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Biharmonic system with Hartree-type critical nonlinearity

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Abstract. In this article, we investigate the multiplicity results of the following biharmonic Choquard system involving critical nonlinearities with sign-changing weight function:

$$\begin{cases} \Delta^2 u = \lambda F(x)|u|^{r-2}u + H(x)\left(\displaystyle\int_\Omega \frac{H(y)|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u & \text{in } \Omega,\\ \Delta^2 v = \mu G(x)|v|^{r-2}v + H(x)\left(\displaystyle\int_\Omega \frac{H(y)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|v|^{2^*_\alpha-2}v & \text{in } \Omega,\\ u = v = \nabla u = \nabla v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 5$, $1 < r < 2$, $0 < \alpha < N$, $2^*_\alpha = \frac{2N-\alpha}{N-4}$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality and Δ^2 denotes the biharmonic operator. The functions F, G and $H : \overline{\Omega} \to \mathbb{R}$ are sign-changing weight functions satisfying $F, G \in L^{\frac{2^*}{2^*-r}}(\Omega)$ and $H \in L^\infty(\Omega)$ respectively. By adopting Nehari manifold and fibering map technique, we prove that the system admits at least two nontrivial solutions with respect to parameter $(\lambda,\mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$.

Keywords: biharmonic system, sign-changing weight function, Nehari manifold, Hatree-type critical nonlinearity.

2020 Mathematics Subject Classification: 35A15, 35B33.

1 Introduction

We consider the following biharmonic Choquard system involving concave-convex nonlinearities with critical exponent and sign-changing weight functions

$$\begin{cases} \Delta^2 u = \lambda F(x)|u|^{r-2}u + H(x)\left(\displaystyle\int_\Omega \frac{H(y)|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u & \text{in } \Omega,\\ \Delta^2 v = \mu G(x)|v|^{r-2}v + H(x)\left(\displaystyle\int_\Omega \frac{H(y)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|v|^{2^*_\alpha-2}v & \text{in } \Omega,\\ u = v = \nabla u = \nabla v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{D}_{\lambda,\mu})$$

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where Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 5$, $0 < \alpha < N$, $1 < r < 2$, $2^*_\alpha = \frac{2N-\alpha}{N-4}$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality, Δ^2 denotes the biharmonic operator and λ, μ are the parameter such that $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$. We assume the following additive assumptions on the weight functions F, G and H:

$(Z1)$ F, $G \in L^\beta(\Omega)$ with $\beta = \frac{2^*}{2^*-r}$ and $2^* = \frac{2N}{N-4}$, $F^\pm = \max\{\pm F, 0\} \not\equiv 0$ in $\overline{\Omega}$ and $G^\pm = \max\{\pm G, 0\} \not\equiv 0$ in $\overline{\Omega}$.

$(Z2)$ $H \in L^\infty(\Omega)$ and $H^+ = \max\{H, 0\} \not\equiv 0$ in Ω.

Over the last many decades, biharmonic equations have been studied by many authors. These equations have wide application in many physical problems such as phase field models of multi-phase systems, in thin film theory, micro electro-mechanical system, nonlinear surface diffusion on solids, interface dynamics, flow in Hele–Shaw cells, incompressible flows, in theory of elasticity and the deformation of a nonlinear elastic beam (see [16,27,28,33,37]).

In recent years, many researchers are highly attracted to the study of nonlinear Choquard equation because of its applications in physical models (see [35,41]). The origin of nonlinear Choquard equation is related to the work of S. Pekar in 1976 [38] and P. Choquard. They used the elliptic equations with Hardy–Littlewood–Sobolev type nonlinearity to describe the model of an electron trapped in its hole in the Hartree–Fock theory of one component plasma and the quantum theory of a polaron at rest respectively.

Here, we are interested to study the biharmonic system with Choquard type nonlinearity because such type of equations occur in many applications. For this, consider the following Schrödinger–Hartree equation

$$i\partial_t u + \alpha(t)\Delta u + \beta(t)\Delta^2 u = \theta(|x|^{-\lambda} * |u|^2)u = 0, \qquad x \in \mathbb{R}^N, \qquad t \in \mathbb{R}$$
$$u(x, t_0) = u_0(x),\ x \in \mathbb{R}^N,$$

where $u(x,t)$ is a complex valued function in space-time $\mathbb{R}^N \times \mathbb{R}$, $N \geq 1$, α, β are real valued functions denoting the variable dispersion, $\theta \neq 0$ represents the focusing or defocus behaviour and λ is a positive parameter. The above model can be used in nonlinear optics for the electromagnetic wave propagation in optical fibers exhibiting particular nonlinearities, where there exists a repulsive (Hartree) force with strength θ, and when α, β experience variations in time due to the need of balance effect of the nonlinearity and the dispersions ([1,2]).

Towards the study of biharmonic equations, Bernis et al. [5] have examined the following critical biharmonic equation with Dirichlet and Navier boundary conditions

$$\begin{aligned} \Delta^2 u &= \lambda|u|^{q-2}u + |u|^{2^*-2}u, \quad \text{in } \Omega, \\ u &= \frac{\partial u}{\partial n} = 0 \quad \text{or} \quad u = \Delta u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda > 0$, $2^* = \frac{2N}{N-4}$. The authors proved that there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, (1.1) has infinitely many solutions. Moreover, they also showed the existence of at least two positive solutions of (1.1) in the critical case. We suggest some literature ([11,12,15,21,24,32,39]) for reader's convenience and references therein.

Starting with the work of Pekar and Choquard [30,38], there has been a lot of work done involving Laplace, p-Laplace and nonlocal operator with Choquard type nonlinearity (see [9,10,29,36,43]). In [34], Moroz and Schaftingen studied the following Hartree equation (or Choquard equation)

$$-\Delta u + u = (I_\alpha * |u|^p)\,|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \qquad p > 1, \tag{1.2}$$

where I_α denotes the Riesz potential, defined as

$$I_\alpha(x) = \frac{B_\alpha}{|x|^{N-\alpha}}, \quad \text{with} \quad B_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\pi^{\frac{N}{2}}2^\alpha\right)}, \qquad \alpha \in (0, N).$$

and the term $(I_\alpha * |u|^p)\,|u|^{p-2}u$ is also known as Hartree-type nonlinearity. They proved the existence, positivity and radial symmetry of ground state solution. In 2018, Gao and Yang [19] investigated Brézis–Nirenberg type critical Choquard equation regarded as

$$-\Delta u = \left(\int_\Omega \frac{|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dy\right) |u|^{2^*_\alpha - 2}u + \lambda u \quad \text{in } \Omega, \tag{1.3}$$

where Ω is an open and bounded subset in $\mathbb{R}^N$ with Lipschitz boundary, $N \geq 3$, $2^*_\alpha = \frac{2N-\alpha}{N-2}$, $\alpha \in (0, N)$ and λ is a parameter. They established the existence and nonexistence of the nontrivial solution for (1.3) using variational methods. For more literature in this direction, we cite [4, 17, 18, 20, 45] and references therein. Recently, there are few works concerning the system involving nonlinear Choquard term. In [49], You and Zhao studied the following system with critical Choquard type nonlinearity

$$\begin{aligned} -\Delta u + \lambda_1 u &= \mu_1 \left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu}\right) |u|^{2^*_\mu - 1} + \beta \left(\frac{1}{|x|^\mu} * |v|^{2^*_\mu}\right) |u|^{2^*_\mu - 1}, \ x \in \Omega, \\ -\Delta v + \lambda_2 v &= \mu_1 \left(\frac{1}{|x|^\mu} * |v|^{2^*_\mu}\right) |v|^{2^*_\mu - 1} + \beta \left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu}\right) |v|^{2^*_\mu - 1}, \ x \in \Omega, \\ u, v &\geq 0 \quad \text{in } \Omega, \quad u = v = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where $\mu_1, \mu_2 > 0$, $\beta \neq 0$, $-\lambda_1(\Omega) < \lambda_1, \lambda_2 > 0$, $\lambda_1(\Omega)$ is the first eigen value and $2^*_\mu = \frac{2N-\mu}{N-2}$ is the critical exponent in sense of Hardy–Littlewood–Sobolev inequality. The author proved the existence of a positive ground state solution using variational methods. Moreover, for elliptic system involving Laplace and fractional Laplacian with Choquard nonlinearity, we cite [23, 25, 26, 46, 48] and references therein.

Recently, Sang et al. [42] examined the critical Choquard equation with weighted terms and Sobolev-Hardy exponent in the case of Laplacian. They showed the existence of multiple positive solutions corresponding to the problem using variational methods and Lusternik–Schnirelmann category. Afterwards, Rani and Sarika [40] investigated the critical Choquard equation for biharmonic operator involving sign-changing weight functions and proved the multiplicity results analogous to the problem using the method of Nehari manifold and fibering map analysis. Considering all these facts as mentioned above, we have studied the system of critical Choquard equation involving sign-changing weight functions for biharmonic operator and proved the multiplicity results of nontrivial solution related to the system ($\mathcal{D}_{\lambda,\mu}$) with the help of Nehari manifold and fibering map techniques ([7, 8, 13]).

To the best of our knowledge, no work has been done on biharmonic system involving critical Choquard nonlinearity with sign-changing weight function. Apart from that, the minimizers for $S_{H,L}$ demonstrated here are entirely novel in the case of biharmonic system. Moreover, the results obtained in this article are completely fresh and new in the case of Laplacian also however the approach may be familiar.

In this article, we will discuss the existence and multiplicity results of nontrivial solutions for the system ($\mathcal{D}_{\lambda,\mu}$) with respect to parameter λ and μ. Using the Nehari manifold and fibering map analysis [7, 8, 13], we establish the existence of at least two nontrivial solutions for system involving critical Choquard nonlinearities with sign-changing weight functions with respect to the pair of parameters λ, μ belongs to a suitable subset of $\mathbb{R}^2$. The conspicuous aspect of this article is the study of the critical level (c_∞) below which the Palais–Smale condition is satisfied. Altogether, this article amplifies the branch of knowledge and gives a novel addition to the literature of the critical Choquard system.

In order to present our main results, we define the constant Υ_1 as

$$\Upsilon_1 := \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right)^{\frac{2}{2-r}} \left[\frac{2-r}{2(22^*_\alpha - r)}\|H^+\|_\infty^{-2}(\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{1}{2^*_\alpha - 1}} S^{\frac{r}{2-r}},$$

where $\bar{S}_{H,L}$ and S are defined later.

Now we state our following main results.

Theorem 1.1. *If $1 \le r < 2$, $0 < \alpha < N$ and $\lambda, \mu > 0$ satisfy $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, then the system ($\mathcal{D}_{\lambda,\mu}$) has at least one nontrivial solution in $H_0^2(\Omega) \times H_0^2(\Omega)$.*

For multiplicity result, we need the following assumptions on F, G and H respectively:

($Z3$) There exist a_0, b_0 and $r_0 > 0$ such that $B(0, 2r_0) \subset \Omega$ and $F(x) \ge a_0$, $G(x) \ge b_0$ for all $x \in B(0, 2r_0)$.

($Z4$) There exists $\delta_0 > \frac{2N-\alpha}{2}$ such that $\|H^+\|_\infty = H(0) = \max_{x\in\overline{\Omega}} h(x)$, $H(x) > 0$ for all $x \in B(0, 2r_0)$ and
$$H(x) = H(0) + o\left(|x|^{\delta_0}\right) \text{ as } x \to 0.$$

Theorem 1.2. *If $1 \le r < 2$, $0 < \alpha < N$ and $\lambda, \mu > 0$ satisfy $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_2$ (where $\Upsilon_2 \le \Upsilon_1$) , then the system ($\mathcal{D}_{\lambda,\mu}$) has at least two nontrivial solution in $H_0^2(\Omega) \times H_0^2(\Omega)$. Moreover, the solutions corresponding to the system ($\mathcal{D}_{\lambda,\mu}$) are not semi-trivial.*

Remark 1.3. We note that the multiplicity results for the system ($\mathcal{D}_{\lambda,\mu}$) can be generalized to the following polyharmonic system

$$\begin{cases} (-\Delta)^m u = \lambda F(x)|u|^{r-2}u + H(x)\left(\int_\Omega \frac{H(y)|v(y)|^{2^*_{\alpha,m}}}{|x-y|^\alpha}dy\right)|u|^{2^*_{\alpha,m}-2}u & \text{in } \Omega, \\ (-\Delta)^m v = \mu G(x)|v|^{r-2}v + H(x)\left(\int_\Omega \frac{H(y)|u(y)|^{2^*_{\alpha,m}}}{|x-y|^\alpha}dy\right)|v|^{2^*_{\alpha,m}-2}v & \text{in } \Omega, \\ D^k u = D^k v = 0 \quad \text{for all } |k| \le m-1 & \text{on } \partial\Omega, \end{cases}$$

where $(-\Delta)^m$ denotes the polyharmonic operators, $m \in \mathbb{N}$, $N \ge 2m+1$, $0 < \alpha < N$, $1 < r < 2$, $2^*_{\alpha,m} = \frac{2N-\alpha}{N-2m}$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality, and λ, μ are the parameter such that $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$.

Let S be the best Sobolev constant defined as

$$S := \inf_{u \in H_0^m(\Omega)\setminus\{0\}} \frac{\int_\Omega |D^m u|^2 dx}{\left(\int_\Omega |u|^{2^*_m} dx\right)^{\frac{2}{2^*_m}}},$$

where $2^*_m = \frac{2N}{N-2m}$. Then it is well known that S is achieved if and only if $\Omega = \mathbb{R}^N$, by the function

$$U(x) = \frac{C_{N,m}^{\frac{N-2m}{4m}}}{(1+|x|^2)^{\frac{N-2m}{2}}}$$

(see [44]). All the minimizers of S are obtained by

$$U_\epsilon(x) = \epsilon^{\frac{2m-N}{2}} U\left(\frac{x}{\epsilon}\right) = \frac{C_{N,m}^{\frac{N-2m}{4m}} \epsilon^{\frac{N-2m}{2}}}{(\epsilon^2 + |x|^2)^{\frac{N-2m}{2}}},$$

where $\epsilon > 0$ with $C_{N,m} := C(N,m) = \prod_{j=1-m}^{m}(N-2j)$.

Define $S_{H,L}$ to be the best constant as

$$S_{H,L} := \inf_{u \in H_0^m(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |D^m u|^2 dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\alpha,m}}|u(y)|^{2^*_{\alpha,m}}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_{\alpha,m}}}}.$$

One can obtain a family of minimizers for $S_{H,L}$ in the similar manner as shown in section 2 for the case $m = 2$ by taking $\widetilde{U_\epsilon}(x) = S^{\frac{(N-\mu)(2m-N))}{4m(N+2m-\mu)}} (C(N,\alpha))^{\frac{2m-N}{2(N+2m-\mu)}} U_\epsilon(x)$, where $\epsilon > 0$ and $\widetilde{U_\epsilon}(x)$ provides a family of minimizers for $S_{H,L}$. Using the same approach, multiplicity results can be established with respect to parameter λ and μ.

