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Neumann problems of superlinear elliptic systems at resonance

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Abstract. We prove existence of weak solutions of Neumann problem of nonhomogeneous elliptic system with asymmetric nonlinearities that may resonant at $-\infty$ and superlinear at $+\infty$. The proof is based on Mawhin's coincidence theory and the product formula of Brouwer degree.

Keywords: elliptic equation, Neumann problem, weak solution, continuation methods.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a smooth bounded connected domain in real N-dimensional Euclidean space. We are concerned with the existence of weak solutions of the following Neumann problem of semilinear elliptic systems

$$\begin{aligned} &\Delta u + f(v) = h_1(x), && \text{in } \Omega,\\ &\Delta v + g(u) = h_2(x), && \text{in } \Omega,\\ &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions, $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on $\partial\Omega$, the boundary of Ω, and $h_1, h_2 \in L^1(\Omega)$.

The motivation for this work is the paper F. O. de Paiva, W. Rosa [12], in which the authors showed the following resonant Neumann problems

$$\begin{aligned} &-\Delta u = (v^+)^p + h_1(x), && \text{in } \Omega,\\ &-\Delta v = (u^+)^q + h_2(x), && \text{in } \Omega,\\ &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, && \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

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has at least one solution (u, v) in $H^1(\Omega) \times H^1(\Omega)$ under the assumptions $h_1, h_2 \in L^r(\Omega)$, $r > \frac{N}{2}$, $1 < p, q < \frac{N}{N-2}$ and

$$\int_\Omega h_i(x)dx < 0, \qquad i = 1, 2. \tag{1.3}$$

We first define the bilinear form associated with the Laplacian operator. For $u, v \in W^{1,1}(\Omega)$, $\varphi, \psi \in W^{1,\infty}(\Omega)$, we define $B_1(u, \varphi)$ and $B_2(v, \psi)$ by

$$B_1(u, \varphi) = -\sum_{i=1}^{N} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx,$$

$$B_2(v, \psi) = -\sum_{i=1}^{N} \int_\Omega \frac{\partial v}{\partial x_i} \frac{\partial \psi}{\partial x_i} dx,$$

where the derivatives are taken in the distributional sense. By a *weak solution* of (1.1), we mean a pair $(u, v) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega)$, such that $f(v(\cdot)) \in L^1(\Omega)$, $g(u(\cdot)) \in L^1(\Omega)$ and

$$B_1(u, \varphi) + \int_\Omega f(v)\varphi dx = \int_\Omega h_1(x)\varphi dx, \qquad \forall\, \varphi \in W^{1,\infty}(\Omega),$$
$$B_2(v, \psi) + \int_\Omega g(u)\psi dx = \int_\Omega h_2(x)\psi dx, \qquad \forall\, \psi \in W^{1,\infty}(\Omega).$$

Denote

$$f^{-\infty} = \limsup_{s \to -\infty} f(s), \qquad g^{-\infty} = \limsup_{s \to -\infty} g(s),$$
$$f_{+\infty} = \liminf_{s \to +\infty} f(s), \qquad g_{+\infty} = \liminf_{s \to +\infty} g(s).$$

We will make the following assumptions.

(C0) $h_1, h_2 \in L^1(\Omega)$.

(Cl) There are the nonnegative constants $C_1, C_2 \in (0, \infty)$ such that

$$f(t) \geq -C_1, \qquad g(t) \geq -C_2, \qquad t \in \mathbb{R}$$

and for all $t \leq 0$ we have also $|f(t)| \leq C_1$, $|g(t)| \leq C_2$.

(C2) There are the constants $a, b \in \mathbb{R}$ and p with $1 \leq p < N/(N-2)$ for $N \geq 3$ and $1 \leq p < \infty$ for $N = 2$ such that for all $t \geq 0$ the inequality

$$|f(t)|, |g(t)| \leq at^p + b \quad \text{a.e. on } \Omega.$$

(C3) We assume f, g tends to be nondecreasing in that there is a $\gamma \in \mathbb{R}$ and a number $M \geq 0$ such that the inequalities

$$f(t_1) \leq f(t_2) + \gamma, \qquad g(t_1) \leq g(t_2) + \gamma$$

hold a.e. on Ω whenever $t_2 - t_1 \geq M$.

