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Special cases of critical linear difference equations

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Abstract. In this paper, we investigate even-order linear difference equations and their criticality. However, we restrict our attention only to several special cases of the general Sturm–Liouville equation. We wish to investigate on such cases a possible converse of a known theorem. This theorem holds for second-order equations as an equivalence; however, only one implication is known for even-order equations. First, we show the converse in a sense for one term equations. Later, we show an upper bound on criticality for equations with nonnegative coefficients as well. Finally, we extend the criticality of the second-order linear self-adjoint equation for the class of equations with interlacing indices. In this way, we can obtain concrete examples aiding us with our investigation.

Keywords: critical equations, linear difference equations, equations with interlacing indices.

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1 Introduction

The concept of criticality for second-order equations was developed in [15] and for equations of general even-order in [8]. It is established for continuous case as well which the reader can find for example in [14, 16, 27, 32, 36–38] and in other references. This work was intended as an attempt to investigate a converse of the main result obtained in [8] through observing subclasses of the Sturm–Liouville difference equation. We obtain several new properties of said subclasses and concrete examples whose behaviour motivates further research.

Section 2 contains a summary of necessary definitions and theorems together with some minor improvements. Nevertheless, it is worth pointing out that critical linear equations create a subclass of disconjugated equations. When we work with second-order equations we have only two options, that a disconjugated equation is either critical or subcritical. For higher-order equations of order $2k$ we have to separate this approach into subsequent cases, that equations can be p-critical for $0 \leq p \leq k$, $p \in \mathbb{Z}$ and when it is 0-critical we say that the equation is subcritical.

In Section 3 we work with the one term linear equation

$$(-\triangle)^k \left(r_n \triangle^k y_{n-k}\right) = 0, \quad r_n > 0, \quad n \in \mathbb{Z}. \tag{1.1}$$

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Eq. (1.1) gives a subclass of the general Sturm–Liouville equation where only one of the coefficients is non-zero. Our main result of Section 3 considers a situation where we make one of the zero coefficients of Eq. (1.1) arbitrarily smaller. When this change leads to the situation where Eq. (1.1) loses disconjugacy, then the original Eq. (1.1) is at least p-critical where the assumptions give the number p. Later, we extend this approach for equations with more terms, where we use mainly equations with nonnegative coefficients. We will introduce an upper bound on the number p in the p-criticality of such equations. Our approach also partially covers two term equations used in [7,39].

Section 4 focuses on the following class of linear difference equations with interlacing indices

$$a_n y_{n+2} + b_n y_n + a_{n-2} y_{n-2} = 0, \quad n \in \mathbb{Z}. \tag{1.2}$$

The equations with interlacing indices from time to time appear in the literature (see, e.g., [19, 40–42]). They, among others, can be used in getting some counterexamples. Here we describe a space of recessive solutions of Eq. (1.2) at $\pm\infty$ and link the criticality of the second-order self-adjoint equation to the criticality of Eq. (1.2). The important fact to note here is that for even-order equations, we cannot use several tools which are available for second-order equations. Hence, we work with equations with interlacing indices to apply these tools at least on a subclass of the Sturm–Liouville equation. By this, we obtain concrete examples where the possible behaviour of the converse shows clearly.

Overall, we develop a background for further research even though no attempt has been made to postulate the form of the possible converse. Additionally, our results show that there are still many uncharted territories in regard to the criticality of even-order linear equations. For other examples of the recent development in this field, we refer the reader to see, for example, [13,17,22,25,28]. The important point to note here is that the topic of critical equations is also close to the topic of oscillation. Hence, other closely related results about the critical case concerning non-oscillation are stated in [23,24], see also [9].

2 Preliminaries

The article [8] works with linear even-order Sturm–Liouville equation in the form

$$\sum_{i=0}^{k} (-\triangle)^i \left(r_n^{[i]} \triangle^i y_{n-i} \right) = 0, \quad n \in \mathbb{Z}, \tag{2.1}$$

and its criticality is developed. To show this, we have to link solutions of Eq. (2.1) to the solutions of linear Hamiltonian difference system (see for example [1,3,8])

$$\triangle x_n = A x_{n+1} + B_n u_n, \qquad \triangle u_n = C_n x_{n+1} - A^T u_n \tag{2.2}$$

through the substitution

$$x_n = \begin{pmatrix} y_n \\ \triangle y_{n-1} \\ \cdots \\ \triangle^{k-1} y_{n+1-k} \end{pmatrix}, \qquad u_n = \begin{pmatrix} \sum_{i=1}^{k} (-\triangle)^{i-1} \left(r_{n+1}^{[i]} \triangle^i y_{n+1-i} \right) \\ \vdots \\ -\triangle \left(r_{n+1}^{[k]} \triangle^k y_{n+1-k} \right) + r_{n+1}^{[k-1]} \triangle^{k-1} y_{n+2-k} \\ r_{n+1}^{[k]} \triangle^k y_{n+1-k} \end{pmatrix}. \tag{2.3}$$

Here A, B_n, C_n are $k\times k$ matrices $B_n = \operatorname{diag}\left(0,\dots,0,\frac{1}{r^{[k]}_{n+1}}\right)$, $C_n = \operatorname{diag}\left(r^{[0]}_{n+1},\dots,r^{[k-1]}_{n+1}\right)$ and

$$A = a_{ij} = \begin{cases} 1, & i=j, \\ -1, & i+1=j, \\ 0, & \text{otherwise.} \end{cases}$$

A $2k\times k$ matrix solution $\binom{X_n}{U_n}$ of Eq. (2.2) is said to be a conjoined basis when $X_n^T U_n$ is symmetric and rank $\binom{X_n}{U_n} = k$. A conjoined basis $\binom{X_n}{U_n}$ is said to be recessive solution at ∞ provided that for some N sufficiently large holds $X_n X_{n+1}^{-1} A^{-1} B_n \geq 0$, for all $n \geq N$ and

$$\lim_{h\to\infty} \left(\sum_{n=N}^{h} X_{n+1}^{-1} A^{-1} B_n \left(X_n^T\right)^{-1} \right)^{-1} = 0.$$

If matrix solution $\binom{X_n}{U_n}$ is a recessive solution at ∞, then solutions $y_n^1,\dots,y_n^k$ generating columns of X_n form the system of recessive solutions of Eq. (2.1) at ∞. The system of recessive solutions at $-\infty$ is defined similarly. For analysis of recessive solutions of second-order equations, see for example [4,5,35].

