



On the solution manifold of a differential equation with a delay which has a zero

Hans-Otto Walther 

Mathematisches Institut, Universität Gießen, Arndtstr. 2, 35392 Gießen, Germany

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Abstract. For a differential equation with a state-dependent delay we show that the associated solution manifold X_f of codimension 1 in the space $C^1([-r, 0], \mathbb{R}^n)$ is an almost graph over a hyperplane, which implies that X_f is diffeomorphic to the hyperplane. For the case considered previous results only provide a covering by 2 almost graphs.

Keywords: delay differential equation, state-dependent delay, solution manifold, almost graph.

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1 Introduction

Let $r > 0$ be given, choose a norm on \mathbb{R}^n , and let $C_n = C([-r, 0], \mathbb{R}^n)$ and $C_n^1 = C^1([-r, 0], \mathbb{R}^n)$ denote the Banach spaces of continuous and continuously differentiable functions $[-r, 0] \rightarrow \mathbb{R}^n$, respectively, with the norms given by $|\phi|_C = \max_{-r \leq t \leq 0} |\phi(t)|$ and $|\phi| = |\phi|_C + |\phi'|_C$. For a delay differential equation

$$x'(t) = f(x_t)$$

with a vector-valued functional $f : C_n^1 \supset U \rightarrow \mathbb{R}^n$ and with the solution segment $x_t \in U$ defined as $x_t(s) = x(t + s)$, the associated *solution manifold* is the set

$$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}.$$

Assume that f is continuously differentiable and

- (e) each derivative $Df(\phi) : C_n^1 \rightarrow \mathbb{R}^n$, $\phi \in U$, has a linear extension $D_e f(\phi) : C_n \rightarrow \mathbb{R}^n$ so that the map

$$U \times C_n \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous.

The extension property (e) is a relative of the notion of being *almost Fréchet differentiable* from [4] and can be verified for a variety of differential equations with state-dependent delay. – Under the said, mild conditions a non-empty set X_f is a continuously differentiable submanifold of codimension n in the Banach space C_n^1 , and it is on this manifold that the initial value problem associated with $x'(t) = f(x_t)$ is well-posed, with solutions which are continuously differentiable with respect to initial data [1,5].

The present note continues the description of solution manifolds initiated in [6,7]. In [6] we saw that in case f satisfies a condition which in examples with explicit delays corresponds to all of these delays being bounded away from zero the solution manifold is a graph over the closed subspace

$$X_0 = \{\phi \in C_n^1 : \phi'(0) = 0\},$$

which also is the solution manifold for $f = 0$. An example of a scalar equation with a single state-dependent delay which is positive but not bounded away from zero shows that in general solution manifolds do not admit any graph representation [6, Section 3]. However, the main result from [6] guarantees that for a reasonably large class of systems with explicit discrete state-dependent delays which are all positive the solution manifolds are nearly as simple as graphs: They are *almost graphs* over X_0 , in the terminology introduced in [6,7].

Let us recall from [7] that given a continuously differentiable submanifold X of a Banach space E and a closed subspace H with a closed complementary space,

- (i) X is called a *graph* (over H) if there are a closed complementary space Q for H and a continuously differentiable map $\gamma : H \supset \text{dom} \rightarrow Q$ with

$$X = \{\zeta + \gamma(\zeta) \in E : \zeta \in \text{dom}\},$$

and that

- (ii) X is called an *almost graph* (over H) if there is a continuously differentiable map $\alpha : H \supset \text{dom} \rightarrow E$ with

$$\begin{aligned} \alpha(\zeta) &= 0 && \text{on } \text{dom} \cap X, \\ \alpha(\zeta) &\in E \setminus H && \text{on } \text{dom} \setminus X, \end{aligned}$$

so that the map

$$\text{dom} \ni \zeta \mapsto \zeta + \alpha(\zeta) \in E$$

defines a diffeomorphism onto X .

