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Positive ground state of coupled planar systems of nonlinear Schrödinger equations with critical exponential growth

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Abstract. In this paper, we prove the existence of a positive ground state solution to the following coupled system involving nonlinear Schrödinger equations:

$$\begin{cases} -\Delta u + V_1(x)u = f_1(x,u) + \lambda(x)v, & x \in \mathbb{R}^2, \\ -\Delta v + V_2(x)v = f_2(x,v) + \lambda(x)u, & x \in \mathbb{R}^2, \end{cases}$$

where $\lambda, V_1, V_2 \in C(\mathbb{R}^2, (0,+\infty))$ and $f_1, f_2 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ have critical exponential growth in the sense of Trudinger–Moser inequality. The potentials $V_1(x)$ and $V_2(x)$ satisfy a condition involving the coupling term $\lambda(x)$, namely $0 < \lambda(x) \le \lambda_0\sqrt{V_1(x)V_2(x)}$. We use non-Nehari manifold, Lions's concentration compactness and strong maximum principle to get a positive ground state solution. Moreover, by using a bootstrap regularity lifting argument and L^q-estimates we get regularity and asymptotic behavior. Our results improve and extend the previous results.

Keywords: coupled system, nonlinear Schrödinger equations, variational methods, Trudinger–Moser inequality, positive ground state solution, regularity.

2020 Mathematics Subject Classification: 35J10, 35J50, 35J61, 35B33, 35Q55.

1 Introduction and main results

This article is devoted to studying standing waves for the following system of nonlinear Schrödinger equations:

$$\begin{cases} -\Delta u + V_1(x)u = f_1(x,u) + \lambda(x)v, & x \in \mathbb{R}^2, \\ -\Delta v + V_2(x)v = f_2(x,v) + \lambda(x)u, & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\lambda, V_1, V_2 \in C(\mathbb{R}^2, \mathbb{R})$ and $f_1, f_2 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ satisfy the following basic assumptions:

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(V) $V_1(x),\ V_2(x),\ \lambda(x) \in C(\mathbb{R}^2, (0, +\infty))$ are all 1-periodic in each of x_1 and x_2. Moreover, there exists $\lambda_0 \in (0, 1)$ such that

$$0 < \lambda(x) \le \lambda_0 \sqrt{V_1(x)V_2(x)}, \qquad \forall x \in \mathbb{R}^2;$$

(F1) $f_i \in C(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$, $f_i(x,t)$ is 1-periodic in each of x_1 and x_2, and there exists $\alpha_1, \alpha_2 > 0$ such that

$$\lim_{t\to\infty} \frac{|f_i(x,t)|}{e^{\alpha t^2}} = 0, \text{ uniformly on } x \in \mathbb{R}^2 \text{ for all } \alpha > \alpha_i, i = 1, 2;$$

and

$$\lim_{t\to\infty} \frac{|f_i(x,t)|}{e^{\alpha t^2}} = +\infty, \text{ uniformly on } x \in \mathbb{R}^2 \text{ for all } \alpha < \alpha_i, i = 1, 2;$$

(F2) $f_i(x,t) = o(t)$ as $t \to 0$ uniformly on $x \in \mathbb{R}^2$, for $i = 1, 2$. $f_i(x,t) = 0$ for all $x \in \mathbb{R}^2, t \le 0$.

Solutions of system (1.1) are related with standing waves of the following two-component system:

$$\begin{cases} -i\frac{\partial \psi}{\partial t} = \Delta\psi - V_1(x)\psi + f_1(x,\psi) + \lambda(x)\phi, & (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \\ -i\frac{\partial \phi}{\partial t} = \Delta\phi - V_2(x)\phi + f_2(x,\psi) + \lambda(x)\psi, & (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \end{cases} \tag{1.2}$$

where i denotes the imaginary unit. Such class of systems arise in various branches of mathematical physics and nonlinear optics, see [1]. For instance, solutions of (1.1) are related to the existence of solitary wave solutions for nonlinear Schrödinger equations and Klein–Gordon equations, see [4]. For system (1.2), a solution of the form

$$(\psi(x,t), \phi(x,t)) = (e^{-iMt}u(x), e^{-iMt}v(x)),$$

where M is some real constant, called standing wave solution.

In order to motivate our results, we begin by giving a brief survey on this subject. Let us consider the scalar case. Notice that if $\lambda \equiv 0, V_1 \equiv V_2 = V(x), f_1 \equiv f_2 = f$ and $u \equiv v$, system (1.1) reduces to the scalar equation

$$-\Delta u + V(x)u = f(x,u). \tag{1.3}$$

This class of nonlinear Schrödinger equation has been widely studied by many researchers, under various hypotheses on the potential $V(x)$ and nonlinear term $f(x,u)$. Such as coercive potential, axially symmetric potential, positive potential and periodic potential. In particular, Chen and Tang [8] developed a direct approach to get nontrivial solutions and ground state solutions when they considered the equation (1.3) in $\mathbb{R}^2$ where $V(x)$ was a 1-periodic function with respect to x_1 and x_2, 0 lies in the gap of $-\Delta + V$, and the nonlinear term was of Trudinger-Moser critical exponential growth. Using the generalized linking theorem to obtain a Cerami sequence, they showed that the Cerami sequence was bounded and the minimax-level was less than the threshold value by virtue of Moser type functions. Furthermore, they obtained that the Cerami sequence was nonvanishing, which extended and improved the results of [2, 17].

For the system of nonlinear Schrödinger equations, there are some results on the linearly coupled system in subcritical and critical case. Chen and Zou [9] studied the following system

$$\begin{cases} -\Delta u + u = f(x,u) + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + v = g(x,v) + \lambda u, & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $0 < \lambda < 1$. They discussed the system for non-autonomous and autonomous nonlinearities of subcritical growth respectively. When $N \geq 2$, $f(x,u) = (1 + a(x))|u|^{p-1}u$ and $g(x,v) = (1 + b(x))|v|^{p-1}v$, they improved the results of [3] for establishing energy estimates of the ground states. Under some assumptions of potential $a(x)$ and $b(x)$, they obtained not only the existence of positive bound states, but also a precise description of the limit behavior of the bound states as the parameter λ goes to zero. When $N \geq 3$, $f(x,u) = f(u)$, $g(x,v) = g(v)$, and Berestycki–Lions type assumptions were satisfied, they proved system (1.4) had a positive radial ground state, moreover, the behavior and energy estimates of the bound states as $\lambda \to 0$ were also obtained.

Later, Chen and Zou [10] investigated the following coupled systems with critical power-type nonlinearity:

$$\begin{cases} -\Delta u + \mu u = |u|^{p-1}u + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{2^*-2}v + \lambda u, & x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $0 < \lambda < \sqrt{\mu\nu}, 1 < p < 2^* - 1$ and $N \geq 3$. They proved the existence of positive ground states for system (1.5) when $0 < \mu \leq \mu_0$, where $\mu_0 \in (0,1)$ was some critical value. When μ and λ were both large, system (1.5) had a positive ground state also. While, when μ was large but λ was small, the system (1.5) had no ground state solutions. In addition, when $p = 2^* - 1$, system (1.5) had no nontrivial solutions by the Pohozaev identity. Motivated by [10], Li and Tang [14] considered system (1.5) in $\mathbb{R}^N$, $N \geq 3$, when $\mu = a(x) > 0$, $\nu = b(x) > 0$ and $\lambda = \lambda(x)$ were continuous functions, 1-periodic in each of $x_1, x_2, \ldots, x_N$, and satisfied $\lambda(x) < \sqrt{a(x)b(x)}$, they proved system (1.5) had a Nehari-type ground state solution when $0 < a(x) < \mu_0$ for some $\mu_0 \in (0,1)$. Some related linearly coupled systems were also studied in [3,11,12] and the references therein.

In the above references we refer to, it is noticed that the nonlinearities were only considered the polynominal growth of subcritical or critical type in terms of the Sobolev embedding. As we all know, the Trudinger–Moser inequality in $\mathbb{R}^2$ with critical exponential growth instead of the Sobolev inequality in $\mathbb{R}^N$ with critical polynominal growth, which was first established by Cao in [5], reads as follows.

Lemma 1.1 ([5]).

i) If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)\mathrm{d}x < \infty;$$

ii) if $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \leq 1$, $\|u\|_2^2 \leq M < \infty$, and $\alpha < 4\pi$. then there exists a constant $C(M,\alpha)$, which depends only on M and α, such that

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)\mathrm{d}x \leq C(M,\alpha).$$

By virtue of the Trudinger–Moser inequality, do Ó and de Albuquerque [16] investigated the following linear coupled system with constant potential in $\mathbb{R}^2$,

$$\begin{cases} -\Delta u + u = f_1(u) + \lambda(x)v, & x \in \mathbb{R}^2, \\ -\Delta v + v = f_2(v) + \lambda(x)u, & x \in \mathbb{R}^2. \end{cases} \tag{1.6}$$

By using the minimization technique over the Nehari manifold and strong maximum principle, the existence of positive ground state solution and the corresponding asymptotic behavior were obtained.

In the paper [15], do Ó and de Albuquerque used the same idea as [16] to investigate the existence of positive ground state solution and asymptotic behaviors for the coupled system (1.1) with nonnegative variable potentials. The main problem they faced was to overcome the difficulty originated from the lack of compactness when the nonlinear terms had critical exponential growth in $\mathbb{R}^2$. Based on this, they considered the following weighted Sobolev space defined by

$$H_{V_i}(\mathbb{R}^2)=\left\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}V_i(x)u^2\mathrm{d}x<\infty\right\},$$

endowed with the norm

$$\|u\|_{V_i}=\left(\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x+\int_{\mathbb{R}^2}V_i(x)u^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

They assumed the following conditions on the potential $V_i(x),\ i=1,2$.

(V1′) $V_i(x)\geq 0$, for all $x\in\mathbb{R}^2$ and $V_i\in L^\infty_{loc}(\mathbb{R}^2)$;

(V2′) The infimum

$$\inf_{u\in H_{V_i}(\mathbb{R}^2)}\left\{\int_{\mathbb{R}^2}(|\nabla u|^2+V_i(x)u^2)\mathrm{d}x:\int_{\mathbb{R}^2}u^2\mathrm{d}x=1\right\}$$

is positive;

(V3′) There exists $s\in[2,+\infty)$ such that

$$\lim_{R\to\infty}\nu_s^i(\mathbb{R}^2\setminus\overline{B_R})=\infty,$$

here,

$$\nu_s^i(\Omega)=\begin{cases}\inf\limits_{u\in H_0^1(\Omega)\setminus\{0\}}\dfrac{\int_\Omega(|\nabla u|^2+V_i(x)u^2)\mathrm{d}x}{(\int_\Omega|u|^s\mathrm{d}x)^{\frac{2}{s}}}, & \Omega\neq\emptyset,\\ \infty, & \Omega=\emptyset;\end{cases}$$

(V4′) There exists functions $A_i(x)\in L^\infty_{loc}(\mathbb{R}^2)$, with $A_i(x)\geq 1$, and constants $\beta_i>1, C_0, R_0>0$ such that

$$A_i(x)\leq C_0[1+V_i(x)^{\frac{1}{\beta_i}}],\quad \text{for all } |x|\geq R_0.$$

Here, (V1′) and (V2′) is assumed to ensure that $H_{V_i}(\mathbb{R}^2)$ is a Hilbert space, (V3′) and (V4′) play a crucial role in overcoming the lack of compactness.