Here and subsequently, we denote the spaces of recessive solutions at $\pm\infty$ as $\nu^\pm$, i.e.

$$\nu^\pm = \text{Lin}\{\text{recessive solution of Eq. (2.1) at} \pm\infty\}.$$

With this notation we shall call a disconjugate Eq. (2.1) as p-critical on $\mathbb{Z}$ when $\dim \nu^+ \cap \nu^- = p$. The main result of [8] reads as follows.

Theorem 2.1. *Let disconjugate Eq. (2.1) be p-critical on $\mathbb{Z}$, and let $H \in \mathbb{Z}$, $\varepsilon > 0$ be arbitrary. Furthermore, let arbitrary $J \subset \{0,\dots,n-1\}$ satisfy $|J| = k-p+1$ and consider the sequences*

$$s_H^{[j]} = \begin{cases} r_H^{[j]} - \varepsilon, & \textit{for } j \in J, \\ r_H^{[j]}, & \textit{otherwise,} \end{cases}$$

and $s_n^{[i]} = r_n^{[i]}$, for all i and $n \neq H$. Then the equation

$$\sum_{i=0}^{k} (-\triangle)^i \left(s_n^{[i]} \triangle^i y_{n-i} \right) = 0$$

is not disconjugate.

This theorem has been later extended in [26,44] and shows that critical equations create a borderline where appears a bifurcation with respect to disconjugacy. Nevertheless, Theorem 2.1 holds for second-order equations as an equivalence. One may ask whether this is still true if we consider a general even-order equation. Such question also serves as the primary motivation for our work.

Final conjecture of [8] is proved in [20] and they both focus on the one term equation

$$(-\triangle)^k \left(r_n \triangle^k y_{n-k} \right) = 0, \quad r_n > 0, \quad n \in \mathbb{Z}, k \in \mathbb{N}. \tag{2.4}$$

With a notation that $n^{[p]} = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-p+1)$, $p \in \mathbb{N}$, the results state the following.

Theorem 2.2. *Let $p \in \{1, \dots, k\}$ and suppose that*

$$\sum_{j=-\infty}^{0} \frac{j^{2(k-p)}}{r_{j+k}} = \infty = \sum_{j=0}^{\infty} \frac{j^{2(k-p)}}{r_{j+k}}.$$

Then $\mathrm{Lin}\{1, \dots, n^{[p-1]}\} \subset \nu^+ \cap \nu^-$ *and* (2.4) *is at least p-critical. Moreover, if either*

$$\sum_{j=-\infty}^{0} \frac{j^{2(k-1)}}{r_{j+k}} < \infty \quad or \quad \sum_{j=0}^{\infty} \frac{j^{2(k-1)}}{r_{j+k}} < \infty$$

then $\nu^+ \cap \nu^- = \emptyset$.

The converse of Theorem 2.2 can be found in [21]. Eq. (2.4) will be the main objective of the following section and so let us mention that when dealing with Eq. (2.4) it is useful to utilize the fact that if $\triangle^{j+1} y_{n-j-1} = z_n$ then

$$y_n = \frac{1}{j!} \sum_{i=-\infty}^{n-1} (n-i-1)^{[j]} z_{i+j+1}. \tag{2.5}$$

Another useful result of [8] is the following lemma. However, first of all, let us mention that we follow the notation of [8] and by $l_0^2(\mathbb{Z})$ we denote the set of sequences

$$l_0^2(\mathbb{Z}) = \{\{u_n\} \mid \text{ only for finitely many } n \in \mathbb{Z} \text{ is } u_n \neq 0\}.$$

Lemma 2.3. *Suppose that Eq.* (2.1) *is p-critical for some $p \in \{1, \dots, k\}$. Then for every $\varepsilon > 0$ there exists a sequence $u_n \in l_0^2(\mathbb{Z})$ such that*

$$F(u) = \sum_{n=-\infty}^{\infty} \sum_{i=0}^{k} r_n^{[i]} \left(\triangle^i u_{n-i} \right) < \varepsilon.$$

Proof of Lemma 2.3 obtains for any $y_n \in \nu^+ \cap \nu^-$ such an $u_n \in l_0^2(\mathbb{Z})$ that $y_n = u_n$ on arbitrary compact $[A, B]$ and which satisfies that $F(u) < \varepsilon$, for arbitrary small $\varepsilon > 0$. In light of this, we can reformulate ideas of the proof of Theorem 2.1 to obtain the following theorem.

Theorem 2.4. *Let Eq.* (2.1) *be disconjugate and p-critical on $\mathbb{Z}$, and let $\varepsilon > 0$ be arbitrary. For any $H \in \mathbb{Z}$ there is $J \subset \{0, \dots, n-1\}$ with $|J| \geq p$ such that if for any $j \in J$ we replace*

$$s_H^{[i]} = \begin{cases} r_H^{[i]} - \varepsilon, & \text{for } i = j \\ r_H^{[i]}, & \text{for } i \neq j \end{cases}$$

and $s_n^{[i]} = r_n^{[i]}$ for all i, $n \neq H$, then the equation

$$\sum_{i=0}^{k} (-\triangle)^i \left(s_n^{[i]} \triangle^i y_{n-i} \right) = 0$$

is not disconjugate.

