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Asymptotic behavior of multiple solutions for quasilinear Schrödinger equations

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Abstract. This paper establishes the multiplicity of solutions for a class of quasilinear Schrödinger elliptic equations:

$$-\Delta u + V(x)u - \frac{\gamma}{2}\Delta(u^2)u = f(x,u), \qquad x \in \mathbb{R}^3,$$

where $V(x) : \mathbb{R}^3 \to \mathbb{R}$ is a given potential and $\gamma > 0$. Furthermore, by the variational argument and L^∞-estimates, we are able to obtain the precise asymptotic behavior of these solutions as $\gamma \to 0^+$.

Keywords: quasilinear Schrödinger equations, variational methods, L^∞-estimate, asymptotic behavior.

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1 Introduction

This paper deals with multiplicity and asymptotic behavior of solitary wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t z = -\Delta z + W(x)z - l(x,|z|^2)z - \frac{\gamma}{2}[\Delta\rho(|z|^2)]\rho'(|z|^2)z, \tag{1.1}$$

where $z : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$, $W : \mathbb{R}^3 \to \mathbb{R}$ is a given potential, γ is a real constant and l, ρ are real functions. Quasilinear equations of the form (1.1) have been established in the past in several areas of physics with different types of ρ. For example, the case $\rho(t) = t$ was used in [18] for the superfluid film equation in plasma physics; the case $\rho(t) = (1+t)^{1/2}$ was considered for the self-channeling of a high-power ultrashort laser in matter, see [11] and [12]. These types of equations also appear in fluid mechanics [19], in the theory of Heidelberg ferromagnetism and magnus [20], in dissipative quantum mechanics [17] and in condensed matter theory [27].

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We now consider the case of the superfluid film equation in plasma physics, namely $\rho(t) = t$. If we look for standing waves, that is, solutions of the form $z(t,x) := \exp(-iEt)u(x)$ with $E > 0$, we are lead to investigate the following elliptic equation

$$-\Delta u + V(x)u - \frac{\gamma}{2}\Delta(u^2)u = f(x,u), \qquad x \in \mathbb{R}^3, \tag{1.2}$$

with $V(x) = W(x) - E$ and $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ given by $f(x,t) := l(x,|t|^2)t$ is a new nonlinear term. Later on, we shall pose precisely the hypotheses on V and f.

Taking $\gamma = 0$, the equation (1.2) is a semilinear case, scholars have obtained a large number of existence and multiplicity results based on variational methods, see e.g. [10, 14, 21, 22]. When $\gamma > 0$, the first existence of positive solutions is proved by Poppenberg, Schmitt and Wang in [28] with a constrained minimization argument. While a general existence result for (1.1) is due to Liu et al. in [25] through using of a change of variable to reformulate the quasilinear problem (1.2) to a semilinear one in an Orlicz space framework. Colin and Jeanjean in [13] used the same method of changing variables, but the classical Sobolev space $H^1(\mathbb{R}^N)$ was chosen. We refer the readers to [5, 26, 31, 33, 34] for more results. Recently, in [23], by using perturbation methods, Liu et al. proved the existence of nodal solutions for the general quasilinear problem in bounded domains.

In the above references mentioned, the γ in the quasilinear problem (1.2) was assumed to be a fixed constant. While, the constant γ represents several physical effect and is assumed to be small in some situation. This indicates the importance of the study of the asymptotic behavior of ground states as $\gamma \to 0^+$. But, asymptotic behavior of solutions for quasilinear Schrödinger equations is much less studied. In [1], Adachi et al. considered the problem for $N = 3$, $\lambda > 0$, $\gamma > 0$ and $f(x,s) = |s|^{p-2}s$ $(4 < p < 6)$:

$$-\Delta u + \lambda u - \frac{\gamma}{2}\Delta(u^2)u = |u|^{p-2}u, \qquad x \in \mathbb{R}^3. \tag{1.3}$$

They showed the ground states u_γ of (1.3) satisfies $u_\gamma \to u_0$ in $H^2(\mathbb{R}^3) \cap C^2(\mathbb{R}^3)$ as $\gamma \to 0^+$, where u_0 is a unique ground state of

$$-\Delta u + \lambda u = |u|^{p-2}u, \qquad x \in \mathbb{R}^3.$$

Then, in [34], Wang and Shen proved the asymptotic behavior of positive solutions for (1.3) when $p \in (2,4)$, which complemented the result given by Adachi et al. in [1]. By applying the blow-up analysis and the variational methods, in [2–4] Adachi et al. obtained the precise asymptotic behavior of ground states when $N \geq 3$ and the nonlinear term has H^1-critical growth or H^1-supercritical growth.

However, the work in the literature always assumed that $V(x) \equiv \lambda > 0$ and studied the asymptotic behavior of one ground state solution for (1.4). We are interested in the problem that whether or not we can find the multiplicity of solutions for (1.4) with some suitable potential conditions. Furthermore, as $\gamma \to 0^+$, whether these solutions have any asymptotic behavior. Specifically, the main purpose of the present paper is to solve the following three problems:

(Q_1) We have the multiplicity of solutions for (1.4) in unbounded domains, which complements the results given by Liu et al. in [23].

(Q_2) We obtain the asymptotic properties of solutions for (1.4) under some suitable potential conditions. Our result, in the sense that we do not need the restrictive conditions $V(x) \equiv \lambda > 0$, improves the one obtained in [1].

(Q_3) All the papers mentioned above only studied the asymptotic behavior of a positive ground state solution for (1.4). In this paper, we explore the asymptotic behavior of multiple solutions for quasilinear Schrödinger equations. More precisely, we can obtain the asymptotic behavior of sign-changing solution for (1.4).

For this purpose, we consider the multiplicity and asymptotic behavior of solutions for the following one-parameter family of elliptic equations with general nonlinearities:

$$-\Delta u + V(x)u - \frac{\gamma}{2}\Delta(u^2)u = f(x,u), \qquad x \in \mathbb{R}^3, \tag{1.4}$$

where $\gamma > 0$ and $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ satisfying:

(V_0) : $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^3$;

(V_1) : For any $M, r > 0$, there is a ball $B_r(y)$ centered at y with radius r such that

$$\mu(\{x \in B_r(y) : V(x) \leq M\}) \to 0, \quad \text{as } |y| \to \infty.$$

Remark 1.1. The condition (V_1) was firstly introduced by Bartsch, Pankov and Wang [8] to guarantee the compactness of embeddings of the work space. The limit of condition (V_1) can be replaced by one of the following simpler conditions:

(V_2) : $V(x) \in C(\mathbb{R}^3)$, $\mu(\{x \in \mathbb{R}^3 : V(x) \leq M\}) < \infty$ for any $M > 0$ (see [9]);

(V_3) : $V(x) \in C(\mathbb{R}^3)$, $V(x)$ is coercive, i.e., $\lim_{|x|\to\infty} V(x) = \infty$.

For the continuous nonlinearity f, we suppose that it satisfies the following conditions:

(f_1) : there exist a constant C and $p \in (4,6)$ such that

$$|f(x,t)| \leq C(1 + |t|^{p-1}), \quad \text{for all } x \in \mathbb{R}^3, t \in \mathbb{R};$$

(f_2) : $\lim_{t\to 0} \frac{f(x,t)}{t} = 0$ uniformly with respect to $x \in \mathbb{R}^3$;

(f_3) : there exists $\theta > 4$ such that

$$0 < \theta F(x,t) \leq t f(x,t), \quad \text{for all } x \in \mathbb{R}^3, t \neq 0,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

Note that (1.4) is the Euler–Lagrange equation associated to the natural energy functional:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (1 + \gamma u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx,$$

which is not well defined in $H^1(\mathbb{R}^3)$. Due to this fact, the usual variational methods can not be applied directly. This difficulty makes problem like (1.4) interesting and challenging. Inspired by the work of Shen [29], we first establish the existence of signed solutions for a modified quasilinear Schrödinger equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x,u), \qquad x \in \mathbb{R}^3, \tag{1.5}$$

where $g(t) = \sqrt{1 + \gamma t^2}$.

In what follows, instead of using the dual method, we search the existence of sign-changing solutions for the problem (1.4) via the perturbation method and invariant sets of descending flow.

For asymptotic behavior of solutions for the problem (1.4), arguments we apply are rather standard. Using a bootstrap argument, we obtain the uniform boundedness of L^∞-norm of u_γ. Then we apply the uniform estimates for the energies to show the strong convergence in $H^1_V(\mathbb{R}^3)$ ($H^1_V(\mathbb{R}^3)$ will be defined in Section 2), this is a key problem to the study.

Next, we give our main results.

Theorem 1.2. *Assume that* (V_0), (V_1), *and* (f_1)–(f_3) *hold. Then, for fixed* $\gamma \in (0,1]$, *the problem* (1.4) *has at least three solutions: a positive solution* $u_{\gamma,1}$, *a negative solution* $u_{\gamma,2}$ *and a sign-changing solution* $u_{\gamma,3}$.

Theorem 1.3. *For fixed* $\gamma \in (0,1]$, $u_{\gamma,i}$ $(i = 1,2,3)$ *are solutions of the problem* (1.4). *As* $\gamma \to 0^+$, *then passing to a subsequence, there exist* $u_i \in H^1_V(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ $(i = 1,2,3)$ *such that* $u_{\gamma,i} \to u_i$ *strongly in* $H^1_V(\mathbb{R}^3)$, *where* u_1 *is a positive solution of problem*

$$-\Delta u + V(x)u = f(x,u), \qquad x \in \mathbb{R}^3. \tag{1.6}$$

u_2 *is a negative solution of the problem* (1.6) *and* u_3 *is a sign-changing solution of the problem* (1.6).

Remark 1.4. In order to prove the existence of a sign-changing solution, we need a restriction $p > 4$ because of the degeneracy of the quasilinear term. Moreover we require that p is H^1-subcritical to prove the L^∞-norm of the solutions of (1.5) are uniformly bounded. Since $4 < \frac{2N}{N-2}$ if and only if $N < 4$. Hence we only show the asymptotic behavior of multiple solutions for the quasilinear Schrödinger for $N = 3$.

This paper is organized as follows. In Section 2, we describe the variational framework associated with the problem (1.4). We give the proofs of existence of signed and sign-changing solutions in Sections 3–4, respectively. Section 5 is devoted to the study of asymptotic behavior of solutions.

In what follows, C and C_i $(i = 1,2,\dots)$ denote positive generic constants. In this paper, the norms of $L^s(\mathbb{R}^N)(s \geq 1)$ is denoted by $|\cdot|_s$.

2 The modified problem

Let

$$H^1_V(\mathbb{R}^3) = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx < +\infty \right\}$$

with the inner product

$$\langle u,v\rangle_{H^1_V(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)\, dx$$

and the norm

$$\|u\|^2_{H^1_V} = \langle u,u\rangle_{H^1_V(\mathbb{R}^3)}.$$

From [9], we know that under the assumptions (V_0) and (V_1), the embedding $H^1_V(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is compact for each $s \in [2,6)$.

Note that (1.4) is the Euler–Lagrange equation associated to the natural energy functional:

$$I_\gamma(u) = \frac{1}{2}\int_{\mathbb{R}^3}(1+\gamma u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx,$$

which is not well defined in $H^1(\mathbb{R}^3)$ or $H^1_V(\mathbb{R}^3)$. Inspired by [13, 29, 30], we consider the following quasilinear Schrödinger equation:

$$-\mathrm{div}(g_\gamma^2(u)\nabla u) + g_\gamma(u)g_\gamma'(u)|\nabla u|^2 + V(x)u = f(x,u), \qquad x \in \mathbb{R}^3. \tag{2.1}$$

Here we choose $g_\gamma(t) : \mathbb{R} \to \mathbb{R}$ given by

$$g_\gamma(t) = \sqrt{1+\gamma t^2}.$$

It follows that $g_\gamma(t) \in C^1(\mathbb{R}, [1,\infty))$, increases in $[0,+\infty)$ and decreases in $(-\infty, 0]$.

Next, we set

$$G_\gamma(t) = \int_0^t g_\gamma(s)ds.$$

It is well known that $G_\gamma(t)$ is an odd function and inverse function $G_\gamma^{-1}(t)$ exists. Moreover, we summarize some properties of $G_\gamma^{-1}(t)$ as follows.