In terms of nonlinearities, they defined $f_i:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ had α_0^i-critical growth at $+\infty$ involving the term $A_i(x)$, such as

$$\lim_{t\to+\infty}\frac{|f_i(x,t)|}{A_i(x)e^{\alpha t^2}}=0,\ \text{uniformly on } x\in\mathbb{R}^2 \text{ for all } \alpha>\alpha_0^i.$$

$$\lim_{t\to+\infty}\frac{|f_i(x,t)|}{A_i(x)e^{\alpha t^2}}=+\infty,\ \text{uniformly on } x\in\mathbb{R}^2 \text{ for all } \alpha<\alpha_0^i.$$

Here, $A_i(x)$ was defined in (V4′). When $A_i(x)=1$, (F1) holds. In addition, they assumed the following hypotheses:

(F1′) $f_i : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is C^1, $f_i(x,t) = 0$ for all $x \in \mathbb{R}^2, t \le 0$, and

$$\lim_{t\to 0} \frac{|f_i(x,t)|}{A_i(x)|t|} = 0, \quad \text{uniformly on } x \in \mathbb{R}^2;$$

(F2′) $f_i(x,t)$ is locally bounded in t, that is, for any bounded interval $\Lambda \subset \mathbb{R}$, there exists $C > 0$ such that $f_i(x,t) \le C$, for all $(x,t) \in \mathbb{R}^2 \times \Lambda$;

(F3′) There exists $\mu_i > 2$ such that

$$tf_i(x,t) \ge \mu_i F_i(x,t) := \mu_i \int_0^t f_i(x,s)\mathrm{d}s > 0, \qquad \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+;$$

(F4′) For each fixed $x \in \mathbb{R}^2$ the function $t \mapsto \frac{f_i(x,t)}{t}$ is increasing for $t > 0$;

(F5′) There exists $q > 2$ such that

$$F_1(x,s) + F_2(x,t) \ge \vartheta(s^q + t^q)$$

for all $x \in \mathbb{R}^2$ and $s, t \ge 0$, $\vartheta > 0$ is a constant.

In [15], jointly with (V4′), one can find that the growth of f_i are controlled by the growth of $V_i(x)$, $i = 1,2$ form (F1′). Moreover, the condition $f_i \in C^1$ in (F1′) plays a crucial role to obtain the Nehari-type ground state solutions for (1.1) via the Nehari manifold method. (F3′) is the well-known Ambrosetti–Rabinowitz condition ((AR) condition), which ensures that the functional associated with the problem has a mountain pass geometry and guarantees the boundedness of the Palais–Smale sequence. (F4′) is the Nehari monotonic condition. (F5′) needs that the nonlinearities are super-q growth at zero, $q > 2$. It is noticed that sufficiently large ϑ in (F5′) is very crucial in their arguments. In fact, by virtue of this condition, the minimax-level for the energy functional can be choosen sufficiently small, therefore the difficult arising from the critical growth of Trudinger–Moser type is easily overcome. But this result has no relationship with the exponential velocity α_0^i, $i = 1,2$, hence it does not reveal the essential characteristics with the critical growth of Trudinger–Moser type.

Recently, Wei, Lin and Tang [20] used non-Nehari manifold methods (see [19]), Lions's concentration compactness and a direct approach derived from [7] for obtaining the minimax estimate to investigate system (1.6) in the non-autonomous case. They proved that (1.6) still possessed a Nehari-type ground state solution and a nontrivial solution. Their results improved the existence results of [16] by weakening the nonlinearities to be continuous, and only needed to satisfy the weaker Nehari monotonic condition, even without (AR) condition. Additionally, since the generalized linking theorem did not work for the strongly indefinite Hamiltonian elliptic system with critical exponential growth in $\mathbb{R}^2$, Qin, Tang and Zhang [18] developed a new approach to seek Cerami sequences for the energy functional and estimated the minimax levels of these sequences. Furthermore, they used non-Nehari manifold method to obtain the existence of ground state solutions without (AR) condition.

It is interesting to ask if the existence of positive ground state solutions for linearly coupled systems with variable potentials is preserved without (AR) condition. Our aim in this paper is to prove the existence of positive Nehari-type ground state solution of (1.1) and obtain the asymptotic behaviors of ground states with some mild assumptions. This work is motivated by the results of [15, 18, 20]. Our main result below (Theorem 1.2) can handle the case of

$f_i(x,t)$ with less restrictions, which are in the true sense of critical exponential growth, and are independent of (F5′) with some large constant ϑ (see [15, Theroem 1.1]).

To this end, we emphasize that we need refinements in order to treat the different setting from the constant potentials to the variable ones. Indeed, it is easy to get the mountain pass geometry for the problem with the constant potentials, while for variable potentials, some new analysis techniques and imbedding inequalities such as (2.2) are needed. We borrow the ideas from [18, 20] to look for the minimizing Cerami sequence for the energy functional associated with (1.1) by using the non-Nehari manifold approach. By means of slightly weaker monotonic conditions, we show the boundedness of the Cerami sequence. Furthermore, to recover the compactness of the minimizing Cerami sequence, we estimate an accurate threshold for the minimax-level, meanwhile, we use Lions's concentration compactness principle and the invariance of the energy functional by translation to show that the sequence does not vanish. Then by using a standard bootstrap argument and L^q-estimates we get regularity and asymptotic behavior of the ground state solution.

To state our main results, in addition to (F1) and (F2), we also introduce the following assumptions:

(F3) There exists $M_0 > 0$ and $t_0 > 0$ such that for every $x \in \mathbb{R}^2$,

$$F_i(x,t) \leq M_0|f_i(x,t)|, \qquad \forall |t| \geq t_0;$$

(F4) For every $x \in \mathbb{R}^2$, $\frac{f_i(x,t)}{t}$ is non-decreasing on $(0,\infty)$;

(F5) $\liminf_{|t|\to\infty} \frac{t^2 F_i(x,t)}{e^{\alpha_0 t^2}} \geq \kappa > \frac{V_M}{\alpha_0^2}$ uniformly on $x \in \mathbb{R}^2$, where $\alpha_0 = \max\{\alpha_1, \alpha_2\}$, $V_M = \max_{\mathbb{R}^2}\{V_1, V_2\}$.

In view of Lemma 1.1 i), under assumption (V), (F1) and (F2), the weak solutions of (1.1) correspond to the critical points of the energy functional defined by

$$\Phi(u,v) = \frac{1}{2}\left[\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv\mathrm{d}x\right] - \int_{\mathbb{R}^2}[F_1(x,u) + F_2(x,v)]\mathrm{d}x. \tag{1.7}$$

where $\|\cdot\|$ is defined in Section 2, (2.3).

Now our main results can be stated as follow.

Theorem 1.2. *Let (V), $(F1)$–$(F5)$ be satisfied. Then (1.1) has a solution $(\bar{u},\bar{v}) \in \mathcal{N}$ with $\bar{u} > 0$ and $\bar{v} > 0$ such that $\Phi(\bar{u},\bar{v}) = b := \inf_{\mathcal{N}} \Phi$, where*

$$\mathcal{N} := \{u \in E \setminus \{(0,0)\} : \langle \Phi'(u,v), (u,v)\rangle = 0\}, \tag{1.8}$$

where E is defined in Section 2. Moreover, $(\bar{u},\bar{v}) \in C^{1,\beta}_{loc}(\mathbb{R}^2) \times C^{1,\beta}_{loc}(\mathbb{R}^2)$ for some $\beta \in (0,1)$ with the following asymptotic behavior

$$\|\bar{u}\|_{C^{1,\beta}(\overline{B_R})} \to 0 \quad \textit{and} \quad \|\bar{v}\|_{C^{1,\beta}(\overline{B_R})} \to 0 \quad \textit{as } |x| \to \infty. \tag{1.9}$$

Remark 1.3. Theorem 1.2 improves and extends the results in [15, Theorem 1.1]. In the sense of the conditions of nonlinearities, f_1 and f_2 are continuous and the growth of them are independent on $V_i(x)$. For obtaining the boundedness of Cerami sequence, we only need the condition (F3) used for the exponential growth problems instead of (F3′). When it comes to the minimax level estimates of the energy functional, the authors in [15] made use of a rigorous limitation on the norm of the minimizing sequence by the the polynomial controlled condition (F5′), while we use the direct calculation argument with the exponential controlled condition (F5). Moreover, we use the weaker monotonicity condition (F4) to replace (F4′).

Remark 1.4. There are many functions satisfying the conditions (F1)–(F5) of the nonlinearities in this paper, but not satisfying the conditions (F4′) and (F5′) in [15]. For example, for$a_1, a_2 > 0$,

$$f_1(x,t) = \begin{cases} a_1(e^{2t^2} - 1), & t > 0, \\ 0, & t \le 0, \end{cases}$$

$$f_2(x,t) = \begin{cases} a_2|t|te^{t^2}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

The paper is organized as follows. In Section 2, we give the variational setting and preliminaries. In Section 3, we establish the minimax estimates of the energy functional. The proof of ground state solution will be stated in Section 4. Then in Section 5, we give the proof of regularity and asymptotic behavior.

Throughout the paper, we make use of the following notations:

- $L^s(\mathbb{R}^2)(1 \le s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^2} |u|^s \mathrm{d}x)^{1/s}$;
- $\forall x \in \mathbb{R}^2$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^2 : |y - x| < r\}$;
- $C_1, C_2, C_3, \ldots$ denote positive constants possibly different in different places.

2 Variational setting and preliminaries

Consider that the potentials are positive, we define the inner product in $H^1(\mathbb{R}^2)$ and the associated norm as follows,

$$(u,v) := \int_{\mathbb{R}^2} [\nabla u \nabla v + V(x)uv]\mathrm{d}x, \qquad \|u\|^2 := (u,u), \quad \forall u,v \in H^1(\mathbb{R}^2). \tag{2.1}$$

For any $s \in [2, +\infty)$, the Sobolev embedding theorem yields the existence of $\gamma_s \in (0, +\infty)$ such that

$$\|u\|_s \le \gamma_s \|u\|, \qquad \forall u \in H^1(\mathbb{R}^2). \tag{2.2}$$

Under (V), let $H_{V_1}(\mathbb{R}^2)$ and $H_{V_2}(\mathbb{R}^2)$ be endowed with the norm

$$\|u\|_{V_1} = \left(\int_{\mathbb{R}^2} |\nabla u|^2 + V_1(x)u^2 \mathrm{d}x \right)^{\frac{1}{2}}, \qquad \|v\|_{V_2} = \left(\int_{\mathbb{R}^2} |\nabla v|^2 + V_2(x)v^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Define $E := H_{V_1}(\mathbb{R}^2) \times H_{V_2}(\mathbb{R}^2)$ and

$$((u,v),(\phi,\psi)) := \int_{\mathbb{R}^2} (\nabla u \nabla \phi + \nabla v \nabla \psi + V_1(x)u\phi + V_2(x)v\psi)\mathrm{d}x, \quad \forall (u,\phi) \in H_{V_1}, (v,\psi) \in H_{V_2}.$$

Then E is a Hilbert space on the above inner product. The induced norm

$$\|(u,v)\|^2 := \int_{\mathbb{R}^2} (|\nabla u|^2 + |\nabla v|^2 + V_1(x)u^2 + V_2(x)v^2)\mathrm{d}x, \qquad \forall (u,v) \in E. \tag{2.3}$$

That is $\|(u,v)\|^2 = \|u\|_{V_1}^2 + \|v\|_{V_2}^2$. By (V), (1.7) and Lemma 1.1, we know that the functional $\Phi(u,v)$ is well defined on E. Moreover, by standard arguments, $\Phi \in C^1(E, \mathbb{R})$ and its derivative is given by

$$\langle \Phi'(u,v), (\phi,\psi) \rangle = ((u,v),(\phi,\psi)) - \int_{\mathbb{R}^2} \lambda(x)(u\psi + v\phi)\mathrm{d}x - \int_{\mathbb{R}^2} [f_1(x,u)\phi + f_2(x,v)\psi]\mathrm{d}x \tag{2.4}$$

and

$$\langle \Phi'(u,v),(u,v)\rangle = \|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv\mathrm{d}x - \int_{\mathbb{R}^2} [f_1(x,u)u + f_2(x,v)v]\mathrm{d}x. \tag{2.5}$$

For any $\varepsilon > 0, \alpha > \alpha_0$ and $\bar{q} > 0$, it follows from (F1) and (F2) that there exists $C = C(\varepsilon, \alpha, \bar{q}) > 0$ such that

$$|F_i(x,t)| \le \varepsilon t^2 + C|t|^{\bar{q}} e^{\alpha t^2}. \tag{2.6}$$

Now we choose $(u_0, v_0) \in E \setminus \{(0,0)\}$, it is easy to show that $\lim_{t\to\infty} \Phi(tu_0, tv_0) = -\infty$ due to (V) and (F1).