Proof. The proof of Theorem 2.1 (see also [8, 10]) shows that there are p solutions $y_n^1, \dots, y_n^p$ of Eq. (2.1) with the following property. For any $H \in \mathbb{Z}$ there is $J \subset \{0, \dots, n-1\}$ with $|J| = p$

such that there is a surjection from J to y_n^j where for any $j \in J$ holds $\triangle^j y^j_{H-j} = 1$. Hence, for any $\varepsilon > 0$, $H \in \mathbb{Z}$ and $j \in J$ we replace $r_n^{[i]}$ by $s_n^{[i]}$ to obtain

$$\sum_{n=-\infty}^{\infty} \sum_{i=0}^{k} s_n^{[i]} \left(\triangle^i u_{n-i} \right) = -\varepsilon \left(\triangle^j u_{H-j} \right) + \sum_{n=-\infty}^{\infty} \sum_{i=0}^{k} r_n^{[i]} \left(\triangle^i u_{n-i} \right) = -\varepsilon \left(\triangle^j u_{H-j} \right) + F(u).$$

However, from the proof of Lemma 2.3 we have that we can choose such u_n which satisfies $F(u) < \frac{\varepsilon}{2}$ and that $\triangle^j u_{H-j} = \triangle^j y^j_{H-j} = 1$. □

The principal difference between Theorems 2.1 and 2.4 is that in Theorem 2.1 we make $k - p + 1$ coefficients arbitrarily smaller, and then we lose disconjugancy. On the other hand, in Theorem 2.4 it is enough to make only one of p coefficients smaller to obtain the same. The problem in Theorem 2.4 is identifying the right coefficients. In contrast, because conditions of Theorem 2.4 are less restrictive, we assume that we could find a converse of Theorem 2.4 in the future.

We would like to also remind the reader about the following results concerning the self-adjoint second-order linear equation

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = 0. \tag{2.6}$$

In [15] it is shown, that Eq. (2.6) is disconjugate if and only if there are positive solutions $u_n^{\pm}$, which are recessive at $\pm\infty$. Moreover, in [35] (see also [15]) appears the following theorem.

Theorem 2.5. *If Eq. (2.6) is disconjugate, then*

$$\sum_{n}^{\infty} \frac{1}{(-a_n) u_n^+ u_{n+1}^+} = \infty = \sum_{n=-\infty} \frac{1}{(-a_n) u_n^- u_{n+1}^-}.$$

Additionally, Eq. (2.6) is critical if and only if $u_n^+ = u_n^-$ and Theorem 2.1 and 2.4 are for Eq. (2.6) the same. They hold as an equivalence for the second-order equations and therefore we have another way how to define criticality of Eq. (2.6). Other equivalent ways to define critical equations can be found in [15] or [29].

3 One term even-order linear equations

Following section deals with one term difference equation

$$(-\triangle)^k \left(r_n \triangle^k y_{n-k} \right) = 0, \quad r_n > 0, \quad n \in \mathbb{Z}, k \in \mathbb{N}. \tag{3.1}$$

Such equation is investigated in [20] and according to [2] Eq. (3.1) is disconjugate if and only if

$$\sum_{n=-\infty}^{\infty} r_n \left(\triangle^k u_{n-k} \right)^2 > 0, \quad \text{for all } u_n \in l_0^2(\mathbb{Z}),\, u_n \neq 0.$$

Of course, this sum can be rewritten in different shapes and forms, as we can see for example in [8]. Our main result is the following theorem. For simplicity of formulas, we denote in the proof $|0|^{k-p} = 1$, because otherwise, we would have to define a new sequence

$$\chi_n = \begin{cases} |n|^{k-p}, & n \neq 0, \\ 1, & n = 0. \end{cases}$$

Theorem 3.1. *Assume that for any $\varepsilon > 0$ and $H \in \mathbb{Z}$ exists nontrivial $u_n \in l_0^2(\mathbb{Z})$ such that*

$$\sum_{n=-\infty}^{\infty} r_n \left(\triangle^k u_{n-k}\right)^2 < \varepsilon \left(\triangle^{p-1} u_{H-(p-1)}\right)^2. \tag{3.2}$$

Then Eq. (3.1) is at least p-critical and $\operatorname{Lin}\{1, \dots, n^{[p-1]}\} \subset \nu^+ \cap \nu^-$.

Proof. We first start by a series of substitution. Let us set $v_n = \triangle^{p-1} u_{n-p+1}$ and then (3.2) transforms as

$$\sum_{n=-\infty}^{\infty} r_n \left(\triangle^{k-p+1} v_{n-(k-p+1)}\right)^2 < \varepsilon \left(v_H\right)^2.$$

Because $u_n \in l_0^2(\mathbb{Z})$ and because differencing a zero sequence gives us only a zero sequence then also $v_n \in l_0^2(\mathbb{Z})$ and additionally $\triangle^{k-p+1} v_{n-(k-p+1)} \in l_0^2(\mathbb{Z})$. Bearing that in mind consider also $x_n = |n|^{k-p} \frac{1}{v_H} \triangle^{k-p+1} v_{n-(k-p+1)}$ to obtain that $x_n \in l_0^2(\mathbb{Z})$ as well and that

$$\sum_{n=-\infty}^{\infty} \frac{r_n}{n^{2(k-p)}} x_n^2 < \varepsilon. \tag{3.3}$$

