data example

Oguntunde et al. [14] used the following data of survival times (in days) of a group of patients suffering from head and neck cancer diseases and treated using a combination of radiotherapy and chemotherapy (see [7]):

12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776.

Oguntunde et al. [14] modeled these survival times using the exponential inverse exponential (EIE) distribution that has the following PDF and CDF, respectively,

$$f_{\theta,\alpha}(x) = \alpha \frac{\theta}{x^2} e^{-\frac{\theta}{x}} \frac{1}{\left(1 + e^{-\frac{\theta}{x}}\right)^2} e^{-\alpha \frac{e^{-\frac{\theta}{x}}}{1 - e^{-\frac{\theta}{x}}}} \quad (4)$$

$$F_{\theta,\alpha}(x) = 1 - e^{-\alpha \frac{e^{-\frac{\theta}{x}}}{1 - e^{-\frac{\theta}{x}}}}, \quad (5)$$

where $x, \alpha, \theta > 0$. The values of the maximum likelihood estimations $\hat{\theta}$ and $\hat{\alpha}$ for the parameters θ and α , respectively, the maximum value of the log-likelihood function, the value of the Akaike information criterion (AIC) and the value of the Bayesian information criterion (BIC) are shown in Table 1.

For this data set, Table 1 shows the maximum likelihood estimation results for the inverse exponential (IE) distribution as well.

Here, we computed the maximum likelihood estimations of the parameters for the inverse epsilon distribution as follows. Let the random variable X be the

Table 1: Estimation results

Distribution	Parameters		Log-likelihood	AIC	BIC
EIE	$\hat{\theta} = 33.4469$	$\hat{\alpha} = 0.1609$	-280.4043	564.8086	568.3770
IE	$\hat{\lambda} = 76.7000$		-279.5773	561.1546	562.9389
Inverse epsilon	$\hat{\lambda} = 76.7000$	$\hat{d} = 11.9953$	-279.5773	563.1547	566.7231

survival time of patients, $X \sim \bar{\varepsilon}(\lambda, d)$, and let x_1, x_2, \dots, x_n be independent observations on X . Then, the likelihood function $L: (0, \infty)^2 \rightarrow (0, \infty)$ for the sample x_1, x_2, \dots, x_n is given by

$$L(\lambda, d; x_1, x_2, \dots, x_n) = \prod_{i=1}^n \left(\lambda \frac{d^2}{d^2 x_i^2 - 1} \left(\frac{d + x_i^{-1}}{d - x_i^{-1}} \right)^{-\lambda \frac{d}{2}} \right),$$

where $d > \frac{1}{\min_{i=1,2,\dots,n}(x_i)}$. The log-likelihood function $l = \ln \circ L$ is given by

$$l(\lambda, d; x_1, x_2, \dots, x_n) = n \ln(\lambda) + \sum_{i=1}^n \ln \left(\frac{d^2}{d^2 x_i^2 - 1} \right) - \lambda \frac{d}{2} \sum_{i=1}^n \ln \left(\frac{d + x_i^{-1}}{d - x_i^{-1}} \right),$$

where $d > \frac{1}{\min_{i=1,2,\dots,n}(x_i)}$. We used the GLOBAL method, which is a stochastic global optimization procedure introduced by Csendes et al. [4], to find the maxima of the log-likelihood function. The estimations of λ and d , respectively, $\hat{\lambda}$ and \hat{d} , and the maximal value of the log-likelihood function are shown in Table 1.

Based on the maximum likelihood estimation results, we can summarize our findings as follows.

- (a) For the studied survival times, the inverse epsilon distribution gives better maximal log-likelihood value and better AIC and BIC values than the exponential inverse exponential distribution. At the same time, the PDF and the CDF of the inverse exponential distribution have much simpler formulas (see Eq. (1) and Eq. (2)) than those of the exponential inverse exponential distribution (see Eq. (4) and Eq. (5)).
- (b) The inverse epsilon distribution and the inverse exponential distribution result the same maximal log-likelihood value. That is, in line with the finding of Proposition 3, these two distributions coincide if $d \rightarrow \infty$. Notice that in our case, these two distributions may be viewed as being identical already for $d = 11.9953$.
- (c) The inverse epsilon distribution has two parameters (λ and d), while the inverse exponential distribution has only one parameter (λ). Therefore, as in our case these two distributions result the same maximal log-likelihood value, the AIC and BIC values for the inverse exponential distribution (561.1546 and 562.9389, respectively) are lower than those for the inverse epsilon distribution

(563.1547 and 566.7231, respectively). It should be added that by fixing the value of parameter d at a large value (e.g. $d = 100$), the inverse epsilon distribution may be treated as a one-parameter distribution, which coincides with the inverse exponential distribution.

- (d) The PDFs and the CDFs of the exponential distribution and the exponential inverse exponential distribution contain exponential terms, while the PDF and the CDF of the inverse epsilon distribution do not contain any exponential term.

Figure 4 shows the plots of the empirical CDF, EIE CDF, inverse exponential CDF and inverse epsilon CDF with the parameter values listed in Table 1.

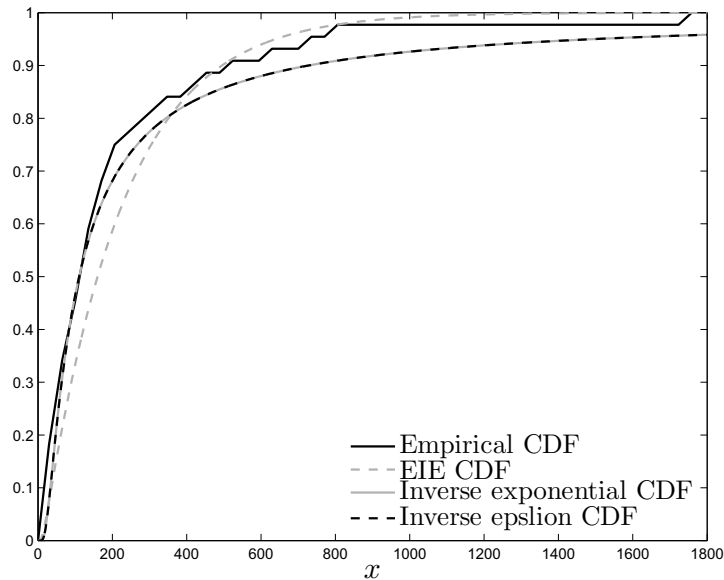


Figure 4: Empirical CDF, EIE CDF, Inverse exponential CDF and Inverse epsilon CDF

4 Conclusions

In this paper, we study the possibilities offered by a new two-parameter lower-truncated distribution constructed from the inversion of the so-called epsilon distribution. Here, diverse motivations for this new distribution are provided. We have studied in depth the shapes of the probability density and hazard rate functions, determined the quantile function and discussed the moments. The theory

is illustrated by a complete graphical analysis. Through the maximum likelihood approach, the new model is derived and an application with real data is also given.

We further plan to use the inverse epsilon distribution in an applied regression setting, and to investigate some of its natural generalizations through standard schemes (Marshall-Olkin, transmuted, type I half-logistic, etc).

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