(C4)

$$\int_\Omega f^{-\infty} < \int_\Omega h_1(x)dx < \int_\Omega f_{+\infty}, \qquad \int_\Omega g^{-\infty} < \int_\Omega h_2(x)dx < \int_\Omega g_{+\infty}.$$

Our main result is the following

Theorem 1.1. *Under assumptions (C0)–(C4) the Neumann problem* (1.1) *has a weak solution* $(u,v) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega)$. *Moreover the solution* $(u,v) \in W^{1,q}(\Omega) \times W^{1,q}(\Omega)$ *for all* $1 \le q < N/(N-1)$.

Remark 1.2. Obviously, (1.3) in F. O. de Paiva, W. Rosa [12] are the special case of (C0) and (C4).

Remark 1.3. Our proof is based upon ideas found in Ward Jr [16]. He used the well-known Mawhin's continuation theorem to get a weak solution of the scale elliptic equation

$$\begin{aligned} &\Delta u + f(x,u) = k(x), && \text{in } \Omega, \\ &\frac{\partial u}{\partial \nu} = 0, && \text{on } \partial\Omega \end{aligned} \tag{1.4}$$

under the conditions $k \in L^1(\Omega)$,

$$|f(x,t)| \le \alpha(x)|t|^p + \beta(x), \qquad x \in \Omega,$$

where $p \in [1, \frac{N}{N-2}), \alpha \in L^\infty(\Omega), \beta \in L^1(\Omega)$, and Landesman–Lazer condition

$$\int_\Omega f^{-\infty} < \int_\Omega k(x)dx < \int_\Omega f_{+\infty}.$$

Remark 1.4. Similar problems, under Dirichlet and Neumann boundary condition, can be found in D. Arcoya and S. Villegas [2], M. Cuesta and C. De Coster [3], F. M. Ferreira, F. O. de Paiva [4], R. Kannan and R. Ortega [6,7], S. Kyritsi and N. S. Papageorgiou [8], D. Motreanu, V. Motreanu, N. S. Papageorgiou [10], K. Perera [14], N. S. Papageorgiou and V. D. Rădulescu [13], F. O. de Paiva and A. E. Presoto [11], L. Recova and A. Rumbos [15], J. R. Ward [16].

2 The preliminaries

Before proving Theorem 1.1 we will need a lemma. In the following we will write L^p for $L^p(\Omega)$ and $W^{1,p}$ for $W^{1,p}(\Omega)$. We denote the norm in L^p by $|\cdot|_p$, that of $W^{1,p}$ by $|\cdot|_{1,p}$. For $h \in L^1$. Let Qh be the projection

$$Qh = |\Omega|^{-1} \int_\Omega h dx.$$

Lemma 2.1 ([16])**.** *For each* $h \in L^1(\Omega)$ *with* $Qh = 0$. *There is a unique* $w \in W^{1,1}(\Omega)$ *with* $Qw = 0$ *such that*

$$B(w,\varphi) = \int_\Omega h(x)\varphi dx,$$

for all $\varphi \in W^{1,\infty}$, *where* $B(w,\varphi) = -\sum_{i=1}^N \int_\Omega \frac{\partial w}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx$. *Moreover* $w \in W^{1,q}$ *for each* q *satisfying* $1 \le q < N/(N-1)$ *and there is a constant* $C(q)$ *such that*

$$|w|_{1,q} \le C(q)|h|_1.$$

By the Rellich–Kondrachov theorem $W^{1,q}$ is compactly imbedded in L^p for $1 \le p < \frac{Nq}{N-q}$ since $q < N/(N-1) \le N$ for all $N \ge 2$. (e.g. see [1, p. 144]). Assume that the number p in condition (C2) is fixed hereafter, satisfying $1 \le p < N/(N-2)$ if $N \ge 3$ and $1 \le p < \infty$ if $N = 2$.

Choose q so that

$$p < \frac{Nq}{N-q} \quad \text{and} \quad 1 < q < \frac{N}{N-1}.$$

We have that $W^{1,q}$ is compactly imbedded in L^p.

