Acta Cybernetica **25** (2022) 703–716.

The Generalized Epsilon Function: An Alternative to the Exponential Function

Tamás Jónás[a]

Abstract

It is well known that the exponential function plays an extremely important role in many areas of science. In this study, a generator function-based mapping, called the generalized epsilon function is presented. Next, we demonstrate that the exponential function is an asymptotic generalized epsilon function. Exploiting this result and the fact that this new function is generator function-dependent, it can be utilized as a very flexible alternative to the exponential function in a wide range of applications. We should add that if the generator is a rational function, then the generalized epsilon function is rational as well. In this case, the generalized epsilon function is computationally simple and it may be treated as an easy-to-compute alternative to the exponential function. In this paper, we briefly present two applications of this novel function: an approximation to the exponential probability distribution, and an alternative to the sigmoid function on a bounded domain.

Keywords: exponential function, approximation, epsilon function, exponential distribution, sigmoid function

1 Preliminaries

In [9], Dombi *et al.* introduced the epsilon function and by using this mapping the authors constructed the epsilon probability distribution that may be viewed as an alternative to the exponential probability distribution. The epsilon function is defined as follows.

Definition 1. *The epsilon function* $\varepsilon_d^{(\lambda)}(x)$ *is given by*

$$\varepsilon_d^{(\lambda)}(x) = \left(\frac{d+x}{d-x}\right)^{\lambda\frac{d}{2}}, \tag{1}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, $d > 0$, $x \in (-d, +d)$.

[a]Faculty of Economics, Eötvös Loránd University, Rákóczi út 7, H-1088 Budapest, Hungary, E-mail: `jonas@gtk.elte.hu`, ORCID: `0000-0001-8241-2321`

DOI: `10.14232/actacyb.294699`

In [9], we proved the following theorem, which states an important asymptotic property of the epsilon function.

Theorem 1. *For any $x \in (-d, +d)$, if $d \to \infty$, then*

$$\varepsilon_d^{(\lambda)}(x) \to \mathrm{e}^{\lambda x}. \tag{2}$$

It should be mentioned that this results was also utilized for constructing an effective approximation to the normal probability distribution (see [8]).

The epsilon function given in Definition 1 may be treated as an alternative of the exponential function on the domain $(-d, d)$. The exponential function has an extremely wide range of applications in many areas of science including mathematics, physics, chemistry, computer science, economics and biology (see, e.g., [19, 1, 4, 15, 5]). Our motivation was to generalize the epsilon function such that it can be used to approximate the exponential function with an even higher level of flexibility.

In this paper, we will present the generalized epsilon function, which is a generator function-based mapping from the domain $[-d, d]$ to the non-negative extended real line ($d > 0$). We will prove that if $d \to \infty$, then the generalized epsilon function coincides with the exponential function. This result allows us to treat the generalized epsilon function as an alternative to exponential function on a bounded domain. Since this new function is generator function-dependent, it is very flexible and it can be utilized in a wide range of applications. Here, we will briefly present two applications of the generalized epsilon function: an approximation to the exponential probability distribution, and an alternative to the sigmoid function on a bounded domain.

We will use the common notation $\mathbb{R}$ for the real line and $\overline{\mathbb{R}}$ for the extended real line, i.e., $\overline{\mathbb{R}} = [-\infty, \infty]$. Also, $\overline{\mathbb{R}}_+$ will denote the non-negative extended real line, i.e., $\overline{\mathbb{R}}_+ = [0, \infty]$. We will consider the arithmetic operations on the extended real line according to Klement *et al.* [13] and Grabisch *et al.* [10].

2 The generalized epsilon function

First, we will construct a generator function-based mapping, called the general epsilon function, which we can use to approximate the exponential function. Then, we will prove that the exponential function is an asymptotic generalized epsilon function.

2.1 Construction

Let $d \in \mathbb{R}$, $d > 0$ and let $\lambda \in \mathbb{R} \setminus \{0\}$. Our aim is to construct a function $f_d^{(\alpha)} \colon [-d, d] \to \overline{\mathbb{R}}_+$ in the form

$$f_d^{(\alpha)}(x) = c\, h_d^{\alpha}(x), \tag{3}$$

where $h_d\colon [-d,d] \to \overline{\mathbb{R}}_+$ is a continuous and strictly monotonic mapping, $c > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$, such that $f_d^{(\alpha)}$ approximates the exponential function $\mathrm{e}^{\lambda x}$ on the domain $[-d,d]$. Noting the basic properties of the exponential function $\mathrm{e}^{\lambda x}$, we set the following requirements for $f_d^{(\alpha)}$:

(a) For any $x \in [-d,d]$, $f_d^{(\alpha)}(x) \in \overline{\mathbb{R}}_+$.

(b) If $\lambda > 0$ ($\lambda < 0$, respectively), then $f_d^{(\alpha)}$ is strictly increasing (decreasing, respectively).