Lemma 2.1 ([30]).

(1) $\lim_{t\to 0} \frac{G_\gamma^{-1}(t)}{t} = 1$;

(2) $\lim_{t\to +\infty} \frac{G_\gamma^{-1}(t)}{t} = 0$;

(3) $\lim_{t\to +\infty} \frac{|G_\gamma^{-1}(t)|^2}{t} = \frac{2}{\sqrt{\gamma}}$;

(4) for all $t, s \in \mathbb{R}$, *then*

$$G_\gamma(s) \le g_\gamma(s)s, \qquad |G_\gamma^{-1}(t)| \le |t|;$$

(5) $0 \le \frac{s}{g_\gamma(s)} g_\gamma'(s) \le 1$, *for all* $s \in \mathbb{R}$;

(6) there exists a positive constant C *independent of* γ *such that*

$$|G_\gamma^{-1}(t)| \ge \begin{cases} C|t| & \text{if } |t| \le 1, \\ C|t|^{1/2} & \text{if } |t| \ge 1; \end{cases}$$

(7) there exists $\theta > 4$ *such that*

$$0 < \frac{\theta}{2} F(x,t) g_\gamma(t) \le G_\gamma(t) f(x,t), \quad \text{for all } x \in \mathbb{R}^3, t \ne 0.$$

In what follows, taking the change variable

$$v = G_\gamma(u) = \int_0^u g_\gamma(s)ds,$$

we observe that the functional $I_\gamma(u)$ can be written of the following way

$$J_\gamma(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G_\gamma^{-1}(v)|^2 dx - \int_{\mathbb{R}^3} F(x, G_\gamma^{-1}(v))dx.$$

From Lemma 2.1 and conditions (V_0), (V_1) and (f_1)–(f_3), we obtain the functional $J_\gamma(v)$ is well-defined in $H^1_V(\mathbb{R}^3)$, $J_\gamma \in C^1(H^1_V(\mathbb{R}^3), \mathbb{R})$ and

$$J_\gamma'(v)\varphi = \int_{\mathbb{R}^3} \nabla v \nabla \varphi dx + \int_{\mathbb{R}^3} V(x) \frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} \varphi dx - \int_{\mathbb{R}^3} \frac{f(x, G_\gamma^{-1}(v))}{g_\gamma(G_\gamma^{-1}(v))} \varphi dx,$$

for all $\varphi \in H_V^1(\mathbb{R}^3)$.

Moreover, the critical points of the functional J_γ correspond to the weak solutions of the following equation

$$-\Delta v + V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} = \frac{f(x, G_\gamma^{-1}(v))}{g_\gamma(G_\gamma^{-1}(v))}, \qquad x \in \mathbb{R}^3. \tag{2.2}$$

It is clear that if v is a critical point of J_γ, $u = G_\gamma^{-1}(v)$ is a critical point of I_γ, i.e. $u = G_\gamma^{-1}(v)$ is a solution of (1.4).

3 The existence of signed solutions

In this section we fix $1 \geq \gamma > 0$. Let $u_+ = \max\{u, 0\}$ and $u_- = \min\{u, 0\}$. Set

$$I_\gamma^{\pm}(u) = \frac{1}{2}\int_{\mathbb{R}^3}(1+\gamma u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx - \int_{\mathbb{R}^3} F(x, u_\pm)dx$$

and

$$J_\gamma^{\pm}(v) := I_\gamma^{\pm}(G_\gamma^{-1}(v)) = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G_\gamma^{-1}(v)|^2 dx - \int_{\mathbb{R}^3} F(x, (G_\gamma^{-1}(v))_\pm)dx.$$

Lemma 3.1. *Assume that (f_1)–(f_3), (V_0) and (V_1) hold. Then there exist $\rho > 0$ and $e \in H_V^1(\mathbb{R}^3)$ such that*

$$J_\gamma^{+}(v) > 0, \quad \textit{for } \|v\|_{H_V^1} = \rho,$$

and $J_\gamma^{+}(e) < 0$.

Proof. By conditions (f_1), (f_2) and $|G_\gamma^{-1}(s)| \leq |s|$, for $\delta > 0$ small enough, there exists $C_\delta > 0$ such that

$$|F(x, G_\gamma^{-1}(v)_+)| \leq \delta V(x)v^2 + C_\delta |v|^p, \quad \text{for all } x \in \mathbb{R}^3,$$

since we have

$$\lim_{|t|\to 0}\frac{G_\gamma^{-1}(t)}{t} = 1,$$

and

$$\lim_{|t|\to \infty}\frac{G_\gamma^{-1}(t)}{t} = 0.$$

Then, setting $H_\gamma(x,t) := -\frac{1}{2}V(x)|G_\gamma^{-1}(t)|^2 + F(x, (G_\gamma^{-1}(t))_+)$, it follows that

$$\lim_{t\to 0}\frac{H_\gamma(x,t)}{t^2} = -\frac{1}{2}V(x) < 0, \qquad \lim_{t\to +\infty}\frac{H_\gamma(x,t)}{t^6} = 0, \quad \text{for all } x \in \mathbb{R}^3$$

and we have

$$\begin{aligned}
J_\gamma^{+}(v) &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G_\gamma^{-1}(v)|^2 dx - \int_{\mathbb{R}^3} F(x, (G_\gamma^{-1}(v))_+)dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2 dx - \int_{\mathbb{R}^3} H_\gamma(x, v)dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2 dx + (\frac{1}{2} - \delta)\int_{\mathbb{R}^3} V(x)|v|^2 dx - C_\delta \int_{\mathbb{R}^3}|v|^6 dx \\
&\geq C\|v\|_{H_V^1}^2 - C\|v\|_{H_V^1}^6,
\end{aligned}$$

where we need sufficiently small $\delta > 0$ and the Sobolev inequality. Thus, it implies $J_\gamma^+(v)$ has local minimum at $v = 0$.

On the other hand, the condition (f_3) implies that

$$F(x,t) \geq Ct^\theta - C, \quad \text{for all } t > 0, x \in \mathbb{R}^3.$$

For $w \in C_0^\infty(\mathbb{R}^3)$ with $\text{supp}(w) = \overline{B_1}$ and $w(x) \geq 0$,

$$\begin{aligned} J_\gamma^+(tw) &= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla w|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G_\gamma^{-1}(tw)|^2 dx - \int_{\mathbb{R}^3} F(x,(G_\gamma^{-1}(tw))_+)dx \\ &\leq \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla w|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^3} V(x)|w|^2 dx - Ct^{\frac{\theta}{2}}\int_{\mathbb{R}^3} |w|^{\frac{\theta}{2}} dx - C. \end{aligned}$$

Since $\theta > 4$, it follows that $J_\gamma^+(tw) \to -\infty$ as $t \to \infty$. □

As a consequence of Lemma 3.1 and the Ambrosetti–Rabinowitz Mountain Pass Theorem, for the constant

$$d_\gamma = \inf_{\eta \in \Gamma} \sup_{t \in [0,1]} J_\gamma^+(\eta(t)),$$

where

$$\Gamma = \{\eta : \ \eta \in C([0,1], H_V^1(\mathbb{R}^3)), \eta(0) = 0, J_\gamma^+(\eta(1)) < 0\},$$

there exists a Palais–Smale sequence $\{v_n\}$ at level d_γ, that is $J_\gamma^+(v_n) \to d_\gamma$ and $(J_\gamma^+)'(v_n) \to 0$, as $n \to \infty$.

Lemma 3.2. *Assume that (f_1)–(f_3), (V_0) and (V_1) hold. Then the Palais–Smale sequence of J_γ^+ is bounded.*

Proof. Let $\{v_n\} \subset H_V^1(\mathbb{R}^3)$ be a Palais–Smale sequence. Then

$$\begin{aligned} J_\gamma^+(v_n) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G_\gamma^{-1}(v_n)|^2 dx - \int_{\mathbb{R}^3} F(x,(G_\gamma^{-1}(v_n))_+)dx \\ &= d_\gamma + o_n(1) \end{aligned} \tag{3.1}$$

and for any $\varphi \in H_V^1(\mathbb{R}^3)$, $\langle (J_\gamma^+)'(v_n), \varphi\rangle = o_n(1)\|\varphi\|_{H_V^1}$, that is

$$\int_{\mathbb{R}^3}\left(\nabla v_n \nabla\varphi + V(x)\frac{G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}\varphi\right)dx - \int_{\mathbb{R}^3}\frac{f(x,(G_\gamma^{-1}(v_n))_+)}{g(G_\gamma^{-1}(v_n))}\varphi dx = o_n(1)\|\varphi\|_{H_V^1}. \tag{3.2}$$

Fixing $\varphi = v_n$, we deduce that

$$\begin{aligned} o_n(1)\|v_n\|_{H_V^1} &= \langle (J_\gamma^+)'(v_n), v_n\rangle \\ &= \int_{\mathbb{R}^3}\left(|\nabla v_n|^2 + V(x)\frac{G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}v_n\right)dx \\ &\quad - \int_{\mathbb{R}^3}\frac{f(x,(G_\gamma^{-1}(v_n))_+)}{g_\gamma(G_\gamma^{-1}(v_n))}v_n dx. \end{aligned} \tag{3.3}$$

Therefore, by (3.1)–(3.3) and Lemma 2.1-(7), we have

$$\begin{aligned}\frac{\theta}{2}d_\gamma + o_n(1) + o_n(1)\|v_n\|_{H^1_V} &= \frac{\theta}{2}J^+_\gamma(v_n) - \langle (J^+_\gamma)'(v_n), v_n\rangle \\ &\geq \frac{\theta-4}{4}\int_{\mathbb{R}^3}|\nabla v_n|^2dx \\ &\quad + \int_{\mathbb{R}^3} V(x)G^{-1}_\gamma(v_n)\left(\frac{\theta G^{-1}_\gamma(v_n)}{4} - \frac{1}{g_\gamma(G^{-1}_\gamma(v_n))}v_n\right)dx \\ &\quad - \int_{\mathbb{R}^3}\left(\frac{\theta}{2}F(x,(G^{-1}_\gamma(v_n))^+) - \frac{f(x,(G^{-1}_\gamma(v_n))^+)}{g_\gamma(G^{-1}_\gamma(v_n))}v_n\right)dx \\ &\geq \frac{\theta-4}{4}\left(\int_{\mathbb{R}^3}|\nabla v_n|^2dx + \int_{\mathbb{R}^3}V(x)(G^{-1}_\gamma(v_n))^2dx\right).\end{aligned}$$

Next, we will prove that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^3}\left(|\nabla v_n|^2 + V(x)(G^{-1}_\gamma(v_n))^2\right)dx \geq C\|v_n\|^2_{H^1_V}.$$

Otherwise, there exists a sequence $\{v_{n_k}\} \subset H^1_V(\mathbb{R}^3)$ such that

$$A_k^2 := \int_{\mathbb{R}^3}\left(|\nabla v_{n_k}|^2 + V(x)(G^{-1}_\gamma(v_{n_k}))^2\right)dx < \frac{1}{k}\|v_{n_k}\|^2_{H^1_V}. \tag{3.4}$$

Hence, by (3.4), $\frac{A_k^2}{\|v_{n_k}\|^2_{H^1_V}} \to 0$. Consequently, in Lemma 2.4 of [30], we get a contradiction. This shows that $\|v_n\|_{H^1_V} < +\infty$. □

Lemma 3.3. *Assume that (f_1)–(f_3), (V_0) and (V_1) hold. Then J^+_γ has a positive critical point.*

Proof. First, we show that the sequence $\{v_n\}$ possesses a convergent subsequence in $H^1_V(\mathbb{R}^3)$. Indeed, by the boundedness of $\{v_n\}$ and the compactness of embedding $H^1_V(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 \leq s < 6)$, up to subsequence, one has $v_n \rightharpoonup v$ weakly in $H^1_V(\mathbb{R}^3)$, $v_n \to v$ strongly in $L^s(\mathbb{R}^3)$ for all $s \in [2, 6)$ and $v_n(x) \to v(x)$ a.e. on $\mathbb{R}^3$.