Furthermore,

- (iii) a diffeomorphism A from an open neighbourhood \mathcal{O} of X in E onto an open subset of E is called an *almost graph diffeomorphism* (associated with X and H) if

$$A(X) \subset H$$

and

$$A(\zeta) = \zeta \quad \text{on } X \cap H.$$

In [7, Section 1] it is verified that in case there is an almost graph diffeomorphism A associated with X and H the submanifold X is an almost graph over H .

An example of an almost graph in finite dimension is the unit circle in the plane without the point on top. The inverse of the stereographic projection onto the real line serves as the map $\zeta \mapsto \zeta + \alpha(\zeta)$ in Part (ii) of the previous definition.

We return to the results in [6] and [7]. It is not difficult to see that the approach used in [6] for an almost graph representation of a solution manifold fails in case some of the delays in the system considered have zeros. For the solution manifold of such a system, with k delays some of which have zeros, the approach can be used, however, in order to obtain a finite atlas of manifold charts whose domains are almost graphs over X_0 . This is achieved in [7], with the size of the atlas independent of the number of equations in the system considered, solely determined by the zerosets of the delays, and not exceeding 2^k .

The immediate question with regard to the topological properties of these solution manifolds is whether the size of the atlas found in [7] is minimal. The result of the present note shows that for a prototype of the systems studied in [6,7] this is not the case: The entire solution manifolds of the prototype equation is in fact an almost graph over X_0 . The proof relies on a major modification of the approach taken in [7].

The prototype equation belongs to the simplest cases of the systems studied in [6,7] which are scalar equations with a single delay ($k=1$) and have the form

$$x'(t) = g(x(t - d(Lx_t))) \quad (1.1)$$

with a continuously differentiable map $g : \mathbb{R} \rightarrow \mathbb{R}$, a continuously differentiable *delay function* $d : F \rightarrow [0, r]$ defined on a finite-dimensional topological vectorspace F , and a surjective continuous linear map $L : C \rightarrow F$. We abbreviate $C = C_1$ and $C^1 = C_1^1$. For $f : C^1 \rightarrow \mathbb{R}$ given by

$$f(\phi) = g(\phi(-d(L\phi)))$$

Eq. (1.1) takes the form $x'(t) = f(x_t)$. Proposition 2.1 from [7] applies and shows that f is continuously differentiable with property (e). Proposition 2.3 from [7] yields that the associated solution manifold

$$X_f = \{\phi \in C^1 : \phi'(0) = g(\phi(-d(L\phi)))\}$$

is non-empty, and it follows that it is a continuously differentiable submanifold of codimension 1 in C^1 .

If $d(\xi) > 0$ everywhere than we know from [6] that X_f is an almost graph over $X_0 = \{\chi \in C^1 : \chi'(0) = 0\}$. If d has zeros then the result of [7] yields an atlas of $2 = 2^1$ manifold charts whose domains are almost graphs over X_0 .

In [8] an adaptation of the approach from [6,7] to (1.1) with a linear map L for which, loosely spoken, $L\phi$ does not depend on $\phi(0)$ is used to prove that the associated solution manifold is an almost graph over X_0 , no matter whether d has zeros or not.

In the sequel we consider a prototype for the remaining, critical cases, namely, Eq. (1.1) for

$$F = \mathbb{R}, \quad L\phi = \phi(0) \quad \text{for all } \phi \in C,$$

and for

$$d \quad \text{with a single zero } \eta_0 \in \mathbb{R}.$$

We find an almost graph diffeomorphism $A : C^1 \rightarrow C^1$ which maps X_f onto X_0 .

A part of the construction of the diffeomorphism A uses the technique developed in [6,7]. For another part it was helpful to have in mind an idea of Krisztin [2] which yields graph representations of solution manifolds from bounds on extended derivatives as in property (e), compare the proof of Lemma 1 in [3].

The assumption that d has a single zero is for simplicity and may be relaxed in future work.

2 Preliminaries

Differentiable maps are always defined on open subsets of Banach spaces or Banach manifolds.