Organization of the article is as follows: In Section 2, variational setting for the problem $(\mathcal{D}_{\lambda,\mu})$ and some essential results are proved. Besides this, we show various asymptotic estimates which perform a vital role in the study of a second solution for the critical case. In Section 3, we discuss that the Palais–Smale condition holds for the energy functional associated with $(\mathcal{D}_{\lambda,\mu})$ at energy level in a suitable range related to the best Sobolev constant. Further, Nehari manifold and fibering map analysis are discussed precisely in Section 4. In Section 5, we prove the existence of Palais–Smale sequences and showed the existence of first nontrivial solution by the proof of Theorem 1.1. In Section 6, we give the detail of proof of the Theorem 1.2.

2 Preliminaries and some important results

We are using Sobolev space $\mathcal{H} := H_0^2(\Omega) \times H_0^2(\Omega)$ as a function space with standard norm $\|(u,v)\| = \left(\|u\|^2 + \|v\|^2\right)^{\frac{1}{2}}$, where $\|u\| = \left(\int_\Omega |\Delta u|^2 dx\right)^{\frac{1}{2}}$ and $\|u\|_p = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$ be the usual $L^p(\Omega)$ norm.

Now, we state the well known Hardy–Littlewood–Sobolev inequality that plays a crucial role in solving the problem involving Choquard type nonlinearity.

Proposition 2.1 (Hardy–Littlewood–Sobolev inequality [31])**.** *Let t, $q > 1$ and $0 < \alpha < N$ with $1/t + \alpha/N + 1/q = 2$, $g \in L^t(\mathbb{R}^N)$ and $h \in L^q(\mathbb{R}^N)$. Then there exists a sharp constant $C(t,N,\alpha,q)$, independent of g, h such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^\alpha} dxdy \leq C(t,N,\alpha,q)\|g\|_{L^t(\mathbb{R}^N)}\|h\|_{L^q(\mathbb{R}^N)}. \tag{2.1}$$

If $t = q = \frac{2N}{2N-\alpha}$ then

$$C(t,N,\alpha,q) = C(N,\alpha) = \pi^{\frac{\alpha}{2}} \frac{\Gamma\left(\frac{N}{2} - \frac{\alpha}{2}\right)}{\Gamma\left(N - \frac{\alpha}{2}\right)} \left\{\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right\}^{-1+\frac{\alpha}{N}}.$$

In this case there is equality in (2.1) if and only if $g \equiv Ch$ and

$$h(x) = A(b^2 + |x-a|^2)^{-\frac{(2N-\alpha)}{2}},$$

for some $A \in \mathbb{C}$, $0 \neq b \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Thus, if $|u|^s \in L^t(\mathbb{R}^N)$ for $t > 1$ such that $\frac{2}{t} + \frac{\alpha}{N} = 2$, then by the Hardy–Littlewood–Sobolev inequality, the integral $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^s|u(y)|^s}{|x-y|^\alpha}dxdy$ is well defined. Hence for $u \in H^2(\mathbb{R}^N)$, by Sobolev embedding theorems, we obtain

$$2_\alpha := \frac{2N-\alpha}{N} \leq s \leq \frac{2N-\alpha}{N-4} =: 2^*_\alpha,$$

where 2_α and 2^*_α are known as lower and upper critical exponent respectively in the sense of Hardy–Littlewood–Sobolev inequality.

Therefore, for all $u \in H^2(\mathbb{R}^N)$, by the Hardy–Littlewood–Sobolev inequality, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy \leq C(N,\alpha)\|u\|_{2^*}^{22^*_\alpha},$$

where $C(N,\alpha)$ is same as defined in Proposition 2.1. One can easily see that in the Hardy–Littlewood–Sobolev inequality, equality takes place if and only if

$$h(x) = C\left(\frac{k}{k^2+|x-a|^2}\right)^{\frac{2N-\alpha}{2}},$$

where $C > 0$ is fixed constant. Thus, $u = C\left(\frac{k}{k^2+|x-a|^2}\right)^{\frac{N-4}{2}}$ if and only if

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}} = (C(N,\alpha))^{\frac{1}{2^*_\alpha}}\left(\int_{\mathbb{R}^N}|u(x)|^{2^*}dx\right)^{\frac{2}{2^*}}. \tag{2.2}$$

Let S be the best Sobolev constant defined as

$$S = \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^2 dx}{(\int_\Omega |u(x)|^{2^*}dx)^{\frac{2}{2^*}}}.$$

The best constant S is attained by the function $U(x) = \frac{[N(N+2)(N-2)(N-4)]^{\frac{N-4}{8}}}{(1+|x|^2)^{\frac{N-4}{2}}}$ and all the minimizers of S are obtained by

$$U_\epsilon(x) = \epsilon^{\frac{4-N}{2}} U\left(\frac{x}{\epsilon}\right), \quad \text{where } \epsilon > 0, \tag{2.3}$$

which satisfies the equation $\Delta^2 u = |u|^{2^*-2}u$ in $\mathbb{R}^N$, with

$$\|U_\epsilon(x)\|^2 = \|U_\epsilon(x)\|_{2^*}^{2^*} = S^{\frac{N}{4}}.$$

Further, we define $S_{H,L}$ to be the best constant as

$$S_{H,L} := \inf_{u \in D^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\Delta u|^2 dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}}.$$

Next, we show the relation between S and $S_{H,L}$ by the following lemma in which the leading concept is taken from [19].

Theorem 2.2. *The constant* $S_{H,L}$ *is achieved if and only if*

$$u = C\left(\frac{k}{k^2+|x-a|^2}\right)^{\frac{N-4}{2}},$$

where $C > 0$ *is a constant,* $a \in \mathbb{R}^N$ *and* $k \in \mathbb{R}^+$*. Furthermore*

$$S_{H,L} = \frac{S}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}}}. \tag{2.4}$$

Proof. The Hardy–Littlewood–Sobolev inequality yields that

$$S_{H,L} \geq \frac{1}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}}} \inf_{u\in D^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\Delta u|^2dx}{\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{\frac{2}{2^*}}} = \frac{S}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}}}.$$

Further, it follows by the definition of $S_{H,L}$ and (2.2) that

$$S_{H,L} \leq \frac{\int_{\mathbb{R}^N}|\Delta u|^2dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}} \leq \frac{\int_{\mathbb{R}^N}|\Delta u|^2dx}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}}\left(\int_{\mathbb{R}^N}|u(x)|^{2^*}dx\right)^{\frac{2}{2^*}}} \leq \frac{S}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}}}.$$

In conclusion, we obtain the required result. □

Take $\widetilde{U_\epsilon}(x) = S^{\frac{(N-\alpha)(4-N)}{8(N+4-\alpha)}}(C(N,\alpha))^{\frac{4-N}{2(N+4-\alpha)}}U_\epsilon(x)$, then $\widetilde{U_\epsilon}$ gives a family of minimizers for $S_{H,L}$ and satisfies the equation

$$\Delta^2 u = \left(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u \quad \text{in } \mathbb{R}^N.$$

Moreover,

$$\int_{\mathbb{R}^N}|\Delta\widetilde{U_\epsilon}|^2dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\widetilde{U_\epsilon}(x)|^{2^*_\alpha}|\widetilde{U_\epsilon}(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy = (S_{H,L})^{\frac{2N-\alpha}{N+4-\alpha}}.$$

Consider the best constant $\bar{S}_{H,L}$ given as

$$\bar{S}_{H,L} := \inf_{u\in\mathcal{H}\setminus\{(0,0)\}} \frac{\|(u,v)\|^2}{\left(\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}}.$$

Now, we state an important lemma which is used to show the relation between $\bar{S}_{H,L}$ and $S_{H,L}$.

Lemma 2.3. *For* $u,v \in L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$, $0<\alpha<N$ *and* $s\in[2_\alpha, 2^*_\alpha]$*, the following inequality holds true*

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^s|v(y)|^s}{|x-y|^\alpha}dxdy\\ &\quad\leq\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^s|u(y)|^s}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)|^s|v(y)|^s}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2}}.\end{aligned}$$

Proof. The proof is similar as given in [22]. □

Afterwards, we build a relation that connecting $\bar{S}_{H,L}$ and $S_{H,L}$ by adopting an idea from [3].

Lemma 2.4. *The following relation holds*

$$\bar{S}_{H,L} = 2S_{H,L}.$$

Proof. Let $\{k_n\} \subset H_0^2(\Omega)$ be a minimizing sequence for $S_{H,L}$. Choose the sequences $\{u_n = sk_n\}$ and $\{v_n = tk_n\}$ in $H_0^2(\Omega)$, where $s, t > 0$. Then the definition of $\bar{S}_{H,L}$ implies that

$$\bar{S}_{H,L} \leq \frac{\|(u_n, v_n)\|^2}{\left(\int_\Omega \int_\Omega \frac{|u_n(x)|^{2_\alpha^*}|v_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2_\alpha^*}}} = \left(\frac{s}{t} + \frac{t}{s}\right) \frac{\|k_n\|^2}{\left(\int_\Omega \int_\Omega \frac{|k_n(x)|^{2_\alpha^*}|k_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2_\alpha^*}}}. \tag{2.5}$$

Further, define a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x) = x + \frac{1}{x}$. Then $f(\frac{s}{t}) = \frac{s}{t} + \frac{t}{s}$ and f achieves its minimum at $x_0 = 1$. Thus, we have

$$\min_{x \in \mathbb{R}^+} f(x) = f(x_0) = 2.$$

Now, choose s, t in such a way that $s = t$ and taking $n \to \infty$ in (2.5), we obtain

$$\bar{S}_{H,L} \leq 2S_{H,L}. \tag{2.6}$$

At the same time, let $\{(u_n, v_n)\}$ be a minimizing sequence of $\bar{S}_{H,L}$. Take $a_n = s_n v_n$ for some $s_n > 0$ such that $\int_\Omega \int_\Omega \frac{|u_n(x)|^{2_\alpha^*}|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy = \int_\Omega \int_\Omega \frac{|a_n(x)|^{2_\alpha^*}|a_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy$.

This together with Lemma 2.3 implies that

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|u_n(x)|^{2_\alpha^*}|a_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy \\
&\quad \leq \left(\int_\Omega \int_\Omega \frac{|u_n(x)|^{2_\alpha^*}|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2}} \left(\int_\Omega \int_\Omega \frac{|a_n(x)|^{2_\alpha^*}|a_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2}} \\
&\quad = \int_\Omega \int_\Omega \frac{|u_n(x)|^{2_\alpha^*}|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{\|(u_n, v_n)\|^2}{\left(\int_\Omega \int_\Omega \frac{|u_n(x)|^{2_\alpha^*}|v_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2_\alpha^*}}} &= s_n \frac{\|(u_n, v_n)\|^2}{\left(\int_\Omega \int_\Omega \frac{|u_n(x)|^{2_\alpha^*}|a_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2_\alpha^*}}} \\
&\geq s_n \frac{\|u_n\|^2}{\left(\int_\Omega \int_\Omega \frac{|u_n(x)|^{2_\alpha^*}|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}\right)^{\frac{1}{2_\alpha^*}}} + s_n^{-1} \frac{\|a_n\|^2}{\left(\int_\Omega \int_\Omega \frac{|a_n(x)|^{2_\alpha^*}|a_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}\right)^{\frac{1}{2_\alpha^*}}} \\
&\geq \left(s_n + s_n^{-1}\right) S_{H,L} \\
&\geq \gamma(x_0) S_{H,L}.
\end{aligned}$$

Now passing the limit as $n \to \infty$

$$\bar{S}_{H,L} \geq 2S_{H,L}. \tag{2.7}$$

We desire our result after combining (2.6) and (2.7). □

Now, we prove some estimates, which are useful to obtain the critical level. Without loss of generality, we may assume that $0 \in \Omega$ and $B(0,2\gamma) \subset \Omega$. Let $\phi \in C_c^\infty(\Omega)$ be a fixed cut-off function such that $0 \le \phi \le 1$ in $\mathbb{R}^N$, $\phi(x) = 1$ on $B_\gamma = B(0,\gamma)$ and $\phi(x) = 0$ in $\mathbb{R}^N \setminus B_{2\gamma}$ with $|\nabla \phi| \le C, |\Delta \phi| \le C$. Define

$$\overline{U}_\epsilon(x) = \phi U_\epsilon(x),$$

where $U_\epsilon(x)$ is define in (2.3). Accordingly, we have the following norm estimates (see[12]).

Lemma 2.5. *The following estimates are true for $\epsilon > 0$ small enough.*

$$\|\overline{U}_\epsilon(x)\|^2 = S^{\frac{N}{4}} + o(\epsilon^{N-4}).$$

$$\int_\Omega |\overline{U}_\epsilon(x)|^{2^*} = S^{\frac{N}{4}} + o(\epsilon^N).$$

$$\int_\Omega |\overline{U}_\epsilon(x)|^r dx = \begin{cases} o\left(\epsilon^{\frac{N-4}{2}r}\right), & r < \frac{N}{N-4} \\ o\left(\epsilon^{N-\frac{N-4}{2}r}|\ln \epsilon|\right), & r = \frac{N}{N-4} \\ o\left(\epsilon^{N-\frac{N-4}{2}r}\right), & r > \frac{N}{N-4}. \end{cases} \tag{2.8}$$

Lemma 2.6. *For Choquard term, the following estimate is true:*

$$\begin{aligned} 0 &\le \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N-4}{4}} - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right) \\ &\le \left(\int_\Omega \int_\Omega H(x)H(y) \frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha} |\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}} \\ &\le \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N-4}{4}}. \end{aligned} \tag{2.9}$$

Proof. By assumption $(Z2)$, there exists $0 < \gamma \le r_0$ such that for all $x \in B(0,2\gamma)$ with $\delta_0 > \frac{2N-\alpha}{2}$

$$H(x) = H(0) + o(|x|^{\delta_0}), \quad \text{as } x \to 0. \tag{2.10}$$

Using the Hardy–Littlewood–Sobolev inequality and (2.4), we have

$$\begin{aligned} \left(\int_\Omega \int_\Omega H(x)H(y) \frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha} |\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}} &\le \|H^+\|_\infty^{\frac{2}{2^*_\alpha}} (C(N,\alpha))^{\frac{1}{2^*_\alpha}} \|\overline{U}_\epsilon(x)\|_{2^*}^2 \\ &= \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N-4}{4}} + o(\epsilon^{N-4}). \end{aligned}$$

Thus

$$\int_\Omega \int_\Omega H(x)H(y) \frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha} |\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy \le \|H^+\|_\infty^2 (C(N,\alpha))^{\frac{N}{4}} S_{H,L}^{\frac{2N-\alpha}{4}} + o(\epsilon^{2N-\alpha}).$$

Consider

$$\begin{aligned} &\epsilon^{\alpha-2N} \|H^+\|_\infty^2 (C(N,\alpha))^{\frac{N}{4}} S_{H,L}^{\frac{2N-\alpha}{4}} - \epsilon^{\alpha-2N} \int_\Omega \int_\Omega H(x)H(y) \frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha} |\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy \\ &= A^{2N-\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(H(0))^2}{(\epsilon^2 + |x|^2)^{\frac{2N-\alpha}{2}} (\epsilon^2 + |y|^2)^{\frac{2N-\alpha}{2}} |x-y|^\alpha} \\ &\quad - \epsilon^{\alpha-2N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} H(x)H(y) \frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha} |\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} \end{aligned}$$

$$
\begin{aligned}
&= A^{2N-\alpha}\left[\int_{\mathbb{R}^N\setminus B_\gamma}\frac{H(0)\left(H(0)-H(x)|\phi(x)|^{2^*_\alpha}\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N}\frac{1}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\right.\\
&\qquad+\int_{\mathbb{R}^N\setminus B_\gamma}\frac{H(x)|\phi(x)|^{2^*_\alpha}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N}\frac{H(0)-H(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\\
&\qquad+\int_{B_\gamma}\frac{H(0)\left(H(0)-H(x)\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N}\frac{1}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\\
&\qquad\left.+\int_{B_\gamma}\frac{H(x)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N}\frac{H(0)-H(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\right]\\
&= E_1+E_2+E_3+E_4, \qquad (2.11)
\end{aligned}
$$

where $A=[N(N+2)(N-2)(N-4)]^{\frac{N-4}{8}}$.