By conditions (f_1), (f_2), Lemma 2.1-(4) and $g_\gamma(s) \geq 1$, one has

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\left(\frac{f(x,(G^{-1}_\gamma(v_n))_+)}{g_\gamma(G^{-1}_\gamma(v_n))} - \frac{f(x,(G^{-1}_\gamma(v))_+)}{g_\gamma(G^{-1}_\gamma(v))}\right)(v_n - v)dx\right| \\ &\quad\leq C\int_{\mathbb{R}^3}\left(|G^{-1}_\gamma(v_n)| + |G^{-1}_\gamma(v_n)|^{p-1} + |G^{-1}_\gamma(v)| + |G^{-1}_\gamma(v)|^{p-1}\right)|v_n - v|dx \\ &\quad\leq C\int_{\mathbb{R}^3}\left(|v_n| + |v_n|^{p-1} + |v| + |v|^{p-1}\right)|v_n - v|dx \\ &\quad\leq C\left((|v_n|_2 + |v|_2)|v_n - v|_2 + (|v_n|_p^{p-1} + |v|_p^{p-1})|v_n - v|_p\right).\end{aligned} \tag{3.5}$$

On the other hand, as in Lemma 2.5 of [30], we know that

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(|\nabla(v_n - v)|^2 + V(x)\left(\frac{G^{-1}_\gamma(v_n)}{g_\gamma(G^{-1}_\gamma(v_n))} - \frac{G^{-1}_\gamma(v)}{g_\gamma(G^{-1}_\gamma(v))}\right)(v_n - v)\right)dx \\ &\quad\geq C\|v_n - v\|^2_{H^1_V}.\end{aligned} \tag{3.6}$$

By virtue of (3.5) and (3.6), we have

$$
\begin{aligned}
o(1) &= \langle (J_\gamma^+)'(v_n) - (J_\gamma^+)'(v), v_n - v\rangle \\
&= \int_{\mathbb{R}^3} \left(|\nabla (v_n - v)|^2 + V(x) \left(\frac{G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))} - \frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} \right) (v_n - v) \right) dx \\
&\quad - \int_{\mathbb{R}^3} \left(\frac{f(x,(G_\gamma^{-1}(v_n))_+)}{g_\gamma(G_\gamma^{-1}(v_n))} - \frac{f(x,(G_\gamma^{-1}(v))_+)}{g_\gamma(G_\gamma^{-1}(v))} \right) (v_n - v) dx \\
&\geq C\|v_n - v\|_{H_V^1}^2 + o(1).
\end{aligned}
$$

This implies $v_n \to v$ strongly in $H_V^1(\mathbb{R}^3)$. By standard regular arguments, the weak limit v of $\{v_n\}$ is a critical point of J_γ^+. Furthermore, from $v_n \to v$ strongly in $H_V^1(\mathbb{R}^3)$ and v can be shown to be positive critical point of J_γ by applying the maximum principle in [16]. Hence, $u = G_\gamma^{-1}(v)$ is a positive weak solution of (1.4). By the similar argument, we know that the equation (1.4) also has a negative weak solution. □

The next two results establish the uniform boundedness of H_V^1-norm of v_γ. This important estimate will be used in Section 5.

Lemma 3.4. *Assume that (f_1)–(f_3), (V_0) and (V_1) hold. Let v_γ be a critical point of J_γ^+ with $J_\gamma^+(v_\gamma) = d_\gamma$. Then there exists $C > 0$ (independent of γ) such that*

$$\|v_\gamma\|_{H_V^1}^2 \leq C d_\gamma. \tag{3.7}$$

Proof. Let v_γ be a critical point of J_γ^+. Similar with Lemma 3.2, we get the following estimates

$$
\begin{aligned}
\frac{\theta}{2} d_\gamma &= \frac{\theta}{2} J_\gamma^+(v_\gamma) - \langle (J_\gamma^+)'(v_\gamma), v_\gamma\rangle \\
&\geq \frac{\theta - 4}{4} \int_{\mathbb{R}^3} |\nabla v_\gamma|^2 dx \\
&\quad + \int_{\mathbb{R}^3} V(x) G_\gamma^{-1}(v_\gamma) \left(\frac{\theta G_\gamma^{-1}(v_\gamma)}{4} - \frac{1}{g_\gamma(G_\gamma^{-1}(v_\gamma))} v_\gamma \right) dx \\
&\quad - \int_{\mathbb{R}^3} \left(\frac{\theta}{2} F(x,(G_\gamma^{-1}(v_\gamma))_+) - \frac{f(x,(G_\gamma^{-1}(v_\gamma))_+)}{g_\gamma(G_\gamma^{-1}(v_\gamma))} v_\gamma \right) dx \\
&\geq \frac{\theta - 4}{4} \left(\int_{\mathbb{R}^3} |\nabla v_\gamma|^2 dx + \int_{\mathbb{R}^3} V(x) (G_\gamma^{-1}(v_\gamma))^2 dx \right) \\
&\geq C\|v_\gamma\|_{H_V^1}^2,
\end{aligned}
$$

which implies $\|v_\gamma\|_{H_V^1}^2 \leq C d_\gamma$. □

Lemma 3.5. *Assume $\gamma \in [0, 1]$. Then there exist positive constants m_1, m_2 (independent on γ), such that*

$$m_1 \leq J_\gamma^+(v_\gamma) \leq m_2,$$

where v_γ is a positive critical point of J_γ^+.

Proof. For $\rho > 0$, let

$$\Sigma_\rho = \left\{ v \in H_V^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(|\nabla v|^2 + V(x) v^2 \right) dx \leq \rho^2 \right\}.$$

Similar with Lemma 3.1, we have

$$\begin{aligned} J_\gamma^+(v) &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V(x)|G_\gamma^{-1}(v)|^2dx - \int_{\mathbb{R}^3}F(x,(G_\gamma^{-1}(v))_+)dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx - \int_{\mathbb{R}^3}H_\gamma(x,v)dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx + \left(\frac{1}{2}-\delta\right)\int_{\mathbb{R}^3}V(x)|v|^2dx - C_\delta\int_{\mathbb{R}^3}|v|^6dx \\ &\geq C\|v\|_{H_V^1}^2 - C\|v\|_{H_V^1}^6, \end{aligned}$$

where we need sufficiently small $\delta > 0$ and the Sobolev inequality. Thus, if $v \in \partial\Sigma_\rho$, take ρ small enough, it implies that $J_\gamma^+(v) \geq C\rho^2 := m_1$, where m_1 does not depend on γ.

Note that

$$J_\gamma^+(v_\gamma) = \inf_{\eta\in\Gamma}\sup_{t\in[0,1]} J_\gamma^+(\eta(t)),$$

where

$$\Gamma = \{\eta:\ \eta \in C([0,1], H_V^1(\mathbb{R}^3)),\ \eta(0) = 0,\ J_\gamma^+(\eta(1)) < 0\}.$$

Since any path $\eta(t) \in \Gamma$ always passes though $\partial\Sigma_\rho$, then

$$J_\gamma^+(v_\gamma) = \inf_{\eta\in\Gamma}\sup_{t\in[0,1]} J_\gamma^+(\eta(t)) \geq \inf_{v\in\partial\Sigma_\rho} J_\gamma^+(v) \geq m_1.$$

Take $\varphi \in C_0^\infty(\mathbb{R}^3)$, $\varphi \geq 0$, and define a path $h : [0,1] \to H_V^1(\mathbb{R}^3)$ by $h(t) = tT\varphi$, where the constant $T > 0$. For T large enough, we have

$$J_\gamma^+(h(1)) \leq J_1^+(h(1)) < 0, \qquad \int_{\mathbb{R}^3}|\nabla h(1)|^2 + V(x)(G_\gamma^{-1}(h(1)))^2dx > \rho^2.$$

Due to $h(t) \in \Gamma$, then we get

$$J_\gamma^+(v_\gamma) \leq \sup_{t\in[0,1]} J_\gamma^+(h(t)) \leq \sup_{t\in[0,1]} J_1^+(h(t)) := m_2,$$

where m_2 does not depend on γ. □

4 The existence of sign-changing solutions

The goal of this section is to consider the existence of sign-changing solutions. To do this, we define the work space E as follows

$$E = W^{1,4}(\mathbb{R}^3) \cap H_V^1(\mathbb{R}^3),$$

where

$$H_V^1(\mathbb{R}^3) := \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2dx < +\infty\right\},$$

which endowed with the norm

$$\|u\|_{H_V^1} = \left(\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)dx\right)^{1/2}$$

and $W^{1,4}(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_W = \left(\int_{\mathbb{R}^3} (|\nabla u|^4 + u^4) dx \right)^{1/4}.$$

The norm of E is denoted by

$$\|u\|_E = \|u\|_W + \|u\|_{H^1_V}.$$

Remark 4.1. It is noteworthy that the embedding from $H^1_V(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$ is compact (see [9]). Applying the interpolation inequality, we obtain that the embedding from E into $L^s(\mathbb{R}^3)$ for $2 \leq s < 12$ is compact.

In what follows, we formally formulate (1.4) in variational structure as follows

$$I_\gamma(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 + \gamma u^2 |\nabla u|^2 \right) dx - \int_{\mathbb{R}^3} F(x,u) dx. \tag{4.1}$$

If $u \in H^1_V(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is a weak solution of (1.4), that is, for all $\varphi \in C_0^\infty(\mathbb{R}^3)$ the following equation holds

$$\int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(x) u \varphi) \, dx + \gamma \int_{\mathbb{R}^3} u^2 \nabla u \nabla \varphi dx + \gamma \int_{\mathbb{R}^3} |\nabla u|^2 u \varphi dx - \int_{\mathbb{R}^3} f(x,u) \varphi dx = 0. \tag{4.2}$$

Notice that I_γ is an ill-behaved functional in $H^1_V(\mathbb{R}^3)$. To avoid this difficulty, in the sequel, for each $\mu, \gamma > 0$ fixed, let us consider the perturbation functional $I_{\mu,\gamma} : E \to \mathbb{R}$ associated with (1.4) given by

$$I_{\mu,\gamma}(u) = \frac{\mu}{4} \int_{\mathbb{R}^3} \left(|\nabla u|^4 + u^4 \right) dx + I_\gamma(u). \tag{4.3}$$

By deducing as in [15] (see also [23]), it is normal to verify that $I_{\mu,\gamma} \in C^1(E, \mathbb{R})$ and for each $\varphi \in E$, we get

$$\begin{aligned}\langle I'_{\mu,\gamma}(u), \varphi \rangle = \mu \int_{\mathbb{R}^3} \left(|\nabla u|^2 \nabla u \nabla \varphi + u^3 \varphi \right) dx + \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(x) u \varphi) \, dx \\ + \gamma \int_{\mathbb{R}^3} u^2 \nabla u \nabla \varphi dx + \gamma \int_{\mathbb{R}^3} |\nabla u|^2 u \varphi dx - \int_{\mathbb{R}^3} f(x,u) \varphi dx.\end{aligned} \tag{4.4}$$

In the following, we prove a compactness condition for $I_{\mu,\gamma}$.

Lemma 4.2. *For $\mu, \gamma > 0$ fixed, then $I_{\mu,\gamma}$ satisfies the (PS) conditions.*

Proof. Let $\{u_n\} \subset E$ be a (PS) sequence for $I_{\mu,\gamma}$, that is $\{u_n\}$ satisfies:

$$|I_{\mu,\gamma}(u_n)| \leq c \quad \text{and} \quad I'_{\mu,\gamma}(u_n) \to 0 \quad \text{as } n \to \infty.$$

Consider

$$\begin{aligned} & I_{\mu,\gamma}(u_n) - \frac{1}{\theta} \langle I'_{\mu,\gamma}(u_n), u_n \rangle \\ &= \left(\frac{\mu}{4} - \frac{\mu}{\theta} \right) \|u_n\|_W^4 + \left(\frac{1}{2} - \frac{1}{\theta} \right) \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + V(x) u_n^2 \right) dx \\ &\quad + \left(\frac{1}{2} - \frac{2}{\theta} \right) \gamma \int_{\mathbb{R}^3} |\nabla u_n|^2 u_n^2 dx + \int_{\mathbb{R}^3} \left(\frac{1}{\theta} u_n f(x,u_n) - F(x,u_n) \right) dx \\ &\geq \left(\frac{\mu}{4} - \frac{\mu}{\theta} \right) \|u_n\|_W^4 + \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u_n\|^2_{H^1_V}, \end{aligned}$$

which deduces that $\{u_n\}$ is bounded in E.