Let X_1 denote the closed subspace of L^1 defined by $h \in X_1$ if and only if

$$Qh = 0.$$

Let T denotes the operator mapping X_1 into $W^{1,q} \cap X_1$ given by $h \to w$ where w is the unique weak solution to

$$\begin{aligned} \Delta w &= h \qquad \text{in } \Omega, \qquad Qu = 0, \\ \frac{\partial w}{\partial \nu} &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Note that $W^{1,q} = (W^{1,q} \cap X_1) \oplus \mathbb{R}$. T maps X_1 into $W^{1,q}$ and we see that $\Psi \circ T$ maps X_1 compactly into L^p if Ψ is the imbedding of $W^{1,q}$ into L^p. Let

$$K = \Psi \circ T,$$

and define an operator $L : L^p \to L^1$. Because L^1 is not the dual space to L^∞, we do not use the usual method of defining L. Instead, we let

$$D(L) = \text{Range}\, K \oplus \mathbb{R}$$

and

$$L(w_1 + \tilde{\alpha}) = h,$$

where $h \in X_1$ and $Kh = w_1$, for $w_1 \in \text{Range}\, K$ and $\tilde{\alpha} \in \mathbb{R}$. It is easy to see that L is a Fredholm operator: it has closed range X_1 and since $\ker(L) = \mathbb{R}$ and the dimension of $L^1 \setminus X_1$ is clearly 1, the index of L is 0,

$$\text{index}(L) = \dim \ker L - \dim \operatorname{coker} L.$$

We now define the substitution operators $N_1, N_2 : L^p \to L^1$ by

$$N_1 v(x) = f(v(x)) - h_1(x), \qquad v \in L^p \text{ and } x \in \Omega.$$

$$N_2 u(x) = g(u(x)) - h_2(x), \qquad u \in L^p \text{ and } x \in \Omega.$$

It is well known that the conditions on f and g imply that N_j maps L^p into L^1 continuously and N_j obviously takes sets bounded in L^p into sets bounded in L^1 for $j = 1, 2$.

A function $(u, v) \in W^{1,1} \times W^{1,1}$ is a weak solution of (1.1) if and only if $(u, v) \in D(L) \times D(L)$ and

$$\begin{aligned} Lu + N_1 v &= 0, \\ Lv + N_2 u &= 0. \end{aligned} \tag{2.1}$$

Recalling that for $u \in L^1$ we have defined Qu to be the mean value of u, we have from our remarks above that $K(I - Q)N_j : L^p \to L^p$ is compact and continuous, clearly QN_j is also compact and continuous for $j = 1, 2$. Thus N_j is L-compact (see [5]) on $\bar{G}$ for any open bounded set $\bar{G}$ in L^p for $j = 1, 2$. We will use a well known continuation theorem of Mawhin (see [5] and [9]).

3 Proof of the main result

We are in the position to prove our main result.

Proof of Theorem 1.1. By one of Mawhin's continuation theorems (see [5, p. 40] or [9, Theorem 7.2]) and our remarks above, if we can show the existence of a bounded open set $G := \bar{G} \times \bar{G}$ in $L^p \times L^p$ such that conditions (i) and (ii) below hold, then (2.1) has a solution. The conditions are:

(i) For each $\lambda \in (0,1)$ and each $(u,v) \in (D(L) \times D(L)) \cap \partial G$,

$$\begin{aligned} Lu + \lambda N_1 v &\neq 0, \\ Lv + \lambda N_2 u &\neq 0. \end{aligned} \tag{3.1}$$

(ii) $QN_j w \neq 0$ for each $j = 1, 2$, $w \in \ker L \cap \partial\bar{G}$ and

$$d(\Gamma, G \cap (\ker L \times \ker L), 0) \neq 0,$$

where $\Gamma := (JQN_1, JQN_2)$, $J : \operatorname{Im} Q \to \ker L$ is an isomorphism, and d is the Brouwer topological degree.

We first verify (i). We consider

$$\begin{aligned} Lu + \lambda N_1 v &= 0, \\ Lv + \lambda N_1 u &= 0 \end{aligned} \tag{3.2}$$

for $0 < \lambda < 1$. If $((u,v),\lambda)$ is a solution of (3.2) then

$$\begin{aligned} B_1(u,\varphi) + \lambda \int_\Omega f(v)\varphi &= \lambda \int_\Omega h_1\varphi, \qquad \forall\ \varphi \in W^{1,\infty}, \\ B_2(v,\psi) + \lambda \int_\Omega g(u)\psi &= \lambda \int_\Omega h_2\psi, \qquad \forall\ \psi \in W^{1,\infty}. \end{aligned}$$