Lemma 2.1. *Assume that* (V), (F1) *and* (F2) *hold. Then there exists a sequence* $(u_n, v_n) \subset E$ *satisfying*

$$\Phi(u_n, v_n) \to c^*, \quad \|\Phi'(u_n, v_n)\|(1 + \|(u_n, v_n)\|) \to 0. \tag{2.7}$$

where c^* *is given by*

$$c^* = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \Phi(\gamma(1)) < 0\}.$$

Proof. By (2.6), one has for some constants $\alpha > \alpha_0$ and $C_1 > 0$

$$F_i(x,t) \le \frac{1-\lambda_0}{4\gamma_2^2} t^2 + C_1|t|^3(e^{\alpha t^2} - 1). \tag{2.8}$$

From (2.8) and Lemma 1.1 ii), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} F_1(x,u)\mathrm{d}x &\le \frac{1-\lambda_0}{4\gamma_2^2}\|u\|_2^2 + C_1\int_{\mathbb{R}^2}(e^{\alpha u^2} - 1)|u|^3\mathrm{d}x \\
&\le \frac{1-\lambda_0}{4\gamma_2^2}\|u\|_2^2 + C_1\Big[\int_{\mathbb{R}^2}(e^{2\alpha u^2} - 1)\mathrm{d}x\Big]^{\frac{1}{2}}\|u\|_6^3 \\
&\le \frac{1-\lambda_0}{4}\|u\|_{V_1}^2 + C_2\|u\|_{V_1}^3, \qquad \forall \|(u,v)\| \le \sqrt{\pi/\alpha}.
\end{aligned} \tag{2.9}$$

Similarly, we have

$$\int_{\mathbb{R}^2} F_2(x,v)\mathrm{d}x \le \frac{1-\lambda_0}{4}\|v\|_{V_2}^2 + C_2\|v\|_{V_2}^3, \qquad \forall \|(u,v)\| \le \sqrt{\pi/\alpha}. \tag{2.10}$$

Hence, it follows from (V), (1.7), (2.9) and (2.10) that

$$\begin{aligned}
\Phi(u,v) &= \frac{1}{2}\Big[\|(u,v)\|^2 - 2\int_{\mathbb{R}^2}\lambda(x)uv\mathrm{d}x\Big] - \int_{\mathbb{R}^2}[F_1(x,u) + F_2(x,v)]\mathrm{d}x \\
&\ge \frac{1}{2}\Big[\|(u,v)\|^2 - \lambda_0\int_{\mathbb{R}^2}(V_1(x)u^2 + V_2(x)v^2)\mathrm{d}x\Big] - \frac{1-\lambda_0}{4}(\|u\|_{V_1}^2 + \|v\|_{V_2}^2) \\
&\quad - C_2(\|u\|_{V_1}^3 + \|v\|_{V_2}^3) \\
&\ge \frac{1-\lambda_0}{4}\|(u,v)\|^2 - C_3\|(u,v)\|^3.
\end{aligned} \tag{2.11}$$

Therefore, there exists $\kappa_0 > 0$ and $0 < \rho < \sqrt{\pi/\alpha}$ such that

$$\Phi(u,v) \ge \kappa_0, \qquad \forall (u,v) \in S := \{(u,v) \in E : \|(u,v)\| = \rho\}. \tag{2.12}$$

Since $\lim_{t\to\infty} \Phi(tu_0, tv_0) = -\infty$, we can choose $T > 0$ such that $e = (Tu_0, Tv_0) \in \{(u,v) \in E : \|(u,v)\| \ge \rho\}$ and $\Phi(e) < 0$, then according to the mountain pass lemma, we deduce that there exists $c^* \in [\kappa_0, \sup_{t\ge 0}\Phi(tu_0, tv_0)]$ and a sequence $\{(u_n, v_n)\} \subset E$ satisfying (2.7). □

Lemma 2.2. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then*

$$\Phi(u,v) \geq \Phi(tu,tv) + \frac{1-t^2}{2}\left\langle \Phi'(u,v),(u,v)\right\rangle, \qquad \forall t>0. \tag{2.13}$$

Proof. It is obvious that (F4) implies the following inequality:

$$\frac{1-t^2}{2}f_i(x,s)s + F_i(x,ts) - F_i(x,s) = \int_t^1 \left[\frac{f_i(x,s)}{s} - \frac{f_i(x,\tau s)}{\tau s}\right]\tau s^2 \mathrm{d}\tau \geq 0. \tag{2.14}$$

From (1.7), (2.5) and (2.14), we have

$$\begin{aligned}
\Phi(u,v) - \Phi(tu,tv) &= \frac{1}{2}\left[\|(u,v)\|^2 - \int_{\mathbb{R}^2}\lambda(x)uv\mathrm{d}x\right] - \int_{\mathbb{R}^2}[F_1(x,u)+F_2(x,v)]\mathrm{d}x \\
&\quad - \left\{\frac{t^2}{2}\|(u,v)\|^2 - t^2\int_{\mathbb{R}^2}\lambda(x)uv\mathrm{d}x - \int_{\mathbb{R}^2}F_1(x,tu)+F_2(x,tv)\mathrm{d}x\right\} \\
&= \frac{1-t^2}{2}\left\langle \Phi'(u,v),(u,v)\right\rangle + \int_{\mathbb{R}^2}\left[\frac{1-t^2}{2}f_1(x,u)u + F_1(x,tu) - F_1(x,u)\right]\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2}\left[\frac{1-t^2}{2}f_2(x,v)v + F_2(x,tv) - F_2(x,v)\right]\mathrm{d}x \\
&\geq \frac{1-t^2}{2}\left\langle \Phi'(u,v),(u,v)\right\rangle.
\end{aligned}$$

□

From Lemma 2.2, we get the following corollary easily.

Corollary 2.3. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then*

$$\Phi(u,v) \geq \max_{t\geq 0}\Phi(tu,tv), \qquad \forall (u,v)\in\mathcal{N}. \tag{2.15}$$

Lemma 2.4. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then for any* $(u,v)\in E\setminus\{(0,0)\}$, *there exists a unique* $t_{(u,v)}>0$ *such that* $(t_{(u,v)}u, t_{(u,v)}v)\in\mathcal{N}$.

Proof. Let $(u,v)\in E\setminus\{(0,0)\}$ be fixed and define a function $\zeta(t) := \Phi(tu,tv)$ on $[0,\infty)$. Clearly, by (2.5), we have

$$\begin{aligned}
\zeta'(t) = 0 &\Leftrightarrow t^2\|(u,v)\|^2 - 2t^2\int_{\mathbb{R}^2}\lambda(x)uv\mathrm{d}x - \int_{\mathbb{R}^2}[f_1(x,tu)tu + f_2(x,tv)tv]\mathrm{d}x = 0 \\
&\Leftrightarrow \left\langle \Phi'(tu,tv),(tu,tv)\right\rangle = 0 \Leftrightarrow (tu,tv)\in\mathcal{N}.
\end{aligned}$$

By (2.11) and (F1), one has $\zeta(0)=0$ and $\zeta(t)>0$ for $t>0$ small and $\zeta(t)<0$ for t large. Therefore, $\max_{t\in(0,\infty)}\zeta(t)$ is achieved at some $t_0 = t_{(u,v)}>0$, so that $\zeta'(t_0)=0$ and $t_{(u,v)}(u,v)\in\mathcal{N}$.

Next we claim that $t_{(u,v)}$ is unique for any $(u,v)\in E\setminus\{(0,0)\}$, let $t_1,t_2>0$ such that $\zeta'(t_1)=\zeta'(t_2)=0$. Then $\langle\Phi'(t_1u,t_1v),(t_1u,t_1v)\rangle = \langle\Phi'(t_2u,t_2v),(t_2u,t_2v)\rangle = 0$. Jointly with (2.13) , we have

$$\Phi(t_1u,t_1v) \geq \Phi(t_2u,t_2v) + \frac{1-t^2}{2}\left\langle \Phi'(t_1u,t_1v),(t_1u,t_1v)\right\rangle \tag{2.16}$$

and

$$\Phi(t_2u,t_2v) \geq \Phi(t_1u,t_1v) + \frac{1-t^2}{2}\left\langle \Phi'(t_2u,t_2v),(t_2u,t_2v)\right\rangle. \tag{2.17}$$

By (2.16) and (2.17), it is obvious that $t_1=t_2$. Therefore $t_{(u,v)}>0$ is unique for any $(u,v)\in E\setminus\{(0,0)\}$. □

From Corollary 2.3 and Lemma 2.4, we directly have the following lemma about minimax characterization of $\inf_{\mathcal{N}} \Phi$.

Lemma 2.5. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then*

$$b := \inf_{\mathcal{N}} \Phi = \inf_{(u,v)\in E\setminus\{(0,0)\}} \max_{t\geq 0} \Phi(tu, tv). \tag{2.18}$$

Lemma 2.6. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then there exist a constant $\bar{c} \in (0, b]$ and a sequence $\{(u_n, v_n)\} \subset E$ satisfying*

$$\Phi(u_n, v_n) \to \bar{c}, \qquad \|\Phi'(u_n, v_n)\|_{E^*}(1 + \|(u_n, v_n)\|) \to 0. \tag{2.19}$$

Similarly with [20, Lemma 2.6], the proof is omitted here.

Lemma 2.7. *Assume that* (V), (F1)–(F4) *hold. Then any sequence $\{(u_n, v_n)\}$ satisfying* (2.19) *is bounded.*

Proof. Arguing by contradiction, suppose that $\|(u_n, v_n)\| \to \infty$ as $n \to \infty$. Let $(\tilde{u}_n, \tilde{v}_n) = (u_n, v_n)/\|(u_n, v_n)\|$. Then $1 = \|(\tilde{u}_n, \tilde{v}_n)\|^2$. By (F2) and (F4), we have

$$\frac{f_i(x, \theta t)\theta t}{\theta^2} \geq f_i(x, t)t \geq 2F_i(x, t) \geq 0, \qquad \forall x \in \mathbb{R}^2, t \in \mathbb{R}, \theta \geq 1, i = 1, 2. \tag{2.20}$$

It follows from (F3) and (2.20) that there exists $R > t_0$ such that

$$f_i(x, t)t \geq 4F_i(x, t), \qquad \forall |t| \geq R. \tag{2.21}$$

From (1.7), (2.5), (2.19), (2.20), and (2.21), we have

$$\begin{aligned}
\bar{c} + o(1) &= \Phi(u_n, v_n) - \frac{1}{2}\left\langle \Phi'(u_n, v_n), (u_n, v_n) \right\rangle \\
&= \int_{\mathbb{R}^2} \left[\frac{1}{2} f_1(x, u_n)u_n - F_1(x, u_n)\right] \mathrm{d}x + \int_{\mathbb{R}^2} \left[\frac{1}{2} f_2(x, v_n)v_n - F_2(x, v_n)\right] \mathrm{d}x \\
&\geq \int_{|u_n|\leq R} \left[\frac{1}{2} f_1(x, u_n)u_n - F_1(x, u_n)\right] \mathrm{d}x + \int_{|v_n|\leq R} \left[\frac{1}{2} f_2(x, v_n)v_n - F_2(x, v_n)\right] \mathrm{d}x \\
&\quad + \frac{1}{4}\int_{|u_n|>R} f_1(x, u_n)u_n \mathrm{d}x + \frac{1}{4}\int_{|v_n|>R} f_2(x, v_n)v_n \mathrm{d}x \\
&\geq \frac{1}{4}\int_{|u_n|>R} f_1(x, u_n)u_n \mathrm{d}x + \frac{1}{4}\int_{|v_n|>R} f_2(x, v_n)v_n \mathrm{d}x.
\end{aligned} \tag{2.22}$$