By a standard argument, we can prove that every bounded (PS) sequence $\{u_n\} \subset E$ of $I_{\mu,\gamma}$ possesses a convergent subsequence, cf. [15]. This completes the proof. □

In the following, we would like to construct a descending flow guaranteeing existence of desired invariant sets for the functional $I_{\mu,\gamma}$. For this purpose, we introduce an auxiliary operator $\mathcal{A} : E \to E$, $u \mapsto \mathcal{A}u := v$ satisfies

$$\langle J'_{\mu,\gamma}(v), \omega \rangle = C_0 \int_{\mathbb{R}^3} u^3 \omega dx + \int_{\mathbb{R}^3} f(x,u)\omega dx, \quad \text{for all } \omega \in E, \tag{4.5}$$

where

$$J_{\mu,\gamma}(v) = \frac{\mu}{4} \int_{\mathbb{R}^3} \left(|\nabla v|^4 + v^4 \right) dx + \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla v|^2 + V(x)v^2 + \gamma v^2 |\nabla v|^2 \right) dx + \frac{C_0}{4} \int_{\mathbb{R}^3} v^4 dx,$$

and $C_0 > 0$ large enough. It is normal to verify that $J_{\mu,\gamma} \in C^1(E, \mathbb{R})$ and for all $\omega \in E$ we have

$$\begin{aligned} \langle J'_{\mu,\gamma}(v), \omega \rangle = \mu \int_{\mathbb{R}^3} \left(|\nabla v|^2 \nabla v \nabla \omega + v^3 \omega \right) dx + \int_{\mathbb{R}^3} \left(\nabla v \nabla \omega + V(x) v \omega \right) dx \\ + \gamma \int_{\mathbb{R}^3} \left(|\nabla v|^2 v \omega + v^2 \nabla v \nabla \omega \right) dx + C_0 \int_{\mathbb{R}^3} v^3 \omega dx. \end{aligned}$$

Clearly, we notice that the following two statements are equivalent:

$$u \text{ is a fixed point of } \mathcal{A} \text{ and } u \text{ is a critical point of } I_{\mu,\gamma}.$$

Lemma 4.3. *For fixed $\mu \in (0,1]$ and $\gamma > 0$, the operator $u \mapsto v = \mathcal{A}u$ is well defined and continuous. Moreover, there exist constants $c_1, c_2, c_3 > 0$ such that*

(1) $\|I'_{\mu,\gamma}(u)\|_{E^*} \le c_1(\|u\|_W^2 + \|\mathcal{A}u\|_W^2)\|u - \mathcal{A}u\|_W + c_2\|u - \mathcal{A}u\|_{H^1_V}$;

(2) $\langle I'_{\mu,\gamma}(u), u - \mathcal{A}u \rangle \ge c_3(\|u - \mathcal{A}u\|_W^4 + \|u - \mathcal{A}u\|_{H^1_V}^2)$;

(3) for all $u \in I^{-1}_{\mu,\gamma}([a,b])$, if $\|I'_{\mu,\gamma}(u)\|_{E^*} \ge \alpha > 0$, then there exists $\delta > 0$ such that $\|u - \mathcal{A}u\|_E \ge \delta$.

Proof. To prove the operator $u \mapsto v = \mathcal{A}u$ is well defined and continuous, we consider

$$\begin{aligned} \Phi_{\mu,\gamma}(v) = \frac{\mu}{4} \int_{\mathbb{R}^3} \left(|\nabla v|^4 + v^4 \right) dx + \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla v|^2 + V(x)v^2 + \gamma v^2 |\nabla v|^2 \right) dx \\ + \frac{C_0}{4} \int_{\mathbb{R}^3} v^4 dx - \frac{C_0}{4} \int_{\mathbb{R}^3} u^3 v dx - \int_{\mathbb{R}^3} f(x,u) v dx, \quad \text{for all } v \in E. \end{aligned}$$

Obviously, $\Phi_{\mu,\gamma} \in C^1(E, \mathbb{R})$. And one can see that $\Phi_{\mu,\gamma}$ is weakly lower semicontinuous.

From conditions (f_1), (f_2) and the Sobolev embeddings theorem, for any $\delta > 0$, there exists C_δ, such that

$$\int_{\mathbb{R}^3} \left(\frac{C_0}{4} u^3 + f(x,u) \right) v dx \le \frac{C_0}{4} |u|_6^3 |v|_2 + \delta |u|_2 |v|_2 + C_\delta |u|_p^{p-1} |v|_p \le C\|v\|_E.$$

This deduces

$$\Phi_{\mu,\gamma}(v) \ge C(\|v\|_W^4 + \|v\|_{H^1_V}^2) - C\|v\|_E \to +\infty, \quad \text{as } \|v\|_E \to +\infty.$$

Therefore, the functional $\Phi_{\mu,\gamma}$ is coercive. We can see that the functional $\Phi_{\mu,\gamma}$ is bounded from below and maps bounded sets into bounded sets. In the following, we shall prove that the functional $\Phi_{\mu,\gamma}$ is also strictly convex. In fact, since

$$\begin{aligned}&\langle \Phi'_{\mu,\gamma}(v)-\Phi'_{\mu,\gamma}(\omega), v-\omega\rangle\\ &\quad= 3\mu\int_0^1\int_{\mathbb{R}^3}|\nabla\theta_t|^2|\nabla(v-\omega)|^2dxdt+3\mu\int_0^1\int_{\mathbb{R}^3}\theta_t^2(v-\omega)^2dxdt\\ &\qquad+\int_{\mathbb{R}^3}(|\nabla(v-\omega)|^2+V(x)(v-\omega)^2)dx+4\gamma\int_0^1\int_{\mathbb{R}^3}\nabla\theta_t\nabla(v-\omega)\theta_t(v-\omega)dxdt\\ &\qquad+\gamma\int_0^1\int_{\mathbb{R}^3}|\nabla\theta_t|^2(v-\omega)^2dxdt+\gamma\int_0^1\int_{\mathbb{R}^3}\theta_t^2|\nabla(v-\omega)|^2dxdt\\ &\qquad+3C_0\int_0^1\int_{\mathbb{R}^3}\theta_t^2(v-\omega)^2dxdt,\end{aligned}$$

where $\theta_t = tv+(1-t)\omega$ $(t\in(0,1))$. By Young's inequality, for any $\delta>0$, there exists $C_\delta>0$, such that

$$\begin{aligned}&\left|4\gamma\int_0^1\int_{\mathbb{R}^3}\nabla\theta_t\nabla(v-\omega)\theta_t(v-\omega)dxdt\right|\\ &\qquad\le\delta\int_0^1\int_{\mathbb{R}^3}|\nabla\theta_t|^2|\nabla(v-\omega)|^2dxdt+C_\delta\int_0^1\int_{\mathbb{R}^3}\theta_t^2(v-\omega)^2dxdt.\end{aligned}$$

Taking $\delta=\frac{3\mu}{2}$ and choosing $C_0>\frac{C_{\frac{3\mu}{2}}}{3}$, if $v\neq\omega$, we get

$$\begin{aligned}&\langle \Phi'_{\mu,\gamma}(v)-\Phi'_{\mu,\gamma}(\omega), v-\omega\rangle\\ &\quad\ge\frac{\mu}{2}\int_{\mathbb{R}^3}\big((|\nabla v|^2\nabla v-|\nabla\omega|^2\nabla\omega)\nabla(v-\omega)+(v^3-\omega^3)(v-\omega)\big)\,dx\\ &\qquad+\int_{\mathbb{R}^3}\big(|\nabla(v-\omega)|^2+V(x)(v-\omega)^2\big)\,dx\\ &\quad\ge C(\|v-\omega\|_W^4+\|v-\omega\|_{H_V^1}^2)\\ &\quad>0.\end{aligned}\tag{4.6}$$

From the above analysis, we obtain that the functional $\Phi_{\mu,\gamma}$ is coercive, bounded below, weakly lower semicontinuous and strictly convex. Thus, the functional $\Phi_{\mu,\gamma}$ admits a unique minimizer $v=\mathcal{A}(u)$. Moreover, the operator $\mathcal{A}$ maps bounded sets into bounded sets.

Next, we will verify the continuity of the operator $\mathcal{A}$ on E. To prove this, let

$$K(u)=\frac{C_0}{4}\int_{\mathbb{R}^3}u^4dx+\int_{\mathbb{R}^3}F(x,u)dx.$$

If $\{u_n\}\subset E$ satisfying $u_n\to u$ strongly in E, setting $v=\mathcal{A}(u)$ and $v_n=\mathcal{A}(u_n)$, then we can obtain

$$\langle J'_{\mu,\gamma}(v_n)-J'_{\mu,\gamma}(v),\omega\rangle=\langle K'(u_n)-K'(u),\omega\rangle,\quad\text{for all }\omega\in E.\tag{4.7}$$

Furthermore, by the similar estimates of (4.6), for C_0 large enough, we get

$$\begin{aligned}&\langle J'_{\mu,\gamma}(v_n)-J'_{\mu,\gamma}(v), v_n-v\rangle\\ &\quad\ge\frac{\mu}{2}\int_{\mathbb{R}^3}\big((|\nabla v_n|^2\nabla v_n-|\nabla v|^2\nabla v)\nabla(v_n-v)+(v_n^3-v^3)(v_n-v)\big)\,dx\\ &\qquad+\int_{\mathbb{R}^3}\big(|\nabla(v_n-v)|^2+V(x)(v_n-v)^2\big)\,dx\\ &\quad\ge C(\|v_n-v\|_W^4+\|v_n-v\|_{H_V^1}^2).\end{aligned}\tag{4.8}$$

Then, combining (4.7) with (4.8), we have

$$\begin{aligned} C(\|v_n - v\|_W^4 + \|v_n - v\|_{H_V^1}^2) &\le \langle J'_{\mu,\gamma}(v_n) - J'_{\mu,\gamma}(v), v_n - v\rangle \\ &= \langle K'(u_n) - K'(u), v_n - v\rangle \\ &\le \|K'(u_n) - K'(u)\|_{E^*}\|v_n - v\|_E. \end{aligned}$$

Since $K \in C^1(E, \mathbb{R})$ and $u_n \to u$ strongly in E, we get that $v_n \to v$ strongly in E and the operator $\mathcal{A}$ is continuous.