It is clear from the sum (3.3) that $\lim_{\varepsilon \to 0} x_n = 0$ pointwise, for all $n \in \mathbb{Z}$. Through (2.5) we get via $x_n = |n|^{k-p} \frac{1}{v_H} \triangle^{k-p+1} v_{n-(k-p+1)}$ that

$$v_n = \frac{v_H}{(k-p)!} \sum_{j=-\infty}^{n-1} \frac{(n-j-1)^{[k-p]}}{|j+(k-p+1)|^{k-p}} x_{j+(k-p+1)}.$$

Hence, for all $\varepsilon > 0$ it has to hold that

$$1 = \frac{1}{(k-p)!} \sum_{j=-\infty}^{H-1} \frac{(H-j-1)^{[k-p]}}{|j+(k-p+1)|^{k-p}} x_{j+(k-p+1)} = \frac{1}{(k-p)!} \sum_{i=-\infty}^{H+k-p} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} x_i. \tag{3.4}$$

Next, we claim that we can obtain easily that

$$\lim_{i \to -\infty} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} = 1.$$

Therefore, for some $\omega > 0$ and some i_0 is eventually

$$1 - \omega < \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} < 1 + \omega, \quad \text{for all } i \leq i_0. \tag{3.5}$$

Having disposed of the preliminary steps, we can now assume for contradiction that it holds $\sum_{n=-\infty} \frac{n^{2(k-p)}}{r_n} < \infty$. However, this would mean that

$$\lim_{\substack{n \to -\infty \\ \varepsilon \to 0}} \frac{r_n}{n^{2(k-p)}} x_n \neq 0.$$

Otherwise, we get for arbitrarily small $\delta > 0$ some ε_0, n_0 such that $\frac{r_n}{n^{2(k-p)}} x_n < \delta$, for any $n \leq n_0$ and $\varepsilon < \varepsilon_0$. It is a simple fact that because of

$$\sum_{n=-\infty}^{n_0} x_n < \delta \sum_{n=-\infty}^{n_0} \frac{n^{2(k-p)}}{r_n}$$

is the sum $\sum_{n=-\infty}^{n_0} x_n$ arbitrarily small. However, such situation cannot happen because by (3.4) and (3.5) we get that

$$\begin{aligned}1 &= \frac{1}{(k-p)!}\sum_{i=-\infty}^{H+k-p}\frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}}x_i\\ &< \frac{(1+\omega)\delta}{(k-p)!}\sum_{i=-\infty}^{\min\{n_0,i_0\}}\frac{i^{2(k-p)}}{r_i}+\frac{1}{(k-p)!}\sum_{i=\min\{n_0,i_0\}+1}^{H+k-p}\frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}}x_i\\ &\xrightarrow{\varepsilon\to 0}\frac{(1+\omega)\delta}{(k-p)!}\sum_{i=-\infty}^{\min\{n_0,i_0\}}\frac{i^{2(k-p)}}{r_i}<1,\quad \text{for } \delta \text{ sufficiently small.}\end{aligned}$$

Therefore,

$$\lim_{\substack{n\to-\infty\\ \varepsilon\to 0}}\frac{r_n}{n^{2(k-p)}}x_n\neq 0$$

and by the definition of the limit we can find a positive constant C for which there is a sequence $\varepsilon_k\to 0$ with the following property. For any given ε_k there is a subsequence $n_l\to-\infty$ such that

$$\frac{r_{n_l}}{n_l^{2(k-p)}}\left|x_{n_l}(\varepsilon_k)\right|>C.$$

Before we proceed any further, let us consider, that for ε_k there can also be a subsequence $n_{\hat{l}}$ for which is

$$\frac{r_{n_{\hat{l}}}}{n_{\hat{l}}^{2(k-p)}}\left|x_{n_{\hat{l}}}(\varepsilon_k)\right|<\delta.$$

Altogether, we obtain the inequality

$$\begin{aligned}1 &= \frac{1}{(k-p)!}\sum_{i=-\infty}^{H+k-p}\frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}}x_i\\ &< \frac{(1+\omega)\delta}{(k-p)!}\sum_{i\in\{n_{\hat{l}}\}}\frac{i^{2(k-p)}}{r_i}+\frac{1}{(k-p)!}\sum_{i\notin\{n_{\hat{l}}\}}^{H+k-p}\frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}}x_i\\ &\le \frac{(1+\omega)\delta}{(k-p)!}\sum_{i\in\{n_{\hat{l}}\}}\frac{i^{2(k-p)}}{r_i}+\frac{1+\omega}{(k-p)!}\sum_{i\notin\{n_{\hat{l}}\}}^{i_0}|x_i|+\frac{1}{(k-p)!}\sum_{i=i_0+1}^{H+k-p}\frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}}x_i.\end{aligned}$$

We continue in this fashion by singling out

$$\begin{aligned}\sum_{i\notin\{n_{\hat{l}}\}}^{i_0}|x_i| &> \frac{(k-p)!}{1+\omega}-\delta\sum_{i\in\{n_{\hat{l}}\}}\frac{i^{2(k-p)}}{r_i}-\frac{1}{1+\omega}\sum_{i=i_0+1}^{H+k-p}\frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}}x_i\\ &\ge \frac{(k-p)!}{1+\omega}-\delta\sum_{i=-\infty}^{H+k-p}\frac{i^{2(k-p)}}{r_i}-\frac{1}{1+\omega}\sum_{i=i_0+1}^{H+k-p}\frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}}x_i.\end{aligned}$$

Because x_n converges pointwise to the zero sequence, then the sum

$$\frac{1}{1+\omega}\sum_{i=i_0+1}^{H+k-p}\frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}}x_i$$

can be arbitrarily small if we make given ε_k sufficiently small. Hence, by letting $\varepsilon_k \to 0$ we can find δ sufficiently small so that

$$\sum_{i \notin \{n_{\hat{\imath}}\}}^{i_0} |x_i| > \delta \quad \text{and} \quad \frac{r_n}{n^{2(k-p)}} |x_n(\varepsilon_k)| > \delta, \text{ for all } n \notin \{n_{\hat{\imath}}\}.$$