Differentiation $\partial : C^1 \rightarrow C$, $\partial\phi = \phi'$, is linear and continuous, and the evaluation map $ev : C \times [-r, 0] \ni (\phi, t) \mapsto \phi(t) \in \mathbb{R}$ is continuous but not locally Lipschitz continuous. The composition

$$ev(\cdot, 0) \circ \partial : C^1 \ni \phi \mapsto ev(\partial\phi, 0) \in \mathbb{R}$$

is linear and continuous.

The restriction ev_1 of ev to $C^1 \times (-r, 0)$ is continuously differentiable with

$$D ev_1(\phi, t)(\chi, s) = \chi(t) + \phi'(t) s,$$

and the composition

$$h : C^1 \ni \phi \mapsto ev(\phi, -d(L\phi)) \in \mathbb{R}$$

is continuously differentiable with

$$Dh(\phi)\chi = \chi(-d(L\phi)) - \phi'(-d(L\phi))d'(L\phi)L\chi,$$

see Part 2.1 of the proof of [7, Proposition 2.1].

In Section 1 we quoted [7, Proposition 2.3] for $X_f \neq \emptyset$. Using the choice of L this also follows directly from the fact that for every $\xi \in \mathbb{R}$ each $\phi \in C^1$ with $L(\phi) = \phi(0) = \xi$, $\phi(-d(\xi)) = \xi$, and $\phi'(0) = g(\xi)$ belongs to X_f .

The tangent space $T_\phi X_f$ of the solution manifold X_f at $\phi \in X_f$ consists of the vectors $c'(0) = Dc(0)1$ of differentiable curves $c : I \rightarrow C^1$ with $0 \in I \subset \mathbb{R}$, $c(0) = \phi$, $c(I) \subset X_f$. We have

$$\begin{aligned} T_\phi X_f &= \{\chi \in C^1 : \chi'(0) = Df(\phi)\chi\} \\ &= \{\chi \in C^1 : \chi'(0) = g'(\phi(-d(L\phi)))[\chi(-d(L\phi)) - \phi'(-d(L\phi))d'(L\phi)L\chi]\}. \end{aligned}$$

3 A map A taking X_f into X_0

As d is minimal at η_0 we have $d'(\eta_0) = 0$.

Notice that for $\phi \in C$ with $L\phi = \eta_0$,

$$\phi(-d(L\phi)) = \phi(-d(\eta_0)) = \phi(0) = L\phi = \eta_0. \quad (3.1)$$

For reals η we introduce the continuous linear maps

$$L_\eta : C \rightarrow \mathbb{R}$$

given by $L_\eta\phi = \phi(-d(\eta))$.

In order to develop a bit of intuition about the shape of X_f observe that the sets

$$\begin{aligned} X_{f\eta} &= X_f \cap L^{-1}(\eta) \\ &= \{\phi \in C^1 : L\phi = \eta \text{ and } \phi'(0) = g(\phi(-d(L\phi)))\} \\ &= \{\phi \in C^1 : L\phi = \eta \text{ and } \phi'(0) = g(\phi(-d(\eta)))\} \end{aligned}$$

are mutually disjoint and decompose X_f , and that

$$X_{f\eta_0} = \{\phi \in C^1 : L\phi = \eta_0 \text{ and } \phi'(0) = g(\eta_0)\}$$

(with Eq. (3.1) for $L\phi = \eta_0$) is a closed affine subspace of codimension 2 in C^1 .