Next, we shall verify (1) and (2) as follows. By (4.5), we get

$$\langle I'_{\mu,\gamma}(u), \varphi\rangle = \langle J'_{\mu,\gamma}(u) - J'_{\mu,\gamma}(v), \varphi\rangle, \quad \text{for } \varphi \in E. \tag{4.9}$$

Furthermore, we have the following estimates

$$\begin{aligned} &\langle J'_{\mu,\gamma}(u) - J'_{\mu,\gamma}(v), \varphi\rangle \\ &\quad = 3\mu \int_0^1 \int_{\mathbb{R}^3} |\nabla \omega_t|^2 \nabla(u - v)\nabla\varphi dxdt + 3\mu \int_0^1 \int_{\mathbb{R}^3} \omega_t^2 (u - v)\varphi dxdt \\ &\qquad + \int_{\mathbb{R}^3} (\nabla(u - v)\nabla\varphi + V(x)(u - v)\varphi)dx + 2\gamma \int_0^1 \int_{\mathbb{R}^3} \nabla\omega_t \nabla(u - v)\omega_t\varphi dxdt \\ &\qquad + \gamma \int_0^1 \int_{\mathbb{R}^3} |\nabla\omega_t|^2 (u - v)\varphi dxdt + 2\gamma \int_0^1 \int_{\mathbb{R}^3} \omega_t (u - v)\nabla\omega_t \nabla\varphi dxdt \\ &\qquad + \gamma \int_0^1 \int_{\mathbb{R}^3} \omega_t^2 \nabla(u - v)\nabla\varphi dxdt + 3C_0 \int_0^1 \int_{\mathbb{R}^3} \omega_t^2 (u - v)\varphi dxdt, \end{aligned} \tag{4.10}$$

where $\omega_t = tu + (1 - t)v$. By $|\omega_t| \le |u| + |v|$, $|\nabla\omega_t| \le |\nabla u| + |\nabla v|$, the Hölder inequality and (4.9), we can get

$$|\langle I'_\lambda(u), \varphi\rangle| \le c_1(\|u\|_W^2 + \|v\|_W^2)\|u - v\|_W \|\varphi\|_E + c_2\|u - v\|_{H_V^1}\|\varphi\|_E.$$

In fact, there hold

$$\begin{aligned} &3\mu \int_0^1 \int_{\mathbb{R}^3} |\nabla\omega_t|^2 \nabla(u - v)\nabla\varphi dxdt + 3\mu \int_0^1 \int_{\mathbb{R}^3} \omega_t^2 (u - v)\varphi dxdt \\ &\quad \le C(|\nabla u|_4^2 + |\nabla v|_4^2)|\nabla(u - v)|_4 |\nabla\varphi|_4 + C(|u|_4^2 + |v|_4^2)|u - v|_4 |\varphi|_4 \\ &\quad \le C(\|u\|_W^2 + \|v\|_W^2)\|u - v\|_W \|\varphi\|_E \end{aligned}$$

and

$$\int_{\mathbb{R}^3} (\nabla(u - v)\nabla\varphi + V(x)(u - v)\varphi)dx \le C\|u - v\|_{H_V^1}\|\varphi\|_E.$$

Using similar methods, we can also estimate other terms in (4.10). Hence

$$\|I'_{\mu,\gamma}(u)\|_{E^*} \le c_1(\|u\|_W^2 + \|v\|_W^2)\|u - v\|_W + c_2\|u - v\|_{H_V^1}.$$

For (2), by the similar estimates of (4.6), set $\varphi = u - v$, we have

$$\begin{aligned} \langle I'_{\mu,\gamma}(u), u - v\rangle &= \langle J'_{\mu,\gamma}(u) - J'_{\mu,\gamma}(v), u - v\rangle \\ &\ge \frac{\mu}{2} \int_{\mathbb{R}^3} \left((|\nabla u|^2 \nabla u - |\nabla v|^2 \nabla v)\nabla(u - v) + (u^3 - v^3)(u - v)\right) dx \\ &\quad + \int_{\mathbb{R}^3} \left(|\nabla(u - v)|^2 + V(x)(u - v)^2\right) dx \\ &\ge c_3(\|u - v\|_W^4 + \|u - v\|_{H_V^1}^2). \end{aligned}$$

In order to prove (3), we consider

$$
\begin{aligned}
& I_{\mu,\gamma}(u) - \frac{1}{\theta}\langle I'_{\mu,\gamma}(u), u\rangle \\
&= \left(\frac{\mu}{4} - \frac{\mu}{\theta}\right)\|u\|_W^4 + \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)dx \\
&\quad + \left(\frac{\gamma}{2} - \frac{2\gamma}{\theta}\right)\int_{\mathbb{R}^3}|\nabla u|^2u^2dx + \int_{\mathbb{R}^3}\left(\frac{1}{\theta}uf(x,u) - F(x,u)\right)dx \\
&\geq \left(\frac{\mu}{4} - \frac{\mu}{\theta}\right)\|u\|_W^4 + \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2_{H^1_V}.
\end{aligned}
$$

Hence, for any $\delta > 0$, there exists C_δ, such that

$$
\begin{aligned}
\|u\|_W^4 + \|u\|^2_{H^1_V} &\leq C(|I_{\mu,\gamma}(u)| + \|I'_{\mu,\gamma}(u)\|_{E^*}\|u\|_E) \\
&= C(|I_{\mu,\gamma}(u)| + \|I'_{\mu,\gamma}(u)\|_{E^*}(\|u\|_W + \|u\|_{H^1_V})) \\
&\leq C(|I_{\mu,\gamma}(u)| + C_\delta\|I'_{\mu,\gamma}(u)\|^{4/3}_{E^*} + \delta\|u\|_W^4 + C_\delta\|I'_{\mu,\gamma}(u)\|^2_{E^*} + \delta\|u\|^2_{H^1_V}).
\end{aligned}
$$

Taking $\delta > 0$ small enough, by direct calculation, we obtain the following estimates

$$
\|u\|_W^2 \leq C(1 + |I_{\mu,\gamma}(u)|^{1/2} + \|I'_{\mu,\gamma}(u)\|_{E^*}) \tag{4.11}
$$

Combining (4.11) and Lemma 4.3-(1), we can obtain

$$
\begin{aligned}
\|I'_{\mu,\gamma}(u)\|_{E^*} &\leq c_1(\|u\|_W^2 + \|v\|_W^2)\|u - v\|_W + c_2\|u - v\|_{H^1_V} \\
&\leq C(1 + \|u\|_W^2 + \|u - v\|_E^2)\|u - v\|_E \\
&\leq \widetilde{C}(1 + |I_{\mu,\gamma}(u)|^{1/2} + \|I'_{\mu,\gamma}(u)\|_{E^*} + \|u - v\|_E^2)\|u - v\|_E.
\end{aligned}
$$

For $u \in I^{-1}_{\mu,\gamma}([a,b])$ and $\|I'_{\mu,\gamma}(u)\|_{E^*} \geq \alpha > 0$, without loss of generality, let $\|u - v\|_E \leq \frac{1}{2\widetilde{C}}$, we obtain

$$
\|I'_{\mu,\gamma}(u)\|_{E^*} \leq \widetilde{C}\left(1 + b^{1/2} + \frac{1}{(2\widetilde{C})^2}\right)\|u - v\|_E + \frac{1}{2}\|I'_{\mu,\gamma}(u)\|_{E^*},
$$

and

$$
\|u - v\|_E \geq C\|I'_\lambda(u)\|_{E^*} \geq C\alpha.
$$

□

Consider a positive cone P in E defined by $P := \{u \in E : \ u \geq 0 \ a.e. \text{ on } x \in \mathbb{R}^3\}$. For an arbitrary $\varepsilon > 0$, let

$$
P^\pm_\varepsilon = \left\{u \in E : \ V_0\int_{\mathbb{R}^3}u^2_\mp dx + S\left(\int_{\mathbb{R}^3}|u_\mp|^6dx\right)^{\frac{1}{3}} < \varepsilon\right\},
$$

where $S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3}|\nabla u|^2dx}{(\int_{\mathbb{R}^3}|u|^6dx)^{1/3}}$, $u_+ = \max\{u, 0\}$, $u_- = \min\{u, 0\}$.

Lemma 4.4. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, then*

$$
\mathcal{A}(\partial P^+_\varepsilon) \subset P^+_\varepsilon \quad \textit{and} \quad \mathcal{A}(\partial P^-_\varepsilon) \subset P^-_\varepsilon.
$$

Proof. Since the proofs of the two conclusions are similar, we just give the proof of $\mathcal{A}(\partial P_\varepsilon^+) \subset P_\varepsilon^+$.

Let $u \in E$, $v = \mathcal{A}(u)$, v satisfying (4.5). Taking $\omega = v_-$, we have

$$\begin{aligned} &\mu \int_{\mathbb{R}^3} \left(|\nabla v_-|^4 + v_-^4\right) dx + \int_{\mathbb{R}^3} \left(|\nabla v_-|^2 + V(x)v_-^2\right) dx \\ &\quad + 2\gamma \int_{\mathbb{R}^3} |\nabla v_-|^2 v_-^2 \, dx + C_0 \int_{\mathbb{R}^3} v_-^4 \, dx \\ &= C_0 \int_{\mathbb{R}^3} u^3 v_- \, dx + \int_{\mathbb{R}^3} f(x,u) v_- \, dx. \end{aligned} \tag{4.12}$$

Next, we will give the estimates of both sides of above equality. On one hand, we have

$$\begin{aligned} &\mu \int_{\mathbb{R}^3} \left(|\nabla v_-|^4 + v_-^4\right) dx + \int_{\mathbb{R}^3} \left(|\nabla v_-|^2 + V(x)v_-^2\right) dx \\ &\quad + 2\gamma \int_{\mathbb{R}^3} |\nabla v_-|^2 v_-^2 \, dx + C_0 \int_{\mathbb{R}^3} v_-^4 \, dx \\ &\geq V_0 \int_{\mathbb{R}^3} v_-^2 \, dx + S \left(\int_{\mathbb{R}^3} |v_-|^6 dx \right)^{1/3}. \end{aligned} \tag{4.13}$$

On the other hand, by Young inequality, we obtain

$$\begin{aligned} &C_0 \int_{\mathbb{R}^3} u^3 v_- \, dx + \int_{\mathbb{R}^3} f(u) v_- \, dx \\ &\quad \leq \delta \int_{\mathbb{R}^3} u_- v_- \, dx + C_\delta \int_{\mathbb{R}^3} u_-^5 v_- \, dx \\ &\quad \leq \frac{1}{2}\delta \int_{\mathbb{R}^3} (u_-^2 + v_-^2) dx + \frac{S}{2} \left(\int_{\mathbb{R}^3} |v_-|^6 dx \right)^{1/3} \\ &\qquad + C_\delta \left(\int_{\mathbb{R}^3} |u_-|^6 dx \right)^{5/3}, \quad \text{for any } \delta > 0. \end{aligned} \tag{4.14}$$

Fix $\delta = V_0$ and choose ε_0 such that $C_\delta (\frac{\varepsilon_0}{S})^4 \leq \frac{S}{2}$. For $0 < \varepsilon < \varepsilon_0$ and $u \in P_\varepsilon^+$, we have

$$C_\delta \left(\int_{\mathbb{R}^3} |u_-|^6 dx \right)^{4/3} \leq C_\delta \left(\frac{\varepsilon}{S} \right)^4 \leq \frac{S}{2}. \tag{4.15}$$

By (4.13)–(4.15), we get

$$V_0 \int_{\mathbb{R}^3} v_-^2 \, dx + S \left(\int_{\mathbb{R}^3} |v_-|^6 dx \right)^{\frac{1}{3}} \leq V_0 \int_{\mathbb{R}^3} u_-^2 \, dx + S \left(\int_{\mathbb{R}^3} |u_-|^6 dx \right)^{\frac{1}{3}}.$$

Therefore, for $u \in \partial P_\varepsilon^+$, $u \neq 0$, we have

$$V_0 \int_{\mathbb{R}^3} v_-^2 \, dx + S \left(\int_{\mathbb{R}^3} |v_-|^6 dx \right)^{\frac{1}{3}} < \varepsilon,$$

which implies $v \in P_\varepsilon^+$. This completes the proof. □

From the above analysis, we know that $\mathcal{A}$ is merely continuous. But $\mathcal{A}$ itself is not applicable to construct a descending flow for $I_{\mu,\gamma}$, and we have to construct a locally Lipschitz continuous operator $\mathcal{B}$ which inherits the main properties of $\mathcal{A}$.

Lemma 4.5. *Let $E_0 = E \setminus K$, $K = \{u \in E : I'_{\mu,\gamma}(u) = 0\}$. There exist a locally Lipschitz continuous operator $\mathcal{B} : E_0 \to E$ such that*

(1) $\frac{1}{2}\|u - \mathcal{B}(u)\|_E \leq \|u - \mathcal{A}(u)\|_E \leq 2\|u - \mathcal{B}(u)\|_E$ *for all* $u \in E_0$*;*

(2) $\langle I'_{\mu,\gamma}(u), u - \mathcal{B}(u)\rangle \geq c_3^*(\|u - \mathcal{B}u\|_W^4 + \|u - \mathcal{B}u\|^2_{H^1_V})$ *for all* $u \in E_0$*;*

(3) $\|I'_{\mu,\gamma}(u)\|_{E^*} \leq c_1^*(\|u\|_W^2 + \|\mathcal{B}u\|_W^2)\|u - \mathcal{B}u\|_W + c_2^*\|u - \mathcal{B}u\|_{H^1_V}$ *for all* $u \in E_0$*;*

(4) $\mathcal{B}(\partial P_\varepsilon^+) \subset P_\varepsilon^+$, $\mathcal{B}(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ *for* $\varepsilon \in (0, \varepsilon_0)$*,*

where c_1^*, c_2^*, c_3^* *are different constants.*

Proof. The proof is similar to the proofs in [6] and [7]. We omit the details. □

From the above discussions, it is worth pointing that P_ε^+ and P_ε^- are invariant sets of descending flow τ, where $\varepsilon \in (0, \varepsilon_0)$ and τ satisfies the following initial value problem

$$\begin{cases} \frac{d}{dt}\tau(t,u) = -(id - \mathcal{B})\tau(t,u), \\ \tau(0,u) = u. \end{cases}$$

By applying invariant sets of descending flow, we can find one sign-changing critical point of the functional $I_{\mu,\gamma}$. For this purpose, we adapt some abstract results in [24].