(c) $f_d^{(\alpha)}$ is differentiable on $(-d,d)$; and $f_d^{(\alpha)}(x)$ and $\mathrm{e}^{\lambda x}$ are identical to first order at $x = 0$, i.e.,

(c1)
$$f_d^{(\alpha)}(0) = 1 \quad \text{and}$$

(c2)
$$\left. \frac{\mathrm{d}f_d^{(\alpha)}(x)}{\mathrm{d}x} \right|_{x=0} = \lambda.$$

As $c > 0$ and $\alpha \neq 0$, there exists a $\hat{c} > 0$ such that $c = \hat{c}^{\alpha}$. Using this substitution, Eq. (3) can be written as
$$f_d^{(\alpha)}(x) = \left(\hat{c}\, h_d(x)\right)^{\alpha}.$$

Since we wish $f_d^{(\alpha)}$ to be a generator function-based mapping from $[-d,d]$ to $\overline{\mathbb{R}}_+$ (see requirement (a)), let h_d have the form

$$h_d(x) = g\left(\frac{x+d}{2d}\right), \quad x \in [-d,d],$$

where $g\colon [0,1] \to \overline{\mathbb{R}}_+$ is a continuous and strictly monotonic function. This means that
$$f_d^{(\alpha)}(x) = \left(\hat{c}\, g\left(\frac{x+d}{2d}\right)\right)^{\alpha} \tag{4}$$

for any $x \in [-d,d]$. We will call the function g the generator of $f_d^{(\alpha)}$.

Taking into account requirement (b), we have that if $\lambda > 0$, then for a strictly increasing (decreasing, respectively) generator g, α has to be positive (negative, respectively). Similarly, noting requirement (b), we have that if $\lambda < 0$, then for a strictly increasing (decreasing, respectively) generator g, α has to be negative (positive, respectively).

Using Eq. (4), the requirement (c1) leads us to

$$\left(\hat{c}\, g\left(\frac{1}{2}\right)\right)^{\alpha} = 1,$$

from which

$$\hat{c} = \left(g\left(\frac{1}{2}\right)\right)^{-1}$$

and so Eq. (4) can be written as

$$f_d^{(\alpha)}(x) = \left(\frac{g\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)}\right)^{\alpha} \tag{5}$$

for any $x \in [-d, d]$.

Next, considering requirement (c2), we get that g has to be a differentiable function on $(0, 1)$ and

$$\left(f_d^{(\alpha)}(x)\right)'\Bigg|_{x=0} = \left(\left(\frac{g\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)}\right)^{\alpha}\right)'\Bigg|_{x=0} = \lambda. \tag{6}$$

The first derivative of $f_d^{(\alpha)}$ is

$$\left(f_d^{(\alpha)}(x)\right)' = \alpha\left(\frac{g\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)}\right)^{\alpha-1}\frac{g'\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)}\frac{1}{2d}$$

and so from Eq. (6), we get

$$\alpha = 2\lambda d\frac{g\left(\frac{1}{2}\right)}{g'\left(\frac{1}{2}\right)}.$$

Hence, using Eq. (5), $f_d^{(\alpha)}(x)$ can be written as

$$f_d^{(\alpha)}(x) = \left(\frac{g\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)}\right)^{2\lambda d\frac{g\left(\frac{1}{2}\right)}{g'\left(\frac{1}{2}\right)}}$$

for any $x \in [-d, d]$. Notice that the generator function g needs to meet the criterion $g'\left(\frac{1}{2}\right) \neq 0$ as well.

Remark 1. If $\lambda > 0$ and g is strictly increasing, then $g'\left(\frac{1}{2}\right) > 0$ and so $\alpha > 0$, which means that $f_d^{(\alpha)}$ is strictly increasing on $[-d, d]$. Similarly, if $\lambda > 0$ and g is strictly decreasing, then $g'\left(\frac{1}{2}\right) < 0$, which implies $\alpha < 0$ and so $f_d^{(\alpha)}$ is strictly increasing on $[-d, d]$. Therefore, if $\lambda > 0$, then $f_d^{(\alpha)}$ is strictly increasing on $[-d, d]$ regardless if g is a strictly increasing or a strictly decreasing function. Based on similar considerations, we get that if $\lambda < 0$, then $f_d^{(\alpha)}$ is strictly decreasing on $[-d, d]$ independently of the monotonicity of function g. Therefore, $f_d^{(\alpha)}$ satisfies the requirement (b).

Now, using the construction presented so far, we will introduce the so-called generalized epsilon function, which can be used to approximate the exponential function. In this definition, we will utilize the following class of functions.

Definition 2. *Let $\mathcal{G}$ be the set of all functions $g\colon [0,1] \to \overline{\mathbb{R}}_+$ that are strictly monotonic and differentiable on $(0,1)$ with $g'\left(\frac{1}{2}\right) \neq 0$, where g' denotes the first derivative of g, and g' is continuous on $(0,1)$.*

For a strictly increasing $g \in \mathcal{G}$, $g(1) = \infty$ will mean the limit $\lim_{x\to 1} g(x) = \infty$. Similarly, for a strictly decreasing $g \in \mathcal{G}$, $g(0) = \infty$ will stand for the limit $\lim_{x\to 0} g(x) = \infty$.