In particular by taking $\varphi = \psi = 1$, then

$$\int_\Omega f(v) = \int_\Omega h_1,$$

$$\int_\Omega g(u) = \int_\Omega h_2.$$

It follows from (Cl) that for each $t \in \mathbb{R}$

$$|f(t)| \leq f(t) + 2C_1, \qquad |g(t)| \leq g(t) + 2C_2.$$

Thus

$$\begin{aligned} |N_1 v|_1 &= \int_\Omega |f(v) - h_1(x)| dx \\ &\leq \int_\Omega \Big(f(v) + 2C_1 + |h_1(x)| \Big) dx \\ &\leq \int_\Omega h_1 dx + 2|C_1| \cdot |\Omega| + \int_\Omega |h_1(x)| dx =: d_1, \end{aligned}$$

$$\begin{aligned}|N_2u|_1 &= \int_\Omega |g(u) - h_2(x)|dx \\ &\leq \int_\Omega \Big(g(u) + 2C_2 + |h_2(x)|\Big)dx \\ &\leq \int_\Omega h_2 dx + 2|C_2| \cdot |\Omega| + \int_\Omega |h_2(x)|dx =: d_2.\end{aligned}$$

Writing $u = u_1 + \alpha,\ v = v_1 + \beta$ with $u_1, v_1 \in \text{Range}\, K$ and $\alpha, \beta \in \mathbb{R}$ by Lemma 2.1 we have

$$|u_1|_{1,q} \leq C(q)d_1 =: m_1,$$

$$|v_1|_{1,q} \leq C(q)d_2 =: m_2,$$

where m_1 and m_2 are independently of $\lambda \in (0, 1)$. By the Sobolev imbedding theorem

$$|u_1|_p \leq m_3, \qquad |v_1|_p \leq m_4$$

for some constants m_3 and m_4.

We now show that for solutions $((u, v), \lambda) = \big((u_1 + \alpha, v_1 + \beta), \lambda\big)$ that α and β are also bounded independently of $\lambda \in (0, 1)$.

Suppose this is not the case. Then there is a sequence $((u_n, v_n), \lambda_n)$ of solutions to (3.2) with

$$u_n = u_{1n} + \alpha_n, \qquad v_n = v_{1n} + \beta_n$$

and

$$|\alpha_n| + |\beta_n| \to \infty, \quad \text{as } n \to \infty.$$

Suppose first that a subsequence of $\{\alpha_n\}$, relabeled as $\{\alpha_n\}$, tends to $+\infty$. Then using $|u_{1n}|_{1,q} \leq m_1$ is easy to show that

$$\lim_{n\to\infty} u_n(x) = +\infty \quad \text{a.e.} \tag{3.3}$$

For otherwise there is a constant $k_1 > 0$ and sets $\Omega(n)$ in Ω for infinitely many n (without loss of generality we assume for all n) such that $|\Omega(n)| \geq \delta > 0$ and $u_n(x) \leq k_1$ for $x \in \Omega(n)$. We have $u_{1n} + \alpha_n \leq k_1$ implies

$$\begin{aligned}k_1|\Omega| &\geq \int_{\Omega(n)} k_1 dx \geq \int_{\Omega(n)} u_{1n} + \alpha_n dx \\ &\geq \alpha_n|\Omega(n)| - \int_\Omega |u_{1n}| \\ &\geq \alpha_n\delta - C\end{aligned}$$

for C, a constant, which contradicts $\alpha_n \to \infty$. Thus (3.3) holds and

$$\liminf_{n\to\infty} g(u_n(x)) = g_{+\infty} \quad \text{a.e.}$$

Since $g(u_n(x)) \geq -C_2$ for all n and $C_2 \in \mathbb{R}$ we have by Fatou's lemma

$$\int_\Omega h_2 = \liminf_{n\to\infty} \int_\Omega g(u_n(x))dx \geq \int_\Omega g_{+\infty} dx$$

which contradicts (C4). Thus the $\{\alpha_n\}$ must be bounded above.

Suppose $\alpha_n \to -\infty$. It follows as for (3.3) that

$$\lim_{n\to\infty} u_n(x) = -\infty \quad \text{a.e.}$$

Because $g(t)$ is not everywhere bounded above by an L^1 function, we cannot use the simple Fatou's lemma argument as in the case of $\alpha_n \to -\infty$.

We proceed as follows. Since $|u_{1n}|_{1,q} \le m_1$, we may without loss of generality assume the existence of $\tilde{u}_1 \in L^p$ such that $u_{1n} \to \tilde{u}_1$ in L^p.

Let $0 < \epsilon < |\Omega|$ be given. Then $\tilde{u}_1 \in L^p$ implies that there exists an integer $n(\epsilon)$ and a measurable set $E \subseteq \Omega$ such that if $F = \Omega - E$ then $|F| < \epsilon$ and

$$u_n(x) \le 0, \qquad x \in E, \quad n \ge n(\epsilon),$$

hence

$$g(u_n(x)) \le C_2, \qquad x \in E, \quad n \ge n(\epsilon).$$

Moreover there exists another integer m such that for $n \ge m$ we have $\alpha_n \le -\overline{M}$, where $\overline{M}$ is a positive constant.