Let $\tau \geq \left(\frac{4(\bar{c}+1)}{1-2\lambda_0}\right)^{\frac{1}{2}}$ and $t_n = \tau/\|(u_n, v_n)\|$. Then $t_n \to 0$ as $n \to \infty$. It follows from (F2), (2.21) and (2.22) that

$$
\begin{aligned}
&\int_{\mathbb{R}^2}[F_1(x,t_nu_n)+F_2(x,t_nu_n)]\mathrm{d}x\\
&\quad=\int_{|u_n|\le R}F_1(x,t_nu_n)\mathrm{d}x+\int_{|u_n|>R}F_1(x,t_nu_n)\mathrm{d}x\\
&\qquad+\int_{|v_n|\le R}F_2(x,t_nv_n)\mathrm{d}x+\int_{|v_n|>R}F_2(x,t_nv_n)\mathrm{d}x\\
&\quad\le\frac{1}{4\gamma_2^2}\int_{|u_n|\le R}|t_nu_n|^2\mathrm{d}x+\frac{1}{4\gamma_2^2}\int_{|v_n|\le R}|t_nv_n|^2\mathrm{d}x\\
&\qquad+\frac{t_n^2}{4}\int_{|u_n|>R}f_1(x,u_n)u_n\mathrm{d}x+\frac{t_n^2}{4}\int_{|v_n|>R}f_2(x,v_n)v_n\mathrm{d}x\\
&\quad\le\frac{t_n^2}{4\gamma_2^2}\int_{|u_n|\le R}|u_n|^2\mathrm{d}x+\frac{t_n^2}{4\gamma_2^2}\int_{|v_n|\le R}|v_n|^2\mathrm{d}x+\frac{\tau^2(\bar{c}+1)}{\|(u_n,v_n)\|^2}\\
&\quad\le\frac{\tau^2}{4}+o(1).
\end{aligned}\tag{2.23}
$$

Hence, from (2.13), (2.19) and (2.23), we have

$$
\begin{aligned}
\bar{c}+o(1)&=\Phi(u_n,v_n)\\
&\ge\Phi(t_nu_n,t_nv_n)+\frac{1-t_n^2}{2}\left\langle\Phi'(u_n,v_n),(u_n,v_n)\right\rangle\\
&=\frac{t_n^2}{2}\left[\|(u_n,v_n)\|^2-2\int_{\mathbb{R}^2}\lambda(x)u_nv_n\mathrm{d}x\right]-\int_{\mathbb{R}^2}[F_1(x,t_nu_n)+F_2(x,t_nv_n)]\mathrm{d}x+o(1)\\
&\ge\frac{(1-2\lambda_0)\tau^2}{4}+o(1)\\
&\ge\bar{c}+1+o(1).
\end{aligned}\tag{2.24}
$$

This contradiction shows that $\{(u_n,v_n)\}$ is bounded. □

3 Minimax estimates

In this section, we give a accurate estimation about the minimax level c^* defined by Lemma 2.1.

At first, we define a Moser type function $w_n(x)$ supported in $B_{\sqrt{2/V_M}}:=B_{\sqrt{2/V_M}}(0)$ as follows:

$$
w_n(x)=\frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\log n}, & 0\le|x|\le\sqrt{2}/(\sqrt{V_M}n);\\ \dfrac{\log(\sqrt{2}/\sqrt{V_M}|x|)}{\sqrt{\log n}}, & \sqrt{2}/(\sqrt{V_M}n)\le|x|\le\sqrt{2/V_M};\\ 0, & |x|\ge\sqrt{2/V_M}.\end{cases}\tag{3.1}
$$

By an elementary computation, we have

$$
\|\nabla w_n\|_2^2=\int_{\mathbb{R}^2}|\nabla w_n|^2\mathrm{d}x=1,\tag{3.2}
$$

and

$$
\|w_n\|_2^2=\int_{\mathbb{R}^2}|w_n|^2\mathrm{d}x=\frac{2\delta_n}{V_M},\tag{3.3}
$$

where

$$\delta_n := \frac{1}{4\log n} - \frac{1}{4n^2\log n} - \frac{1}{2n^2} > 0. \tag{3.4}$$

Lemma 3.1. *Assume that* (V), (F1), (F2), *and* (F5) *hold. Furthermore, suppose that* $F_i(x,t) \geq 0$ *for all* $t \in \mathbb{R}$. *Then there exists* $\bar{n} \in \mathbb{N}$ *such that*

$$c^* \leq \max_{t\geq 0} \Phi(tw_{\bar{n}}, tw_{\bar{n}}) < \frac{4\pi}{\alpha_0}. \tag{3.5}$$

Proof. By (F5), we can choose $\varepsilon > 0$ and $t_\varepsilon > 0$ such that

$$\log \frac{V_M(1+\epsilon)^2}{(2-\epsilon)(\kappa-\epsilon)\alpha_0^2} < -\epsilon \tag{3.6}$$

and

$$t^2 F_i(x,t) \geq (\kappa-\epsilon)e^{\alpha_0 t^2}, \quad \forall x \in \mathbb{R}^2, \quad |t| \geq t_\epsilon, \quad i = 1,2. \tag{3.7}$$

From (1.7), (3.2) and (3.3), we have

$$\begin{aligned}\Phi(tw_n, tw_n) &= \frac{t^2}{2}\int_{\mathbb{R}^2}[|\nabla w_n|^2 + |\nabla w_n|^2 + V_1(x)w_n^2 + V_2(x)w_n^2]\mathrm{d}x \\ &\quad - t^2\int_{\mathbb{R}^2}\lambda(x)w_n^2\mathrm{d}x - \int_{\mathbb{R}^2}[F_1(x,tw_n)+F_2(x,tw_n)]\mathrm{d}x \\ &\leq t^2(1+2\delta_n) - \int_{\mathbb{R}^2}[F_1(x,tw_n)+F_2(x,tw_n)]\mathrm{d}x. \end{aligned}\tag{3.8}$$

There are four cases to distinguish. In the sequel, we agree that all inequalities hold for large $n \in \mathbb{N}$ without mentioning.

Case i). $t \in \left[0, \sqrt{\frac{2\pi}{\alpha_0}}\right]$. Then it follows from (3.8) that

$$\begin{aligned}\Phi(tw_n, tw_n) &\leq t^2(1+2\delta_n) - \int_{\mathbb{R}^2}[F_1(x,tw_n)+F_2(x,tw_n)]\mathrm{d}x \\ &\leq t^2\left(1+\frac{1}{2\log n}\right) + O\left(\frac{1}{n\log n}\right) \\ &\leq \frac{2\pi}{\alpha_0} + O\left(\frac{1}{n\log n}\right). \end{aligned}\tag{3.9}$$

Clearly, there exists $\bar{n} \in \mathbb{N}$ such that (3.5) hold.

Case ii). $t \in \left[\sqrt{\frac{2\pi}{\alpha_0}}, \sqrt{\frac{4\pi}{\alpha_0}}\right]$. Then $tw_n(x) \geq t_\epsilon$ for $x \in B_{\sqrt{2/(V_M n)}}$ and for large $n \in \mathbb{N}$, it follows (3.1) and (3.7) that

$$\begin{aligned}\int_{\mathbb{R}^2} F_1(x,tw_n)\mathrm{d}x &\geq \int_{B_{\sqrt{2/(V_M n)}}} F_1(x,tw_n)\mathrm{d}x \\ &\geq \int_{B_{\sqrt{2/(V_M n)}}} \frac{(\kappa-\epsilon)e^{\alpha_0 t^2 w_n^2}}{t^2 w_n^2}\mathrm{d}x \\ &\geq \frac{(\kappa-\epsilon)\alpha_0}{2\log n}\int_{B_{\sqrt{2/(V_M n)}}} e^{\alpha_0 t^2 w_n^2}\mathrm{d}x \\ &= \frac{\pi(\kappa-\epsilon)\alpha_0}{V_M n^2\log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2\log n} + 2n^2\log n\int_{1/2}^1 n^{(2\pi)^{-1}\alpha_0 t^2 s^2 - 2s}\mathrm{d}s\right] \end{aligned}$$

$$\geq \frac{\pi(\kappa-\epsilon)\alpha_0}{V_M n^2 \log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + 2n^2 \log n \int_{1/2}^{1} n^{(2\pi)^{-1}\alpha_0 t^2 s - 2} \mathrm{d}s\right]$$
$$= \frac{\pi(\kappa-\epsilon)\alpha_0}{V_M n^2 \log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + \frac{4\pi}{\alpha_0 t^2}\left(n^{(2\pi)^{-1}\alpha_0 t^2} - n^{(4\pi)^{-1}\alpha_0 t^2}\right)\right]$$
$$\geq \frac{2\pi(\kappa-\epsilon)\alpha_0}{V_M n^2 \log n} e^{(2\pi)^{-1}\alpha_0 t^2 \log n} - O\left(\frac{1}{n\log n}\right). \tag{3.10}$$

Similarly, we have

$$\int_{\mathbb{R}^2} F_2(x, tw_n)\mathrm{d}x \geq \frac{2\pi(\kappa-\epsilon)\alpha_0}{V_M n^2 \log n} e^{(2\pi)^{-1}\alpha_0 t^2 \log n} - O\left(\frac{1}{n\log n}\right). \tag{3.11}$$

It follows from (3.8), (3.10) and (3.11) that

$$\Phi(tw_n, tw_n) \leq t^2\left(1 + \frac{1}{2\log n}\right) - \frac{4\pi(\kappa-\epsilon)\alpha_0}{V_M n^2 \log n} e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + 2O\left(\frac{1}{n\log n}\right)$$
$$=: \varphi_n(t) + 2O\left(\frac{1}{n\log n}\right). \tag{3.12}$$

Let $t_n > 0$ such that $\varphi_n'(t_n) = 0$. Then

$$1 + \frac{1}{2\log n} = \frac{2(\kappa-\epsilon)\alpha_0^2}{V_M n^2} e^{(2\pi)^{-1}\alpha_0 t_n^2 \log n}. \tag{3.13}$$

It follows that

$$\lim_{n\to\infty} t_n^2 = \frac{4\pi}{\alpha_0}. \tag{3.14}$$

From (3.13) and (3.14), we have

$$t_n^2 = \frac{4\pi}{\alpha_0}\left[1 + \frac{\log\left(V_M + \frac{V_M}{2\log n}\right) - \log(2(\kappa-\epsilon)\alpha_0^2)}{2\log n}\right]$$
$$\leq \frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0 \log n}\log\frac{V_M(1+\epsilon)}{2(\kappa-\epsilon)\alpha_0^2}. \tag{3.15}$$

and

$$\varphi_n(t) \leq \varphi_n(t_n) = t_n^2\left(1 + \frac{1}{2\log n}\right) - \frac{2\pi}{\alpha_0 \log n}\left(1 + \frac{1}{2\log n}\right). \tag{3.16}$$

From (3.14), (3.15) and (3.16), we have

$$\varphi_n(t) \leq t_n^2\left(1 + \frac{1}{2\log n}\right) - \frac{2\pi}{\alpha_0 \log n}\left(1 + \frac{1}{2\log n}\right)$$
$$\leq \left[\frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0 \log n}\log\frac{V_M(1+\epsilon)}{2(\kappa-\epsilon)\alpha_0^2}\right]\left(1 + \frac{1}{2\log n}\right) - \frac{2\pi}{\alpha_0 \log n}\left(1 + \frac{1}{2\log n}\right)$$
$$\leq \left[\frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0 \log n}\log\frac{V_M(1+\epsilon)}{2(\kappa-\epsilon)\alpha_0^2}\right]\left(1 + \frac{1}{2\log n}\right) - \frac{2\pi}{\alpha_0 \log n}(1-\epsilon)$$
$$\leq \frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0 \log n}\left[\log\frac{V_M(1+\epsilon)}{2(\kappa-\epsilon)\alpha_0^2} + \epsilon\right] + O\left(\frac{1}{\log^2 n}\right). \tag{3.17}$$

Hence, combining (3.12) with (3.17), one has

$$\Phi(tw_n, tw_n) \leq \frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0 \log n}\left[\log\frac{V_M(1+\epsilon)}{2(\kappa-\epsilon)\alpha_0^2} + \epsilon\right] + O\left(\frac{1}{\log^2 n}\right). \tag{3.18}$$

Obviously, (3.6) and (3.18) imply that there exists $\bar{n} \in \mathbb{N}$ such that (3.5) hold.