Definition 3 (Generalized epsilon function)**.** *Let $g \in \mathcal{G}$, let $\lambda \in \mathbb{R} \setminus \{0\}$ and $d > 0$. We say that the function $\varepsilon_{d,g}^{(\lambda)}\colon [-d,d] \to \overline{\mathbb{R}}_+$, which is given by*

$$\varepsilon_{d,g}^{(\lambda)}(x) = \left(\frac{g\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)}\right)^{2\lambda d \frac{g\left(\frac{1}{2}\right)}{g'\left(\frac{1}{2}\right)}}, \tag{7}$$

is a generalized epsilon function (GEF) with the parameters λ and d induced by the generator function g.

Later, we will show that the epsilon function, which was first introduced by Dombi *et al.* in [9], is just a special case of function $\varepsilon_{d,g}^{(\lambda)}$. That is, $\varepsilon_{d,g}^{(\lambda)}$ may be viewed as a generalization of the epsilon function.

2.2 Identicality of two generalized epsilon functions

As a GEF is generator function-dependent, the question when two GEFs are identical naturally arises. The following proposition gives a sufficient condition for the equality of two generalized epsilon functions that are induced by two generator functions.

Proposition 1. *The GEF is uniquely determined up to any transformation*

$$t(x) = \alpha x^{\beta}, \quad x \in \overline{\mathbb{R}}_+ \tag{8}$$

on its generator function, if $\alpha > 0$ and $\beta \in \mathbb{R} \setminus \{0\}$.

Proof. Let $\lambda \in \mathbb{R} \setminus \{0\}$, $d > 0$ and let the GEF $\varepsilon_{d,g}^{(\lambda)}$ be induced by the generator function $g \in \mathcal{G}$. Furthermore, let $\alpha > 0$ and $\beta \in \mathbb{R} \setminus \{0\}$. Now, let the function $t\colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ be by given by Eq. (8), and let $h(x) = t(g(x))$ for any $x \in [0,1]$. Then, $h \in \mathcal{G}$ and for any $x \in [-d,d]$, the GEF induced by h can be written as

$$\varepsilon_{d,h}^{(\lambda)}(x) = \left(\frac{h\left(\frac{x+d}{2d}\right)}{h\left(\frac{1}{2}\right)}\right)^{2\lambda d \frac{h\left(\frac{1}{2}\right)}{h'\left(\frac{1}{2}\right)}} = \left(\frac{\alpha g^{\beta}\left(\frac{x+d}{2d}\right)}{\alpha g^{\beta}\left(\frac{1}{2}\right)}\right)^{2\lambda d \frac{\alpha g^{\beta}\left(\frac{1}{2}\right)}{\alpha\beta g^{\beta-1}\left(\frac{1}{2}\right) g'\left(\frac{1}{2}\right)}} =$$

$$= \left(\frac{g\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)}\right)^{2\lambda d \frac{g\left(\frac{1}{2}\right)}{g'\left(\frac{1}{2}\right)}} = \varepsilon_{d,g}^{(\lambda)}(x).$$

$\square$

3 The exponential function as an asymptotic generalized epsilon function

Let $\varepsilon_{d,g}^{(\lambda)}$ be a GEF induced by the generator function $g \in \mathcal{G}$, where $\lambda \in \mathbb{R} \setminus \{0\}$ and $d > 0$. Due to the construction of $\varepsilon_{d,g}^{(\lambda)}$, it has the following properties:

- For any $x \in [-d, d]$, $\varepsilon_{d,g}^{(\lambda)}(x) \in \overline{\mathbb{R}}_+$.
- If $\lambda > 0$ ($\lambda < 0$, respectively), then $\varepsilon_{d,g}^{(\lambda)}$ is strictly increasing (decreasing, respectively).
- $\varepsilon_{d,g}^{(\lambda)}(x)$ and $\mathrm{e}^{\lambda x}$ are identical to first order at $x = 0$.

Here, we will demonstrate an important asymptotic property of the generalized epsilon function.

Theorem 2. *Let $g \in \mathcal{G}$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $d > 0$. Let $\varepsilon_{d,g}^{(\lambda)} \colon [-d, d] \to \overline{\mathbb{R}}_+$ be a GEF induced by g according to Eq. (7). Then, for any $x \in (-d, +d)$,*

$$\lim_{d \to \infty} \varepsilon_{d,g}^{(\lambda)}(x) = \mathrm{e}^{\lambda x}. \tag{9}$$

Proof. Using the definition of $\varepsilon_{d,g}^{(\lambda)}$, for any $x \in (-d, +d)$, Eq. (9) is equivalent to

$$\lim_{d \to \infty} \left(2\lambda d \frac{g\left(\frac{1}{2}\right)}{g'\left(\frac{1}{2}\right)} \ln \left(\frac{g\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)} \right) \right) = \lambda x. \tag{10}$$

The left hand side of Eq. (10) can be written as

$$2\lambda \frac{g\left(\frac{1}{2}\right)}{g'\left(\frac{1}{2}\right)} \lim_{d \to \infty} \left(\frac{\ln\left(\frac{g\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)} \right)}{\frac{1}{d}} \right). \tag{11}$$