On taking E_1, we have

$$
\begin{aligned}
E_1 &= A^{2N-\alpha}\left[\int_{\mathbb{R}^N\setminus B_\gamma}\frac{H(0)\left(H(0)-H(x)|\phi(x)|^{2^*_\alpha}\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{1}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\right.\\
&\quad\left.+\int_{\mathbb{R}^N\setminus B_\gamma}\frac{H(0)\left(H(0)-H(x)|\phi(x)|^{2^*_\alpha}\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{B_\gamma}\frac{1}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\right]\\
&= E_{1,1}+E_{1,2}.
\end{aligned}
$$

Applying the Hardy–Littlewood–Sobolev inequality on $E_{1,1}$ and $E_{1,2}$ respectively, we get

$$
\begin{aligned}
E_{1,1} &\le C_1\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{dx}{(\epsilon^2+|x|^2)^N}\right)^{\frac{2N-\alpha}{2N}}\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{dy}{(\epsilon^2+|y|^2)^N}\right)^{\frac{2N-\alpha}{2N}}\\
&= C_1\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{dt}{(\epsilon^2+|t|^2)^N}\right)^{\frac{2N-\alpha}{N}}\le C_1\left(\int_\gamma^\infty\frac{r^{N-1}}{r^{2N}}dr\right)^{\frac{2N-\alpha}{N}}=C_2.
\end{aligned}
$$

and

$$
\begin{aligned}
E_{1,2} &\le C_3\int_{\mathbb{R}^N\setminus B_\gamma}\int_{B_\gamma}\frac{1}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dxdy\\
&\le C_3\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{dx}{(\epsilon^2+|x|^2)^N}\right)^{\frac{2N-\alpha}{2N}}\left(\int_{B_\gamma}\frac{dy}{(\epsilon^2+|y|^2)^N}\right)^{\frac{2N-\alpha}{2N}}\\
&\le C_4\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{dx}{|x|^{2N}}\right)^{\frac{2N-\alpha}{2N}}\left(\int_0^\gamma\frac{r^{N-1}dr}{(\epsilon^2+r^2)^N}\right)^{\frac{2N-\alpha}{2N}}\\
&\le o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right)\left(\int_0^{\frac{\gamma}{\epsilon}}\frac{t^{N-1}dt}{(1+t^2)^N}\right)^{\frac{2N-\alpha}{2N}}\\
&\le o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right)\left(\int_0^\infty\frac{t^{N-1}dt}{(1+t^2)^N}\right)^{\frac{2N-\alpha}{2N}}=o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).
\end{aligned}
$$

Thus

$$
E_1=C_2+o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).
$$

Further on taking E_2, we obtain

$$\begin{aligned} E_2 = A^{2N-\alpha} \Bigg[&\int_{\mathbb{R}^N \setminus B_\gamma} \frac{H(x)|\phi(x)|^{2^*_\alpha}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}} \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{H(0)-H(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dy \right) dx \\ &+ \int_{\mathbb{R}^N \setminus B_\gamma} \frac{H(x)|\phi(x)|^{2^*_\alpha}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}} \left(\int_{B_\gamma} \frac{H(0)-H(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dy \right) dx \Bigg] \\ = E_{2,1} + E_{2,2}. \end{aligned}$$

Now estimating $E_{2,1}$ same as $E_{1,1}$, we have

$$E_{2,1} \leq C_5 \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dy}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} = C_6.$$

Using the Hardy–Littlewood–Sobolev inequality $E_{2,2}$ and (2.10), we get

$$\begin{aligned} E_{2,2} &\leq C_7 \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{B_\gamma} \frac{|y|^{\frac{2N\delta_0}{2N-\alpha}}}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\ &\leq C_8 \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{B_\gamma} \frac{|y|^{\frac{2N\delta_0}{2N-\alpha}}}{|y|^{2N}} \right)^{\frac{2N-\alpha}{2N}} \\ &= C_9. \end{aligned}$$

Hence

$$E_2 = C_6 + C_9.$$

For E_3, we use the Hardy–Littlewood–Sobolev inequality with (2.10) which implies that

$$\begin{aligned} E_3 = A^{2N-\alpha} \Bigg[&\int_{B_\gamma} \int_{\mathbb{R}^N \setminus B_\gamma} \frac{H(0)\,(H(0)-H(x))}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dxdy \\ &+ \int_{B_\gamma} \int_{B_\gamma} \frac{H(0)\,(H(0)-H(x))}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dxdy \Bigg] \\ = E_{3,1} + E_{3,2}. \end{aligned}$$

$$\begin{aligned} E_{3,1} &\leq A^{2N-\alpha} \int_{B_\gamma} \int_{\mathbb{R}^N \setminus B_\gamma} \frac{H(0)|x|^{\delta_0}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dxdy \\ &\leq A^{2N-\alpha} \left(\int_{B_\gamma} \frac{|x|^{\frac{2N\delta_0}{2N-\alpha}}dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dy}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\ &\leq C_{10} \left(\int_{B_\gamma} \frac{|x|^{\frac{2N\delta_0}{2N-\alpha}}dx}{|x|^{2N}} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dy}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\ &= C_{11}. \end{aligned}$$

$$\begin{aligned} E_{3,2} &\leq A^{2N-\alpha} \int_{B_\gamma}\int_{B_\gamma} \frac{H(0)|x|^{\delta_0}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dxdy \\ &\leq A^{2N-\alpha}\left(\int_{B_\gamma} \frac{|x|^{\frac{2N\delta_0}{2N-\alpha}}dx}{(\epsilon^2+|x|^2)^N}\right)^{\frac{2N-\alpha}{2N}} \left(\int_{B_\gamma}\frac{dy}{(\epsilon^2+|y|^2)^N}\right)^{\frac{2N-\alpha}{2N}} \\ &\leq o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right)\left(\int_{B_\gamma}\frac{|x|^{\frac{2N\delta_0}{2N-\alpha}}dx}{|x|^{2N}}\right)^{\frac{2N-\alpha}{2N}}\left(\int_0^{\frac{\gamma}{\epsilon}}\frac{r^{N-1}dr}{(1+r^2)^N}\right)^{\frac{2N-\alpha}{2N}} \\ &\leq o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right)\left(\int_0^\infty \frac{r^{N-1}dr}{r^{2N}}\right)^{\frac{2N-\alpha}{2N}} = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right). \end{aligned}$$

Thus

$$E_3 = C_{11} + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).$$

Similarly on taking E_4, we have

$$\begin{aligned} E_4 = A^{2N-\alpha}&\left[\int_{B_\gamma}\frac{H(x)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{H(0)-H(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\right. \\ &\left. + \int_{B_\gamma}\int_{B_\gamma}\frac{H(x)\left(H(0)-H(y)\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dxdy\right] \\ = E_{4,1}&+E_{4,2}. \end{aligned}$$

By the same approach used in $E_{1,2}$ and $E_{3,2}$ respectively, we obtain

$$E_{4,1} = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) \text{ and } E_{4,2} = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).$$

Hence

$$E_4 = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).$$

Therefore

$$E_1+E_2+E_3+E_4 = \widehat{C} + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right),$$

where $\widehat{C} = C_2 + C_9 + C_{12}$.

Using (2.11), we obtain

$$\begin{aligned} 0 &\leq \epsilon^{\alpha-2N}\|H^+\|_\infty^2 (C(N,\alpha))^{\frac{N}{4}} S_{H,L}^{\frac{2N-\alpha}{4}} - \epsilon^{\alpha-2N}\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy \\ &\leq \widehat{C} + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right). \end{aligned}$$

This implies that

$$\begin{aligned} 0 &\leq 1 - \|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}} S_{H,L}^{-\frac{2N-\alpha}{4}} \int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy \\ &\leq \epsilon^{2N-\alpha}\|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}} S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C} + o\left(\epsilon^{\frac{2N-\alpha}{2}}\right). \end{aligned}$$

Furthermore

$$
\begin{aligned}
0 &\leq 1-\epsilon^{2N-\alpha}\|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C}-o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\\
&\leq \|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\\
&\leq 1.
\end{aligned}
$$

Now, choose $\epsilon>0$ such that $\epsilon^{2N-\alpha}\|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C}<1$. Thus

$$
\begin{aligned}
0 &\leq 1-\epsilon^{2N-\alpha}\|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C}-o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\\
&\leq \left(1-\epsilon^{2N-\alpha}\|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C}-o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\right)^{\frac{1}{2^*_\alpha}}\\
&\leq \|H^+\|_\infty^{-\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{-\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{-\frac{N-4}{4}}\left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}\\
&\leq 1.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
0 &\leq \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N-4}{4}}\\
&\quad -\epsilon^{2N-\alpha}\|H^+\|_\infty^{\frac{2(4-N+\alpha)}{2N-\alpha}}(C(N,\alpha))^{\frac{(\alpha-N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{\alpha-N-4}{4}}\widehat{C}-o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\\
&\leq \left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}\leq \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N-4}{4}}.
\end{aligned}
$$

Thus, we can write

$$
\begin{aligned}
0 &\leq \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N-4}{4}}-o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\\
&\leq \left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}\\
&\leq \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N-4}{4}}.
\end{aligned}
$$

Thus, the proof is complete. □

Definition 2.7. A pair of functions $(u,v)\in\mathcal{H}$ is said to be a weak solution of the system ($\mathcal{D}_{\lambda,\mu}$) if for all $(\phi_1,\phi_2)\in\mathcal{H}$, the following holds

$$
\begin{aligned}
&\int_\Omega \Delta u\Delta\phi_1 dx+\int_\Omega\Delta v\Delta\phi_2 dx-\lambda\int_\Omega F(x)|u|^{r-2}u\phi_1 dx-\mu\int_\Omega G(x)|v|^{r-2}v\phi_2 dx\\
&\quad-\int_\Omega\int_\Omega H(x)H(y)\left(\frac{|v(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha-2}u(y)\phi_1(y)+|u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha-2}v(y)\phi_2(y)}{|x-y|^\alpha}\right)=0.
\end{aligned}
$$

In order to prove the Palais–Smale condition, we need the following lemma which is inspired by the Brézis–Lieb convergence lemma (see [6]).

Lemma 2.8. *Let $N\geq 5$, $0<\alpha<N$ and $\{u_n\}$ be a bounded sequence in $L^{\frac{2N}{N-4}}(\mathbb{R}^N)$. If $u_n\to u$ a.e. in $\mathbb{R}^N$ as $n\to\infty$, then*

$$
\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}(|x|^{-\alpha}*|u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha}-\int_{\mathbb{R}^N}(|x|^{-\alpha}*|u_n-u|^{2^*_\alpha})|u_n-u|^{2^*_\alpha}\right)=\int_{\mathbb{R}^N}(|x|^{-\alpha}*|u|^{2^*_\alpha})|u|^{2^*_\alpha}.
$$

Proof. The proof is similar to the proof of the Brézis–Lieb Lemma (see [6]) or Lemma 2.2 [19]. But for completeness, we give the detail. Consider

$$\begin{aligned}\int_{\mathbb{R}^N}(|x|^{-\alpha}*|u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha}-\int_{\mathbb{R}^N}(|x|^{-\alpha}*|u_n-u|^{2^*_\alpha})|u_n-u|^{2^*_\alpha}\\ =\int_{\mathbb{R}^N}(|x|^{-\alpha}*(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha}))(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha})\\ +2\int_{\mathbb{R}^N}(|x|^{-\alpha}*(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha}))|u_n-u|^{2^*_\alpha}.\end{aligned}\tag{2.12}$$

Now by using [34, Lemma 2.5], for $q=2^*_\alpha=\frac{2N-\alpha}{N-4}$ and $r=\frac{2N}{2N-\alpha}2^*_\alpha$, then we obtain

$$|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha}\to|u|^{2^*_\alpha}\quad\text{in }\ L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)\ \text{ as }\ n\to\infty.\tag{2.13}$$

Also the Hardy–Littlewood–Sobolev inequality implies that

$$|x|^{-\alpha}*(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha})\to|x|^{-\alpha}*|u|^{2^*_\alpha}\quad\text{in }\ L^{\frac{2N}{\alpha}}(\mathbb{R}^N)\ \text{ as }\ n\to\infty.\tag{2.14}$$

Hence with the help of [47, Proposition 5.4.7], we obtain $|u_n-u|^{2^*_\alpha}\rightharpoonup 0$ weakly in $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$ as $n\to\infty$. So using this together with (2.13), (2.14), in (2.12), we obtain the required result. □

Now, we define the energy functional $I_{\lambda,\mu}:\mathcal{H}\to\mathbb{R}$ associated with the system ($\mathcal{D}_{\lambda,\mu}$) as

$$\begin{aligned}I_{\lambda,\mu}(u,v)&=\frac{1}{2}\|(u,v)\|^2-\frac{1}{r}\int_\Omega(\lambda F(x)|u|^r+\mu G(x)|v|^r)\\ &\quad-\frac{1}{2^*_\alpha}\int_\Omega\int_\Omega H(x)H(y)\left(\frac{|u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right).\end{aligned}\tag{2.15}$$

Then $I_{\lambda,\mu}(u,v)$ is C^1 function on $\mathcal{H}$. Moreover, the critical points of the functional $I_{\lambda,\mu}$ are the solutions of ($\mathcal{D}_{\lambda,\mu}$). For convenience, we define $P_{\lambda,\mu}(u,v)$ and $Q(u,v)$ as

$$\begin{aligned}P_{\lambda,\mu}(u,v)&:=\int_\Omega(\lambda F(x)|u|^r+\mu G(x)|v|^r)dx,\\ Q(u,v)&:=\int_\Omega\int_\Omega H(x)H(y)\left(\frac{|u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)dxdy,\end{aligned}$$

throughout the article. Then we obtain the estimates on $P_{\lambda,\mu}(u,v)$ and $Q(u,v)$ by using Hölder's inequality, Sobolev's embedding theorem and the definition of $\bar{S}_{H,L}$ as follows

$$\begin{aligned}P_{\lambda,\mu}(u,v)&=\int_\Omega(\lambda F(x)|u|^r+\mu G(x)|v|^r)dx\\ &\le S^{-\frac{r}{2}}\left(\lambda\|F\|_\beta\|u\|^r+\mu\|G\|_\beta\|v\|^r\right)\\ &\le S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}}+(\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\|(u,v)\|^r.\end{aligned}\tag{2.16}$$

$$Q(u,v)\le\|H^+\|^2_\infty(\bar{S}_{H,L})^{-2^*_\alpha}\|(u,v)\|^{22^*_\alpha}.\tag{2.17}$$

Definition 2.9. Let $J:X\to\mathbb{R}$ be a C^1 functional on a Banach space X.