Case iii). $t \in \left[\sqrt{\frac{4\pi}{\alpha_0}}, \sqrt{\frac{4\pi}{\alpha_0}(1+\epsilon)}\right]$. Then $tw_n(x) \geq t_\epsilon$ for $x \in B_{\sqrt{2/(V_M n)}}$ and for large $n \in \mathbb{N}$, it follows (3.7) that

$$\begin{aligned}
\int_{\mathbb{R}^2} F_1(x, tw_n)\mathrm{d}x \geq & \int_{B_{\sqrt{2/(V_M n)}}} F_1(x, tw_n)\mathrm{d}x \\
\geq & \int_{B_{\sqrt{2/(V_M n)}}} \frac{(\kappa-\epsilon)e^{\alpha_0 t^2 w_n^2}}{t^2 w_n^2}\mathrm{d}x \\
\geq & \frac{(\kappa-\epsilon)\alpha_0}{2(1+\epsilon)\log n}\int_{B_{\sqrt{2/(V_M n)}}} e^{\alpha_0 t^2 w_n^2}\mathrm{d}x \\
= & \frac{\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)V_M n^2 \log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + 2n^2 \log n \int_{1/2}^{1} n^{(2\pi)^{-1}\alpha_0 t^2 s^2 - 2s}\mathrm{d}s\right] \\
\geq & \frac{\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)V_M n^2 \log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + 2\log n \int_{1-\epsilon}^{1} n^{[(1-\epsilon)(2\pi)^{-1}\alpha_0 t^2 + 2\epsilon]s}\mathrm{d}s\right] \\
= & \frac{\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)V_M n^2 \log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + \frac{1}{1+\epsilon}e^{[(1-\epsilon)(2\pi)^{-1}\alpha_0 t^2 + 2\epsilon]\log n}\right] \\
& - O\left(\frac{1}{n^{2\epsilon^2}\log n}\right) \\
\geq & \frac{2\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)^{3/2}V_M n^{2-\epsilon}\log n}e^{(2-\epsilon)(4\pi)^{-1}\alpha_0 t^2 \log n} - O\left(\frac{1}{n^{2\epsilon^2}\log n}\right).
\end{aligned} \tag{3.19}$$

Similarly, we have

$$\int_{\mathbb{R}^2} F_2(x, tw_n)\mathrm{d}x \geq \frac{2\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)^{3/2}V_M n^{2-\epsilon}\log n}e^{(2-\epsilon)(4\pi)^{-1}\alpha_0 t^2 \log n} - O\left(\frac{1}{n^{2\epsilon^2}\log n}\right). \tag{3.20}$$

It follows from (3.6) , (3.19) and (3.20) that

$$\begin{aligned}
\Phi(tw_n, tw_n) \leq & t^2\left(1 + \frac{1}{2\log n}\right) - \frac{4\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)^{3/2}V_M n^{2-\epsilon}\log n}e^{(2-\epsilon)(4\pi)^{-1}\alpha_0 t^2 \log n} + 2O\left(\frac{1}{n^{2\epsilon^2}\log n}\right) \\
=: & \psi_n(t) + 2O\left(\frac{1}{n^{2\epsilon^2}\log n}\right).
\end{aligned} \tag{3.21}$$

Let $\hat{t}_n > 0$ such that $\psi_n'(\hat{t}_n) = 0$. Then

$$1 + \frac{1}{2\log n} = \frac{(\kappa-\epsilon)(2-\epsilon)\alpha_0^2}{(1+\epsilon)^{3/2}V_M n^{2-\epsilon}}e^{(2-\epsilon)(4\pi)^{-1}\alpha_0 \hat{t}_n^2 \log n}. \tag{3.22}$$

It follows that

$$\lim_{n\to\infty} \hat{t}_n^2 = \frac{4\pi}{\alpha_0}. \tag{3.23}$$

From (3.22) and (3.23), we have

$$\begin{aligned}
\hat{t}_n^2 = & \frac{4\pi}{\alpha_0}\left[1 + \frac{(1+\epsilon)^{3/2}V_M\left(1 + \frac{1}{2\log n}\right) - \log((2-\epsilon)(\kappa-\epsilon)\alpha_0^2)}{(2-\epsilon)\log n}\right] \\
\leq & \frac{4\pi}{\alpha_0} + \frac{4\pi}{\alpha_0(2-\epsilon)\log n}\log\frac{V_M(1+\epsilon)^2}{(2-\epsilon)(\kappa-\epsilon)\alpha_0^2}.
\end{aligned} \tag{3.24}$$

It follows from (3.21), (3.23) and (3.24), we have

$$
\begin{aligned}
\psi_n(t) \leq \psi_n(\hat{t}_n) &= \hat{t}_n^2\left(1+\frac{1}{2\log n}\right) - \frac{4\pi}{(2-\epsilon)\alpha_0\log n}\left(1+\frac{1}{2\log n}\right)\\
&\leq \left[\frac{4\pi}{\alpha_0} + \frac{4\pi}{\alpha_0(2-\epsilon)\log n}\log\frac{V_M(1+\epsilon)^2}{(2-\epsilon)(\kappa-\epsilon)\alpha_0^2}\right]\left(1+\frac{1}{2\log n}\right) - \frac{4\pi(1-\epsilon)}{(2-\epsilon)\alpha_0\log n}\\
&\leq \frac{4\pi}{\alpha_0} + \frac{4\pi}{(2-\epsilon)\alpha_0\log n}\left[\epsilon + \log\frac{V_M(1+\epsilon)^2}{(2-\epsilon)(\kappa-\epsilon)\alpha_0^2}\right] + O\left(\frac{1}{\log^2 n}\right). \qquad (3.25)
\end{aligned}
$$

Hence, combining (3.21) with (3.25), one has

$$
\Phi(tw_n, tw_n) \leq \frac{4\pi}{\alpha_0} + \frac{4\pi}{(2-\epsilon)\alpha_0\log n}\left[\epsilon + \log\frac{V_M(1+\epsilon)^2}{(2-\epsilon)(\kappa-\epsilon)\alpha_0^2}\right] + O\left(\frac{1}{\log^2 n}\right). \qquad (3.26)
$$

Clearly, (3.6) and (3.26) imply that there exists $\bar{n} \in \mathbb{N}$ such that (3.5) hold.

Case iv). $t \in \left(\sqrt{\frac{4\pi}{\alpha_0}(1+\epsilon)}, +\infty\right)$. Then $tw_n(x) \geq t_\epsilon$ for $x \in B_{\sqrt{2}/(\sqrt{V_M}n)}$ and for large $n \in \mathbb{N}$, it follows (3.1) and (3.8) that

$$
\begin{aligned}
\Phi(tw_n, tw_n) &\leq t^2\left(1+\frac{1}{2\log n}\right) - \int_{\mathbb{R}^2}[F_1(x, tw_n) + F_2(x, tw_n)]\mathrm{d}x\\
&\leq t^2\left(1+\frac{1}{2\log n}\right) - \frac{8\pi^2(\kappa-\epsilon)}{V_M n^2 t^2\log n}e^{(2\pi)^{-1}\alpha_0 t^2\log n} + 2O\left(\frac{1}{n\log n}\right)\\
&\leq \frac{4\pi(1+\epsilon)}{\alpha_0}\left(1+\frac{1}{2\log n}\right) - \frac{2\alpha_0\pi(\kappa-\epsilon)}{V_M(1+\epsilon)\log n}e^{2\epsilon\log n} + 2O\left(\frac{1}{n\log n}\right)\\
&\leq \frac{3\pi}{\alpha_0}. \qquad (3.27)
\end{aligned}
$$

which implies that there exists $\bar{n} \in \mathbb{N}$ such that (3.5) hold. In the above derivation process, we use the fact that the function

$$
t^2\left(1+\frac{1}{2\log n}\right) - \frac{8\pi^2(\kappa-\epsilon)}{V_M n^2 t^2\log n}e^{(2\pi)^{-1}\alpha_0 t^2\log n} + O\left(\frac{1}{n\log n}\right) \qquad (3.28)
$$

is decreasing on $t \in \left(\sqrt{\frac{4\pi}{\alpha_0}(1+\epsilon)}, +\infty\right)$, since its stagnation tend to $\sqrt{\frac{4\pi}{\alpha_0}}$ as $n \to \infty$. □

From Lemma 2.5 and 3.1, we have the following corollary immediately.

Corollary 3.2. *Assume that* (V), (F1), (F2), (F4) *and* (F5) *hold. Then*

$$
b := \inf_{\mathcal{N}} \Phi < \frac{4\pi}{\alpha_0}. \qquad (3.29)
$$

4 Proofs of the main results

Lemma 4.1. *The weak solution $(\tilde{u}, \tilde{v})$ is nontrival.*

Proof. By Lemmas 2.1 and 2.7, there exist a subsequence $\{u_n, v_n\} \subset E$ satisfying (2.7) and $\|(u_n, v_n)\| + \|u_n\|_2 + \|v_n\|_2 \leq C$ for some constant $C_4 > 0$, it follows from (2.5) and (2.7) that

$$
\int_{\mathbb{R}^2} f_1(x, u_n)u_n\mathrm{d}x \leq C_5, \qquad \int_{\mathbb{R}^2} f_2(x, v_n)v_n\mathrm{d}x \leq C_5. \qquad (4.1)
$$

We may assume, passing to a subsequence if necessary, that $(u_n, v_n) \rightharpoonup (\tilde{u}, \tilde{v})$ in E, $(u_n, v_n) \to (\tilde{u}, \tilde{v})$ in $L^s_{loc}(\mathbb{R}^2)$ for $s \in [1, +\infty)$ and $(u_n, v_n) \to (\tilde{u}, \tilde{v})$ a.e. on $\mathbb{R}^2$.