Choose $\rho > 0$ and set

$$c = \max_{|\xi - \eta_0| \leq \rho} |g(\xi)| + \max_{|\xi - \eta_0| \leq \rho} |g'(\xi)|,$$

$$c_* = \frac{\rho}{4(c+1)(\rho+3)}.$$

The map which we are going to construct relies on vectors $\psi_\eta \in C^1$ which are transversal to the solution manifold at points in $X_{f\eta}$. We begin with the case $\eta = \eta_0$ and choose $\psi_{\eta_0} \in C^1$ with the properties

$$\psi_{\eta_0}(0) = 0, \quad \psi'_{\eta_0}(0) = 1, \quad |\psi_{\eta_0}|_C \leq c_*.$$

For each $\phi \in X_f$ with $L\phi = \eta_0$ (or, for each $\phi \in X_{f\eta_0}$) we have

$$\begin{aligned} \psi'_{\eta_0}(0) &= 1 > c c_* \geq |g'(\eta_0)| |\psi_{\eta_0}|_C \\ &= |g'(\phi(-d(L\phi)))| |\psi_{\eta_0}|_C \geq |g'(\phi(-d(L\phi)))| \psi_{\eta_0}(-d(L\phi))| \\ &= |g'(\phi(-d(L\phi)))| [\psi_{\eta_0}(-d(L\phi)) - \phi'(-d(L\phi))d'(L\phi)L\psi_{\eta_0}] \\ &\quad (\text{with } d'(L\phi) = d'(\eta_0) = 0) \end{aligned}$$

which means

$$\psi_{\eta_0} \in C^1 \setminus T_\phi X_f.$$

Proposition 3.1. *There exists a continuously differentiable map*

$$\mathbb{R} \setminus \{\eta_0\} \ni \eta \mapsto \psi_\eta \in C^1$$

so that for every $\eta \in \mathbb{R} \setminus \{\eta_0\}$ we have

$$\psi_\eta(t) = 0 \quad \text{on } [-r, -d(\eta)] \cup \{0\}, \quad \psi'_\eta(0) = 1, \quad |\psi_\eta|_C \leq c_*,$$

and for all $\phi \in X_f$ with $L\phi = \eta$,

$$\psi_\eta \in C^1 \setminus T_\phi X_f.$$

Proof. 1. For each $z \in [-r, 0)$ choose $\psi_z \in C^1$ with $\psi_z(t) = 0$ on $[-r, z] \cup \{0\}$, $\psi'_z(0) = 1$, and $|\psi_z|_C \leq c_*$. Then proceed as in the proof of [7, Proposition 4.1], with $\mathcal{F} = \mathbb{R}$, $\mathcal{W} = \mathbb{R} \setminus \{\eta_0\}$, $\lambda = L$, and construct the desired map $\mathbb{R} \setminus \{\eta_0\} \rightarrow C^1$ from a sequence of maps ψ_{z_m} , $m \in \mathbb{N}$, with $z_m \rightarrow \min d = 0$ as $m \rightarrow \infty$. Observe that $|\psi_\eta|_C \leq c_*$ is achieved.

2. For $\eta \in \mathbb{R} \setminus \{\eta_0\}$ and $\phi \in X_f$ with $L\phi = \eta$ the function ψ_η does not satisfy the equation characterizing the tangent space $T_\phi X_f$, due to $\psi'_\eta(0) = 1$ and $\psi_\eta(-d(L\phi)) = \psi_\eta(-d(\eta)) = 0 = \psi_\eta(0) = L\psi_\eta$. \square

The map from Proposition 3.1 has no continuous extension to \mathbb{R} . Nevertheless, for all $\eta \in \mathbb{R}$,

$$L\psi_\eta = \psi_\eta(0) = 0. \tag{3.2}$$

Also, for each $\eta \in \mathbb{R}$,

$$L_\eta \psi_\eta = 0, \tag{3.3}$$

which in case $\eta = \eta_0$ holds with $L_{\eta_0} = L$, and,

$$\psi'_\eta(0) = 1. \tag{3.4}$$

It follows from Eq. (3.4) that for all $\eta \in \mathbb{R}$,

$$C^1 = X_0 \oplus \mathbb{R}\psi_\eta,$$

and the continuous linear projection $P_\eta : C^1 \rightarrow C^1$ along $\mathbb{R}\psi_\eta$ onto X_0 is given by

$$P_\eta\phi = \phi - \phi'(0)\psi_\eta.$$

Now we are ready for the definition of the map $A : C^1 \rightarrow C^1$ which in the next section will be shown to be an almost graph diffeomorphism associated with X_f and X_0 . Let

$$a : \mathbb{R} \rightarrow [0, 1]$$

be a continuously differentiable map with $a(\xi) = 1$ for $|\xi - \eta_0| \leq \frac{\rho}{2}$, $a(\xi) = 0$ for $|\xi - \eta_0| \geq \rho$, and $|a'(\xi)| \leq \frac{3}{\rho}$ for $\frac{\rho}{2} \leq |\xi - \eta_0| \leq \rho$.