1. For $c\in\mathbb{R}$, a sequence $\{u_k\}\subset X$ is a Palais–Smale sequence at level c ($(PS)_c$) in X for J if $J(u_k)=c+o_k(1)$ and $J'(u_k)\to 0$ in X^{-1} as $k\to\infty$.
2. We say J satisfies $(PS)_c$-condition if for any Palais–Smale sequence $\{u_k\}$ in X for J has a convergent subsequence.

3 The Palais–Smale condition

In this section, we show that the energy functional $\mathcal{I}_\lambda$ satisfies the Palais–Smale condition below a certain level i.e. c_∞, which is used to prove the existence of second solution.

Lemma 3.1. *Consider* $(Z1)$ *and* $(Z2)$ *are true. Suppose* $\{(u_n, v_n\} \subset \mathcal{H}$ *is a* $(PS)_c$*-sequence for* $I_{\lambda,\mu}$ *such that* $(u_n, v_n) \rightharpoonup (u, v)$ *weakly in* $\mathcal{H}$*. Then* $I'_{\lambda,\mu}(u, v) = 0$*. Furthermore, there exists a positive constant* K_0 *depending on* $r, \alpha, N, 2^*_\alpha$ *and* S *such that*

$$I_{\lambda,\mu}(u, v) \geq -K_0 \left((\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \right),$$

where $K_0 = \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) \left(\frac{2-r}{2}\right) \left[\left(\frac{2N-\alpha}{N+4-\alpha}\right) \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}} \right]^{\frac{r}{2-r}} S^{-\frac{r}{2}}.$

Proof. If $\{(u_n, v_n)\}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ with $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$, then by using the standard argument, we get $I'_{\lambda,\mu}(u, v) = 0$. i.e.

$$\|(u, v)\|^2 - P_{\lambda,\mu}(u, v) - 2Q(u, v) = 0.$$

Above with Hölder's inequality, Sobolev embedding theorem and Young's inequality in (2.15) implies that

$$\begin{aligned}
I_{\lambda,\mu}(u, v) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u, v)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) \int_\Omega (\lambda F(x)|u|^r + \mu G(x)|v|^r) \, dx \\
&\geq \frac{N+4-\alpha}{2(2N-\alpha)} \|(u, v)\|^2 - \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) \left((\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \right)^{\frac{2-r}{2}} S^{-\frac{r}{2}} \|(u, v)\|^r \\
&\geq \frac{N+4-\alpha}{2(2N-\alpha)} \|(u, v)\|^2 \\
&\quad - \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}} \left[\frac{2-r}{2} l^{\frac{2}{2-r}} \left((\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \right) + \frac{r}{2} l^{-\frac{2}{r}} \|(u, v)\|^2 \right] \\
&= K_0 \left((\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \right),
\end{aligned}$$

where, $K_0 = \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) \left(\frac{2-r}{2}\right) \left[\left(\frac{2N-\alpha}{N+4-\alpha}\right) \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}} \right]^{\frac{r}{2-r}} S^{-\frac{r}{2}}$ and $l = \left[\left(\frac{2N-\alpha}{N+4-\alpha}\right) \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}} \right]^{\frac{r}{2}}$. This completes the proof. □

Lemma 3.2. *Assume* $\{(u_n, v_n)\} \subset \mathcal{H}$ *is a* $(PS)_c$*-sequence for* $I_{\lambda,\mu}$*, then* $\{(u_n, v_n)\}$ *is bounded in* $\mathcal{H}$*.*

Proof. Let $\{(u_n, v_n)\}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ in $\mathcal{H}$, then as per the definition of $(PS)_c$-sequence, $I_{\lambda,\mu}(u_n, v_n) \to c$ and $I'_{\lambda,\mu}(u_n, v_n) \to 0$ in $\mathcal{H}^{-1}$ i.e.

$$\frac{1}{2} \|(u_n, v_n)\|^2 - \frac{1}{r} P_{\lambda,\mu}(u_n, v_n) - \frac{1}{2^*_\alpha} Q(u_n, v_n) = c + o_n(1), \tag{3.1}$$

$$\|(u_n, v_n)\|^2 - P_{\lambda,\mu}(u_n, v_n) - Q(u_n, v_n) = o_n(1). \tag{3.2}$$

Now, our aim is to show that $\{(u_n, v_n)\}$ is bounded. On contrary, assume that $\|(u_n, v_n)\| \to \infty$ as $n \to \infty$ and take $(\widehat{u}_n, \widehat{v}_n) := \frac{(u_n, v_n)}{\|(u_n, v_n)\|}$. It follows that $\{(\widehat{u}_n, \widehat{v}_n)\}$ is a bounded sequence. Consequently, up to a subsequence $(\widehat{u}_n, \widehat{v}_n) \rightharpoonup (\widehat{u}, \widehat{v})$ weakly in $\mathcal{H}$, $(\widehat{u}_n, \widehat{v}_n) \to (\widehat{u}, \widehat{v})$ strongly in $L^m(\Omega)$ for all $1 \leq m < 2^*$ and $(\widehat{u}_n(x), \widehat{v}_n(x)) \to (\widehat{u}(x), \widehat{v}(x))$ pointwise a.e. in $\Omega \times \Omega$.
Using (3.1) and (3.2), we have

$$\frac{1}{2} \|(\widehat{u}_n, \widehat{v}_n)\|^2 - \frac{1}{r} \|(u_n, v_n)\|^{r-2} P_{\lambda,\mu}(\widehat{u}_n, \widehat{v}_n) - \frac{1}{2^*_\alpha} \|(u_n, v_n)\|^{22^*_\alpha - 2} Q(\widehat{u}_n, \widehat{v}_n) = o_n(1), \tag{3.3}$$

and

$$\|(\widehat{u}_n,\widehat{v}_n)\|^2 - \|(u_n,v_n)\|^{r-2}P_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n) - \|(u_n,v_n)\|^{22^*_\alpha-2}Q(\widehat{u}_n,\widehat{v}_n) = o_n(1). \quad (3.4)$$

From (3.3) and (3.4), we can deduce that

$$\|(\widehat{u}_n,\widehat{v}_n)\|^2 = \frac{2(2^*_\alpha - r)}{r(2^*_\alpha-2)}\|(u_n,v_n)\|^{r-2}P_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n) + o_n(1). \quad (3.5)$$

Since $1 \le r < 2$ and $\|(u_n,v_n)\| \to \infty$, then (3.5) implies $\|(\widehat{u}_n,\widehat{v}_n)\|^2 \to 0$ as $n \to \infty$, which is a contradiction to the fact that $\|(\widehat{u}_n,\widehat{v}_n)\| = 1$. Thus, proof is completed. □

Lemma 3.3. *There exists*

$$c_\infty := \frac{N+4-\alpha}{2(2N-\alpha)}\left(\frac{\|H^+\|_\infty^{-2}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - K_0\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right),$$

such that the energy functional $I_{\lambda,\mu}$ satisfies the $(PS)_c$-condition with $c \in (-\infty, c_\infty)$ and K_0 is defined in Lemma 3.1.

Proof. Let $\{(u_n,v_n)\} \subset \mathcal{H}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ with $0 < c < c_\infty$. Then by Lemma 3.2, $\{(u_n,v_n)\}$ is a bounded sequence in $\mathcal{H}$. Thus, up to a subsequence, $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $\mathcal{H}$. So $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $H^2_0(\Omega)$, $u_n \to u$ and $v_n \to v$ strongly in $L^m(\Omega)$ for all $1 \le m < 2^*$ and $u_n \to u$, $v_n \to v$ pointwise a.e. in Ω. Therefore

$$P_{\lambda,\mu}(u_n,v_n) = P_{\lambda,\mu}(u,v) + o_n(1). \quad (3.6)$$

Also, $I'_{\lambda,\mu}(u,v) = 0$, follows from Lemma 3.1. Now, define $(\tilde{u}_n,\tilde{v}_n)$, where $\tilde{u}_n = u_n - u$, $\tilde{v}_n = v_n - v$. Then by the Brézis–Lieb lemma [6] and Lemma 2.8, we have

$$\begin{aligned}\|(\tilde{u}_n,\tilde{v}_n)\|^2 &= \|(u_n,v_n)\|^2 - \|(u,v)\|^2 + o_n(1),\\ Q(u_n,v_n) &= Q(\tilde{u}_n,\tilde{v}_n) + Q(u,v) + o_n(1). \end{aligned}\quad (3.7)$$

Using $I_{\lambda,\mu}(u_n,v_n) = c + o_n(1)$, $I'_{\lambda,\mu}(u_n,v_n) = o_n(1)$ and (3.6)–(3.7), we obtain

$$\frac{1}{2}\|(\tilde{u}_n,\tilde{v}_n)\|^2 - \frac{1}{2^*_\alpha}Q(\tilde{u}_n,\tilde{v}_n) = c - I_{\lambda,\mu}(u,v) + o_n(1), \quad (3.8)$$

and

$$\|(\tilde{u}_n,\tilde{v}_n)\|^2 - 2Q(\tilde{u}_n,\tilde{v}_n) = \langle I'_{\lambda,\mu}(u,v),(u_n-u,v_n-v)\rangle + o_n(1) = o_n(1).$$

Therefore, we may assume that

$$\|(\tilde{u}_n,\tilde{v}_n)\|^2 \to d, \quad \text{and} \quad 2\int_\Omega\int_\Omega H(x)H(y)\frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{v}_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} \to d. \quad (3.9)$$

It follows from the definition of $\bar{S}_{H,L}$ that

$$\begin{aligned}\|(\tilde{u}_n,\tilde{v}_n)\|^2 &\ge \bar{S}_{H,L}\left(\int_\Omega\int_\Omega \frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{v}_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)^{\frac{1}{2^*_\alpha}}\\ &\ge \bar{S}_{H,L}\|H^+\|_\infty^{-\frac{2}{2^*_\alpha}}\left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{v}_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)^{\frac{1}{2^*_\alpha}}. \end{aligned}\quad (3.10)$$

On combining (3.9) and (3.10), we have

$$d \geq \bar{S}_{H,L} \|H^+\|_\infty^{-\frac{2}{2^*_\alpha}} \left(\frac{d}{2}\right)^{\frac{1}{2^*_\alpha}},$$

which gives either

$$d = 0 \quad \text{or} \quad d \geq \left(\frac{\|H^+\|_\infty^{-\frac{2}{2^*_\alpha}}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}.$$

Further, if $d = 0$ then the proof is complete. If

$$d \geq \left(\frac{\|H^+\|_\infty^{-\frac{2}{2^*_\alpha}}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}},$$

then according to (3.8), (3.9) and Lemma 3.1, we get

$$\begin{aligned} c &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) d + I_{\lambda,\mu}(u,v) \\ &\geq \frac{N+4-\alpha}{2(2N-\alpha)} \left(\frac{\|H^+\|_\infty^{-2}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - K_0 \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right) =: c_\infty, \end{aligned}$$

a contradiction to $c < c_\infty$. Hence, $d = 0$ and with this we end the proof. □

4 Nehari manifold and fibering map analysis

In this section, we elaborate some important results for Nehari manifold and analysis of fibering map on $I_{\lambda,\mu}$. Notice that the energy functional $I_{\lambda,\mu}$ is unbounded below on $\mathcal{H}$. So we restrict $I_{\lambda,\mu}$ on an appropriate subset $\mathcal{N}_{\lambda,\mu}$ of $\mathcal{H}$, called Nehari manifold and defined as

$$\mathcal{N}_{\lambda,\mu} := \left\{(u,v) \in \mathcal{H} \setminus \{(0,0)\} : \langle I'_{\lambda,\mu}(u,v), (u,v)\rangle = 0\right\}.$$

Thus, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\langle I'_{\lambda,\mu}(u,v), (u,v)\rangle = \|(u,v)\|^2 - P_{\lambda,\mu}(u,v) - 2Q(u,v) = 0. \tag{4.1}$$

Next, we see that $I_{\lambda,\mu}$ is bounded from below on $\mathcal{N}_{\lambda,\mu}$ in the following lemma.

Lemma 4.1. *The energy functional $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Let $(u,v) \in \mathcal{N}_{\lambda,\mu}$ for $\lambda, \mu > 0$, then using (4.1) and (2.16), we have

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u,v)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u,v) \\ &\geq \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u,v)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) S^{-\frac{r}{2}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} \|(u,v)\|^r, \end{aligned} \tag{4.2}$$

Since $1 < r < 2$. Therefore, $I_{\lambda,\mu}$ is coercive.

Now, consider the function $\varrho : \mathbb{R} \to \mathbb{R}$ as $\varrho(t) = b_1 t^2 - b_2 t^r$. Then one can easily see that $\varrho'(t) = 0$ if and only if $t = \left(\frac{b_2 r}{2b_1}\right)^{\frac{1}{2-r}} =: t^*$ and $\varrho''(t^*) > 0$. So ϱ attains its minimum at t^*. Moreover,

$$\varrho(t) \geq \varrho(t^*) := -(2-r)\left(\frac{b_2}{2}\right)^{\frac{2}{2-r}}\left(\frac{r}{b_1}\right)^{\frac{r}{2-r}}.$$

Taking $b_1 = \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right)$, $b_2 = \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right)S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}$ and $t = \|(u,v)\|$ in the function ϱ, we obtain

$$I_{\lambda,\mu}(u,v) \geq \varrho(\|(u,v)\|) \geq \varrho(t^*).$$

which yields the required assertion. □

The Nehari manifold is intently related to the behaviour of map $\Psi_{u,v} : t \to I_{\lambda,\mu}(tu,tv)$ for $t > 0$, defined as

$$\Psi_{u,v}(t) := I_{\lambda,\mu}(tu,tv) = \frac{t^2}{2}\|(u,v)\|^2 - \frac{t^r}{r}P_{\lambda,\mu}(u,v) - \frac{t^{22^*_\alpha}}{2^*_\alpha}Q(u,v).$$

These maps are known as fibering maps which were introduced by Drábek and Pohozaev in [13]. Thus, $(tu,tv) \in \mathcal{N}_{\lambda,\mu}$ iff $\Psi'_{u,v}(t) = 0$. Furthermore

$$\Psi'_{u,v}(t) = t\|(u,v)\|^2 - t^{r-1}P_{\lambda,\mu}(u,v) - 2t^{2^*_\alpha - 1}Q(u,v),$$
$$\Psi''_{u,v}(t) = \|(u,v)\|^2 - (r-1)t^{r-2}P_{\lambda,\mu}(u,v) - 2(22^*_\alpha - 1)t^{22^*_\alpha - 2}Q(u,v).$$

In particular, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\Psi'_{u,v}(1) = 0$. Therefore it is obvious to split $\mathcal{N}_{\lambda,\mu}$ into three parts namely $\mathcal{N}^+_{\lambda,\mu}$, $\mathcal{N}^-_{\lambda,\mu}$ and $\mathcal{N}^0_{\lambda,\mu}$ corresponding to local minima, local maxima and point of inflexion respectively as:

$$\mathcal{N}^\pm_{\lambda,\mu} := \left\{(u,v) \in \mathcal{N}_{\lambda,\mu} : \Psi''_{u,v}(1) \gtrless 0\right\}, \qquad \mathcal{N}^0_{\lambda,\mu} := \left\{(u,v) \in \mathcal{N}_{\lambda,\mu} : \Psi''_{u,v}(1) = 0\right\}.$$

We note that, for $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\Psi''_{u,v}(1) = \begin{cases} (2 - 22^*_\alpha)\|(u,v)\|^2 - (r - 22^*_\alpha)P_{\lambda,\mu}(u,v) \\ (2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v). \end{cases} \tag{4.3}$$

In next lemma, we will show that the local minimizers of $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ are the critical points of $I_{\lambda,\mu}$.