If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_1(y)} (|u_n|^2 + |v_n|^2) \mathrm{d}x = 0,$$

then by Lions's concentration compactness principle, $(u_n, v_n) \to (0, 0)$ in $L^s(R^2)$ for $2 < s < \infty$. For any given $\varepsilon > 0$, we choose $M_\varepsilon > M_0 C_5/\varepsilon$, then it follows from (F3) and (4.1) that

$$\int_{|u_n|\geq M_\varepsilon} F_1(x, u_n)\mathrm{d}x \leq M_0 \int_{|u_n|\geq M_\varepsilon} |f_1(x, u_n)|\mathrm{d}x \leq \frac{M_0}{M_\varepsilon} \int_{|u_n|\geq M_\varepsilon} f_1(x, u_n)u_n \mathrm{d}x < \varepsilon. \tag{4.2}$$

$$\int_{|v_n|\geq M_\varepsilon} F_2(x, v_n)\mathrm{d}x \leq M_0 \int_{|v_n|\geq M_\varepsilon} |f_2(x, v_n)|\mathrm{d}x \leq \frac{M_0}{M_\varepsilon} \int_{|v_n|\geq M_\varepsilon} f_2(x, v_n)v_n \mathrm{d}x < \varepsilon. \tag{4.3}$$

By (F2), we can choose $N_\varepsilon \in (0, 1)$ such that

$$\int_{|u_n|\leq N_\varepsilon} F_1(x, u_n)\mathrm{d}x \leq \int_{|u_n|\leq N_\varepsilon} f_1(x, u_n)u_n \mathrm{d}x \leq \frac{\varepsilon}{C_4^2}\|u_n\|_2^2 < \varepsilon. \tag{4.4}$$

$$\int_{|v_n|\leq N_\varepsilon} F_2(x, v_n)\mathrm{d}x \leq \int_{|v_n|\leq N_\varepsilon} f_2(x, v_n)v_n \mathrm{d}x \leq \frac{\varepsilon}{C_4^2}\|v_n\|_2^2 < \varepsilon. \tag{4.5}$$

By (F1), we have

$$\int_{N_\varepsilon\leq|u_n|\leq M_\varepsilon} F_1(x, u_n)\mathrm{d}x \leq C_6\|u_n\|_3^3 = o(1), \quad \int_{N_\varepsilon\leq|v_n|\leq M_\varepsilon} F_2(x, v_n)\mathrm{d}x \leq C_6\|v_n\|_3^3 = o(1), \tag{4.6}$$

$$\int_{N_\varepsilon\leq|u_n|\leq 1} f_1(x, u_n)u_n\mathrm{d}x \leq C_7\|u_n\|_3^3 = o(1), \quad \int_{N_\varepsilon\leq|v_n|\leq 1} f_2(x, v_n)v_n\mathrm{d}x \leq C_7\|v_n\|_3^3 = o(1). \tag{4.7}$$

Due to the arbitrariness of $\varepsilon > 0$, from (4.2), (4.4), (4.6), we obtain

$$\int_{\mathbb{R}^2} F_1(x, u_n)\mathrm{d}x = o(1), \qquad \int_{\mathbb{R}^2} F_2(x, v_n)\mathrm{d}x = o(1). \tag{4.8}$$

Hence, it follows from (V), (1.7), (2.7) and (4.8) that

$$\begin{aligned}
\frac{1}{2}(\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2) &< \frac{1}{2}\|(u_n, v_n)\|^2 - \int_{\mathbb{R}^2} \lambda(x)u_n v_n \mathrm{d}x \\
&= c^* + \int_{\mathbb{R}^2} [F_1(x, u_n) + F_2(x, v_n)]\mathrm{d}x + o(1) \\
&= c^* + o(1).
\end{aligned}$$

Which, together with (3.5), implies that $\limsup_{n\to\infty} \|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 < \frac{8\pi}{\alpha_0}$. Hence, there exist $\bar{\varepsilon} > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 \leq \frac{8\pi}{\alpha_0}(1 - 3\bar{\varepsilon}), \qquad \forall n \geq n_0.$$

Let us choose $q \in (1, 2)$ such that

$$\frac{(1 + \bar{\varepsilon})(1 - 3\bar{\varepsilon})q}{1 - \bar{\varepsilon}} < 1. \tag{4.9}$$

By (F1), there exists $C_8 > 0$ such that

$$|f_i(x, t)|^q \leq C_8\left[e^{\alpha_0(1+\bar{\varepsilon})qt^2} - 1\right], \qquad \forall |t| \geq 1, \quad i = 1, 2. \tag{4.10}$$

It follows from (4.9), (4.10) and Lemma 1.1 ii) that

$$\begin{aligned}\int_{|u_n|\geq 1} f_i(x,u_n)^q \mathrm{d}x &\leq C_8 \int_{\mathbb{R}^2} \left[e^{\alpha_0(1+\bar{\varepsilon})qu_n^2} - 1\right]\mathrm{d}x \\ &= C_8 \int_{\mathbb{R}^2} \left[e^{\alpha_0(1+\bar{\varepsilon})q\|u_n\|^2(u_n/\|u_n\|)^2} - 1\right]\mathrm{d}x \\ &\leq C_9. \end{aligned} \tag{4.11}$$

Let $q' = q/(q-1)$. Then we have

$$\int_{|u_n|\geq 1} f_i(x,u_n)u_n \mathrm{d}x \leq \left[\int_{|u_n|\geq 1} |f_i(x,u_n)|^q \mathrm{d}x\right]^{\frac{1}{q}} \|u_n\|_{q'} = o(1). \tag{4.12}$$

Now we derive

$$\begin{aligned} c^* + o(1) &= \Phi(u_n,v_n) - \frac{1}{2}\left\langle \Phi'(u_n,v_n),(u_n,v_n)\right\rangle \\ &= \int_{\mathbb{R}^2}\left[\frac{1}{2}f_1(x,u_n)u_n - F_1(x,u_n)\right]\mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{2}f_2(x,v_n)v_n - F_2(x,v_n)\right]\mathrm{d}x \\ &< \varepsilon + o(1). \end{aligned} \tag{4.13}$$

This contradiction shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume that there exists $\{l_n\} \subset \mathbb{Z}^2$ such that $\int_{B_{1+\sqrt{2}}(l_n)} |(|u_n|^2 + |v_n|^2)\mathrm{d}x > \frac{\delta}{2}$. Let us define $\tilde{u}_n(x) = u_n(x+l_n)$ and $\tilde{v}_n(x) = v_n(x+l_n)$ so that

$$\int_{B_{1+\sqrt{2}}(0)} (|\tilde{u}_n|^2 + |\tilde{v}_n|^2)\mathrm{d}x \geq \frac{\delta}{2}. \tag{4.14}$$

Since $\lambda(x)$ and $f_i(x,u)$ are 1-periodic on x , we have $\|\tilde{u}_n\|_{V_1} = \|u_n\|_{V_1}$, $\|\tilde{v}_n\|_{V_2} = \|v_n\|_{V_2}$ and

$$\Phi(\tilde{u}_n,\tilde{v}_n) \to c^*, \qquad \|\Phi'(\tilde{u}_n,\tilde{v}_n)\|(1+\|(\tilde{u}_n,\tilde{v}_n)\|) \to 0. \tag{4.15}$$

Passing to a subsequence, we have $(\tilde{u}_n,\tilde{v}_n) \rightharpoonup (\tilde{u},\tilde{v})$ in E, $(\tilde{u}_n,\tilde{v}_n) \to (\tilde{u},\tilde{v})$ in $L^s_{loc}(\mathbb{R}^2), 2 \leq s \leq \infty$ and $(\tilde{u}_n,\tilde{v}_n) \to (\tilde{u},\tilde{v})$ a.e. on $\mathbb{R}^2$. Thus (4.14) implies that $(\tilde{u},\tilde{v}) \neq (0,0)$.

For any $\phi,\psi \in C_0^\infty(\mathbb{R}^2)$, let $\{e_n\}_{n=1}^\infty$ be the complete standard orthogonal basis of $C_0^\infty(\mathbb{R}^2)$, we have

$$\phi = \sum_{j=1}^\infty (\phi,e_j)e_j, \qquad \|\phi\|^2 = \sum_{j=1}^\infty |(\phi,e_j)|^2 \tag{4.16}$$

and

$$\psi = \sum_{j=1}^\infty (\psi,e_j)e_j, \qquad \|\psi\|^2 = \sum_{j=1}^\infty |(\psi,e_j)|^2. \tag{4.17}$$

Let

$$\phi_n = \sum_{j=1}^{k_n} (\phi,e_j)e_j, \qquad \tilde{\phi}_n = \sum_{j=k_n+1}^\infty (\phi,e_j)e_j \tag{4.18}$$

and

$$\psi_n = \sum_{j=1}^{k_n} (\psi,e_j)e_j, \qquad \tilde{\psi}_n = \sum_{j=k_n+1}^\infty (\psi,e_j)e_j. \tag{4.19}$$

For any given $\varepsilon > 0$, there holds

$$\int_{|\tilde{u}_n|\geq C_{10}\|\phi\|_\infty \varepsilon^{-1}} |f_1(x,\tilde{u}_n)\phi_n|\mathrm{d}x \leq \frac{\varepsilon}{C_{10}} \int_{|\tilde{u}_n|\geq C_{10}\|\phi\|_\infty \varepsilon^{-1}} f_1(x,\tilde{u}_n)\tilde{u}_n \mathrm{d}x < \varepsilon. \tag{4.20}$$

On the other hand, it follows from (F1) and (F2) that

$$\begin{aligned}\int_{|\tilde{u}_n|<C_{10}\|\phi\|_\infty\varepsilon^{-1}}|f_1(x,\tilde{u}_n)\tilde{\phi}_n|\mathrm{d}x &\le \int_{|\tilde{u}_n|<C_{10}\|\phi\|_\infty\varepsilon^{-1}}|u_n\tilde{\phi}_n|\mathrm{d}x + C_{11}\int_{|\tilde{u}_n|<C_{10}\|\phi\|_\infty\varepsilon^{-1}}(e^{\alpha u_n^2}-1)|\tilde{\phi}_n|\mathrm{d}x\\
&\le \left\{\|u_n\|_2 + C_{11}\left[\int_{\mathbb{R}^2}(e^{\alpha u_n^2}-1)^2\mathrm{d}x\right]^{\frac{1}{2}}\right\}\left(\int_{\mathbb{R}^2}\tilde{\phi}_n^2\mathrm{d}x\right)^{\frac{1}{2}}\\
&\le \left\{\|u_n\|_2 + C_{11}\left[\int_{\mathbb{R}^2}(e^{2\alpha u_n^2}-1)\mathrm{d}x\right]^{\frac{1}{2}}\right\}\|\tilde{\phi}_n\|_2\\
&\le \left\{\|u_n\|_2 + C_{11}\left[\int_{\mathbb{R}^2}\left(e^{2\alpha\rho_0^2\|u_n\|^2(\frac{u_n}{\rho_0\|u_n\|})^2}-1\right)\mathrm{d}x\right]^{\frac{1}{2}}\right\}\|\tilde{\phi}_n\|\\
&\le C_{12}\|\tilde{\phi}_n\| = o(1).\end{aligned} \tag{4.21}$$

Similarly, we have

$$\int_{|\tilde{v}_n|<C_{10}\|\psi\|_\infty\varepsilon^{-1}}|f_2(x,\tilde{v}_n)\tilde{\psi}_n|\mathrm{d}x = o(1). \tag{4.22}$$

From (4.20), (4.21) and (4.22), one has

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}f_1(x,\tilde{u}_n)\tilde{\phi}_n\mathrm{d}x = 0, \quad \lim_{n\to\infty}\int_{\mathbb{R}^2}f_2(x,\tilde{v}_n)\tilde{\psi}_n\mathrm{d}x = 0. \tag{4.23}$$