Let $I \in C^1(E, \mathbb{R})$, $P, Q \subset E$ be open sets, $M = P \cap Q$, $\Sigma = \partial P \cap \partial Q$ and $W = P \cup Q$. For $c \in \mathbb{R}$, let $K_c = \{u \in E : I(u) = c, I'(u) = 0\}$ and $I^c = \{u \in E : I(u) \leq c\}$.

Definition 4.6. $\{P, Q\}$ is called an admissible family of invariant sets with respect to I at level c, provided that the following deformation property holds: if $K_c \setminus W = \emptyset$, then, there exists $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$, there exists $\eta \in C(E, E)$ satisfying

(1) $\eta(\overline{P}) \subset \overline{P}$, $\eta(\overline{Q}) \subset \overline{Q}$;

(2) $\eta|_{I^{c-2\varepsilon}} = \mathrm{id}$;

(3) $\eta(I^{c+\varepsilon} \setminus W) \subset I^{c-\varepsilon}$.

Theorem 4.7 ([24])**.** *Assume that* $\{P, Q\}$ *is an admissible family of invariant sets with respect to* I *at any level* $c \geq c_* := \inf_{u \in \Sigma} I(u)$ *and there exists a map* $\varphi_0 : \chi \to E$ *satisfying*

(1) $\varphi_0(\partial_1\chi) \subset P$ *and* $\varphi_0(\partial_2\chi) \subset Q$*;*

(2) $\varphi_0(\partial_0\chi) \cap M = \emptyset$*;*

(3) $\sup_{u \in \varphi_0(\partial_0\chi)} I(u) < c_*$*,*

where $\chi = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 \leq 1\}$, $\partial_1\chi = \{0\} \times [0, 1]$, $\partial_2\chi = [0, 1] \times \{0\}$ *and* $\partial_0\chi = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 = 1\}$*. Define*

$$c = \inf_{\varphi \in \Gamma} \sup_{u \in \varphi(\chi) \setminus W} I(u),$$

where $\Gamma := \{\varphi \in C(\chi, E) : \varphi(\partial_1\chi) \subset P,\ \varphi(\partial_2\chi) \subset Q,\ \varphi|_{\partial_0\chi} = \varphi_0|_{\partial_0\chi}\}$*. Then* $c \geq c_*$*, and* $K_c \setminus W \neq \emptyset$*.*

To apply Theorem 4.7 to obtain one sign-changing critical point of $I_{\mu,\gamma}$, we take $P = P_\varepsilon^+$, $Q = P_\varepsilon^-$, $I = I_{\mu,\gamma}$. Then we need to prove the following crucial lemma.

Lemma 4.8. *If $K_c \setminus W = \emptyset$, then there exists $\varepsilon_2 > 0$ such that, for $0 < \varepsilon < \varepsilon' < \varepsilon_2$, there exists a continuous map $\sigma : [0,1] \times E \to E$ satisfying*

(1) $\sigma(0,u) = u$ for $u \in E$;

(2) $\sigma(t,u) = u$ for $t \in [0,1]$, $u \notin I_{\mu,\gamma}^{-1}[c-\varepsilon', c+\varepsilon']$;

(3) $\sigma(1, I_{\mu,\gamma}^{c+\varepsilon} \setminus W) \subset I_{\mu,\gamma}^{c-\varepsilon}$;

(4) $\sigma(t, \overline{P_\varepsilon^+}) \subset \overline{P_\varepsilon^+}$ and $\sigma(t, \overline{P_\varepsilon^-}) \subset \overline{P_\varepsilon^-}$ for $t \in [0,1]$.

Proof. The proof is similar to many existing literature (see [25, 32]). For the readers' convenience, here we give the details.

Let $N_\delta(K_c) := \{u \in E : \ d(E, K_c) < \delta\}$. If $K_c \setminus W = \emptyset$, then $K_c \subset W$. Thus for $\delta > 0$ small enough, we get

$$N_\delta(K_c) \subset W.$$

By Lemma 4.2, we know that $I_{\mu,\gamma}$ satisfies the (PS)-condition. Hence K_c is compact and exist $\varepsilon_2, \alpha > 0$ such that

$$\|I'_{\mu,\gamma}(u)\|_{E^*} \geq \alpha, \quad \text{for all } u \in I_{\mu,\gamma}^{-1}([c-\varepsilon_2, c+\varepsilon_2]) \setminus N_{\delta/2}(K_c).$$

Using Lemma 4.3-(3) and Lemma 4.5-(1),(2), we can find $\beta > 0$ such that

$$\langle I'_{\mu,\gamma}(u), \frac{u - Bu}{\|u - Bu\|_E} \rangle \geq \beta, \quad \text{for all } u \in I_{\mu,\gamma}^{-1}([c-\varepsilon_2, c+\varepsilon_2]) \setminus N_{\delta/2}(K_c).$$

Assume

$$\varepsilon_2 < \min\{\frac{\beta\delta}{4}, \varepsilon_0\},$$

where ε_0 is defined in Lemma 4.4. Defining two Lipschitz continuous functionals $g, q : E \to [0,1]$, satisfying

$$g(u) = \begin{cases} 0, & \text{if } u \in N_{\delta/4}(K_c), \\ 1, & \text{if } u \notin N_{\delta/2}(K_c) \end{cases}$$

and

$$q(u) = \begin{cases} 0, & \text{if } u \notin I_{\mu,\gamma}^{-1}([c-\varepsilon', c+\varepsilon']), \\ 1, & \text{if } u \in I_{\mu,\gamma}^{-1}([c-\varepsilon, c+\varepsilon]). \end{cases}$$

Consider the following initial value problem

$$\begin{cases} \frac{d\tau(t,u)}{dt} = -\Phi(\tau(t,u)), \\ \tau(0,u) = u, \end{cases} \tag{4.16}$$

where $\Phi(u) = g(u)q(u)\frac{u-Bu}{\|u-Bu\|_E}$. Using the existence and uniqueness theory of ODE, we obtain that the problem (4.16) has a unique solution $\tau(\cdot, u) \in C(\mathbb{R}^+, E)$. Let $\sigma(t,u) = \tau(\frac{2\varepsilon}{\beta}t, u)$, then we verify (1)–(3). In fact, (1) and (2) are obvious. It suffices to verify (3). To do this, we consider the following two cases.

Case 1. There exists $t_0 \in [0, \frac{2\varepsilon}{\beta}]$ such that $I_{\mu,\gamma}(\tau(t_0, u)) < c - \varepsilon$. Using Lemma 4.5-(2), we obtain that $I_{\mu,\gamma}(\tau(t, u)$ is decreasing for $t \geq 0$. Therefore, $I_{\mu,\gamma}(\sigma(1, u)) \leq c - \varepsilon$.

Case 2. For $u \in I^{c+\varepsilon}_{\mu,\gamma} \setminus W$ and $t \in [0, \frac{2\varepsilon}{\beta}]$, then $I_{\mu,\gamma}(\tau(t, u)) > c - \varepsilon$. In this case, we claim that $\tau(t, u) \in N_{\delta/2}(K_c)$ for any $t \in [0, \frac{2\varepsilon}{\beta}]$. Indeed, if for some $t_0 \in [0, \frac{2\varepsilon}{\beta}]$ such that $\tau(t_0, u) \in N_{\delta/2}(K_c)$, then

$$\frac{\delta}{2} \leq \|\tau(t_0, u) - u\|_E \leq \int_0^{t_0} \|\tau'(s, u)\|_E ds \leq t_0 < \frac{\delta}{2},$$

which is a contradiction. Thus, $g(\tau(t, u))q(\tau(t, u)) \equiv 1$ for all $t \in [0, \frac{2\varepsilon}{\beta}]$. Hence,

$$\begin{aligned} I_{\mu,\gamma}(\sigma(1, u)) &= I_{\mu,\gamma}(\tau(\frac{2\varepsilon}{\beta}, u)) \\ &= I_{\mu,\gamma}(u) - \int_0^{\frac{2\varepsilon}{\beta}} \langle I'_{\mu,\gamma}(\tau(s, u)), \Phi(\tau(s, u))\rangle ds \\ &\leq c + \varepsilon - 2\varepsilon \\ &= c - \varepsilon. \end{aligned}$$

The proof is completed. □

Next, we will construct φ_0 satisfying the hypotheses in Theorem 4.7. Choose $u_1, u_2 \in C_0^\infty(\mathbb{R}^3)$ which satisfy $\mathrm{supp}(u_1) \cap \mathrm{supp}(u_2) = \emptyset$ and $u_1 \leq 0, u_2 \geq 0$. Let $\varphi_0(t, s) := R(tu_1 + su_2)$ for $(t, s) \in \chi$, where $\chi = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 \leq 1\}$ and R is a positive constant to be determined later. Obviously, for $t, s \in [0, 1]$, $\varphi_0(0, s) = Rsu_2 \in P_\varepsilon^+$ and $\varphi_0(t, 0) = Rtu_1 \in P_\varepsilon^-$.

Lemma 4.9. *Assume that (V_0), (V_1) and (f_1)–(f_3) hold. Then the functional $I_{\mu,\gamma}$ has a sign-changing critical point.*

Proof. It is sufficient to check assumptions (2)–(3) in applying Theorem 4.7.

Notice that $\rho = \min\{|tu_1 + (1-t)u_2|_2 : 0 \leq t \leq 1\} > 0$. Then,

$$|u|_2 \geq \rho R \quad \text{for } u \in \varphi_0(\partial_0 \chi).$$

Furthermore, for $u \in M = P_\varepsilon^+ \cap P_\varepsilon^-$, we have that

$$|u|_2^2 \leq \frac{2}{V_0}\varepsilon.$$

Hence, $\varphi_0(\partial_0 \chi) \cap M = \emptyset$ for R large enough.

To verify (3), for any $u \in \Sigma$, from the conditions (f_1) and (f_2) and the definition of Σ, for all $\delta > 0$, there exists $C_\delta > 0$, such that

$$I_{\mu,\gamma}(u) \geq -\int_{\mathbb{R}^3} F(x, u)dx \geq -\delta \int_{\mathbb{R}^3} u^2 dx - C_\delta \int_{\mathbb{R}^3} u^6 dx \geq -C(\varepsilon + \varepsilon^3),$$

which implies that

$$c_* \geq -C(\varepsilon + \varepsilon^3). \tag{4.17}$$

On the other hand, by the condition (f_3), we have $F(x, t) \geq C|t|^\theta$ for all $x \in \mathbb{R}^3$. For any $u \in \varphi_0(\partial_0 \chi)$, then

$$\begin{aligned} I_{\mu,\gamma}(u) &= \frac{\mu}{4}\|u\|_W^4 + \frac{1}{2}\|u\|_{H_V^1}^2 + \frac{\gamma}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 dx - \int_{\mathrm{supp}(u_1) \cap \mathrm{supp}(u_2)} F(x, u)dx \\ &\leq \frac{\mu}{4}\|u\|_W^4 + \frac{1}{2}\|u\|_{H_V^1}^2 + \frac{\gamma}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 dx - C|u|_\theta^\theta \\ &\leq C\|u\|_E^4 - C|u|_\theta^\theta, \end{aligned} \tag{4.18}$$

which together with (4.17) implies that for R large enough and ε small enough, we obtain

$$\sup_{u\in\varphi_0(\partial_0\chi)} I_{\mu,\gamma}(u) < c_*.$$

Hence, by Theorem 4.7, $I_{\mu,\gamma}$ has at least one critical point u in $E\setminus(P_\varepsilon^+\cup P_\varepsilon^-)$. □

The next result establishes an important estimate associated with critical values.