The maps $A_{\rho/2} : C_{\rho/2}^1 \rightarrow C^1$ and $A_+ : C_+^1 \rightarrow C^1$ given by

$$\begin{aligned} C_{\rho/2}^1 &= \{\phi \in C^1 : |\phi(-d(L\phi)) - \eta_0| < \rho/2\}, \\ A_{\rho/2}(\phi) &= \phi - g(\tau)\psi_{\eta_0}, \\ C_+^1 &= \{\phi \in C^1 : d(L\phi) > 0\}, \\ A_+(\phi) &= \phi - g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_\eta], \end{aligned}$$

with

$$\tau = L_\eta\phi = ev(\phi, -d(L\phi)), \quad \eta = L\phi,$$

are continuously differentiable. On the intersection of their domains they coincide: For $\phi \in C_{\rho/2}^1 \cap C_+^1$ we have

$$|\tau - \eta_0| = |\phi(-d(L\phi)) - \eta_0| < \frac{\rho}{2},$$

which yields $a(\tau) = 1$, and thereby, $A_{\rho/2}(\phi) = A_+(\phi)$.

Also,

$$C^1 = C_{\rho/2}^1 \cup C_+^1,$$

since for $\phi \in C^1 \setminus C_+^1$, $d(L\phi) = 0$, hence $L\phi = \eta_0$, and due to Eq. (3.1), $\phi(-d(L\phi)) = \eta_0$, which means $|\phi(-d(L\phi)) - \eta_0| = 0$, or $\phi \in C_{\rho/2}^1$.

It follows that $A_{\rho/2}$ and A_+ define a continuously differentiable map $A : C^1 \rightarrow C^1$.

Using Eq. (3.4) we infer that for every $\phi \in X_f$,

$$(A(\phi))'(0) = \phi'(0) - g(\tau) \cdot 1 = \phi'(0) - g(\phi(-d(L\phi))) = 0,$$

or, $A(\phi) \in X_0$. Hence

$$A(X_f) \subset X_0.$$

Notice also that due to Eq. (3.2),

$$LA(\phi) = L\phi \quad \text{for all } \phi \in C^1. \tag{3.5}$$

4 A is an almost graph diffeomorphism

In order to find the inverse of A we first consider $\phi \in C_+^1 \cap X_f$ and $\chi = A(\phi) = A_+(\phi) \in X_0$, and compare $\tau = L_\eta \phi$ where $\eta = L\phi$ to $\sigma = L_{\hat{\eta}} \chi$ where $\hat{\eta} = L\chi$. By Eq. (3.5), $\hat{\eta} = \eta$, hence

$$\begin{aligned} \sigma &= L_\eta \chi = L_\eta A_+(\phi) \\ &= L_\eta \phi - g(\tau)[a(\tau)L_\eta \phi_{\eta_0} + (1 - a(\tau))L_\eta \phi_\eta] \\ &= \tau - g(\tau)a(\tau)L_\eta \phi_{\eta_0} \quad (\text{with Eq. (3.3)}). \end{aligned} \tag{4.1}$$

For every $\eta \in \mathbb{R}$ the map

$$h_\eta : \mathbb{R} \ni \tau \mapsto \tau - (ga)(\tau)L_\eta \psi_{\eta_0} \in \mathbb{R}$$

is continuously differentiable.