The result is that for a given ε_k sufficiently small we have through (3.3) that

$$\varepsilon_k > \sum_{j=-\infty}^{i_0} \frac{r_j}{j^{2(k-p)}} x_j^2 > \sum_{i \notin \{n_{\hat{\imath}}\}}^{i_0} \frac{r_i}{i^{2(k-p)}} |x_i| \cdot |x_i| > \delta \sum_{i \notin \{n_{\hat{\imath}}\}}^{i_0} |x_i| > \delta^2.$$

This contradicts our assumption as we have ε_k arbitrarily small and δ is independent from ε_k.

Hence, it has to be $\sum_{n=-\infty} \frac{n^{2(k-p)}}{r_n} = \infty$. Divergence of the other sum $\sum^{\infty} \frac{n^{2(k-p)}}{r_n} = \infty$ is obtained analogously. Only this time we have to use that

$$v_n = \frac{v_H}{(k-p)!} \sum_{j=n-1}^{\infty} \frac{(n-j-1)^{[k-p]}}{|j+(k-p+1)|} x_{j+(k-p+1)}.$$

The rest of the proof follows from Theorem 2.2. □

As an example let us consider the case of $k = 2$ with $r_n = \frac{1}{(n+1)^2}$. We know by Theorem 2.2 that such an equation is 2-critical. Furthermore, from Eq. (3.3) we have that for any $\varepsilon > 0$ there is $x_n \in l_0^2(\mathbb{Z})$ such that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+1)^2} x_n^2 < \varepsilon.$$

It is verified easily that an example of such x_n is the almost zero sequence where only $x_p = 1$, for p sufficiently large.

One question we can ask is whether Eq. (3.1) can be p-critical even when $\{1, \dots, n^{[p-1]}\} \not\subset \nu^+ \cap \nu^-$. However, from Theorem 3.1 we get that this cannot happen.

Corollary 3.2. *If Eq. (3.1) is p-critical, then* $\mathrm{Lin}\{1, \dots, n^{[p-1]}\} \subset \nu^+ \cap \nu^-$.

Proof. Let $H \in \mathbb{Z}$ be arbitrary. Because of Theorem 2.4 there is a set $J \subset \{0, \dots, k-1\}$, $|J| \geq p$ such that for any $j \in J$ is

$$\sum_{n=-\infty}^{\infty} r_n \left(\triangle^k u_{n-k}\right)^2 < \varepsilon \left(\triangle^j u_{H-j}\right)^2.$$

However, because of Theorem 3.1 if $j \in J$, then $\mathrm{Lin}\{1, \dots, n^{[j-1]}\} \subset \nu^+ \cap \nu^-$. This can be satisfied only for $J = \{1, \dots, p-1\}$. □

We will formulate the following theorem to complete in a sense the equivalence with Theorem 3.1.

Theorem 3.3. *Suppose Eq. (3.1) is p-critical and* $\mathrm{Lin}\{1, \dots, n^{[p-1]}\} \subset \nu^+ \cap \nu^-$, *then for any $\varepsilon > 0$ and $H \in \mathbb{Z}$ exists $u_n \in l_0^2(\mathbb{Z})$ such that*

$$\sum_{n=-\infty}^{\infty} r_n \left(\triangle^k u_{n-k}\right)^2 < \varepsilon \left(\triangle^{p-1} u_{H-p+1}\right)^2.$$

Proof. This is a direct result of Theorem 2.4. □

We see one drawback of Theorem 3.1 in that we do not know whether Eq. (3.1) is p-critical or q-critical for some $q \geq p$. We could probably deal with this issue if we formulate Theorem 3.1 in a more precise way and with some workaround through Theorem 2.4. Note also that in Eq. (3.3) it holds for $s < p$ that

$$\sum_{n=-\infty}^{\infty} \frac{r_n}{n^{2(k-s)}} x_n^2 < \sum_{n=-\infty}^{\infty} \frac{r_n}{n^{2(k-p)}} x_n^2 < \varepsilon.$$

3.1 Even-order equations with nonnegative coefficients

The following subsection works with Eq. (2.1) where

$$r_n^{[k]} > 0 \text{ and either } r_n^{[i]} > 0 \text{ for all } n \in \mathbb{Z}, \text{ or } r_n^{[i]} \equiv 0, i \in \{0, \dots, k-1\}. \tag{3.6}$$

Similar ideas as those in the proof of Theorem 3.1 lead us to the following result.

Theorem 3.4. *Assume that Eq. (2.1) satisfies condition (3.6) and that for a given i is $r_n^{[i]}$ a positive sequence. Then Eq. (2.1) is at most i-critical.*

Proof. First consider the situation where $r_n^{[j]} > 0$, for all $j > i$. Then replacing $r_H^{[j]}$ by $r_H^{[j]} - \varepsilon > 0$ for $j \geq i$ does not lose disconjugacy. Hence, it means that Eq. (2.1) is at most i-critical by Theorem 2.4.