Notice that we can use the L'Hospital rule to compute the limit in Eq. (11). Taking into account that g is differentiable on $(0,1)$ and g' is continuous on $(0,1)$, after direct calculation, we get

$$2\lambda \frac{g\left(\frac{1}{2}\right)}{g'\left(\frac{1}{2}\right)} \lim_{d \to \infty} \left(\frac{\ln\left(\frac{g\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)} \right)}{\frac{1}{d}} \right) = 2\lambda \frac{g\left(\frac{1}{2}\right)}{g'\left(\frac{1}{2}\right)} \lim_{d \to \infty} \left(\frac{\left(\ln\left(\frac{g\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)} \right)\right)'}{\left(\frac{1}{d}\right)'} \right) =$$

$$= 2\lambda \frac{g\left(\frac{1}{2}\right)}{g'\left(\frac{1}{2}\right)} \lim_{d \to \infty} \left(\frac{g'\left(\frac{x+d}{2d}\right)}{g\left(\frac{x+d}{2d}\right)} \frac{x}{2} \right) = \lambda x.$$

$\square$

Example 1. Let $g(x) = \cos(x)$, where $x \in [0,1]$. Clearly, $g \in \mathcal{G}$, i.e., g satisfies the requirements for a generator function of a generalized epsilon function. After direct calculation, we get that the GEF induced by the function g is

$$\varepsilon_{d,g}^{(\lambda)}(x) = \varepsilon_{d,g}^{(\lambda)}(x) = \left(\frac{\cos\left(\frac{x+d}{2d}\right)}{\cos\left(\frac{1}{2}\right)}\right)^{-\frac{2\lambda d}{\tan\left(\frac{1}{2}\right)}} \approx \left(1.14 \cdot \cos\left(\frac{x+d}{2d}\right)\right)^{-3.66\cdot\lambda d},$$

where $d > 0$ and $x \in [-d, d]$. Table 1 shows the maximum absolute relative errors, i.e.,

$$\max_{x\in(-\Delta,\Delta)} \left|\frac{\varepsilon_{d,g}^{(\lambda)}(x) - \mathrm{e}^{\lambda x}}{\mathrm{e}^{\lambda x}}\right|,$$

of this approximation for various values of d and Δ, where $\lambda = 1$.

Table 1: The maximum absolute relative errors of the approximations of $\mathrm{e}^{\lambda x}$ using $\varepsilon_{d,g}^{(\lambda)}(x)$, for $\lambda = 1$ and $g(x) = \cos(x)$.

d	$x \in (-2,2)$	$x \in (-5,5)$	$x \in (-10,10)$	$x \in (-20,20)$
100	2.42×10^{-2}	1.62×10^{-1}	8.32×10^{-1}	$1.08 \times 10^{+1}$
1000	2.38×10^{-3}	1.50×10^{-2}	6.13×10^{-2}	2.69×10^{-1}

Based on Table 1, the GEF approximates the exponential function around zero quite well, which is in line with the construction of $\varepsilon_{d,g}^{(\lambda)}$. On the other hand, if $x \gg 0$ or $x \ll 0$, ($x \in [-d,d]$), then the goodness of this approximation considerably decreases. Figure 1 shows the plots of the GEF, exponential function and their absolute relative difference for $\lambda = 1$, $d = 100$, $g(x) = \cos(x)$.

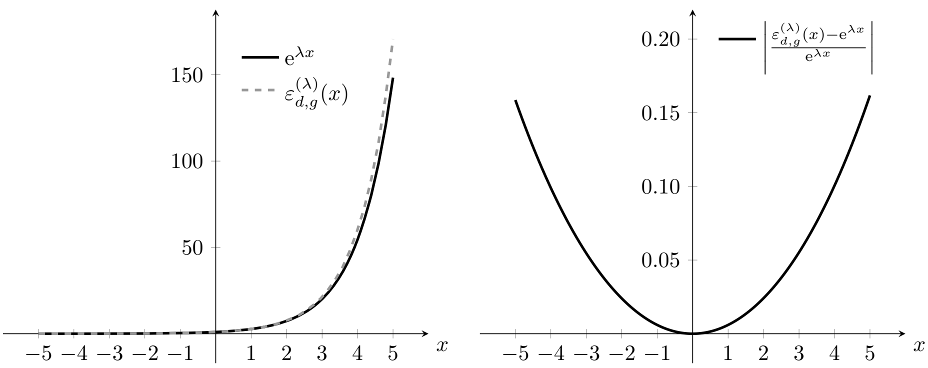

Figure 1: Plots of the GEF, exponential function and their absolute relative difference for $\lambda = 1$, $d = 100$, $g(x) = \cos(x)$.