Lemma 4.2. *If (u,v) is the local minimizer for $I_{\lambda,\mu}$ on subset of $\mathcal{N}_{\lambda,\mu}$, namely $\mathcal{N}^+_{\lambda,\mu}$ or $\mathcal{N}^-_{\lambda,\mu}$ such that $(u,v) \notin \mathcal{N}^0_{\lambda,\mu}$. Then $I'_{\lambda,\mu}(u,v) = 0$ in $\mathcal{H}^{-1}$, where $\mathcal{H}^{-1}$ denotes the dual space of $\mathcal{H}$.*

Proof. Suppose (u,v) is a local minimizer for $I_{\lambda,\mu}$ subject to the constrains $\Phi_{\lambda,\mu}(u,v) : \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = 0$. Then by Lagrange multipliers, there exists $\delta \in \mathbb{R}$ such that $I'_{\lambda,\mu}(u,v) = \delta\Phi'_{\lambda,\mu}(u,v)$. This implies that $\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = \delta\langle\Phi'_{\lambda,\mu}(u,v),(u,v)\rangle$. As $(u,v) \in \mathcal{N}_{\lambda,\mu}$, then $\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle =$ and $\langle\Phi'_{\lambda,\mu}(u,v),(u,v)\rangle \neq 0$ because of $(u,v) \notin \mathcal{N}^0_{\lambda,\mu}$. Therefore $\delta = 0$. This completes the proof. □

Lemma 4.3. *The following hold:*

(i) If $(u,v) \in \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$, then $P_{\lambda,\mu}(u,v) > 0$.

(ii) If $(u,v) \in \mathcal{N}^-_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$, then $Q(u,v) > 0$.

Proof. The proof follows directly from (4.3). □

Before analyzing the fibering map, we define a map $\mathcal{S}_{u,v} : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\mathcal{S}_{u,v}(t) := t^{2-r}\|(u,v)\|^2 - 2t^{22^*_\alpha - r}Q(u,v). \tag{4.4}$$

It is noted that for $t > 0$, $(tu,tv) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\mathcal{S}_{u,v}(t) = P_{\lambda,\mu}(u,v)$. We will check the behaviour of $\mathcal{S}_{u,v}$ near 0 and $+\infty$. Since $1 < r < 2$ and $2 < 22^*_\alpha$, this implies that $\lim_{t\to 0^+} \mathcal{S}_{(u,v)}(t) = 0$ and $\lim_{t\to+\infty} \mathcal{S}_{u,v}(t) = -\infty$. Moreover, for critical points

$$\mathcal{S}'_{u,v}(t) = (2-r)t^{1-r}\|(u,v)\|^2 - 2(22^*_\alpha - r)t^{22^*_\alpha - r - 1}Q(u,v).$$

One can easily see that $\mathcal{S}'_{u,v}(t) = 0$ if and only if $t = t_{\max}$, where

$$t_{\max} = \left(\frac{(2-r)\|(u,v)\|^2}{2(22^*_\alpha - r)Q(u,v)}\right)^{\frac{1}{22^*_\alpha - 2}}.$$

Also, $\mathcal{S}''_{u,v}(t) = (2-r)(1-r)t^{-r}\|(u,v)\|^2 - 2(22^*_\alpha - r)(22^*_\alpha - r - 1)t^{22^*_\alpha - r - 2}Q(u,v)$.

$$\begin{aligned}
\mathcal{S}''_{u,v}(t_{\max}) &= (2-r)(1-r)t_{\max}^{-r}\|(u,v)\|^2 - 2(22^*_\alpha - r)(22^*_\alpha - r - 1)t_{\max}^{22^*_\alpha - r - 2}Q(u,v) \\
&= (2-r)(1-r)\left(\frac{2(22^*_\alpha - r)Q(u,v)}{(2-r)\|(u,v)\|^2}\right)^{\frac{r}{22^*_\alpha - 2}} \|(u,v)\|^2 \\
&\quad - 2(22^*_\alpha - r)(22^*_\alpha - r - 1)\left(\frac{(2-r)\|(u,v)\|^2}{2(22^*_\alpha - r)Q(u,v)}\right)^{\frac{22^*_\alpha - r - 2}{22^*_\alpha - 2}} Q(u,v) \\
&= \frac{\|(u,v)\|^{\frac{2(22^*_\alpha - r - 2)}{22^*_\alpha - 2}}}{(Q(u,v))^{-\frac{r}{22^*_\alpha - 2}}}\left[(2-r)(1-r)\left(\frac{2(22^*_\alpha - r)}{2-r}\right)^{\frac{r}{22^*_\alpha - 2}}\right. \\
&\qquad \left. - 2(22^*_\alpha - r)(22^*_\alpha - r - 1)\left(\frac{2-r}{2(22^*_\alpha - r)}\right)\left(\frac{2(22^*_\alpha - r)}{2-r}\right)^{\frac{r}{22^*_\alpha - 2}}\right] \\
&= \frac{(2-22^*_\alpha)\left(2(22^*_\alpha - r)\right)^{\frac{r}{22^*_\alpha - 2}}}{(2-r)^{\frac{r+2-22^*_\alpha}{22^*_\alpha - 2}}}\|(u,v)\|^{\frac{2(22^*_\alpha - r - 2)}{22^*_\alpha - 2}}\left(Q(u,v)\right)^{\frac{r}{22^*_\alpha - 2}} \\
&< 0.
\end{aligned}$$

Thus, $\mathcal{S}_{u,v}(t)$ has maximum value at $t_{\max}$. Moreover, we have relation

$$\Psi'_{u,v}(t) = t^r\left(\mathcal{S}_{u,v}(t) - P_{\lambda,\mu}(u,v)\right). \tag{4.5}$$

Lemma 4.4. *Assume that $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$ and $(u,v) \in \mathcal{H}$, the following results hold:*

(i) If $Q(u,v) < 0$ and $P_{\lambda,\mu}(u,v) < 0$, then there does not exists any critical point.

(ii) If $Q(u,v) \le 0$ and $P_{\lambda,\mu}(u,v) < 0$, then there exists a unique (t^+u, t^+v) such that $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$ and $I_{\lambda,\mu}(t^+u, t^+v) = \inf_{t\ge 0} I_{\lambda,\mu}(tu,tv)$.

(iii) If $Q(u,v) > 0$ and $P_{\lambda,\mu}(u,v) \le 0$, then there exists a unique $t^- > t_{\max}$ such that $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$ and $I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge t_{\max}} I_{\lambda,\mu}(tu,tv)$.

(*iv*) *If* $Q(u,v) > 0$ *and* $P_{\lambda,\mu}(u,v) > 0$, *then there exists unique* t^+ *and* t^- *satisfying* $0 < t^+ < t_{\max} < t^-$ *such that* $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$ *and* $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$. *Moreover*

$$I_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le t_{\max}} I_{\lambda,\mu}(tu, tv); \qquad I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge t_{\max}} I_{\lambda,\mu}(tu, tv).$$

Proof. Let $(0,0) \neq (u,v) \in \mathcal{H}$, then we have following four possible cases:

(*i*) If $Q(u,v) < 0$ and $P_{\lambda,\mu}(u,v) < 0$, then $\Psi_{u,v}(t) = 0$ at $t = 0$ and $\Psi'_{u,v}(t) > 0$ for all $t > 0$. This implies that Φ_u is strictly increasing and hence no critical point.

(*ii*) If $Q(u,v) < 0$, then from (4.4) $\mathcal{S}_{u,v}$ is strictly increasing for $t > 0$. As $P_{\lambda,\mu}(u,v) \ge 0$, this implies that there exists a unique t^+ such that $\mathcal{S}_{u,v}(t^+) = P_{\lambda,\mu}(u,v)$ with $\mathcal{S}_{u,v}(t^+) > 0$. Using (4.5), we conclude that $(t^+u, t^+v) \in \mathcal{N}_{\lambda,\mu}$. Further, $\Psi'_{u,v}(t) > 0$ and $\Psi'_{u,v}(t) < 0$ for $t > t^+$ and and $t < t^+$ respectively. Also $\Psi''_{u,v}(t^+) = (t^+)^{1+r}\mathcal{S}'_{u,v}(t^+) > 0$. Thus, $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$ and $I_{\lambda,\mu}(t^+u, t^+v) = \inf_{t\ge 0} I_{\lambda,\mu}(tu, tv)$.

(*iii*) If $Q(u,v) > 0$, then $t_{\max}$ is the point at which $\mathcal{S}'_{u,v}(t) > 0$ has maximum. Thus, $\mathcal{S}_{u,v}(t)$ is strictly increasing for $0 \le t < t_{\max}$ and strictly decreasing for $t_{\max} < t < \infty$. As $P_{\lambda,\mu}(u,v) \le 0$, so there is a unique $t^- > t_{\max} > 0$ such that $\mathcal{S}_{u,v}(t^-) = P_{\lambda,\mu}(u,v)$ and $\mathcal{S}_{u,v}(t^-) < 0$. Further, (4.5) gives $\Psi'_{u,v}(t^-) = 0$. Thus $(t^-u, t^-v) \in \mathcal{N}_{\lambda,\mu}$. Also, $\Psi''_{u,v}(t^-) = (t^-)^{1+r}\mathcal{S}'_{u,v}(t^-) < 0$ and $\Psi'_{u,v}(t) < 0$ for $t > t_{\max}$, so $\Psi_{u,v}(t^-) = \sup_{t\ge t_{\max}} \Psi_{u,v}(t)$. Hence, $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$ and $I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge t_{\max}} I_{\lambda,\mu}(tu, tv)$.

(*iv*) Since $Q(u,v) > 0$, $\mathcal{S}_{u,v}(t)$ achieves its maximum at $t = t_{\max}$. Thus

$$\begin{aligned}
\mathcal{S}_{u,v}(t_{\max}) &= \|(u,v)\|^r \left(\frac{2-r}{2(22^*_\alpha - r)}\right)^{\frac{2-r}{22^*_\alpha - 2}} \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right) \left(\frac{\|(u,v)\|^{22^*_\alpha}}{Q(u,v)}\right)^{\frac{2-r}{22^*_\alpha - 2}} \\
&\ge \left[\left(\frac{2-r}{2(22^*_\alpha - r)}\right) \|H^+\|_\infty^{-2} (\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{2-r}{22^*_\alpha - 2}} \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right) \|(u,v)\|^r.
\end{aligned}$$

As $P_{\lambda,\mu}(u,v) > 0$, so

$$\begin{aligned}
\mathcal{S}_{u,v}(t_{\max}) - P_{\lambda,\mu}(u,v) \ge{}& \left[\left(\frac{2-r}{2(22^*_\alpha - r)}\right) \|H^+\|_\infty^{-2} (\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{2-r}{22^*_\alpha - 2}} \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right) \|(u,v)\|^r \\
&- \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} S^{-\frac{r}{2}} \|(u,v)\|^r \\
>{}& 0,
\end{aligned}$$

for $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. Thus, there exists t^+ and t^- with $0 < t^+ < t_{\max} < t^-$ satisfying

$$\mathcal{S}_{u,v}(t^+) = P_{\lambda,\mu}(u,v) = \mathcal{S}_{u,v}(t^-) \quad \text{and} \quad \mathcal{S}'_{u,v}(t^+) < 0 < \mathcal{S}'_{u,v}(t^-).$$

Therefore, $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$, $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$. Furthermore, $\Psi'_{u,v}(t) < 0$ for $t \in (0, t^+)$, $\Psi'_u(t) > 0$ for $t \in (t^+, t^-)$ and $\Psi'_{u,v}(t) < 0$ for $t \in (t^-, \infty)$.

Hence

$$I_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le t_{\max}} I_{\lambda,\mu}(tu, tv); \; I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge t_{\max}} I_{\lambda,\mu}(tu, tv).$$

holds true. This completes the proof. □

Lemma 4.5. *If* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then* $\mathcal{N}^0_{\lambda,\mu}$ *is a null set.*

Proof. We will prove it by contradiction. Let $(u,v) \in \mathcal{N}^0_{\lambda,\mu}$, then (4.3) implies that,

$$\|(u,v)\|^2 = \frac{22^*_\alpha - r}{22^*_\alpha - 2} P_{\lambda,\mu}(u,v) \tag{4.6}$$

and

$$\|(u,v)\|^2 = \frac{2(22^*_\alpha - r)}{2-r} Q(u,v). \tag{4.7}$$

On using (2.16) in (4.6), it is easy to calculate

$$\|(u,v)\| \leq \left(\frac{22^*_\alpha - r}{22^*_\alpha - 2} S^{-\frac{r}{2}}\right)^{\frac{1}{2-r}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{1}{2}}. \tag{4.8}$$

Now, taking (2.17) in (4.7), we find

$$\|(u,v)\|^2 \leq 2\left(\frac{22^*_\alpha - r}{2-r}\right) \|H^+\|^2_\infty (\bar{S}_{H,L})^{-2^*_\alpha} \|(u,v)\|^{22^*_\alpha},$$

or

$$\|(u,v)\| \geq \left[\left(\frac{2-r}{2(22^*_\alpha - r)}\right) \|H^+\|^{-2}_\infty (\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{1}{22^*_\alpha - 2}}. \tag{4.9}$$

Thus, from (4.8) and (4.9), we get

$$(\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} \geq \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right)^{\frac{2}{2-r}} \left[\left(\frac{2-r}{2(22^*_\alpha - r)}\right) \|H^+\|^{-2}_\infty (\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{1}{2^*_\alpha - 1}} S^{\frac{r}{2-r}} =: \Upsilon_1,$$

which contradicts the fact that $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. Hence $\mathcal{N}^0_{\lambda,\mu} = \phi$ which completes the proof. □

Consequently, if $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, then we have

$$\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}.$$

Now, we define

$$k_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(u,v); \qquad k^+_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}} I_{\lambda,\mu}(u,v); \qquad k^-_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}} I_{\lambda,\mu}(u,v).$$

Lemma 4.6. *The following facts hold:*

(*i*) *If* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then* $k_{\lambda,\mu} \leq k^+_{\lambda,\mu} < 0$.