Due to the arbitrariness of $\varepsilon > 0$. Therefore, (2.4), (4.15) and (4.23) yield

$$\begin{aligned}\langle\Phi'(\tilde{u},\tilde{v}),(\phi,\psi)\rangle &= \int_{\mathbb{R}^2}(\nabla\tilde{u}\nabla\phi + V_1(x)\tilde{u}\phi)\,\mathrm{d}x + \int_{\mathbb{R}^2}(\nabla\tilde{v}\nabla\psi + V_2(x)\tilde{v}\psi)\,\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^2}\lambda(x)(\tilde{u}\psi + \tilde{v}\phi)\mathrm{d}x - \int_{\mathbb{R}^2}(f_1(x,\tilde{u})\phi + f_2(x,\tilde{v})\psi)\,\mathrm{d}x\\
&= \lim_{n\to\infty}\left[\int_{\mathbb{R}^2}(\nabla\tilde{u}_n\nabla\phi + V_1(x)\tilde{u}_n\phi)\,\mathrm{d}x + \int_{\mathbb{R}^2}(\nabla\tilde{v}_n\nabla\psi + V_2(x)\tilde{v}_n\psi)\,\mathrm{d}x\right.\\
&\quad\left. - \int_{\mathbb{R}^2}\lambda(x)(\tilde{u}_n\psi + \tilde{v}_n\phi)\mathrm{d}x - \int_{\mathbb{R}^2}(f_1(x,\tilde{u}_n)\phi + f_2(x,\tilde{v}_n)\psi)\,\mathrm{d}x\right]\\
&= \lim_{n\to\infty}\langle\Phi'(\tilde{u}_n,\tilde{v}_n),(\phi,\psi)\rangle\\
&= \lim_{n\to\infty}\left[\langle\Phi'(\tilde{u}_n,\tilde{v}_n)(\phi_n,\psi_n)\rangle + \langle\Phi'(\tilde{u}_n,\tilde{v}_n)(\tilde{\phi}_n,\tilde{\psi}_n)\rangle\right]\\
&= \lim_{n\to\infty}\langle\Phi'(\tilde{u}_n,\tilde{v}_n)(\tilde{\phi}_n,\tilde{\psi}_n)\rangle\\
&= \lim_{n\to\infty}\left\{\int_{\mathbb{R}^2}\left[(\nabla\tilde{u}_n\nabla\tilde{\phi}_n + \nabla\tilde{v}_n\nabla\tilde{\psi}_n) + V_1(x)\tilde{u}_n\tilde{\phi}_n + V_2(x)\tilde{v}_n\tilde{\psi}_n\right]\mathrm{d}x\right\}\\
&\quad - \lim_{n\to\infty}\int_{\mathbb{R}^2}\lambda(x)(\tilde{u}_n\tilde{\psi}_n + \tilde{v}_n\tilde{\phi}_n)\mathrm{d}x - \lim_{n\to\infty}\int_{\mathbb{R}^2}[f_1(x,\tilde{u}_n)\tilde{\phi}_n + f_2(x,\tilde{v}_n)\tilde{\psi}_n]\mathrm{d}x\\
&= -\lim_{n\to\infty}\int_{\mathbb{R}^2}[f_1(x,\tilde{u}_n)\tilde{\phi}_n + f_2(x,\tilde{v}_n)\tilde{\psi}_n]\mathrm{d}x\\
&= 0.\end{aligned}$$

It is easy to show that $\Phi'(\tilde{u},\tilde{v}) = 0$. Since $\lambda \neq 0$, then from (1.1) we know that $(\tilde{u},\tilde{v})$ is a nontrivial solution. □

Lemma 4.2. *The weak solution $(\tilde{u},\tilde{v})$ is a ground state solution.*

Proof. Since that $(\tilde{u},\tilde{v}) \neq (0,0)$ and $\Phi'(\tilde{u},\tilde{v}) = 0$, we have $(\tilde{u},\tilde{v}) \in \mathcal{N}$. Therefore $b \leq \Phi(\tilde{u},\tilde{v})$. On the other hand, it follows from (2.20) and Fatou's lemma that

$$\begin{aligned} b \geq \bar{c} + o(1) &= \Phi(\tilde{u}_n,\tilde{v}_n) - \frac{1}{2}\left\langle \Phi'(\tilde{u}_n,\tilde{v}_n),(\tilde{u}_n,\tilde{v}_n)\right\rangle \\ &= \int_{\mathbb{R}^2}\left[\frac{1}{2}f_1(x,\tilde{u}_n)\tilde{u}_n - F_1(x,\tilde{u}_n)\right]\mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{2}f_2(x,\tilde{v}_n)\tilde{v}_n - F_2(x,\tilde{v}_n)\right]\mathrm{d}x \\ &\geq \int_{\mathbb{R}^2}\left[\frac{1}{2}f_1(x,\tilde{u})\tilde{u} - F_1(x,\tilde{u})\right]\mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{2}f_2(x,\tilde{v})\tilde{v} - F_2(x,\tilde{v})\right]\mathrm{d}x + o_n(1) \\ &= \Phi(\tilde{u},\tilde{v}) - \frac{1}{2}\left\langle \Phi'(\tilde{u},\tilde{v}),(\tilde{u},\tilde{v})\right\rangle + o_n(1) \\ &= \Phi(\tilde{u},\tilde{v}) + o_n(1). \end{aligned} \tag{4.24}$$

Therefore $\Phi(\tilde{u},\tilde{v}) = b$.

We have proved that $(\tilde{u},\tilde{v})$ is a ground state solution for system (1.1). In order to seek a positive ground state, we note by assumptions (F1) and (F2) that

$$F_i(x,s) \leq F_i(x,|s|), \qquad \forall (x,s) \in \mathbb{R}^2 \times \mathbb{R}, \quad i = 1,2.$$

Thus, we can deduce that $\Phi(|\tilde{u}|,|\tilde{v}|) \leq \Phi(\tilde{u},\tilde{v})$. □

5 The regularity and asymptotic behavior

In this section, we use strong maximum principle to get a unique positive ground state solution, we will introduce methods to show that a weak solution of (1.1) is in fact smooth. Moreover, we establish a priori estimate in $W^{2,p}$ for the solution of system (1.1), we show that if the functions $f_1(x,u)$ and $f_2(x,v)$ are in $L^p_{loc}(\mathbb{R}^2)$, then $(u,v) \in W^{2,p}_{loc}(\mathbb{R}^2)$ is a strong solution of (1.1), that is, there exists a constant C such that

$$\|u\|_{W^{2,p}(B_R)} \leq C(\|u\|_{L^p(B_{2R})} + \|p_1(x)\|_{L^p(B_{2R})}), \qquad \|v\|_{W^{2,p}(B_R)} \leq C(\|v\|_{L^p(B_{2R})} + \|p_2(x)\|_{L^p(B_{2R})}),$$

where $p_i(x)$ can be defined as (5.3). We will establish this for a Newtonian potential, finally, we use a bootstrap regularity lifting methods to boost the regularity of solution. The bootstrap method can be found in [6, Subsection 3.3.1], which uses a lot of Sobolev imbedding to enhance the regularity of the weak solution repeatedly, finally, Schauder's estimate will lift the solution to be a classical solution.

Lemma 5.1. *There exists a positive ground state solution* $(\bar{u},\bar{v}) \in C^{1,\beta}_{loc}(\mathbb{R}^2) \times C^{1,\beta}_{loc}(\mathbb{R}^2)$ *for some* $\beta \in (0,1)$ *with the following asymptotic behavior*

$$\|\bar{u}\|_{C^{1,\beta}(\overline{B_R})} \to 0 \quad \text{and} \quad \|\bar{v}\|_{C^{1,\beta}(\overline{B_R})} \to 0, \quad \text{as } |x| \to \infty. \tag{5.1}$$

Proof. Let $(\tilde{u},\tilde{v}) \in E$ be the ground state obtained in Lemma 4.2 It follows from Lemma 2.4 that there exists a unique $t_0 > 0$ such that $(t_0|\tilde{u}|,t_0|\tilde{v}|) \in \mathcal{N}$. Moreover, since $(\tilde{u},\tilde{v}) \in \mathcal{N}$, we point out that $\max\limits_{t\geq 0}\Phi(t\tilde{u},t\tilde{v}) = \Phi(\tilde{u},\tilde{v})$. Thus we have that

$$\Phi(t_0|\tilde{u}|,t_0|\tilde{v}|) \leq \Phi(t_0\tilde{u},t_0\tilde{v}) \leq \max_{t\geq 0}\Phi(t\tilde{u},t\tilde{v}) = \Phi(\tilde{u},\tilde{v}) = b.$$

Therefore, $(t_0|\tilde{u}|,t_0|\tilde{v}|) \in \mathcal{N}$ is a nonnegative ground state solution for (1.1). Next, we denote $(\bar{u},\bar{v}) = (t_0|\tilde{u}|,t_0|\tilde{v}|)$. In order to use the strong maximum principle, we note that $-\bar{u} \in$

$H_{V_1}(\mathbb{R}^2) \setminus \{0\}$ and take $(\varphi, 0)$ as a test function. Here, $\varphi \in C_0^\infty(\mathbb{R}^2)$, $\varphi \geq 0$. Then we have

$$-\int_{\mathbb{R}^2} \nabla(-\bar{u})\nabla\varphi \mathrm{d}x - \int_{\mathbb{R}^2} V_1(x)(-\bar{u})\varphi \mathrm{d}x = \int_{\mathbb{R}^2} f(x,\bar{u})\varphi \mathrm{d}x + \int_{\mathbb{R}^2} \lambda(x)\bar{u}\varphi \mathrm{d}x \geq 0. \tag{5.2}$$

Moreover, since $V_1(x) > 0$, it follows that

$$-\int_{\mathbb{R}^2} V_1(x)\varphi \mathrm{d}x \leq 0, \qquad \forall \varphi \geq 0,\ \varphi \in C_0^1(\mathbb{R}^2).$$

Now suppose by contradiction that there exists $x_0 \in \mathbb{R}^2$ such that $\bar{u}(x_0) = 0$. Thus, since $-\bar{u} \leq 0$ in $\mathbb{R}^2$, for any $R > 0$ we have that

$$0 = \sup_{B_R(x_0)} (-\bar{u}) = \sup_{\mathbb{R}^2} (-\bar{u})$$

By the strong maximum principle we conclude that $-\bar{u} \equiv 0$ in $\mathbb{R}^2$, which is a contradiction. Therefore $\bar{u} > 0$ in $\mathbb{R}^2$. Similarly, we can prove that $\bar{v} > 0$ in $\mathbb{R}^2$. Therefore, $(\bar{u}, \bar{v})$ is positive.

In order to obtain the regularity, we use a bootstrap method. The ground state solution $(\bar{u}, \bar{v})$ is a weak solution of the restricted problem

$$\begin{cases} -\Delta\bar{u} = f_1(x,\bar{u}) + \lambda(x)\bar{v} - V_1(x)\bar{u} = p_1(x), & B_{2R}, \\ -\Delta\bar{v} = f_2(x,\bar{v}) + \lambda(x)\bar{u} - V_2(x)\bar{v} = p_2(x), & B_{2R}, \end{cases} \tag{5.3}$$

where and in the continuation $B_{2R} = B_{2R}(x) \subset \mathbb{R}^2$ denote the ball centered in a fixed point $x \in \mathbb{R}^2$. Since $V_i(x) \in C(\mathbb{R}^2)$, then $V_i(x), \lambda(x) \in L^\infty_{loc}(\mathbb{R}^2)$. For $\bar{u}, \bar{v} \in L^p(\mathbb{R}^2)$, $p \geq 2$, we have that $\lambda(x)\bar{v}, V_1(x)\bar{u} \in L^p(B_{2R})$ for all $p \geq 2$. By (F1) and (F2), for $\varepsilon > 0, p, q \geq 2, r > p$ and $\alpha > \alpha_1$, we have that

$$\begin{aligned} \int_{B_{2R}} |f_1(x,\bar{u})|^p \mathrm{d}x &\leq \int_{B_{2R}} |\varepsilon\bar{u} + C_\varepsilon(e^{\alpha\bar{u}^2} - 1)|\bar{u}|^{q-1}|^p \mathrm{d}x \\ &\leq C_{13}\int_{B_{2R}} \varepsilon^p|\bar{u}|^p \mathrm{d}x + C_{13}\int_{B_{2R}} C_\varepsilon^p(e^{\alpha\bar{u}^2} - 1)^p|\bar{u}|^{p(q-1)} \mathrm{d}x \\ &\leq C_{13}\varepsilon^p\|\bar{u}\|^p_{L^p(B_{2R})} + C_{13}\int_{B_{2R}} C_\varepsilon^p(e^{r\alpha\bar{u}^2} - 1)|\bar{u}|^{p(q-1)-1}|\bar{u}| \mathrm{d}x. \end{aligned} \tag{5.4}$$