We can achieve more effective approximations to the exponential function $\mathrm{e}^{\lambda x}$ on the interval $[-d, d]$, if for a strictly increasing g we require $g(0) = 0$ and for a strictly decreasing g we require $g(1) = 0$. Table 2 summarizes the main properties of a GEF induced by g in these cases. Note that if g is a strictly increasing function with $g(0) = 0$ and $\lambda < 0$, then we interpret $\varepsilon_{d,g}^{(\lambda)}(-d)$ as $\varepsilon_{d,g}^{(\lambda)}(-d) = \lim_{x\to -d^+} \varepsilon_{d,g}^{(\lambda)}(x) = \infty$. Similarly, if g is a strictly decreasing function with $g(1) = 0$ and $\lambda > 0$, then we interpret $\varepsilon_{d,g}^{(\lambda)}(d)$ as $\varepsilon_{d,g}^{(\lambda)}(d) = \lim_{x\to d^-} \varepsilon_{d,g}^{(\lambda)}(x) = \infty$. In Table 2, $< \infty$ stands for a finite value, while $\nearrow$ ($\searrow$, respectively) denotes that a function is strictly increasing (decreasing, respectively).

Table 2: Main properties of the GEF $\varepsilon_{d,g}^{(\lambda)}$ depending on the values of $g(0)$ and $g(1)$.

g	λ	$g(0)$	$g(1)$	$\varepsilon_{d,g}^{(\lambda)}(-d)$	$\varepsilon_{d,g}^{(\lambda)}(d)$	$\varepsilon_{d,g}^{(\lambda)}$
$\nearrow$	>0	0	$<\infty$ (∞)	0	$<\infty$ (∞)	$\nearrow$
$\nearrow$	<0	0	$<\infty$ (∞)	∞	>0 (0)	$\searrow$
$\searrow$	>0	$<\infty$ (∞)	0	>0 (0)	∞	$\nearrow$
$\searrow$	<0	$<\infty$ (∞)	0	$<\infty$ (∞)	0	$\searrow$

Example 2. Let $g(x) = x$, $x \in [0, 1]$. Then, $g\left(\frac{1}{2}\right) = \frac{1}{2}$, $g'\left(\frac{1}{2}\right) = 1$ and via direct calculation, we get that the GEF induced by g is

$$\varepsilon_{d,g}^{(\lambda)}(x) = \left(1 + \frac{x}{d}\right)^{\lambda d},$$

where $d > 0$ and $x \in [-d, d]$. It is well known that $\lim_{d\to\infty} \left(1 + \frac{x}{d}\right)^{\lambda d} = \mathrm{e}^{\lambda x}$, which is in line with the result of Theorem 2.

It should be added that $\varepsilon_{d,g}^{(\lambda)}(x)$ is closely related to the cumulative distribution function of the p-exponential distribution, which is given as

$$F_p(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1 - \left(1 - \frac{x}{p+1}\right)^p, & \text{if } x \in (0, p+1) \\ 1, & \text{if } x \geq p+1, \end{cases}$$

where $p > 0$ (see Sinner *et al.* [18]).

Table 3 shows the maximum absolute relative errors of the approximations for various values of d and Δ ($x \in (-\Delta, \Delta)$), where $\lambda = 1$. Figure 2 shows the plots of the GEF, exponential function and their absolute relative difference for $\lambda = 1$, $d = 100$, $g(x) = x$.

Example 3. Let $g_\alpha(x) = \left(\frac{1-x}{x}\right)^\alpha$, $x \in [0, 1]$ and $\alpha \neq 0$. It should be added that the function g_α is known as the additive generator of the Dombi operators in

Table 3: The maximum absolute relative errors of the approximations of $\mathrm{e}^{\lambda x}$ using $\varepsilon_{d,g}^{(\lambda)}(x)$, for $\lambda = 1$ and $g(x) = x$.

d	$x \in (-2,2)$	$x \in (-5,5)$	$x \in (-10,10)$	$x \in (-20,20)$
100	2.00×10^{-2}	1.21×10^{-1}	4.15×10^{-1}	9.01×10^{-1}
1000	2.00×10^{-3}	1.25×10^{-2}	4.91×10^{-2}	1.83×10^{-1}

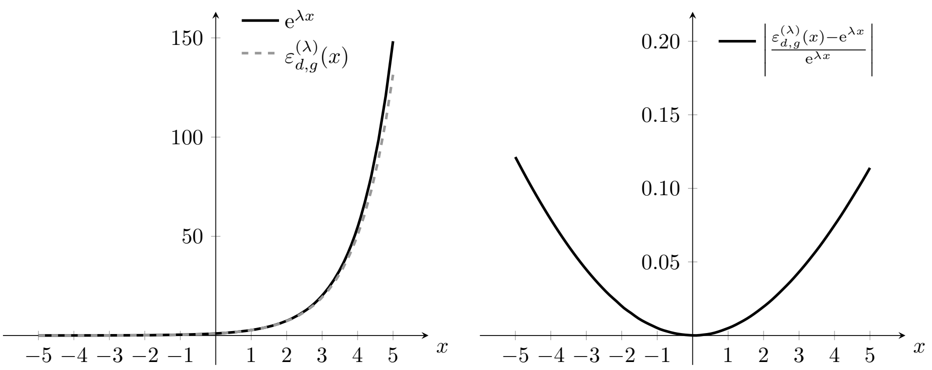

Figure 2: Plots of the GEF, exponential function and their absolute relative difference for $\lambda = 1$, $d = 100$, $g(x) = x$.