Proposition 4.1. *Every map h_η , $\eta \in \mathbb{R}$, is a bijection, the map $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $T(\eta, \sigma) = (h_\eta)^{-1}(\sigma)$ is continuously differentiable, and for all $(\eta, \sigma, \tau) \in \mathbb{R}^3$ Eq. (4.1) is equivalent to $\tau = T(\eta, \sigma)$.*

Proof. 1. For every $\eta \in \mathbb{R}$ the map h_η satisfies $h_\eta(\tau) = \tau$ for $|\tau - \eta_0| \geq \rho$, and for $|\tau - \eta_0| \leq \rho$,

$$\begin{aligned} h'_\eta(\tau) &= 1 - (g'(\tau)a(\tau) + g(\tau)a'(\tau))L_\eta \psi_{\eta_0} \\ &\geq 1 - (c + c(3/\rho))c_* > 0. \end{aligned}$$

It follows that h_η is bijective.

2. The map

$$F : \mathbb{R}^3 \ni (\eta, \sigma, \tau) \mapsto \sigma - h_\eta(\tau) \in \mathbb{R}$$

is continuously differentiable (with $L_\eta \psi_{\eta_0} = \psi_{\eta_0}(-d(\eta))$). For each $(\eta, \sigma, \tau) \in \mathbb{R}^3$ Eq. (4.1) and the relations $F(\eta, \sigma, \tau) = 0$ and $\tau = (h_\eta)^{-1}(\sigma) = T(\eta, \sigma)$ are equivalent. For $|\tau - \eta_0| \leq \rho$ we have

$$\partial_3 F(\eta, \sigma, \tau) = -1 + (ga)'(\tau)L_\eta \psi_{\eta_0} \leq -1 + (c + c(3/\rho))c_* < 0,$$

and

$$\partial_3 F(\eta, \sigma, \tau) = -1 \neq 0$$

for $|\tau - \eta_0| \geq \rho$. Applications of the Implicit Function Theorem to the zerset of F show that the map T is locally given by continuously differentiable maps. \square

For the open subsets $C_{\rho/4}^1 = \{\chi \in C^1 : |\chi(-d(L\chi)) - \eta_0| < \rho/4\}$ and C_+^1 of the space C^1 we have

$$C^1 = C_{\rho/4}^1 \cup C_+^1$$

since for $\chi \in C^1 \setminus C_+^1$, $d(L\chi) = 0$, hence $L\chi = \eta_0$, and due to Eq. (3.1) $\chi(-d(L\chi)) = \eta_0$, which means $|\chi(-d(L\chi)) - \eta_0| = 0$. The maps $B_{\rho/4} : C_{\rho/4}^1 \rightarrow C^1$ and $B_+ : C_+^1 \rightarrow C^1$ given by

$$\begin{aligned} B_{\rho/4}(\chi) &= \chi + g(\tau)\psi_{\eta_0}, \\ B_+(\chi) &= \chi + g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_\eta], \end{aligned}$$

with

$$\tau = T(\eta, \sigma), \quad \eta = L\chi, \quad \sigma = L_\eta \chi = ev(\chi, -d(L\chi))$$

are continuously differentiable. On the intersection $C_{\rho/4}^1 \cap C_+^1$ both maps $B_{\rho/4}$ and B_+ coincide. In order to verify this we need to know $a(\tau) = 1$ for

$$\tau = T(\eta, \sigma), \quad \eta = L\chi, \quad \chi \in C^1, \quad \sigma = L_\eta\chi$$

with $|\chi(-d(L\chi)) - \eta_0| < \rho/4$ and $d(L\chi) > 0$. The equation $a(\tau) = 1$ holds provided $|\tau - \eta_0| \leq \rho/2$, which follows from

$$|\tau - \eta_0| = |\sigma + (ga)(\tau)L_\eta\psi_{\eta_0} - \eta_0| \leq \frac{\rho}{4} + \max_{\xi \in \mathbb{R}} |(ga)(\xi)| \cdot c_*$$

in combination with

$$\begin{aligned} (ga)(\tau) &= 0 \quad \text{for } |\tau - \eta_0| \geq \rho \\ |(ga)(\tau)| &\leq c \quad \text{for } |\tau - \eta_0| \leq \rho, \\ c c_* &< \rho/4. \end{aligned}$$

So $B_{\rho/4}$ and B_+ define a continuously differentiable map $B : C^1 \rightarrow C^1$.