Thus, for $n \ge \max\{n(\epsilon), m\}$,

$$\begin{aligned}\int_\Omega h_2 &= \int_E g(u_{1n}+\alpha_n) + \int_F g(u_{1n}+\alpha_n)\\ &\le \int_E g(u_{1n}+\alpha_n) + \int_F g(u_{1n}) + \int_F \gamma\end{aligned}$$

and

$$\begin{aligned}\int_\Omega h_2 &\le \limsup_{n\to\infty}\left[\int_E g(u_n) + \int_F g(u_{1n})\right] + \int_F \gamma\\ &\le \int_E g^{-\infty}dx + \int_F g(\tilde{u}_1)dx + \int_F \gamma\end{aligned} \tag{3.4}$$

by Fatou's lemma for the integral over E and by convergence in L^1 for the integral over F.

Now choose $\eta > 0$ such that

$$\int_\Omega g^{-\infty}dx + \eta < \int_\Omega h_2 dx. \tag{3.5}$$

We may choose $\epsilon > 0$ so small that, since $|F| < \epsilon$,

$$\left|\int_F g^{-\infty}dx\right| < \frac{\eta}{3}, \qquad \left|\int_F g(\tilde{u}_1)dx\right| < \frac{\eta}{3}, \qquad \left|\int_F \gamma\right| < \frac{\eta}{3}.$$

For such as ϵ we have from (3.4) and (3.5)

$$\int_\Omega h_2 \le \int_\Omega g^{-\infty}dx - \int_F g^{-\infty}dx + \int_F g(\tilde{u}_1)dx + \int_F \gamma \le \int_\Omega g^{-\infty}dx + \eta < \int_\Omega h_2. \tag{3.6}$$

Therefore we cannot have $\alpha_n \to +\infty$ or $\alpha_n \to -\infty$ and this, combined with $|u_1|_p \le m_3$ shows that if $((u,v),\lambda)$ is a solution of (3.2) then $|u|_p = |u_1+\alpha|_p \le m_3 + C_3$ for some constant C_3. Similarly, We can obtain $|v|_p = |v_1+\alpha|_p \le m_4 + C_4$ for some constant C_4.

This verifies condition (i) above for any ball G in $L^1 \times L^1$, centered at the origin and with radius larger than $\rho_1 = \max\{m_3 + C_3, m_4 + C_4\}$.

The verification of condition (ii) is now straightforward. Both the range of Q and the kernel of L may be identified with $\mathbb{R}$, so that the isomorphism J in (ii) we may take to be the identity on $\mathbb{R}$. Now for $\alpha, \beta \in \mathbb{R}$,

$$QN_1\beta = |\Omega|^{-1} \int_\Omega \Big[f(\beta) - h_1(x)\Big] dx, \qquad QN_2\alpha = |\Omega|^{-1} \int_\Omega \Big[g(\alpha) - h_2(x)\Big] dx.$$

We may now make two simple applications of Fatou's lemma using (Cl) to show, using (C4), that there exists an $r > 0$ such that

$$QN_1(\beta) > 0, \qquad QN_1(-\beta) < 0, \quad \text{for } \alpha > r,$$

$$QN_2(\alpha) > 0, \qquad QN_2(-\alpha) < 0, \quad \text{for } \beta > r.$$

Thus for $\bar{r} \geq r \max\{1, |\Omega|\}$,

$$d(QN_j, [-\bar{r}, \bar{r}] \cap \ker L, 0) \neq 0, \qquad j = 1, 2.$$

By the product formula of Brouwer degree, we obtain

$$d(\Gamma, [-\bar{r}, \bar{r}]^2 \cap (\ker L \times \ker L), 0) \neq 0.$$

Now let $\rho := \max\{\rho_1, r \cdot \max\{1, |\Omega|\}\}$. Then we have that both (i) and (ii) are satisfied on $[B_\rho]^2$, where B_ρ is the ball in L^p with radius ρ centered at the origin. Thus (2.1) has a solution $(u, v) \in D(L) \times D(L)$ with

$$|u|_p \leq \rho, \qquad |v|_p \leq \rho,$$

and $(u, v) \in W^{1,p} \times W^{1,p}$ and is a weak solution of (1.1). This completes the proof of Theorem 1.1. □

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