Lemma 4.10. *Assume $0<\mu<1$ and $0<\gamma<1$. Then there exists a positive constant m_3 (independent on μ and γ), such that*

$$I_{\mu,\gamma}(u_{\mu,\gamma}) \le m_3,$$

where $u_{\mu,\gamma}$ is a sign-changing critical point of $I_{\mu,\gamma}$.

Proof. For fixed $0<\mu<1$ and $0<\gamma<1$, take a path $\varphi_{1,1}(s,t):[0,1]\times[0,1]\to E\setminus\{0\}$, $\varphi_{1,1}(t,s):=T(tu_1+su_2)$, where the constant $T>R$ (R is defined in the proof of Lemma 4.9). A simple computation ensures that $\varphi_{1,1}(0,s)\in P_\varepsilon^+$, $\varphi_{1,1}(t,0)\in P_\varepsilon^-$ and $\varphi_{1,1}(\partial_0\chi)\cap M=\emptyset$. By the similar estimates of (4.18), taking T sufficiently large, we obtain

$$I_{1,1}(\varphi_{1,1}(t,s)) \le -C_1 \quad \text{for all } (t,s)\in\partial_0\chi, \tag{4.19}$$

where $C_1>0$ is large enough.

On the other hand, for ε small enough, we have

$$\inf_{u\in\Sigma} I_{\mu,\gamma}(u) > -\sup_{u\in\Sigma}\int_{\mathbb{R}^3} F(x,u)dx \ge -C_2, \tag{4.20}$$

here choose C_1 large enough, such that $0<C_2<C_1$. Then estimates (4.19) and (4.20) ensure that

$$\max_{(t,s)\in\partial_0\chi} I_{\mu,\gamma}(\varphi_{1,1}(t,s)) \le \max_{(t,s)\in\partial_0\chi} I_{1,1}(\varphi_{1,1}(t,s)) \le -C_2 < \inf_{u\in\Sigma} I_{\mu,\gamma}(u).$$

This implies

$$\varphi_{1,1}(s,t)\in\Gamma,$$

where $\Gamma:=\{\varphi\in C(\chi,E):\varphi(\partial_1\chi)\subset P_\varepsilon^+,\ \varphi(\partial_2\chi)\subset P_\varepsilon^-,\ \varphi|_{\partial_0\chi}=\varphi_0|_{\partial_0\chi}\}$, and so

$$I_{\mu,\gamma}(u_{\mu,\gamma}) = \inf_{\varphi\in\Gamma}\sup_{u\in\varphi(\chi)\setminus W} I_{\mu,\gamma}(u) \le \sup_{u\in\varphi_{1,1}(\chi)} I_{\mu,\gamma}(u) \le \max_{(t,s)\in[0,1]\times[0,1]} I_{1,1}(\varphi_{1,1}(t,s)) := m_3,$$

where m_3 is independent on γ and μ. □

Finally, the existence of a sign-changing critical point to the original functional I_γ is based on the following convergence result for the perturbation functional $I_{\mu,\gamma}$.

Proposition 4.11 ([23]). *Let $\mu_i\to 0$ and $\{u_i\}\subset E$ be a sequence of critical points of $I_{\mu_i,\gamma}$ satisfying $I'_{\mu_i,\gamma}(u_i)=0$ and $I_{\mu_i,\gamma}(u_i)\le C$ for some C independent of i. Then as $i\to\infty$, up to a subsequence $u_i\to u_\gamma$ in $H_V^1(\mathbb{R}^3)$, $u_i\nabla u_i\to u_\gamma\nabla u_\gamma$ in $L^2(\mathbb{R}^3)$, $\mu_i\int_{\mathbb{R}^3}\left(|\nabla u_i|^4+u_i^4\right)dx\to 0$, $I_{\mu_i,\gamma}(u_i)\to I_\gamma(u_\gamma)$ and u_γ is a critical point of I_γ.*

Lemma 4.12. *Assume $0<\gamma<1$. Then there exist a positive constant m_3 and a sign-changing critical point u_γ of I_γ, such that*

$$I_\gamma(u_\gamma)\le m_3,$$

where m_3 is independent on γ.

Proof. From Lemma 4.9 and Lemma 4.10, it permits to apply the Proposition 4.11. Therefore, there exists a critical point u_γ of I_γ such that $u_\gamma \in H^1_V(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. In the following, we will show that u_γ is a sign-changing critical point of I_γ. To this end, we need estimate $u_{\gamma+} \neq 0$ as follows. Consider $\langle I'_{\mu_i,\gamma}(u_i), u_{i+}\rangle = 0$, it follows from Sobolev inequality and the conditions (f_1), (f_2) that

$$\begin{aligned} V_0 \int_{\mathbb{R}^3} |u_{i+}|^2 dx + S\left(\int_{\mathbb{R}^3} |u_{i+}|^6 dx\right)^{\frac{1}{3}} &\\ \leq V_0 \int_{\mathbb{R}^3} |u_{i+}|^2 dx + \int_{\mathbb{R}^3} |\nabla u_{i+}|^2 dx &\\ \leq \int_{\mathbb{R}^3} f(x, u_{i+}) u_{i+} dx &\\ \leq \delta \int_{\mathbb{R}^3} |u_{i+}|^2 dx + C_\delta \int_{\mathbb{R}^3} |u_{i+}|^6 dx, & \end{aligned}$$

where $\delta > 0$ small enough. This implies $|u_{i+}|_6 \geq C > 0$. Recall that $u_{i+} \to u_{\gamma+}$ strongly in $L^6(\mathbb{R}^3)$. Therefore, we see that $u_{\gamma+} \neq 0$. By the same argument we can prove that $u_{\gamma-} \neq 0$. Hence we obtain u_γ is a sign-changing critical point of I_γ.

Moreover, by Lemma 4.10, we obtain

$$I_{\mu,\gamma}(u_{\mu,\gamma}) \leq m_3,$$

where m_3 is independent on γ and μ.

Having this in mind, taken $\mu \to 0$, from the Proposition 4.11 we have

$$I_\gamma(u_\gamma) \leq m_3,$$

where u_γ is sign-changing critical point of I_γ. □

Before concluding this section, we would like to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. From Lemma 3.3 and Lemma 4.12, the problem (1.4) has at least three solutions: a positive solution $u_{\gamma,1}$, a negative solution $u_{\gamma,2}$ and a sign-changing solution $u_{\gamma,3}$. □

5 Asymptotic behavior of solutions

In this section, our goal is to study the asymptotic behavior of $u_\gamma = G^{-1}(v_\gamma)$. Having this in mind, we are going to show the L^∞ estimates of the critical points of J_γ.

Lemma 5.1. *If $v_\gamma \in H^1_V(\mathbb{R}^3)$ is a weak solution of problem (2.2), then $v_\gamma \in L^\infty(\mathbb{R}^3)$. Moreover, there exists a constant $C > 0$ independents of γ such that $|v_\gamma|_\infty \leq C\|v_\gamma\|^{\frac{4}{6-p}}_{H^1_V}$.*

Proof. The result can be proved similarly to [5, 14] but we give a proof for the convenience of the readers. In what follows, for simplicity, we denote v_γ by v. Let $v \in H^1_V(\mathbb{R}^3)$ be a weak solution of $-\Delta v + V(x)\frac{G^{-1}_\gamma(v)}{g_\gamma(G^{-1}_\gamma(v))} = \frac{f(x,G^{-1}_\gamma(v))}{g_\gamma(G^{-1}_\gamma(v))}$, i.e.

$$\int_{\mathbb{R}^3} \nabla v \nabla \varphi dx + \int_{\mathbb{R}^3} V(x)\frac{G^{-1}_\gamma(v)}{g_\gamma(G^{-1}_\gamma(v))}\varphi dx = \int_{\mathbb{R}^3} \frac{f(x, G^{-1}_\gamma(v))}{g_\gamma(G^{-1}_\gamma(v))}\varphi dx, \quad \text{for all } \varphi \in H^1_V(\mathbb{R}^3). \tag{5.1}$$

Set $T>0$, and denote

$$v_T=\begin{cases}-T, & \text{if } v\le -T,\\ v, & \text{if } -T<v<T,\\ T, & \text{if } v\ge T.\end{cases}$$

Choosing $\varphi=|v_T|^{2(\eta-1)}v$ in (5.1), where $\eta>1$ to be determined later, we get

$$\begin{aligned}&\int_{\mathbb{R}^3}|\nabla v|^2\cdot|v_T|^{2(\eta-1)}dx+2(\eta-1)\int_{\{x:\ |v(x)|<T\}}|v|^{2(\eta-1)}|\nabla v|^2dx\\ &\quad+\int_{\mathbb{R}^3}V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}vdx\\ &=\int_{\mathbb{R}^3}\frac{f(x,G_\gamma^{-1}(v))}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}vdx.\end{aligned}$$

Combining the fact that the second term in the left side of the above equation is nonnegative and Lemma 2.1-(4), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3}|\nabla v|^2|v_T|^{2(\eta-1)}dx+\int_{\mathbb{R}^3}V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}vdx\\ &\le\int_{\mathbb{R}^3}\frac{f(x,G_\gamma^{-1}(v))}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}vdx\\ &\le\delta\int_{\mathbb{R}^3}\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}vdx+C_\delta\int_{\mathbb{R}^3}\frac{|G_\gamma^{-1}(v)|^{p-1}}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}vdx\\ &\le\delta\int_{\mathbb{R}^3}\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}vdx+C_\delta\int_{\mathbb{R}^3}|v|^p|v_T|^{2(\eta-1)}dx.\end{aligned}\tag{5.2}$$

Taking δ small enough in (5.2), we have

$$\int_{\mathbb{R}^3}|\nabla v|^2|v_T|^{2(\eta-1)}dx\le C\int_{\mathbb{R}^3}|v|^p|v_T|^{2(\eta-1)}dx.\tag{5.3}$$

On the other hand, using the Sobolev inequality, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^3}(|v||v_T|^{\eta-1})^6dx\right)^{\frac13}&\le C\int_{\mathbb{R}^3}|\nabla(vv_T^{\eta-1})|^2dx\\ &\le C\int_{\mathbb{R}^3}|\nabla v|^2|v_T|^{2(\eta-1)}dx+C(\eta-1)^2\int_{\mathbb{R}^3}|\nabla v|^2|v_T|^{2(\eta-1)}dx\\ &\le C\eta^2\int_{\mathbb{R}^3}|\nabla v|^2|v_T|^{2(\eta-1)}dx,\end{aligned}$$

where we used that $(a+b)^2\le 2(a^2+b^2)$ and $\eta^2\ge(\eta-1)^2+1$.