(*ii*) *If* $0 < \lambda < \frac{r}{2}\Upsilon_1$, *then* $k^-_{\lambda,\mu} > d_0$, *where* d_0 *is a positive constant depending on* $\lambda, \mu, \alpha, r, N$, S, $\|F\|_\alpha$, $\|G\|_\alpha$ *and* $\|H^+\|_\infty$.

Proof. (i) Let $(u,v) \in \mathcal{N}^+_{\lambda,\mu}$, then (4.3) gives

$$\frac{2-r}{2(22^*_\alpha - r)}\|(u,v)\|^2 > Q(u,v).$$

This together with (2.15) and (4.1) yield

$$I_{\lambda,\mu}(u,v) = \left(\frac{1}{2}-\frac{1}{r}\right)\|(u,v)\|^2 + 2\left(\frac{1}{r}-\frac{1}{22^*_\alpha}\right)Q(u,v) < -\frac{(2-r)(2^*_\alpha-1)}{r22^*_\alpha}\|(u,v)\|^2 < 0.$$

Thus, by the definition of $k_{\lambda,\mu}$ and $k^+_{\lambda,\mu}$, we conclude that $k_{\lambda,\mu} \le k^+_{\lambda,\mu} < 0$.

(ii) Let $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$. Then using (4.3) and (2.17), we have

$$\frac{2-r}{2(22^*_\alpha - r)}\|(u,v)\|^2 < Q(u,v) \le \|H^+\|^2_\infty \left(\bar{S}_{H,L}\right)^{-2^*_\alpha}\|(u,v)\|^{22^*_\alpha}.$$

This implies that

$$\|(u,v)\| > \left(\frac{2-r}{2(22^*_\alpha - r)}\|H^+\|^{-2}_\infty \left(\bar{S}_{H,L}\right)^{2^*_\alpha}\right)^{\frac{1}{22^*_\alpha-2}}. \tag{4.10}$$

On combining (4.2) and (4.10), we obtain

$$\begin{aligned}
&I_{\lambda,\mu}(u,v)\\
&\ge \|(u,v)\|^r\left[\left(\frac{1}{2}-\frac{1}{22^*_\alpha}\right)\|(u,v)\|^{2-r} - \left(\frac{1}{r}-\frac{1}{22^*_\alpha}\right)S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\right]\\
&> \left(\frac{2-r}{2(22^*_\alpha - r)}\|H^+\|^{-2}_\infty \left(\bar{S}_{H,L}\right)^{2^*_\alpha}\right)^{\frac{r}{22^*_\alpha-2}}\left[\left(\frac{1}{2}-\frac{1}{22^*_\alpha}\right)\left(\frac{2-r}{2(22^*_\alpha - r)}\|H^+\|^{-2}_\infty \left(\bar{S}_{H,L}\right)^{2^*_\alpha}\right)^{\frac{2-r}{22^*_\alpha-r}}\right.\\
&\quad\left. - \left(\frac{1}{r}-\frac{1}{22^*_\alpha}\right)S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\right]
\end{aligned}$$

Thus, if $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Upsilon_1$, then $I_{\lambda,\mu}(u,v) > d_0$ for all $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$, where d_0 is a positive constant depending on $\lambda, \mu, \alpha, r, N, S, \|F\|_\alpha, \|G\|_\alpha$ and $\|H^+\|_\infty$. □

5 Existence of solution in $\mathcal{N}^+_{\lambda,\mu}$

In this section, we show the existence of Palais–Smale sequence corresponding to energy functional $I_{\lambda,\mu}$ in $\mathcal{N}^\pm_{\lambda,\mu}$, by using the implicit function theorem.

Lemma 5.1. *Suppose $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. Then for every $z = (u,v) \in \mathcal{N}_{\lambda,\mu}$, there exist $\epsilon > 0$ and a differentiable mapping $\zeta : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ such that $\zeta(0) = 1$, $\zeta(w)(z-w) \in \mathcal{N}_{\lambda,\mu}$ and for all $w = (w_1,w_2) \in \mathcal{H}$*

$$\langle \zeta'(0), w\rangle = \frac{2\mathcal{B}(z,w) - r\mathcal{P}_{\lambda,\mu}(z,w) - 2\mathcal{Q}(z,w)}{(2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v)},$$

where

$$\begin{aligned}
\mathcal{B}(z,w) &= \int_\Omega (\Delta u\Delta w_1 + \Delta v\Delta w_2)\,dx,\\
\mathcal{P}_{\lambda,\mu}(z,w) &= \int_\Omega \left(\lambda F(x)|u|^{r-2}uw_1 + \mu G(x)|v|^{r-2}vw_2\right)dx,\\
\mathcal{Q}(z,w) &= \int_\Omega\int_\Omega H(x)H(y)\left(\frac{|v(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha-2}u(y)z_1 + |u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha-2}v(y)z_2}{|x-y|^\alpha}\right)dxdy
\end{aligned}$$

Proof. For $z = (u, v) \in \mathcal{N}_{\lambda,\mu}$, define a map $\xi_z : \mathbb{R} \times \mathcal{H} \to \mathbb{R}$ such that

$$\xi_z(\zeta, w) = \langle I'_{\lambda,\mu}(\zeta(z-w)), \zeta(z-w)\rangle = \zeta^2\|(u-w_1, v-w_2)\|^2 - \zeta^r \int_\Omega (\lambda F(x)|u-w_1|^r + \mu G(x)|v-w_2|^r)dx - 2\zeta^{22^*_\alpha} Q(u-w_1, v-w_2)dx$$

Then $\xi_z(1, (0,0)) = \langle I'_{\lambda,\mu}(z), z\rangle = 0$ and

$$\frac{d}{d\zeta}\xi_z(1,(0,0)) = 2\|(u,v)\|^2 - r\int_\Omega (\lambda F(x)|u|^r + \mu G(x)|v|^r)dx - 2(22^*_\alpha)Q(u,v)$$
$$= (2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v) \neq 0.$$

Thus, by the implicit function theorem, there exist $\epsilon > 0$ and a differentiable mapping $\zeta : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ such that $\zeta(0) = 1$,

$$\langle \zeta'(0), w\rangle = \frac{2\mathcal{B}(z,w) - r\mathcal{P}_{\lambda,\mu}(z,w) - 2\mathcal{Q}(z,w)}{(2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v)},$$

$\xi_z(\zeta(w), w) = 0 \ \forall\, w \in B(0,\epsilon)$. Thus,

$$\langle I'_{\lambda,\mu}(\zeta(w)(z-w)), \zeta(w)(z-w)\rangle = 0 \qquad \forall\, w \in B(0,\epsilon).$$

Therefore, $\zeta(w)(z-w) \in \mathcal{N}_{\lambda,\mu}$. □

The similar result is also true for $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$, which is as follows

Lemma 5.2. *Suppose* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. *Then for every* $z = (u,v) \in \mathcal{N}^-_{\lambda,\mu}$, *there exist* $\epsilon > 0$ *and a differentiable mapping* $\zeta^- : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ *such that* $\zeta^-(0) = 1$, $\zeta^-(w)(z-w) \in \mathcal{N}^-_{\lambda,\mu}$ *and for all* $w = (w_1, w_2) \in \mathcal{H}$

$$\langle (\zeta^-)'(0), w\rangle = \frac{2\mathcal{B}(z,w) - r\mathcal{P}_{\lambda,\mu}(z,w) - 2\mathcal{Q}(z,w)}{(2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v)},$$

where $\mathcal{B}(z,w)$, $\mathcal{P}_{\lambda,\mu}(z,w)$ *and* $\mathcal{Q}(z,w)$ *is same as in Lemma 5.1.*

Proof. By the same argument used in Lemma 5.1, there exists $\epsilon > 0$ and a differentiable function $\zeta^- : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ such that $\zeta^-(0) = 1$ and $\zeta^-(w)(z-w) \in \mathcal{N}^-_{\lambda,\mu}$. Since

$$\Psi''_{(u,v)}(1) = (2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v) < 0.$$

By the continuity of Ψ'' and ζ^-, we have

$$\Psi''_{\zeta^-(w)(z-w)}(1) = (2-r)\|\zeta^-(w)(z-w)\|^2 - 2(22^*_\alpha - r)Q(\zeta^-(w)(z-w), \zeta^-(w)(z-w) < 0,$$

for $\epsilon > 0$ is sufficiently small. Thus, $\zeta^-(w)(z-w) \in \mathcal{N}^-_{\lambda,\mu}$. □

Lemma 5.3. *The following statements are true:*

(i) *If* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then there exists a* $(PS)_{k_{\lambda,\mu}}$*-sequence* $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ *in* $\mathcal{H}$ *for* $I_{\lambda,\mu}$.

(ii) *If* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Upsilon_1$, *then there exists a* $(PS)_{k^-_{\lambda,\mu}}$*-sequence* $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$ *in* $\mathcal{H}$ *for* $I_{\lambda,\mu}$.

Proof. (i) According to Lemma 4.1 and Ekeland Variational Principle [14], there exists a minimizing sequence $\{(u_n,v_n)\}\subset\mathcal{N}_{\lambda,\mu}$ such that

$$\begin{aligned} I_{\lambda,\mu}(u_n,v_n) &< k_{\lambda,\mu}+\frac{1}{n},\\ I_{\lambda,\mu}(u_n,v_n) &< I_{\lambda,\mu}(u,v)+\frac{1}{n}\|(u,v)-(u_n,v_n)\|, \text{ for each } (u,v)\in\mathcal{N}_{\lambda,\mu}. \end{aligned} \tag{5.1}$$

Using Lemma 4.6(i) and taking n large, we get

$$\begin{aligned} I_{\lambda,\mu}(u_n,v_n) &= \left(\frac{1}{2}-\frac{1}{22^*_\alpha}\right)\|(u_n,v_n)\|^2-\left(\frac{1}{r}-\frac{1}{22^*_\alpha}\right)P_{\lambda,\mu}(u_n,v_n)\\ &< k_{\lambda,\mu}+\frac{1}{n}<\frac{k_{\lambda,\mu}}{2}. \end{aligned} \tag{5.2}$$

This implies that

$$0<-\frac{r2^*_\alpha k_{\lambda,\mu}}{22^*_\alpha-r}<P_{\lambda,\mu}(u_n,v_n)\le S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}}+(\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\|(u_n,v_n)\|^r. \tag{5.3}$$

Wherefore, $(u_n,v_n)\neq(0,0)$. From (5.2), we have

$$\|(u_n,v_n)\|\le\left[\left(\frac{22^*_\alpha-r}{r(2^*_\alpha-1)}\right)S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}}+(\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\right]^{\frac{1}{2-r}}. \tag{5.4}$$

Further, (5.3) gives us

$$\|(u_n,v_n)\|\ge\left[-\frac{r2^*_\alpha k_{\lambda,\mu}}{22^*_\alpha-r}S^{\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}}+(\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{r-2}{2}}\right]^{\frac{1}{r}}.$$

Now, we will prove that

$$\|I'_{\lambda,\mu}(u_n,v_n)\|_{\mathcal{H}^{-1}}\to 0,\quad \text{as } n\to\infty.$$

Using Lemma 5.1 for each $z_n=(u_n,v_n)$ to obtain the mapping $\zeta_n:B(0,\epsilon_n)\to\mathbb{R}^+$ for some $\epsilon_n>0$ such that $\zeta_n(w)(z_n-w)\in\mathcal{N}_{\lambda,\mu}$. Choose $0<\eta<\epsilon_n$. Let $z=(u,v)\in\mathcal{H}$ with $z\neq 0$ and take $w^*_\eta=\frac{\eta z}{\|z\|}$. We set $w_\eta=\zeta_n(w^*_\eta)(z_n-w^*_\eta)$. Since $w_\eta\in\mathcal{N}_{\lambda,\mu}$, from (5.1), we get

$$I_{\lambda,\mu}(w_\eta)-I_{\lambda,\mu}(z_n)\ge-\frac{1}{n}\|w_\eta-z_n\|.$$

Using mean value theorem, we obtain

$$\langle I'_{\lambda,\mu}(z_n),w_\eta-z_n\rangle+o(\|w_\eta-z_n\|)\ge-\frac{1}{n}\|w_\eta-z_n\|.$$

Therefore,

$$\langle I'_{\lambda,\mu}(z_n),-w^*_\eta\rangle+(\zeta_n(w^*_\eta)-1)\langle I'_{\lambda,\mu}(z_n),z_n-w^*_\eta\rangle\ge-\frac{1}{n}\|w_\eta-z_n\|+o(\|w_\eta-z_n\|). \tag{5.5}$$

Since $\zeta_n(w^*_\eta)(z_n-w^*_\eta)\in\mathcal{N}_{\lambda,\mu}$ and from (5.5), we get

$$-\eta\left\langle I'_{\lambda,\mu}(z_n),\frac{z}{\|z\|}\right\rangle+(\zeta_n(w^*_\eta)-1)\langle I'_{\lambda,\mu}(z_n-w_\eta),z_n-w^*_\eta\rangle\ge-\frac{1}{n}\|w_\eta-z_n\|+o(\|w_\eta-z_n\|).$$

Thus, we have

$$\left\langle I'_{\lambda,\mu}(z_n), \frac{z}{\|z\|} \right\rangle \leq \frac{1}{n\eta}\|w_\eta - z_n\| + \frac{1}{\eta}o(\|w_\eta - z_n\|) + \frac{(\zeta_n(w_\eta^*) - 1)}{\eta}\langle I'_{\lambda,\mu}(z_n - w_\eta), z_n - w_\eta^*\rangle. \tag{5.6}$$

As $\|w_\eta - z_n\| \leq \eta|\zeta_n(w_\eta^*)| + |\zeta_n(w_\eta^*) - 1|\|z_n\|$ and $\lim_{\eta\to 0}\frac{|\zeta_n(w_\eta^*)-1|}{\eta} \leq \|\zeta_n'(0)\|$, if we take $\eta \to 0$ in (5.6) for a fixed $n \in \mathbb{N}$ and using (5.4) we can find a constant $A_1 > 0$, free from η such that

$$\left\langle I'_{\lambda,\mu}(z_n), \frac{z}{\|z\|} \right\rangle \leq \frac{A_1}{n}(1 + \|\zeta_n'(0)\|).$$