Observe that due to Eq. (3.2),

$$LB(\chi) = L\chi \quad \text{for all } \chi \in C^1. \quad (4.2)$$

We have

$$B(X_0) \subset X_f.$$

Proof of this: For $\chi \in X_0$ let $\phi = B(\chi)$ and $\eta = L\chi$, $\sigma = L_\eta\chi$, and $\tau = T(\eta, \sigma) = h_\eta^{-1}(\sigma)$. Then

$$\sigma = h_\eta(\tau) = \tau - (ga)(\tau)L_\eta\psi_{\eta_0}.$$

Using $\chi'(0) = 0$, Eq. (3.4), and the preceding equation we get

$$\phi'(0) = (B(\chi))'(0) = 0 + g(\tau) \cdot 1 = g(\tau)$$

with

$$\begin{aligned} \tau &= \sigma + (ga)(\tau)L_\eta\psi_{\eta_0} \\ &= L_\eta\chi + g(\tau)[a(\tau)L_\eta\psi_{\eta_0} + (1 - a(\tau))L_\eta\psi_\eta] \quad (\text{with Eq. (3.3)}) \\ &= L_\eta B(\chi) = L_\eta\phi. \end{aligned}$$

Eq. (4.2) yields $\eta = L\chi = L\phi$, and we obtain

$$\phi'(0) = g(\tau) = g(L_\eta\phi) = g(\phi(-d(\eta))) = g(\phi(-d(L\phi))),$$

or, $B(\chi) = \phi \in X_f$.

Proposition 4.2. $B(A(\phi)) = \phi$ for all $\phi \in C^1$.

Proof. 1. The case $d(L\phi) > 0$. Consider

$$\chi = A(\phi) = A_+(\phi) = \phi - g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_{L\phi}]$$

with

$$\tau = L_\eta\phi, \quad \eta = L\phi.$$

From Eq. (3.5) we infer

$$L\chi = L\phi, \quad (4.3)$$

hence $d(L\chi) = d(L\phi) > 0$, and thereby

$$B(\chi) = B_+(\chi) = \chi + g(\hat{\tau})[a(\hat{\tau})\psi_{\eta_0} + (1 - a(\hat{\tau}))\psi_{L\chi}]$$

with

$$\hat{\tau} = T(\hat{\eta}, \hat{\sigma}) = h_{\hat{\eta}}^{-1}(\hat{\sigma}), \quad \hat{\eta} = L\chi, \quad \hat{\sigma} = L_{\hat{\eta}}\chi.$$

It follows that

$$B(\chi) = B(A(\phi)) = \{\phi - g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_{L\phi}]\} + g(\hat{\tau})[a(\hat{\tau})\psi_{\eta_0} + (1 - a(\hat{\tau}))\psi_{L\chi}].$$

Using the preceding equation in combination with Eq. (4.3) we obtain $B(A(\phi)) = \phi$ provided we have $\hat{\tau} = \tau$. Proof of this: By Eq. (4.3), $\hat{\eta} = L\chi = L\phi = \eta$. We get

$$\begin{aligned} h_{\hat{\eta}}(\hat{\tau}) &= h_{\hat{\eta}}(\hat{\tau}) = \hat{\sigma} = L_{\hat{\eta}}\chi = L_{\eta}\chi = L_{\eta}A(\phi) \\ &= L_{\eta}\phi - g(\tau)[a(\tau)L_{\eta}\psi_{\eta_0} + (1 - a(\tau))L_{\eta}\psi_{\eta}] \\ &= L_{\eta}\phi - g(\tau)a(\tau)L_{\eta}\psi_{\eta_0} \quad (\text{with Eq. (3.3)}) \\ &= \tau - (ga)(\tau)L_{\eta}\psi_{\eta_0} \\ &= h_{\eta}(\tau), \end{aligned}$$

and the injectivity of h_{η} yields $\hat{\tau} = \tau$.