By (5.3), the Hölder inequality and the Sobolev embedding theorem,

$$\begin{aligned}\left(\int_{\mathbb{R}^3}(|v||v_T|^{\eta-1})^6dx\right)^{\frac13}&\le C\eta^2\int_{\mathbb{R}^3}|v|^{p-2}v^2|v_T|^{2(\eta-1)}dx\\ &\le C\eta^2\left(\int_{\mathbb{R}^3}|v|^6dx\right)^{\frac{p-2}{6}}\left(\int_{\mathbb{R}^3}(|v||v_T|^{\eta-1})^{\frac{12}{8-p}}dx\right)^{\frac{8-p}{6}}\\ &\le C\eta^2\|v\|_{H_V^1}^{p-2}\left(\int_{\mathbb{R}^3}|v|^{\frac{12\eta}{8-p}}dx\right)^{\frac{8-p}{6}},\end{aligned}$$

where we used the fact that $|v_T| \leq |v|$. In what follows, taking $\zeta = \frac{12}{8-p}$, we get

$$\left(\int_{\mathbb{R}^3}(|v||v_T|^{\eta-1})^6 dx\right)^{\frac{1}{3}} \leq C\eta^2\|v\|_{H_V^1}^{p-2}|v|_{\eta\zeta}^{2\eta}.$$

From Fatou's lemma, it follows that

$$|v|_{6\eta} \leq (C\eta^2\|v\|_{H_V^1}^{p-2})^{\frac{1}{2\eta}}|v|_{\eta\zeta}. \tag{5.4}$$

Let us define $\eta_{n+1}\zeta = 6\eta_n$ where $n = 0, 1, 2, \dots$ and $\eta_0 = \frac{8-p}{2}$. By (5.4) we have

$$|v|_{6\eta_1} \leq (C\eta_1^2\|v\|_{H_V^1}^{p-2})^{\frac{1}{2\eta_1}}|v|_{6\eta_0} \leq (C\|v\|_{H_V^1}^{p-2})^{\frac{1}{2\eta_1}+\frac{1}{2\eta_0}}\eta_0^{\frac{1}{\eta_0}}\eta_1^{\frac{1}{\eta_1}}|v|_6.$$

By Moser's iteration method we have

$$|v|_{6\eta_n} \leq (C\|v\|_{H_V^1}^{p-2})^{\frac{1}{2\eta_0}\sum_{i=0}^n(\frac{\zeta}{6})^i}(\eta_0)^{\frac{1}{\eta_0}\sum_{i=0}^n(\frac{\zeta}{6})^i}(\frac{6}{\zeta})^{\frac{1}{\eta_0}\sum_{i=0}^n i(\frac{\zeta}{6})^i}|v|_6.$$

Thus, we have

$$|v|_\infty \leq C\|v\|_{H_V^1}^{\frac{4}{6-p}}.$$

□

Now we are ready to prove H_V^1-strong convergence of the weak solution of problem (1.4).

Lemma 5.2. *Assume u_γ is a solution of (1.4), then $u_\gamma \to u_0$ strongly in $H_V^1(\mathbb{R}^3)$ as $\gamma \to 0^+$, where u_0 is a solution of (1.6).*

Proof. If u_γ is a signed solution of (1.4), Lemma 3.4 and Lemma 3.5 guarantee that

$$\|v_\gamma\|_{H_V^1} < C,$$

for some $C > 0$. This together with the fact that

$$\|u_\gamma\|_{H_V^1} = \|G^{-1}(v_\gamma)\|_{H_V^1} \leq C\|v_\gamma\|_{H_V^1},$$

gives $\{u_\gamma\}$ is uniformly bounded in $H_V^1(\mathbb{R}^3)$, that is

$$\|u_\gamma\|_{H_V^1} < C,$$

where C is independent on γ.

Similarly, if u_γ is a sign-changing solution of (1.4), from Lemma 3.4 and Lemma 4.12, it follows that $\{u_\gamma\}$ is uniformly bounded in $H_V^1(\mathbb{R}^3)$ as well.

Thus, if u_γ is a solution of (1.4), then there exists $u_0 \in H_V^1(\mathbb{R}^3)$ such that, as $\gamma \to 0^+$ passing to a subsequence

$$\begin{aligned} &u_\gamma \rightharpoonup u_0 \quad \text{weakly in } H_V^1(\mathbb{R}^3), \\ &u_\gamma \to u_0 \quad \text{strongly in } L^p(\mathbb{R}^3)\ (p \in [2,6)), \\ &u_\gamma \to u_0 \quad a.e. \text{ on } \mathcal{K} := \text{supp}\,\varphi,\ \varphi \in C_0^\infty(\mathbb{R}^3). \end{aligned}$$

Moreover, there exists a function $\phi \in L^p(\mathbb{R}^3)$ such that $|u_\gamma| \leq \phi$ *a.e.* on $\mathcal{K}$ for all γ.

Since $u_\gamma \rightharpoonup u_0$ weakly in $H_V^1(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3}(\nabla u_\gamma\nabla\varphi + V(x)u_\gamma\varphi)\,dx \to \int_{\mathbb{R}^3}(\nabla u_0\nabla\varphi + V(x)u_0\varphi)\,dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3). \tag{5.5}$$

By conditions (f_1) and (f_2), the Lebesgue dominated theorem and the fact that $u_\gamma \to u_0$ strongly in $L^p(\mathbb{R}^3)$, we get

$$\int_{\mathbb{R}^3} f(x,u_\gamma)\varphi dx \to \int_{\mathbb{R}^3} f(x,u_0)\varphi dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3). \tag{5.6}$$

In what follows, define the following functional:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx - \int_{\mathbb{R}^3} F(x,u)dx.$$

Next we are going to show that $\langle I'(u_0), \varphi\rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^3)$. Indeed, u_γ is a critical point of I_γ, i.e. for all $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} (\nabla u_\gamma \nabla \varphi + V(x)u_\gamma\varphi)\, dx + \gamma\int_{\mathbb{R}^3} \left(|\nabla u_\gamma|^2 u_\gamma \varphi + \nabla u_\gamma \nabla\varphi u_\gamma^2\right) dx& \\ - \int_{\mathbb{R}^3} f(x,u_\gamma)\varphi dx = 0.&\end{aligned} \tag{5.7}$$

On the other hand, by Lemma 5.1,

$$|u_\gamma|_\infty \le C|v_\gamma|_\infty \le C\|v_\gamma\|_{H_V^1}^{\frac{4}{6-p}} \le C$$

and so, from $\|u_\gamma\|_{H_V^1} \le C$,

$$\begin{aligned}&\gamma \int_{\mathbb{R}^3} \left(|\nabla u_\gamma|^2 u_\gamma\varphi + \nabla u_\gamma\nabla\varphi u_\gamma^2\right) dx \\ &\quad\le C\gamma|\varphi|_\infty \int_{\mathbb{R}^3} |\nabla u_\gamma|^2 dx + C\gamma\int_{\mathbb{R}^3} |\nabla u_\gamma||\nabla\varphi| dx \\ &\quad\le C\gamma\left(|\varphi|_\infty |\nabla u_\gamma|_2^2 + |\nabla\varphi|_2|\nabla u_\gamma|_2\right) \to 0, \quad \text{as } \gamma \to 0^+.\end{aligned} \tag{5.8}$$

In view of (5.5)–(5.8), for all $\varphi \in C_0^\infty(\mathbb{R}^3)$, we obtain

$$\int_{\mathbb{R}^3} \left(\nabla u_0 + V(x)u_0 - f(x,u_0)\right)\varphi dx = 0, \tag{5.9}$$

which yields that u_0 is a weak solution of problem (1.6).

Next we will show that the test function φ in (5.7) can be taken as arbitrary functions $\psi \in H_V^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. First, without loss of generality, for $\psi \ge 0$, choose a sequence $\{\varphi_n\} \subset C_0^\infty(\mathbb{R}^3)$ such that $\varphi_n \ge 0$, $\varphi_n \to \psi$ strongly in $H_V^1(\mathbb{R}^3)$, $\varphi_n \to \psi$ a.e. $x \in \mathbb{R}^3$ and $|\varphi_n|_\infty \le |\psi|_\infty + 1$. Take φ_n as the test function in (5.7), letting $n \to \infty$ we know that (5.7) holds for $\varphi = \psi$. Hence we can take $\varphi = u_\gamma$ in (5.7), then

$$\int_{\mathbb{R}^3} \left(|\nabla u_\gamma|^2 + V(x)u_\gamma^2\right) dx + 2\gamma\int_{\mathbb{R}^3} |\nabla u_\gamma|^2 u_\gamma^2 dx - \int_{\mathbb{R}^3} f(x,u_\gamma)u_\gamma dx = 0. \tag{5.10}$$

Since u_0 is a weak solution of (1.6), taking $\varphi = u_0$ in (5.9), we have

$$\int_{\mathbb{R}^3} \left(|\nabla u_0|^2 + V(x)u_0^2\right) dx - \int_{\mathbb{R}^3} f(x,u_0)u_0 dx = 0. \tag{5.11}$$

Similar with (5.6), we obtain

$$\int_{\mathbb{R}^3} f(x,u_\gamma)u_\gamma dx \to \int_{\mathbb{R}^3} f(x,u_0)u_0 dx, \quad \text{as } \gamma \to 0^+. \tag{5.12}$$

By (5.10)–(5.12) and the lower semicontinuity of $\|u_\gamma\|_{H^1_V}$, we get

$$\gamma \int_{\mathbb{R}^3} |\nabla u_\gamma|^2 u_\gamma^2 dx \to 0, \quad \text{as } \gamma \to 0^+$$

and

$$\int_{\mathbb{R}^3} (|\nabla u_\gamma|^2 + V(x) u_\gamma^2)\, dx \to \int_{\mathbb{R}^3} (|\nabla u_0|^2 + V(x) u_0^2)\, dx, \quad \text{as } \gamma \to 0^+.$$

This combined with the fact that $u_\gamma \rightharpoonup u_0$ weakly in $H^1_V(\mathbb{R}^3)$ gives

$$u_\gamma \to u_0 \quad \text{strongly in } H^1_V(\mathbb{R}^3) \quad \text{as } \gamma \to 0^+. \qquad \square$$

Proof of Theorem 1.3. From Lemma 3.3, we know that for all $\gamma \in (0,1]$, there exists a positive critical point $u_{\gamma,1}$. Then, by Lemma 5.2, we obtain $u_{\gamma,1} \to u_1$ strongly in $H^1_V(\mathbb{R}^3)$ as $\gamma \to 0^+$, where u_1 is critical point of I. Note that at this stage, we do not know whether $u_1 \neq 0$. To this end, by Lemma 3.5, we know that

$$0 < m_1 \le I^+_\gamma(u_{\gamma,1})$$

and so, by $u_{\gamma,1} \to u_1$ strongly in $H^1_V(\mathbb{R}^3)$ as $\gamma \to 0^+$,

$$I^+_\gamma(u_1) \ge m_1 > 0.$$

Consequently, $u_1 \neq 0$, then u_1 can be shown to be positive critical point of I^+_γ by applying the maximum principle in [16], that is, u_1 is a positive solution of (1.6). Similarly, we can show u_2 is a negative solution of problem (1.6).

On the other hand, by Lemma 4.12, for all $\gamma \in (0,1]$, there exists a positive constant m_3 such that I_γ has a sign-changing solution $u_{\gamma,3}$ with $I_\gamma(u_{\gamma,3}) \le m_3$. By Lemma 5.2, as $\gamma_i \to 0^+$, there exists a sequence of sign-changing critical points $\{u_{\gamma_i,3}\}$ of I_{γ_i}, converges to a critical point $u_3 \in H^1_V(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ of I. Next, we will show u_3 is a sign-changing critical point of I. Taking $\varphi = (u_{\gamma,3})_+ := u^+_{\gamma,3}$ in the equation $\langle I'_\gamma(u_{\gamma,3}), \varphi\rangle = 0$, by the conditions $(f_1), (f_2)$ and Poincare inequalities and Sobolev inequalities we have

$$\begin{aligned} C\int_{\mathbb{R}^3} (u^+_{\gamma,3})^2 dx + C\left(\int_{\mathbb{R}^3} (u^+_{\gamma,3})^6 dx\right)^{1/3} &\le \int_{\mathbb{R}^3} \left(|\nabla u^+_{\gamma,3}|^2 + V(x)(u^+_{\gamma,3})^2\right) dx \\ &\le \int_{\mathbb{R}^3} f(x, u^+_{\gamma,3}) u^+_{\gamma,3} dx \\ &\le \delta \int_{\mathbb{R}^3} (u^+_{\gamma,3})^2 dx + C_\delta \int_{\mathbb{R}^3} (u^+_{\gamma,3})^6 dx. \end{aligned}$$

This implies that there exists $C > 0$ such that $\int_{\mathbb{R}^3} (u^+_{\gamma,3})^6 dx \ge C$ for $\gamma \in (0,1]$. Now by Lemma 5.2, we have $u_{\gamma,3} \to u_3$ strongly in $H^1_V(\mathbb{R}^3)$ as $\gamma \to 0^+$. This combined with the Sobolev embedding gives

$$\int_{\mathbb{R}^3} (u_{3+})^6 dx = \lim_{\gamma \to 0^+} \int_{\mathbb{R}^3} (u^+_{\gamma,3})^6 dx \ge C > 0.$$

Thereby, we can infer that $u_{3+} \neq 0$. By the same argument we can show $u_{3-} \neq 0$. This completes the proof. $\square$

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