Next, for contradiction assume that Eq. (2.1) is at least $(i+1)$-critical. Therefore, for some $j > i$ and any $\varepsilon > 0$ there is $H \in \mathbb{Z}$ such that $r_H^{[j]} = 0$ and

$$\sum_{n=-\infty}^{\infty} r_n^{[i]} \left(\triangle^i u_{n-i}\right)^2 < \sum_{n=-\infty}^{\infty} \sum_{l=0}^{k} r_n^{[l]} \left(\triangle^l u_{n-l}\right)^2 < \varepsilon \left(\triangle^j u_{H-j}\right)^2, \quad u_n \in l_0^2(\mathbb{Z}).$$

With convenient substitution $v_n = \triangle^i u_{n-i}$ we can rewrite this inequality as

$$\sum_{n=-\infty}^{\infty} r_n^{[i]} (v_n)^2 < \varepsilon \left(\triangle^{j-i} v_{H-j+i}\right)^2, \quad \text{for some } v_n \in l_0^2(\mathbb{Z}).$$

Another substitution

$$\left(\triangle^{j-i} v_{H-j+i}\right) x_n = v_n, \tag{3.7}$$

yields

$$\sum_{n=-\infty}^{\infty} r_n^{[i]} (x_n)^2 < \varepsilon, \quad \text{for some } x_n \in l_0^2(\mathbb{Z}).$$

It is clear that letting $\varepsilon \to 0$ gives that $x_n \to 0$ pointwise, for all $n \in \mathbb{Z}$. On the other side, by differentiating (3.7) with respect to n for all $\varepsilon > 0$ we obtain

$$\left(\triangle^{j-i} v_{H-j+i}\right) \triangle^{j-i} x_n = \triangle^{j-i} v_n.$$

Note that $\triangle^{j-i} v_{H-j+i}$ is independent on n. And then by putting $n = H - j + i$ we obtain that $\triangle^{j-i} x_{H-j+i} = 1$. However, we can rewrite (see for example [30]) the equality for all $\varepsilon > 0$ as

$$1 = \triangle^{j-i} x_{H-j+i} = \sum_{q=0}^{j-i} (-1)^q \binom{j-i}{q} x_{H-q}.$$

Taking $\varepsilon \to 0$ together with the fact that we have a finite sum yields

$$1 = \lim_{\varepsilon \to 0} \sum_{q=0}^{j-i} (-1)^q \binom{j-i}{q} x_{H-q} = \sum_{q=0}^{j-i} (-1)^q \binom{j-i}{q} \lim_{\varepsilon \to 0} x_{H-q} = \sum_{q=0}^{j-i} (-1)^q \binom{j-i}{q} \cdot 0 = 0.$$

This contradicts our assumption. □

As a simple example take the equation

$$-2\triangle^2 y_n + \triangle^4 y_{n-1} = 0, \tag{3.8}$$

which can be by Theorem 3.4 at most 1-critical. In fact, results of [29] show that such an equation is 1-critical. However, [29] works only with equations of fourth-order and we do not have any results about equation

$$2\triangle^4 y_n - \triangle^6 y_{n-1} = 0. \tag{3.9}$$

As a result, we can only say that Eq. (3.9) is at most 2-critical and everything else we would have to work through its recessive solutions.

Corollary 3.5. *Assume condition* (3.6). *If for a given* i *is* $r_n^{[i]}$ *a positive sequence and Eq.* (2.1) *is* p*-critical, then*

$$\sum^{\infty} \frac{n^{2(i-p)}}{r_n^{[i]}} = \infty = \sum_{-\infty} \frac{n^{2(i-p)}}{r_n^{[i]}}.$$

Proof. First, because of Theorem 2.4 there is $j \geq p$ such that

$$\sum_{n=-\infty}^{\infty} r_n^{[i]} \left(\triangle^i u_{n-i}\right)^2 < \varepsilon \left(\triangle^{j-1} u_{H-j+1)}\right)^2, \quad \text{for some } u_n \in l_0^2(\mathbb{Z}).$$

Then in the same way as was done in Theorem 3.1 we see that

$$\sum^{\infty} \frac{n^{2(i-j)}}{r_n^{[i]}} = \infty = \sum_{-\infty} \frac{n^{2(i-j)}}{r_n^{[i]}}.$$

However, it holds

$$\infty = \sum^{\infty} \frac{n^{2(i-j)}}{r_n^{[i]}} \leq \sum^{\infty} \frac{n^{2(i-p)}}{r_n^{[i]}},$$
$$\infty = \sum_{-\infty} \frac{n^{2(i-j)}}{r_n^{[i]}} \leq \sum_{-\infty} \frac{n^{2(i-p)}}{r_n^{[i]}}.$$
□

For introducing a nonhomogeneity into studied equations, we could use, for example, results obtained in [33, 34]. Other possible ways forward may be hidden in extending the concept of criticality for half-linear difference equations. See for example [11, 12] together with [44]. For symplectic systems, see also [43].

4 A class of linear equations with interlacing indices

To better understand critical equations of higher-order, we can consider other special cases. In the next part we utilize the second-order linear equation with interlacing indices

$$a_n y_{n+2} + b_n y_n + a_{n-2} y_{n-2} = 0, \quad n \in \mathbb{Z}, \tag{4.1}$$

where $b_n > 0$, $a_n < 0$, for all $n \in \mathbb{Z}$. Through the relations

$$\begin{aligned} r_n^{[2]} &= a_{n-2}, \\ r_n^{[1]} &= -2a_{n-1} - 2a_{n-2}, \\ r_n^{[0]} &= b_n + a_n + a_{n-2}, \end{aligned}$$

we directly link Eq. (2.1) and Eq. (4.1). For equations of general even-order we can find such formulas in [31]. On top of that, Eq. (4.1) has the functional

$$F(u) = \sum_{n=-\infty}^{\infty} a_n u_{n+2} u_n + b_n u_n^2 + a_{n-2} u_n u_{n-2} = \sum_{n=-\infty}^{\infty} b_n u_n^2 + 2a_{n-2} u_n u_{n-2}, \quad \text{for } u_n \in l_0^2(\mathbb{Z}).$$

Eq. (4.1) consists of two equations of the second-order, where we separate Eq. (4.1) into two cases for even and odd n, i.e.