continuous-valued logic (see [7]). Clearly, $g_\alpha \in \mathcal{G}$, i.e., g_α satisfies the requirements for a generator of a GEF. Exploiting Proposition 1, we get that for any $\alpha \neq 0$, g_α induces the same GEF independently of the value of α. Let $\alpha = 1$, and let $g(x) = g_\alpha(x) = \frac{1-x}{x}$, $x \in [0,1]$. Then, $g\left(\frac{1}{2}\right) = 1$, $g'\left(\frac{1}{2}\right) = -4$ and via direct calculation, we get that the GEF induced by g is

$$\varepsilon_{d,g}^{(\lambda)}(x) = \left(\frac{d+x}{d-x}\right)^{\lambda \frac{d}{2}}, \tag{12}$$

where $d > 0$ and $x \in [-d, d]$. Note that this generalized epsilon function is identical to the epsilon function given in Definition 1, which was introduced in [9]. Table 4 shows the maximum absolute relative errors of the approximations for various values of d and Δ ($x \in (-\Delta, \Delta)$), where $\lambda = 1$.

Figure 3 shows the plots of the GEF, exponential function and their absolute relative difference for $\lambda = 1$, $d = 100$, $g(x) = \frac{1-x}{x}$.

Notice that in Example 2, $g(0) = 0$ and $g(1) = 1$, i.e., $g(1)$ is finite, while in Example 3, $g(0) = \infty$ (more precisely, $\lim_{x \to 0} g(x) = \infty$) and $g(1) = 0$. We can see that in this latter case, we obtained a much lower maximum absolute relative approximation error. It is worth noting that based on Proposition 1, the generator $g(x) = \frac{x}{1-x}$ induces the same GEF as that in Eq. (12).

Table 4: The maximum absolute relative errors of the approximations of $\mathrm{e}^{\lambda x}$ using $\varepsilon_{d,g}^{(\lambda)}(x)$, for $\lambda = 1$ and $g(x) = \frac{1-x}{x}$.

d	$x \in (-2,2)$	$x \in (-5,5)$	$x \in (-10,10)$	$x \in (-20,20)$
100	2.67×10^{-4}	4.18×10^{-3}	3.41×10^{-2}	3.14×10^{-1}
1000	2.67×10^{-6}	4.17×10^{-5}	3.33×10^{-4}	2.67×10^{-3}

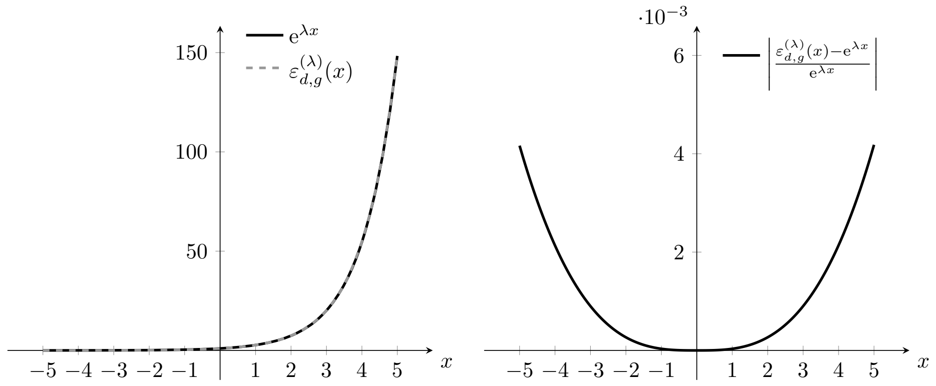

Figure 3: Plots of the GEF, exponential function and their absolute relative difference for $\lambda = 1$, $d = 100$, $g(x) = \frac{1-x}{x}$.

Remark 2. It should be added that if the generator of a GEF is a rational function, then the GEF is rational as well (see, e.g., Eq. (12)). In such a case, the generalized epsilon function is computationally simple, it may be treated as an easy-to-compute alternative to the exponential function.

4 Some applications of the generalized epsilon function

Since the exponential function may be viewed as an asymptotic generalized epsilon function, this latter may have a considerable application potential in many areas of science. Here, we will briefly present two particular applications: the first one is an approximation to the exponential distribution, the second one is an approximation to the sigmoid function.

4.1 An approximation to the exponential probability distribution

The exponential probability distribution plays an important role in probability theory and mathematical statistics (see, e.g., [6, 5, 2, 3]). Now, we will demonstrate how the cumulative distribution function (CDF) of the random variable, which has an exponential probability distribution with a $\lambda > 0$ parameter value, can be approximated using the generalized epsilon function.