2. The case $d(L\phi) = 0$. Then $\phi(0) = L\phi = \eta_0$. Choose a sequence of points $\phi_j \in C^1$, $j \in \mathbb{N}$, with $\phi_j(0) \neq \eta_0$ for all $j \in \mathbb{N}$ and $\phi_j \rightarrow \phi$ in C^1 as $j \rightarrow \infty$. For all $j \in \mathbb{N}$, $B(A(\phi_j)) = \phi_j$, due to Part 1 of the proof, and continuity yields $B(A(\phi)) = \phi$. \square

Proposition 4.3. $A(B(\chi)) = \chi$ for all $\chi \in C^1$.

Proof. 1. The case $d(L\chi) > 0$. Then $\chi \in C_+^1$ and

$$B(\chi) = B_+(\chi) = \chi + g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_{\eta}]$$

with $\tau = T(\eta, \sigma) = h_{\eta}^{-1}(\sigma)$, $\eta = L\chi$, $\sigma = L_{\eta}\chi$. Set $\phi = B(\chi)$. By Eq. (4.2), $L\phi = L\chi = \eta$. Hence $d(L\phi) = d(L\chi) > 0$, or $\phi \in C_+^1$, and

$$A(\phi) = A_+(\phi) = \phi - g(\hat{\tau})[a(\hat{\tau})\psi_{\eta_0} + (1 - a(\hat{\tau}))\psi_{\hat{\eta}}]$$

with

$$\hat{\tau} = L_{\hat{\eta}}\phi, \quad \hat{\eta} = L\phi \quad (= \eta).$$

It follows that

$$\begin{aligned} A(B(\chi)) &= A(\phi) = \{\chi + g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_{\eta}]\} - g(\hat{\tau})[a(\hat{\tau})\psi_{\eta_0} + (1 - a(\hat{\tau}))\psi_{\hat{\eta}}] \\ &= \{\chi + g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_{\eta}]\} - g(\hat{\tau})[a(\hat{\tau})\psi_{\eta_0} + (1 - a(\hat{\tau}))\psi_{\eta}] \quad (\text{with } \hat{\eta} = \eta), \end{aligned}$$

and for $A(B(\chi)) = \chi$ it remains to show $\hat{\tau} = \tau$. Proof of this:

$$\begin{aligned} \hat{\tau} &= L_{\hat{\eta}}\phi = L_{\eta}\phi = L_{\eta}(\chi + g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_{\eta}]) \\ &= L_{\eta}\chi + g(\tau)[a(\tau)L_{\eta}\psi_{\eta_0} + (1 - a(\tau))L_{\eta}\psi_{\eta}] \\ &= \sigma + (ga)(\tau)L_{\eta}\psi_{\eta_0} \quad (\text{with Eq. (3.3)}) \\ &= h_{\eta}(\tau) + (ga)(\tau)L_{\eta}\psi_{\eta_0} = \tau. \end{aligned}$$

2. In case $d(L\chi) = 0$ use the result of Part 1 above and continuity as in Part 2 of the proof of Proposition 4.2. \square

Corollary 4.4. *The map A is an almost graph diffeomorphism associated with X_f and X_0 .*

Proof. Propositions 4.2 and 4.3 yield that A is a diffeomorphism onto C^1 with inverse B . Using $A(X_f) \subset X_0$ and $B(X_0) \subset X_f$ one finds $A(X_f) = X_0$. For $\phi \in X_f \cap X_0$, $\tau = L_\eta\phi$, and $\eta = L\phi$, we have

$$g(\tau) = g(L_\eta\phi) = g(\phi(-d(\eta))) = g(\phi(-d(L\phi))) = \phi'(0) = 0.$$

This yields $A(\phi) = \phi$. □

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