$$a_n y_{n+2} + b_n y_n + a_{n-2} y_{n-2} = 0, \quad n = 2k+1,\ k \in \mathbb{Z}, \tag{4.2}$$

$$a_n y_{n+2} + b_n y_n + a_{n-2} y_{n-2} = 0, \quad n = 2k,\ k \in \mathbb{Z}. \tag{4.3}$$

This property is useful because there are more known results about second-order equations, and through them, we can extend some known results for higher-order equations. Moreover, we have corresponding functionals $F_1(u)$ for Eq. (4.2) and $F_2(u)$ for Eq. (4.3). It holds that

$$\begin{aligned} F(u) &= \sum_{k=-\infty}^{\infty} b_{2k+1} u_{2k+1}^2 + 2a_{2k-1} u_{2k+1} u_{2k-1} + \sum_{k=-\infty}^{\infty} b_{2k} u_{2k}^2 + 2a_{2k-2} u_{2k} u_{2k-2} \\ &= F_1(u^1) + F_2(u^2), \end{aligned}$$

where $u_k^1 = u_{2k+1}$ and $u_k^2 = u_{2k}$. It is clear that if $u_{2k} = 0$, for all $k \in \mathbb{Z}$ then $F(u) = F_1(u^1)$ and vice versa for $F_2(u^2)$. By these arguments, Eq. (4.1) is disconjugate if and only if Eq. (4.2) and (4.3) are both disconjugate. See also [2] and [30].

Theorem 4.1. *Assume that Eq. (4.1) is disconjugate then Eq. (4.1) is p-critical, for $p \in \{1, 2\}$ if and only if p of the equations (4.2), (4.3) are critical. Additionally, disconjugated Eq. (4.1) is subcritical if and only if neither of the equations (4.2), (4.3) is critical.*

Proof. Because of [15] Eq. (4.2) has a positive solutions $u_n^{\pm}$, for $n = 2k+1$, $k \in \mathbb{Z}$ and Eq. (4.3) has a positive solutions $v_n^{\pm}$, for $n = 2k$, $k \in \mathbb{Z}$. Both $u_n^{\pm}$, $v_n^{\pm}$ are recessive at $\pm\infty$. Let us define two solutions of Eq. (4.1) as

$$\alpha_n^{\pm} = \begin{cases} u_n^{\pm}, & n = 2k+1, \\ 0, & n = 2k, \end{cases} \quad \text{and} \quad \beta_n^{\pm} = \begin{cases} v_n^{\pm}, & n = 2k, \\ 0, & n = 2k+1. \end{cases}$$

Through substitution (2.3) we obtain for $n = 2k+1$ odd a matrix solution

$$X_n^{\pm} = \begin{pmatrix} u_n^{\pm} & 0 \\ u_n^{\pm} & -v_{n-1}^{\pm} \end{pmatrix},$$
$$U_n^{\pm} = \begin{pmatrix} -r_{n+1}^{[2]} u_{n+2}^{\pm} - \left(r_{n+1}^{[2]} + 2r_n^{[2]} + r_n^{[1]}\right) u_n^{\pm} & \left(2r_{n+1}^{[2]} + r_n^{[2]} + r_n^{[1]}\right) v_{n+1}^{\pm} + r_n^{[2]} v_{n-1}^{\pm} \\ -2r_n^{[2]} u_n^{\pm} & r_n^{[2]} \left(v_{n+1}^{\pm} + v_{n-1}^{\pm}\right) \end{pmatrix}.$$

For $n = 2k$ even, we get

$$X_n^{\pm} = \begin{pmatrix} 0 & v_n^{\pm} \\ -u_{n-1}^{\pm} & v_n^{\pm} \end{pmatrix},$$
$$U_n^{\pm} = \begin{pmatrix} \left(2r_{n+1}^{[2]} + r_n^{[2]} + r_n^{[1]}\right) u_{n+1}^{\pm} + r_n^{[2]} u_{n-1}^{\pm} & -r_{n+1}^{[2]} v_{n+2}^{\pm} - \left(r_{n+1}^{[2]} + 2r_n^{[2]} + r_n^{[1]}\right) v_n^{\pm} \\ r_n^{[2]} \left(u_{n+1}^{\pm} + u_{n-1}^{\pm}\right) & -2r_n^{[2]} v_n^{\pm} \end{pmatrix}.$$

Such matrix solution is a conjoined basis because $X_n^{\pm}$ will always have rank 2 and it holds for n odd that

$$\left(X_n^{\pm}\right)^T U_n = \begin{pmatrix} \text{something} & \underbrace{\left(2r_{n+1}^{[2]} + 2r_n^{[2]} + r_n^{[1]}\right)}_{=0} v_{n+1}^{\pm} u_n^{\pm} + 2r_n^{[2]} u_n^{\pm} v_{n-1}^{\pm} \\ 2r_n^{[2]} u_n^{\pm} v_{n-1}^{\pm} & \text{something} \end{pmatrix},$$

is symmetrical. For n even is the situation the same.