Proposition 2. *Let $g \in \mathcal{G}$ such that g is either strictly increasing with $g(0) = 0$ and $g(1) = \infty$, or it is strictly decreasing with $g(1) = 0$ and $g(0) = \infty$. Furthermore, let $\lambda > 0$, $d > 0$ and let $\varepsilon_{d,g}^{(\lambda)} \colon [-d, d] \to \overline{\mathbb{R}}_+$ be a GEF induced by g according to Eq. (7). Then the function $F_{d,g}^{(\lambda)} \colon \mathbb{R} \to [0, 1]$ given by*

$$F_{d,g}^{(\lambda)}(x) = \begin{cases} 0, & \textit{if } x \leq 0 \\ 1 - \varepsilon_{d,g}^{(-\lambda)}(x), & \textit{if } 0 < x \leq d \\ 1, & \textit{if } x > d \end{cases} \tag{13}$$

is a CDF of a continuous random variable and for any $x \in \mathbb{R}$,

$$\lim_{d\to\infty} F_{d,g}^{(\lambda)}(x) = 1 - \mathrm{e}^{-\lambda x}. \tag{14}$$

Proof. Clearly, $F_{d,g}^{(\lambda)}(x)$ is continuous and it satisfies the requirements for a CDF, while Eq. (14) is an immediate consequence of Theorem 2. $\square$

Remark 3. Utilizing the generator $g_\alpha(x) = \left(\frac{1-x}{x}\right)^\alpha$, $x \in [0, 1]$ and $\alpha \neq 0$, we get

$$F_{d,g_\alpha}^{(\lambda)}(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1 - \left(\frac{d+x}{d-x}\right)^{-\lambda\frac{d}{2}}, & \text{if } 0 < x < d \\ 1, & \text{if } x \geq d, \end{cases}$$

which is the CDF of the epsilon probability distribution (see [9]). Therefore, the CDF given in Eq. (13) may be treated as a generator function-based generalization of the CDF of the epsilon probability distribution. It is worth noting that $F_{d,g_\alpha}^{(\lambda)}$ approximates the exponential CDF quite well even for small values of the parameter d. For example, for $\lambda = 1$ and $d = 10$, we have

$$\max_{x\in(0,10)} \left|1 - \mathrm{e}^{-\lambda x} - F_{d,g_\alpha}^{(\lambda)}(x)\right| < 4.53 \times 10^{-3}$$

and

$$\max_{x\in(0,10)} \left|\frac{1 - \mathrm{e}^{-\lambda x} - F_{d,g_\alpha}^{(\lambda)}(x)}{1 - \mathrm{e}^{-\lambda x}}\right| < 2.03 \times 10^{-3}.$$

4.2 An approximation to the sigmoid function

It is well-known that the sigmoid function $\sigma^{(\lambda)}\colon \mathbb{R} \to (0,1)$, which is given by

$$\sigma^{(\lambda)}(x) = \frac{1}{1+\mathrm{e}^{-\lambda x}}, \tag{15}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, has a lot of applications in many areas including computer science, engineering, biology and economics (see, e.g., [11, 12, 17, 14, 16]). It should be noted that the sigmoid function is also known as the logistic function. For example, in probability theory and mathematical statistics the logistic function can be utilized as a cumulative distribution function (logistic distribution) or as a regression function (logistic regression). The following proposition is an immediate consequence of Theorem 2.

Proposition 3. *Let $g \in \mathcal{G}$ such that g is either strictly increasing with $g(0) = 0$, or it is strictly decreasing with $g(1) = 0$. Furthermore, let $\lambda \in \mathbb{R} \setminus \{0\}$, $d > 0$ and let $\varepsilon^{(\lambda)}_{d,g}\colon [-d,d] \to \overline{\mathbb{R}}_+$ be a GEF induced by g according to Eq. (7). Then, for any $x \in \mathbb{R}$,*

$$\lim_{d\to\infty} \frac{1}{1+\varepsilon^{(-\lambda)}_{d,g}(x)} = \frac{1}{1+\mathrm{e}^{-\lambda x}}. \tag{16}$$

Exploiting the result of Proposition 3, the function $S^{(\lambda)}_{d,g}\colon [-d,d] \to [0,1]$, which is given by

$$S^{(\lambda)}_{d,g}(x) = \frac{1}{1+\varepsilon^{(-\lambda)}_{d,g}(x)}.$$

may be viewed as a viable alternative to the sigmoid function on the bounded domain $[-d,d]$.

Remark 4. More generally, if g is either strictly increasing with $g(0) = 0$ and $g(1) = \infty$, or g is strictly decreasing with $g(1) = 0$ and $g(0) = \infty$, then the function

$$\sigma^{(\lambda)}_{d,g}(x) = g^{-1}\left(\left(\frac{g\left(\frac{x+d}{2d}\right)}{g\left(\frac{1}{2}\right)}\right)^{-2\lambda d \frac{g\left(\frac{1}{2}\right)}{g'\left(\frac{1}{2}\right)}}\right),$$

may be treated as an alternative to the sigmoid function on the bounded domain $[-d,d]$. Clearly, with the choice $g(x) = \frac{1-x}{x}$, $x \in [0,1]$, $\sigma^{(\lambda)}_{d,g}(x) = S^{(\lambda)}_{d,g}(x)$ for any $x \in [-d,d]$.

5 Conclusions

In this study, we presented the generalized epsilon function, which is a generator function-based mapping from the bounded domain $[-d,d]$ to the non-negative extended real line ($d > 0$). We proved that if $d \to \infty$, then the generalized epsilon function coincides with the exponential function. This result allows us to treat the generalized epsilon function as an alternative to exponential function on a bounded domain. Since this new function is generator function-dependent, it is very flexible and it can be utilized in a wide range of applications.