Further, we will show that $\|\zeta_n'(0)\|$ is uniformly bounded. By Hölder's inequality and Sobolev's embedding theorem, we have

$$\begin{aligned}
&\int_\Omega \lambda F(x)|u_n|^{r-1}w_1 + \mu G(x)|v_n|^{r-1}w_2 \\
&\leq \lambda\|F\|_\alpha \left(\int_\Omega \left(|u_n|^{r-1}w_1\right)^{\frac{2^*}{r}}\right)^{\frac{r}{2^*}} + \mu\|G\|_\alpha \left(\int_\Omega \left(|v_n|^{r-1}w_2\right)^{\frac{2^*}{r}}\right)^{\frac{r}{2^*}} \\
&\leq \lambda\|F\|_\alpha\|u_n\|_{2^*}^{r-1}\|w_1\|_{2^*} + \mu\|G\|_\alpha\|v_n\|_{2^*}^{r-1}\|w_2\|_{2^*} \\
&\leq S^{-\frac{r}{2}}\left(\lambda\|F\|_\alpha + \mu\|G\|_\alpha\right)\|(u_n, v_n)\|^{r-1}\|(w_1, w_2)\|
\end{aligned} \tag{5.7}$$

Further, using the Hardy–Littlewood–Sobolev inequality, Hölder's inequality and Sobolev's embedding theorem, we obtain

$$\begin{aligned}
&\int_\Omega\int_\Omega \left(\frac{|u_n|^{2^*_\alpha}}{|x-y|^\alpha}\right)|v_n|^{2^*_\alpha - 1}w_1 dxdy \\
&\leq C(N,\alpha)\left(\int_\Omega |u_n|^{\frac{22^*_\alpha N}{2N-\alpha}}\right)^{\frac{2N-\alpha}{2N}}\left(\int_\Omega \left(|v_n|^{2^*_\alpha-1}w_1\right)^{\frac{2N}{2N-\alpha}}\right)^{\frac{2N-\alpha}{2N}} \\
&= C(N,\alpha)\left(\int_\Omega |u_n|^{2^*}\right)^{\frac{2N-\alpha}{2N}}\left(\int_\Omega \left(|v_n|^{2^*_\alpha-1}w_1\right)^{\frac{2N}{2N-\alpha}}\right)^{\frac{2N-\alpha}{2N}} \\
&\leq C(N,\alpha)\left(\int_\Omega |u_n|^{2^*}\right)^{\frac{2N-\alpha}{2N}}\left(\int_\Omega |v_n|^{2^*}\right)^{\frac{N+4-\alpha}{2N-\alpha}}\left(\int_\Omega |w_1|^{2^*}\right)^{\frac{1}{2^*}} \\
&\leq \left[\left(S^{-1}\int_\Omega |\Delta u_n|^2\right)^{\frac{2^*}{2}}\right]^{\frac{2N-\alpha}{2N}}\left[\left(S^{-1}\int_\Omega |\Delta v_n|^2\right)^{\frac{2^*}{2}}\right]^{\frac{N+4-\alpha}{2N}}\left[\left(S^{-1}\int_\Omega |\Delta w_1|^2\right)^{\frac{2^*}{2}}\right]^{\frac{1}{2^*}} \\
&\leq A_2\|u_n\|^{2^*_\alpha}\|v_n\|^{\frac{N+4-\alpha}{N-4}}\|w_1\| \\
&\leq A_2\|(u_n, v_n)\|^{\frac{3N+4-2\alpha}{N-4}}\|(w_1, w_2)\|.
\end{aligned} \tag{5.8}$$

Using the same idea, we can calculate

$$\int_\Omega\int_\Omega \left(\frac{|v_n|^{2^*_\alpha}}{|x-y|^\alpha}\right)|u_n|^{2^*_\alpha-1}w_2 dxdy \leq A_3\|(u_n, v_n)\|^{\frac{3N+4-2\alpha}{N-4}}\|(w_1, w_2)\|. \tag{5.9}$$

Thus, on combining (5.7)–(5.9) and (5.4), we have

$$\left|(\zeta_n'(0),w)\right| \le \frac{A_4\|(w_1,w_2)\|}{|(2-r)\|(u_n,v_n)\|^2 - 2(22_\alpha^* - r)Q(u_n,v_n)|},$$

where $A_4 > 0$ is a constant.

Now we are left to show that

$$\left|(2-r)\|(u_n,v_n)\|^2 - 2(22_\alpha^* - r)Q(u_n,v_n)\right| \ge A_5,$$

for some $A_5 > 0$ and n is taking large enough. On contradiction argue, suppose there exists a subsequence $\{(u_n,v_n)\}$ such that

$$\left|(2-r)\|(u_n,v_n)\|^2 - 2(22_\alpha^* - r)Q(u_n,v_n)\right| = o_n(1). \tag{5.10}$$

From (5.10) and using $(u_n,v_n) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\|(u_n,v_n)\|^2 = \frac{22_\alpha^* - r}{22_\alpha^* - 2} P_{\lambda,\mu}(u_n,v_n) + o_n(1),$$
$$\|(u_n,v_n)\|^2 = \frac{2(22_\alpha^* - r)}{2-r} Q(u_n,v_n) + o_n(1).$$

By Hölder's inequality, Sobolev embedding theorem and the definition of $\bar{S}_{H,L}$, we obtain

$$\|(u_n,v_n)\| \le \left(\frac{22_\alpha^* - r}{22_\alpha^* - 2} S^{-\frac{r}{2}}\right)^{\frac{1}{2-r}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{1}{2}} + o_n(1),$$
$$\|(u_n,v_n)\| \ge \left[\left(\frac{2-r}{2(22_\alpha^* - r)}\right)\|H^+\|_\infty^{-2}(2S_{H,L})^{2_\alpha^*}\right]^{\frac{1}{22_\alpha^*-2}} + o_n(1).$$

This implies that $(\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} \ge \Upsilon_1$, which is a contradiction to the fact that $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. Hence,

$$\left\langle I_{\lambda,\mu}'(u_n,v_n), \frac{(u,v)}{\|(u,v)\|}\right\rangle \le \frac{A_1}{n}.$$

Thus, proof of (i) is completed.

(ii) Using Lemma 5.2, one can prove (ii) in a similar manner. □

Lemma 5.4. *Let* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then* $I_{\lambda,\mu}$ *has a minimizer* $(u_{\lambda,\mu}^1, v_{\lambda,\mu}^1)$ *in* $\mathcal{N}_{\lambda,\mu}^+$ *which satisfies the following:*

(i) $I_{\lambda,\mu}(u_{\lambda,\mu}^1, v_{\lambda,\mu}^1) = k_{\lambda,\mu} = k_{\lambda,\mu}^+ < 0.$

(ii) $(u_{\lambda,\mu}^1, v_{\lambda,\mu}^1)$ *is a nontrivial solution of the system* $(\mathcal{D}_{\lambda,\mu})$.

(iii) $I_{\lambda,\mu}(u_{\lambda,\mu}^1, v_{\lambda,\mu}^1) \to (0,0)$ *as* $\lambda \to 0^+$, $\mu \to 0^+$.

Proof. By Lemma 5.3 (i), there exists a minimizing sequence $\{(u_n,v_n)\}$ for $I_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n,v_n) = k_{\lambda,\mu} + o_n(1), \; I_{\lambda,\mu}'(u_n,v_n) = o_n(1) \text{ in } \mathcal{H}^{-1}.$$

Lemma 5.1 gives us that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. So up to subsequence $(u_n, v_n) \rightharpoonup (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ weakly in $(u_n, v_n) \to \mathcal{H}$, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $L^m(\Omega)$ $\forall 1 \leq m < 2^*$ and $(u_n(x), v_n(x)) \to (u^1_{\lambda,\mu}(x), v^1_{\lambda,\mu}(x))$ pointwise a.e. in Ω. Then, it is easy to see that

$$\begin{aligned} |u_n|^{2^*_\alpha} \rightharpoonup |u^1_{\lambda,\mu}|^{2^*_\alpha}, \quad |v_n|^{2^*_\alpha} \rightharpoonup |v^1_{\lambda,\mu}|^{2^*_\alpha} \quad \text{in } L^{\frac{2N}{2N-\alpha}}(\Omega) \quad \text{and} \\ |u_n|^{2^*_\alpha-2}u_n \rightharpoonup |u^1_{\lambda,\mu}|^{2^*_\alpha-2}u^1_{\lambda,\mu}, \quad |v_n|^{2^*_\alpha-2}v_n \rightharpoonup |v^1_{\lambda,\mu}|^{2^*_\alpha-2}v^1_{\lambda,\mu} \quad \text{in } L^{\frac{2N}{N+4-\alpha}}(\Omega), \end{aligned} \tag{5.11}$$

as $n \to \infty$. As we know that the Riesz potential defines a continuous linear map from $L^{\frac{2N}{2N-\alpha}}(\Omega)$ to $L^{\frac{2N}{\alpha}}(\Omega)$ which provides

$$|x|^{-\alpha} * |u_n|^{2^*_\alpha} \rightharpoonup |x|^{-\alpha} * |u^1_{\lambda,\mu}|^{2^*_\alpha} \text{ and } |x|^{-\alpha} * |v_n|^{2^*_\alpha} \rightharpoonup |x|^{-\alpha} * |v^1_{\lambda,\mu}|^{2^*_\alpha} \text{ weakly in } L^{\frac{2N}{\alpha}}(\Omega), \tag{5.12}$$

as $n \to \infty$. Thus, (5.11) and (5.12) gives us

$$\left.\begin{aligned} \left(|x|^{-\alpha} * |v_n|^{2^*_\alpha}\right)|u_n|^{2^*_\alpha-2}u_n &\rightharpoonup \left(|x|^{-\alpha} * |v^1_{\lambda,\mu}|^{2^*_\alpha}\right)|u^1_{\lambda,\mu}|^{2^*_\alpha-2}u^1_{\lambda,\mu}, \\ \left(|x|^{-\alpha} * |u_n|^{2^*_\alpha}\right)|v_n|^{2^*_\alpha-2}v_n &\rightharpoonup \left(|x|^{-\alpha} * |u^1_{\lambda,\mu}|^{2^*_\alpha}\right)|v^1_{\lambda,\mu}|^{2^*_\alpha-2}v^1_{\lambda,\mu}, \end{aligned}\right\} \text{ weakly in } L^{\frac{2N}{N+4}}(\Omega), \tag{5.13}$$

as $n \to \infty$. Therefore, for any $(\phi, \psi) \in \mathcal{H}$, we have

$$\begin{aligned} \lim_{n\to\infty} \Bigg[&\int_\Omega (\Delta u_n \Delta\phi + \Delta v_n \Delta\psi)\, dx - \int_\Omega (\lambda F(x)|u_n|^{r-2}u\phi + \mu G(x)|v_n|^{r-2}v\psi)\, dx \\ &- \int_\Omega\int_\Omega H(x)H(y) \left(\tfrac{|v_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha-2}u_n(y)\phi(y) + |u_n(x)|^{2^*_\alpha}|v_n(y)|^{2^*_\alpha-2}v_n(y)\psi(y)}{|x-y|^\alpha} \right) \Bigg] = 0, \end{aligned}$$

because of $\|I'_{\lambda,\mu}(u_n, v_n)\| \to 0$ as $n \to \infty$. Thus, using (5.13), continuity of H and passing the limit as $n \to \infty$, we have

$$\begin{aligned} &\int_\Omega \left(\Delta u^1_{\lambda,\mu}\Delta\phi + \Delta v^1_{\lambda,\mu}\Delta\psi\right) dx - \int_\Omega \left(\lambda F(x)|u^1_{\lambda,\mu}|^{r-2}u^1_{\lambda,\mu}\phi + \mu G(x)|v^1_{\lambda,\mu}|^{r-2}v^1_{\lambda,\mu}\psi\right) dx \\ &- \int_\Omega\int_\Omega H(x)H(y) \left(\tfrac{|v^1_{\lambda,\mu}(x)|^{2^*_\alpha}|u^1_{\lambda,\mu}(y)|^{2^*_\alpha-2}u^1_{\lambda,\mu}(y)\phi(y) + |u^1_{\lambda,\mu}(x)|^{2^*_\alpha}|v^1_{\lambda,\mu}(y)|^{2^*_\alpha-2}v^1_{\lambda,\mu}(y)\psi(y)}{|x-y|^\alpha} \right) = 0, \end{aligned}$$

i.e. $\langle I'_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}), (\phi, \psi)\rangle \to 0$. This implies that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a weak solution of $(\mathcal{D}_{\lambda,\mu})$. Since $(u_n, v_n) \in \mathcal{N}_{\lambda,\mu}$. So, we have

$$\|(u_n, v_n)\|^2 = P_{\lambda,\mu}(u_n, v_n) + 2Q(u_n, v_n),$$

which gives

$$\begin{aligned} I_{\lambda,\mu}(u_n, v_n) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right)\|(u_n, v_n)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right)P_{\lambda,\mu}(u_n, v_n) \\ &\geq -\left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right)P_{\lambda,\mu}(u_n, v_n). \end{aligned}$$

Taking $n \to \infty$ together with λ, $\mu < 0$, we obtain

$$P_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \geq -\frac{22^*_\alpha k_{\lambda,\mu}}{(22^*_\alpha - r)} > 0. \tag{5.14}$$

Therefore, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a nontrivial solution of $(\mathcal{D}_{\lambda,\mu})$. Afterwards, we will show that $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathcal{H}$ and $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = k_{\lambda,\mu}$. Using Fatou's lemma, we obtain

$$\begin{aligned} k_{\lambda,\mu} \le I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \\ &\le \liminf_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u_n, v_n)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u_n, v_n)\right] \\ &= \liminf_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) = k_{\lambda,\mu}. \end{aligned}$$

This implies that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = k_{\lambda,\mu}$ and $\lim_{n\to\infty} \|(u_n, v_n)\|^2 = \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2$. Further, the Brézis–Lieb lemma [6] contributes that $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathcal{H}$.

Now, we are left to show that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$. We prove this by contradiction argument. Suppose $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$ Then, from Lemma 4.3 (ii) and (5.14), we have

$$Q(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > 0 \quad \text{and} \quad P_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > 0.$$

Thus, from Lemma 4.4, there exist unique t_1^+ and t_1^- such that $(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and $(t_1^- u^1_{\lambda,\mu}, t_1^- v^1_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. In particular, we have $t_1^+ < t_1^- = 1$. Since $\Psi'_{(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})}(t_1^+) = 0$ and $\Psi''_{(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})}(t_1^+) > 0$, there exists $t_1^+ < \bar{t} \le t_1^-$ such that $I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar{t} u^1_{\lambda,\mu}, \bar{t} v^1_{\lambda,\mu})$. On using Lemma 4.4, we obtain

$$I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar{t} u^1_{\lambda,\mu}, \bar{t} v^1_{\lambda,\mu}) \le I_{\lambda,\mu}(t_1^- u^1_{\lambda,\mu}, t_1^- v^1_{\lambda,\mu}) = I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = k_{\lambda,\mu},$$

which is a contradiction. Therefore, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$.