By using Hölder's inequality, it follows from Lemma 1.1 that

$$\begin{aligned} \int_{B_{2R}} C_\varepsilon^p(e^{r\alpha\bar{u}^2} - 1)|\bar{u}|^{p(q-1)-1}|\bar{u}| \mathrm{d}x &\leq \left(\int_{B_{2R}} C_\varepsilon^{2p}(e^{r\alpha\bar{u}^2} - 1)^2|\bar{u}|^{2(p(q-1)-1)} \mathrm{d}x\right)^{\frac{1}{2}} \|\bar{u}\|_{L^2(B_{2R})} \\ &\leq C_{14}\|\bar{u}\|_{L^2(B_{2R})}. \end{aligned} \tag{5.5}$$

Thus, we have

$$\int_{B_{2R}} |f_1(x,\bar{u})|^p \mathrm{d}x \leq C_{13}\|\bar{u}\|^p_{L^p(B_{2R})} + C_{14}\|\bar{u}\|_{L^2(B_{2R})}. \tag{5.6}$$

Since the right-hand side is finite for all $p \geq 2$, we have that $f_1(x,\bar{u}) \in L^p(B_{2R})$ for all $p \geq 2$, together with $\lambda(x)\bar{v}, V_1(x)\bar{u} \in L^p(B_{2R})$, we have that $p_1(x) \in L^p(B_{2R})$ for all $p \geq 2$. Let f_{p_1} be the Newtonian potential of $p_1(x)$. In light of L^p-regularity theory [6, Theorem 3.1.1],

$$\Delta f_{p_1} = p_1(x), \quad x \in B_{2R}, \tag{5.7}$$

and $f_{p_1} \in W^{2,p}(B_{2R})$, for all $p \geq 2$. Combining (5.3) and (5.7) we deduce that

$$\int_{B_{2R}} \nabla(\bar{u} - f_{p_1})\phi \mathrm{d}x = 0, \quad \forall \phi \in C_0^\infty(B_{2R}).$$

which implies that $\bar{u} - f_{p1}$ is a weak solution of $-\Delta z = 0$ in B_{2R}. Since $\bar{u} - f_{p_1} \in W^{1,2}(B_{2R})$. It follows from Weyl's Lemma [13, Corollary 1.2.1] that $\bar{u} - f_{p_1} \in C^\infty(B_{2R})$. Therefore, $\bar{u} \in W^{2,p}(B_{2R}), \forall p \geq 2$. Noticing that $2/p < 2$, as $p > 2$. Thus, by Sobolev imbedding we obtain that $\bar{u} \in C^{1,\beta}(B_{2R})$, for some $\beta \in (0,1)$. The same argument can be used to prove that $\bar{v} \in C^{1,\beta}(B_{2R})$. By interior L^p-estimates [6, Theorem 3.1.2], we have that

$$\|\bar{u}\|_{W^{2,p}(B_R)} \leq C_{15}(\|\bar{u}\|_{L^p(B_{2R})} + \|p_1\|_{L^p(B_{2R})}). \tag{5.8}$$

On the other hand, by the Sobolev's imbedding theorem, there exists $C_{16} > 0$ such that

$$\|\bar{u}\|_{C^{1,\beta}(\overline{B_R})} \leq C_{16}\|\bar{u}\|_{W^{2,p}(B_R)}. \tag{5.9}$$

Therefore, it follows from (5.8) and (5.9), we deduce that

$$\|\bar{u}\|_{C^{1,\beta}(\overline{B_R})} \leq C_{17}(\|\bar{u}\|_{L^p(B_{2R})} + \|\bar{u}\|_{L^2(B_{2R})}).$$

Now we show that $\lim_{|x|\to\infty} \bar{u} = 0$. Suppose on the contrary that there exists $\{x_j\} \subset \mathbb{R}^2$ with $|x_j| \to \infty$ as $j \to \infty$ and $\liminf_{j\to\infty} \bar{u}(x_j) > 0$. Letting $w_j(x) = \bar{u}(x + x_j)$, then

$$-\Delta w_j + V_1(x + x_j)w_j = f_1(x + x_j, w_j) + \lambda(x + x_j)\bar{v}(x + x_j), \quad w_j \in H^1(\mathbb{R}^2). \tag{5.10}$$

Assume that $w_j \to w$ weakly in $H^1(\mathbb{R}^2)$. Then, by elliptic estimates we have $w \neq 0$. However, for fixed $R > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^2} \bar{u}^2 \mathrm{d}x &\geq \liminf_{j\to\infty} \Big(\int_{B_R(0)} \bar{u}^2 \mathrm{d}x + \int_{B_R(x_j)} \bar{u}^2 \mathrm{d}x \Big) \\
&= \int_{B_R(0)} \bar{u}^2 \mathrm{d}x + \liminf_{j\to\infty} \int_{B_R(0)} w_j^2 \mathrm{d}x \\
&= \int_{B_R(0)} \bar{u}^2 \mathrm{d}x + \int_{B_R(0)} w^2 \mathrm{d}x \\
&\to \int_{\mathbb{R}^2} \bar{u}^2 \mathrm{d}x + \int_{\mathbb{R}^2} w^2 \mathrm{d}x, \quad \text{as} \quad R \to \infty.
\end{aligned}$$

Which is a contradiction. Thus, letting $|x| \to \infty$, we get $\bar{u} \to 0$, therefore, $\|\bar{u}\|_{C^{1,\beta}(\overline{B_R})} \to 0$ as $|x| \to \infty$. Similarly, we can prove that $\|\bar{v}\|_{C^{1,\beta}(\overline{B_R})} \to 0$ as $|x| \to \infty$. □

Proof of Theorem 1.2. It follows from Lemmas 4.1, 4.2 and 5.1. □

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References

[1] N. Akhmediev, A. Ankiewicz, Novel soliton states and bifurcation phenomena in nonlinear fiber couplers, *Phys. Rev. Lett.* **70**(1993), No. 16, 2395–2398. https://doi.org/10.1103/PhysRevLett.70.2395; MR1212560; Zbl 1063.35526

[2] C. O. Alves, G. F. Germano , Ground state solution for a class of indefinite variational problems with critical growth, *J. Differential Equations* **265**(2018), No. 1, 444–477. https://doi.org/10.1016/j.jde.2018.02.039; MR3782550; Zbl 1394.35151

[3] A. Ambrosetti, G. Cerami, D. Ruiz , Solitons of linearly coupled systems of semilinear non-autonomous equations on $\mathbb{R}^n$, *J. Funct. Anal.* **254**(2008), No. 11, 2816–2845. https://doi.org/10.1016/j.jfa.2007.11.013; MR2414222; Zbl 1148.35080

[4] H. Berestycki, P. L. Lions, Nonlinear scalar field equations, I existence of a ground state, *Arch. Ration Mech. Anal.*, **82**(1983), No. 4, 313–345. https://doi.org/10.1007/BF00250555; MR0695535; Zbl 0533.35029

[5] D. M. Cao, Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$, *Commun. Partial Differential Equations* **17**(1992), No. 3-4, 407–435. https://doi.org/10.1080/03605309208820848; MR1163431; Zbl 0763.35034

[6] W. X. Chen, C. M. Li, *Methods on nonlinear elliptic equations*, AIMS Series on Differential Equations & Dynamical Systems, Vol. 4, American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2010. MR2759774; Zbl 1214.35023

[7] S. T. Chen, J. P. Shi, X. H. Tang, Ground state solutions of Nehari–Pohozaev type for the planar Schrödinger–Poisson system with general nonlinearity, *Discrete Contin. Dyn. Syst.* **39**(2019), No. 10, 58–67. https://doi.org/10.3934/dcds.2019257; MR4027016; Zbl 1425.35014

[8] S. T. Chen, X. H. Tang, On the planar Schrödinger equation with indefinite linear part and critical growth nonlinearity, *Calc. Var. Partial Differential Equations* **60**(2021), No. 3, 1–27. https://doi.org/10.1007/s00526-021-01963-1; MR4249873; Zbl 1465.35133

[9] Z. J. Chen, W. M. Zou, On coupled systems of Schrödinger equations, *Adv. Differential Equations* **16**(2011), No. 7–8, 775–800. MR2829504; Zbl 1232.35063

[10] Z. J. Chen, W. M. Zou, Ground states for a system of Schrödinger equations with critical exponent, *J. Funct. Anal.* **262**(2012), No. 7, 3091–3107. https://doi.org/10.1016/j.jfa.2012.01.001; MR3148116; Zbl 1234.35241

[11] Z. J. Chen, W. M. Zou, Standing waves for linearly coupled Schrödinger equations with critical exponent, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **31**(2014), No. 3, 429–447. https://doi.org/10.1016/j.anihpc.2013.04.003; MR3208449; Zbl 1300.35029

[12] Z. J. Chen, W. M. Zou, Standing waves for a coupled system of nonlinear Schrödinger equations, *Ann. Mat. Pur. Appl.* **194**(2015), No. 1, 183–220. https://doi.org/10.1007/s10231-013-0371-5; MR3303012; Zbl 1319.35236

[13] J. Jost, *Partial differential equations*, Graduate Texts in Mathematics, Vol. 214, Springer, New York, 2007. https://doi.org/10.1007/978-0-387-49319-0; MR1919991; Zbl 1121.35001

[14] G. B. Li, X. H. Tang, Nehari-type ground state solutions for Schrödinger equations including critical exponent, *Appl. Math. Lett.* **37**(2014), 101–106. https://doi.org/10.1016/j.aml.2014.06.003; MR3231735; Zbl 1320.35167

[15] J. M. do Ó, J. C. de Albuquerque, Positive ground state of coupled systems of Schrödinger equations in involving critical exponential growth, *Math. Methods Appl. Sci.* **40**(2017), No. 18, 6864–6879. https://doi.org/10.1002/mma.4498 MR3742101; Zbl 1387.35182

[16] J. M. do Ó, J. C. de Albuquerque, On coupled systems of nonlinear Schrödinger equations with critical exponential growth, *Appl. Anal.* **97**(2018), No. 6, 1000–1015. https://doi.org/10.1080/00036811.2017.1296951 MR3777854; Zbl 1392.35103

[17] J. M. do Ó, B. Ruf, On a Schrödinger equation with periodic potential and critical growth in $\mathbb{R}^2$, *NoDEA Nonlinear Differential Equations Appl.* **13**(2006), No. 2, 167–192 https://doi.org/10.1007/s00030-005-0034-3; MR2243709; Zbl 1146.35363

[18] D. D. Qin, X. H. Tang, J. Zhang, Ground states for planar Hamiltonian elliptic systems with critical exponential growth, *J. Differential Equations* **308**(2022), 763–776. https://doi.org/10.1016/j.jde.2021.10.063; MR4339599; Zbl 1478.35196

[19] X. H. Tang, Non-Nehari manifold method for asymptotically periodic Schrödinger equations, *Sci. China Math.* **58**(2015), No. 4, 715–728. https://doi.org/10.1007/s11425-014-4957-1; MR3319307; Zbl 1321.35055

[20] J. Y. Wei, X. Y. Lin, X. H. Tang, Ground state solutions for planar coupled system involving nonlinear Schrödinger equations with critical exponential growth, *Math. Methods Appl. Sci.* **44**(2021), No. 11, 9062–9078. https://doi.org/10.1002/mma.7335; MR4279837; Zbl 1472.35138