References

[1] Acar, Tuncer, Mursaleen, Mohammad, and Deveci, Serife Nur. Gamma operators reproducing exponential functions. *Advances in Difference Equations*, 2020(1):1–13, 2020. DOI: `10.1186/s13662-020-02880-x`.

[2] Afify, Ahmed Z and Mohamed, Osama Abdo. A new three-parameter exponential distribution with variable shapes for the hazard rate: Estimation and applications. *Mathematics*, 8(1):135, 2020. DOI: `10.3390/math8010135`.

[3] Albanbay, Nurtay, Medetov, Bekbolat, and Zaks, Michael A. Exponential distribution of lifetimes for transient bursting states in coupled noisy excitable systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 31(9):093105, 2021. DOI: `10.1063/5.0059102`.

[4] Aral, Ali, Cardenas-Morales, Daniel, and Garrancho, Pedro. Bernstein-type operators that reproduce exponential functions. *Journal of Mathematical Inequalities*, 12(3):861–872, 2018. DOI: `10.7153/jmi-2018-12-64`.

[5] Bakouch, Hassan S, Hussain, Tassaddaq, Chesneau, Christophe, and Khan, Muhammad Nauman. On a weighted exponential distribution with a logarithmic weight: Theory and applications. *Afrika Matematika*, pages 1–22, 2021. DOI: `10.1007/s13370-020-00861-7`.

[6] Balakrishnan, K. *Exponential distribution: theory, methods and applications*. Routledge, 2019.

[7] Dombi, J. General class of fuzzy operators, the De Morgan class of fuzzy operators and fuzziness included by fuzzy operators. *Fuzzy Sets and Systems*, 8:149–168, 1982.

[8] Dombi, József and Jónás, Tamás. Approximations to the normal probability distribution function using operators of continuous-valued logic. *Acta Cybernetica*, 23(3):829–852, 2018. DOI: `10.14232/actacyb.23.3.2018.7`.

[9] Dombi, József, Jónás, Tamás, and Tóth, Zsuzsanna Eszter. The epsilon probability distribution and its application in reliability theory. *Acta Polytechnica Hungarica*, 15(1):197–216, 2018. DOI: `10.12700/APH.15.1.2018.1.12`.

[10] Grabisch, Michel, Marichal, Jean-Luc, Mesiar, Radko, and Pap, Endre. Aggregation functions: Means. *Information Sciences*, 181(1):1–22, 2011. DOI: `10.1016/j.ins.2010.08.043`.

[11] Iliev, A, Kyurkchiev, Nikolay, and Markov, S. On the approximation of the step function by some sigmoid functions. *Mathematics and Computers in Simulation*, 133:223–234, 2017.

[12] Khairunnahar, Laila, Hasib, Mohammad Abdul, Rezanur, Razib Hasan Bin, Islam, Mohammad Rakibul, and Hosain, Md Kamal. Classification of malignant and benign tissue with logistic regression. *Informatics in Medicine Unlocked*, 16:100189, 2019. DOI: `10.1016/j.imu.2019.100189`.

[13] Klement, E.P., Mesiar, R., and Pap, E. *Triangular Norms*. Trends in Logic. Springer Netherlands, 2013.

[14] Liu, W., Wang, Z., Yuan, Y., Zeng, N., Hone, K., and Liu, X. A novel sigmoid-function-based adaptive weighted particle swarm optimizer. *IEEE Transactions on Cybernetics*, 51(2):1085–1093, 2021. DOI: `10.1109/TCYB.2019.2925015`.

[15] Musa, Salihu S, Zhao, Shi, Wang, Maggie H, Habib, Abdurrazaq G, Mustapha, Umar T, and He, Daihai. Estimation of exponential growth rate and basic reproduction number of the coronavirus disease 2019 (COVID-19) in Africa. *Infectious Diseases of Poverty*, 9(1):1–6, 2020. DOI: `10.1186/s40249-020-00718-y`.

[16] Qiao, Junfei, Li, Sanyi, and Li, Wenjing. Mutual information based weight initialization method for sigmoidal feedforward neural networks. *Neurocomputing*, 207:676–683, 2016. DOI: `10.1016/j.neucom.2016.05.054`.

[17] Qin, Y., Wang, X., and Zou, J. The optimized deep belief networks with improved logistic sigmoid units and their application in fault diagnosis for planetary gearboxes of wind turbines. *IEEE Transactions on Industrial Electronics*, 66(5):3814–3824, 2019. DOI: `10.1109/TIE.2018.2856205`.

[18] Sinner, Corinne, Dominicy, Yves, Ley, Christophe, Trufin, Julien, and Weber, Patrick. An interpolating family of size distributions, 2016. arXiv preprint arXiv:1606.04430.

[19] Yerokhin, Vladimir A, Patkóš, Vojtěch, and Pachucki, Krzysztof. Atomic structure calculations of helium with correlated exponential functions. *Symmetry*, 13(7):1246, 2021. DOI: `10.3390/sym13071246`.

Received: 16th November 2021