Subsequently, we will show that $\binom{X_n^+}{U_n^+}$ is a recessive solution at ∞. If $n = 2k$ is even, then

$$X_n^+ \left(X_{n+1}^+\right)^{-1} A^{-1} B_n = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-u_{n-1}^+}{u_{n+1}^+ a_{n-1}} \end{pmatrix} \geq 0.$$

By properly multiplying matrices we conclude that it holds

$$\left(X_{n+1}^+\right)^{-1} A^{-1} B_n \left(X_n^{+T}\right)^{-1} = \begin{pmatrix} \frac{1}{u_{n+1}^+} & 0 \\ \frac{1}{v_n^+} & \frac{-1}{v_n^+} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{a_{n-1}} \\ 0 & \frac{1}{a_{n-1}} \end{pmatrix} \begin{pmatrix} \frac{1}{u_{n-1}^+} & \frac{1}{v_n^+} \\ \frac{-1}{u_{n-1}^+} & 0 \end{pmatrix} = \begin{pmatrix} \frac{-1}{a_{n-1} u_{n-1}^+ u_{n+1}^+} & 0 \\ 0 & 0 \end{pmatrix}.$$

Combining this with similar equality means for n odd we obtain that

$$\sum_{n=M}^{h} \left(X_{n+1}^+\right)^{-1} A^{-1} B_n \left(X_n^{+T}\right)^{-1} = \begin{pmatrix} \sum_{i=M, i \text{ even}}^{h} \frac{-1}{a_{i-1} u_{i-1}^+ u_{i+1}^+} & 0 \\ 0 & \sum_{j=M, j \text{ odd}}^{h} \frac{-1}{a_{j-1} v_{j-1}^+ v_{j+1}^+} \end{pmatrix}.$$

Hence, because of Theorem 2.5 it holds that

$$\lim_{h \to \infty} \left(\sum_{n=M}^{h} \left(X_{n+1}^+\right)^{-1} A^{-1} B_n \left(X_n^{+T}\right)^{-1} \right)^{-1} = 0$$

and $\binom{X_n^+}{U_n^+}$ is indeed a recessive solution at ∞. Analogously we assert that $\binom{X_n^-}{U_n^-}$ is a recessive solution at $-\infty$. The proof is complete by comparing definitions of criticality for Eq. (2.1) and both Eq. (4.2) and (4.3). □

We assume that Theorem 4.1 can be extended for any tridiagonal equation of any even-order in a similar fashion. Additionally, a system of recessive solutions of Eq. (4.1) is defined in [18] through the relation that, if there are solutions $u_n^1, \dots, u_n^4$ of Eq. (4.1) where for any $C > 0$ there is K such that

$$u_n^{k-1} < C u_n^k, \quad \text{for all } n \geq K,$$

then u_n^1, u_n^2 create a system of recessive solutions. However, this does not work with $\alpha_n^{\pm}, \beta_n^{\pm}$, and we work around that through the recessive solutions of Hamiltonian systems.

4.1 Final remarks and examples

Consider the following example where we set in Eq. (4.1) sequences a_n, b_n as

$$a_n = \begin{cases} -1, & n \text{ even}, \\ -3, & n \text{ odd}, \end{cases} \qquad b_n = \begin{cases} 2, & n \text{ even}, \\ 6, & n \text{ odd}. \end{cases}$$

We know by the results of [29] and Theorem 4.1 that such an equation is 2-critical. It is simple matter to verify that coefficients of (2.1) are $r_n^{[2]} = a_{n-2}$, $r_n^{[1]} \equiv 8$ and $r_n^{[1]} \equiv 0$. And therefore we have a concrete example of 4th order Sturm–Liouville equations which is 2-critical.

Another interesting situation appears provided that Eq. (4.1) is 1-critical. Through Theorem 4.1 we know that in such a case one of the equations (4.2) or (4.3) has to be critical. Without loss of generality let us say that it is Eq. (4.2). Because Theorem 2.1 holds for Eq. (2.6) as an equivalence, thus for any $\varepsilon > 0$ and H odd there is such $u_n \in l_0^2(\mathbb{Z})$ that

$$F(u) = F_1(u^1) < \varepsilon \left(u_H^1\right)^2 = \varepsilon \left(\triangle u_H\right)^2 = \varepsilon \left(\triangle u_{H-1}\right)^2.$$

Hence, we have seen two different behaviours of $F(u)$ in regard to 1-critical equations. We have seen, that 1-critical Eq. (3.1) satisfies $F(u) < \varepsilon u_H^2$ for any $H \in \mathbb{Z}$. On the other hand, 1-critical Eq. (4.1) satisfies that $F(u) < \varepsilon u_H^2$ and $F(u) < \varepsilon \left(\triangle u_{H-1}\right)^2$ for all H either odd or even. We obtain simple example of 1-critical Eq. (4.1) if we take $b_n \equiv 6$ and

$$a_n = \begin{cases} -1, & n \text{ even}, \\ -3, & n \text{ odd}. \end{cases}$$

Such an equation is again 1-critical by the results of [29]. Furthermore, we can compare this equation to Eq. (3.8) which is also 1-critical.

Possible applications of Eq. (4.1) arise when we consider the second-order self-adjoint linear differential equation

$$\left(p(x)z'(x)\right)' + q(x)z(x) = 0, \tag{4.4}$$

where $p(x) > 0$. We usually link Eq. (4.4) to the self-adjoint linear difference equation by approximating

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

for some small h. See for example [30]. However, from numerical analysis we know (see for example [6]), that we can get better numerical results by approximating

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

Using this approximation with a bit of work in Eq. (4.4) we obtain approximation

$$p_n y_{n+2} - (p_n + p_{n-2}) y_n + p_{n-2} y_{n-2} + 4 q_n y_n \approx 0. \tag{4.5}$$

Furthermore, fixing $\triangle_s y_n = \frac{y_{n+1} - y_{n-1}}{2}$ yields

$$\triangle_s \left(p_{n-1} \triangle_s y_n \right) + q_n y_n \approx 0.$$

This way, we arrive to a second-order self-adjoint linear equation with a different definition of $\triangle$. It is a simple matter to link such an equation through (4.5) to Eq. (4.1) by $b_n = p_n + p_{n-2} - 4q_n$, $a_n = -p_n$, for $p_n + p_{n-2} > 4q_n$.

Competing interests

The author declares that he is also affiliated with the University of Defence, Brno, Czech Republic.

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