(iii) Further, from Lemma 4.6 (i) and (4.2), we have

$$\begin{aligned} 0 > k^+_{\lambda,\mu} &\ge k_{\lambda,\mu} = I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \\ &> -\left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) S^{-\frac{r}{2}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^r, \end{aligned}$$

which implies that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \to (0,0)$ as $\lambda \to 0^+$, $\mu \to 0^+$ which completes the proof. □

Proof of Theorem 1.1. From Lemma 5.4, we conclude that $(\mathcal{D}_{\lambda,\mu})$ has a nontrivial solution $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$. □

6 Existence of solution in $\mathcal{N}^-_{\lambda,\mu}$

In this segment, we first prove the critical level by using few estimates which are already proved in Section 1. Then we show the existence of a second weak solution of problem $(\mathcal{D}_{\lambda,\mu})$ under the assumptions (Z1)–(Z4). At the end of this section, we give the proof of Theorem 1.2.

Lemma 6.1. *Assume that* (Z1)–(Z4) *hold and* $\frac{N}{N-4} \le r < 2$, *then there exist* $(u_{\lambda,\mu}, v_{\lambda,\mu})$ *in* $\mathcal{H} \setminus \{(0,0)\}$ *and* $\Upsilon > 0$ *such that for* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon$,

$$\begin{aligned} &\sup_{t\ge 0} I_{\lambda,\mu}(tu_{\lambda,\mu}, tv_{\lambda,\mu}) \\ &\quad < \frac{N+4-\alpha}{2(2N-\alpha)} \left(\frac{\|H^+\|_\infty^{-2}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - K_0 \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right) =: c_\infty. \end{aligned}$$

Furthermore, $k^-_{\lambda,\mu} < c_\infty$ *for all* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon$.

Proof. For this, we first define the functional $\mathcal{E} : \mathcal{H} \to \mathbb{R}$ such that

$$\mathcal{E}(u,v) = \frac{1}{2}\|(u,v)\|^2 - \frac{1}{2^*_\alpha}Q(u,v), \qquad \forall\ (u,v) \in \mathcal{H}.$$

Take $U_0 = V_0 = \overline{U}_\epsilon$ with $(U_0, V_0) \in \mathcal{H}$. We define $\phi(t) = \mathcal{E}(tU_0, tV_0)$. Then $\phi(t)$ satisfies $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$ small and $\phi(t) < 0$ for $t > 0$ large. Further, one can easily verify that $\phi(t)$ attains its maximum at

$$t = \left(\frac{\|(U_0,V_0)\|^2}{2Q(U_0,V_0)}\right)^{\frac{1}{22^*_\alpha-2}} =: t^*.$$

Thus from (2.9), we have

$$\begin{aligned}
\sup_{t\geq 0}\mathcal{E}(tU_0,tV_0) &= \frac{(t^*)^2}{2}\|(U_0,V_0)\|^2 - \frac{(t^*)^{22^*_\alpha}}{2^*_\alpha}Q(U_0,V_0)\\
&= \frac{(N+4-\alpha)}{2N-\alpha}\left[\frac{\|\overline{U}_\epsilon\|^2}{(Q(\overline{U}_\epsilon,\overline{U}_\epsilon))^{\frac{1}{2^*_\alpha}}}\right]^{\frac{2N-\alpha}{N+4-\alpha}}\\
&\leq \frac{(N+4-\alpha)}{2N-\alpha}\left[\frac{(C(N,\alpha))^{\frac{N(N-4)}{4(2N-\alpha}}\, S_{H,L}^{\frac{N}{4}} + o(\epsilon^{N-4})}{\|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}\,(C(N,\alpha))^{\frac{N(N-4)}{4(2N-\alpha}}\, S_{H,L}^{\frac{N-4}{4}} - o(\epsilon^{2N-\alpha}) - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)}\right]^{\frac{2N-\alpha}{N+4-\alpha}}\\
&\leq \frac{(N+4-\alpha)}{2N-\alpha}\left[\frac{\|H^+\|_\infty^{-\frac{2(N-4)}{2N-\alpha}}S_{H,L} + (\epsilon^{N-4})}{1 - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)}\right]^{\frac{2N-\alpha}{N+4-\alpha}}\\
&\leq \frac{(N+4-\alpha)}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}\left[1 + o(\epsilon^{N-4}) + o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\right]^{\frac{2N-\alpha}{N+4-\alpha}}\\
&\leq \frac{(N+4-\alpha)}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{\overline{S}_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} + \begin{cases} o(\epsilon^{N-4}), & \alpha \leq 8\\ o(\epsilon^{\frac{2N-\alpha}{2}}), & \alpha > 8.\end{cases}
\end{aligned} \tag{6.1}$$

Further, $\delta_1 > 0$ is chosen in such a way that $c_\infty > 0$ for all $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \delta_1$. Then, the definition of $I_{\lambda,\mu}$ and $\lambda,\mu > 0$ yield that $I_{\lambda,\mu}(tU_0,tV_0) \leq \frac{t^2}{2}\|(U_0,V_0)\|^2$ for $t \geq 0$. This implies that, there exists $t_0 \in (0,1)$ such that

$$\sup_{t\in[0,t_0]} I_{\lambda,\mu}(tU_0,tV_0) < c_\infty\ \forall\ 0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \delta_1.$$

Moreover,

$$\begin{aligned}
\sup_{t\geq t_0} P_{\lambda,\mu}(tU_0,tV_0) &= \sup_{t\geq t_0}\left(\int_\Omega \lambda F(x)|tU_0|^r + \mu G(x)|tV_0|^r\right)\\
&= \sup_{t\geq t_0}\left(t^r\int_\Omega (\lambda F(x) + \mu G(x))\,|\overline{U}_\epsilon|^r dx\right)\\
&\geq (t_0)^r\,(\lambda a_0 + \mu b_0)\int_{B(0,2r_0)}|\overline{U}_\epsilon|^r dx\\
&\geq \frac{\omega}{r}\,(\lambda+\mu)\begin{cases} o(\epsilon^{N-\frac{N-4}{2}r}|\ln\epsilon|), & r = \frac{N}{N-4}\\ o(\epsilon^{N-\frac{N-4}{2}r}), & r > \frac{N}{N-4},\end{cases}
\end{aligned} \tag{6.2}$$

where $\omega = \min\{a_0, b_0\}$.

Thus, on using (2.8), (6.1) and (6.2), we have

$$\begin{aligned}\sup_{t\geq t_0} I_{\lambda,\mu}(tU_0, tV_0) &= \sup_{t\geq t_0}\left(\mathcal{E}(tU_0, tV_0) - \frac{1}{r}P_{\lambda,\mu}(tU_0, tV_0)\right)\\ &\leq \frac{N+4-\alpha}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{\overline{S}_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} + \begin{cases} o(\epsilon^{N-4}), & \alpha \leq 8\\ o(\epsilon^{\frac{2N-\alpha}{2}}), & \alpha > 8\end{cases}\\ &\quad - \frac{\omega}{r}(\lambda+\mu)\begin{cases} o(\epsilon^{N-\frac{N-4}{2}r}|\ln\epsilon|), & r = \frac{N}{N-4}\\ o(\epsilon^{N-\frac{N-4}{2}r}), & r > \frac{N}{N-4},\end{cases}\end{aligned}$$

or

$$\begin{aligned}\sup_{t\geq t_0} I_{\lambda,\mu}(tU_0, tV_0) &\leq \frac{N+4-\alpha}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{\overline{S}_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} + o(\epsilon^{\rho})\\ &\quad - \frac{\omega}{r}(\lambda+\mu)\begin{cases} o(\epsilon^{N-\frac{N-4}{2}r}|\ln\epsilon|), & r = \frac{N}{N-4}\\ o(\epsilon^{N-\frac{N-4}{2}r}), & r > \frac{N}{N-4},\end{cases}\end{aligned}$$

where $\rho = \min\left\{N-4, \frac{2N-\alpha}{2}\right\}$.

Choose $\delta_2 > 0$ in this way that $0 \leq \epsilon < \delta_2$ and take $\epsilon = \left[(\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right]^{\frac{1}{\rho}}$. Thus, we have

$$\begin{aligned}\sup_{t\geq t_0} I_{\lambda,\mu}(tU_0, tV_0) &\leq \frac{N+4-\alpha}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{\overline{S}_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} + o(\mathcal{D}(\lambda,\mu))\\ &\quad - \frac{\omega}{r}(\lambda+\mu)\begin{cases} o((\mathcal{D}(\lambda,\mu))^{\frac{N}{2\rho}}|\ln\mathcal{D}(\lambda,\mu)|), & r = \frac{N}{N-4}\\ o((\mathcal{D}(\lambda,\mu))^{\frac{N}{\rho}-\frac{N-4}{2\rho}r}), & r > \frac{N}{N-4},\end{cases}\end{aligned}$$

where $\mathcal{D}(\lambda,\mu) = (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}$.

Case (i): When $\alpha \leq 8$, then $\rho = N-4$.

For $r = \frac{N}{N-4}$, we can choose $\delta_3 > 0$ with $0 < \mathcal{D}(\lambda,\mu) < \delta_3$ such that

$$o(\mathcal{D}(\lambda,\mu)) - \frac{\omega(\lambda+\mu)}{r}o\left((\mathcal{D}(\lambda,\mu))^{\frac{N}{2(N-4)}}\left|\ln\left(\mathcal{D}(\lambda,\mu)\right)\right|\right) < -K_0\left(\mathcal{D}(\lambda,\mu)\right),$$

as $\lambda,\mu \to 0$ and $|\ln(\mathcal{D}(\lambda,\mu))| \to +\infty$.

For $r > \frac{N}{N-4}$, we choose $\delta_4 > 0$ with $0 < \mathcal{D}(\lambda,\mu) < \delta_4$ such that

$$o(\mathcal{D}(\lambda,\mu)) - \frac{\omega(\lambda+\mu)}{r}o\left((\mathcal{D}(\lambda,\mu))^{\frac{N}{N-4}-\frac{r}{2}}\right) < -K_0\left(\mathcal{D}(\lambda,\mu)\right),$$

as $1 + \frac{2}{2-r}\left(\frac{N}{N-4} - \frac{r}{2}\right) < \frac{2}{2-r}$ for $r > \frac{N}{N-4}$. Now, we fix $\Upsilon_* = \min\{\delta_1^{\frac{2-r}{2}}, \delta_2^{\frac{(2-r)(N-4)}{2}}, \delta_3^{\frac{2-r}{2}}, \delta_4^{\frac{2-r}{2}}\} > 0$ such that

$$\sup_{t\geq t_0} I_{\lambda,\mu}(tU_0, tV_0) \leq \frac{N+4-\alpha}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{\overline{S}_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} - K_0(\mathcal{D}(\lambda,\mu)) \text{ for } 0 < \mathcal{D}(\lambda,\mu) < \Upsilon_*.$$

Case (ii): When $\alpha > 8$, then $\rho = \frac{2N-\alpha}{2}$.

For $r = \frac{N}{N-4}$, we choose $\delta_5 > 0$ with $0 < \mathcal{D}(\lambda, \mu) < \delta_5$ such that

$$o(\mathcal{D}(\lambda,\mu)) - \frac{\omega(\lambda+\mu)}{r} o\left((\mathcal{D}(\lambda,\mu))^{\frac{N}{2N-\alpha}} \left| \ln\left(\mathcal{D}(\lambda,\mu)\right) \right| \right) < -K_0\left(\mathcal{D}(\lambda,\mu)\right),$$

as $\lambda, \mu \to 0$ and $|\ln(\mathcal{D}(\lambda,\mu))| \to +\infty$.

For $r > \frac{N}{N-4}$, we choose $\delta_6 > 0$ with $0 < \mathcal{D}(\lambda, \mu) < \delta_6$ such that

$$o(\mathcal{D}(\lambda,\mu)) - \frac{\omega(\lambda+\mu)}{r} o\left((\mathcal{D}(\lambda,\mu))^{\frac{2N}{2N-\alpha} - \frac{N-4}{2N-\alpha} r} \right) < -K_0\left(\mathcal{D}(\lambda,\mu)\right),$$

as $1 + \frac{2}{2-r}\left(\frac{2N}{2N-\alpha} - \frac{N-4}{2N-\alpha} r\right) < \frac{2}{2-r}$ for $r > \frac{N}{N-4}$. Fix $\Upsilon_{**} = \min\{\delta_1^{\frac{2-r}{2}}, \delta_2^{\frac{(2-r)(2N-\alpha)}{2}}, \delta_5^{\frac{2-r}{2}}, \delta_6^{\frac{2-r}{2}}\} > 0$ to obtain

$$\begin{aligned} &\sup_{t \geq t_0} I_{\lambda,\mu}(tU_0, tV_0) \\ &\leq \frac{N+4-\alpha}{2N-\alpha} \|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} \left(\frac{S_{H,L}}{2} \right)^{\frac{2N-\alpha}{N+4-\alpha}} - K_0(\mathcal{D}(\lambda,\mu)) \quad \text{for } 0 < \mathcal{D}(\lambda,\mu) < \Upsilon_{**}. \end{aligned} \tag{6.3}$$

Thereafter, we fix $\Upsilon = \min\{\Upsilon_*, \Upsilon_{**}\}$. Thus, we have

$$\begin{aligned} &\sup_{t \geq t_0} I_{\lambda,\mu}(tU_0, tV_0) \\ &\leq \frac{N+4-\alpha}{2N-\alpha} \|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} \left(\frac{S_{H,L}}{2} \right)^{\frac{2N-\alpha}{N+4-\alpha}} - K_0(\mathcal{D}(\lambda,\mu)) =: c_\infty \quad \text{for } 0 < \mathcal{D}(\lambda,\mu) < \Upsilon. \end{aligned}$$

Later, we show that $k^-_{\lambda,\mu} < c_\infty$ for all $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon$. By using (Z2), (Z4) and the definition of (U_0, V_0), we get

$$P_{\lambda,\mu}(U_0, V_0) > 0 \quad \text{and} \quad Q(U_0, V_0) > 0.$$

Further, by Lemma 4.4, definition of $k^-_{\lambda,\mu}$ and (6.3), there exists $t_2(U_0, V_0) \in \mathcal{N}^-_{\lambda,\mu}$ satisfying

$$\begin{aligned} k^-_{\lambda,\mu} &\leq I_{\lambda,\mu}(t_2U_0, t_2V_0)) \leq I_{\lambda,\mu}(tU_0, tV_0) \\ &\leq \frac{N+4-\alpha}{2N-\alpha} \|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} \left(\frac{S_{H,L}}{2} \right)^{\frac{2N-\alpha}{N+4-\alpha}} - K_0(\mathcal{D}(\lambda,\mu)) =: c_\infty, \end{aligned}$$

for each $0 < \mathcal{D}(\lambda, \mu) < \Upsilon$.

Take $(U_0, V_0) = (u_{\lambda,\mu}, v_{\lambda,\mu})$ and with this we complete the proof. □

Lemma 6.2. *Assume that (Z1)–(Z4) hold. Then $I_{\lambda,\mu}$ satisfies the $(PS)_{k^-_{\lambda,\mu}}$ condition for all $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Upsilon_1$ and has a minimizer $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ in $\mathcal{N}^-_{\lambda,\mu}$ and satisfies the following conditions:*

(i) $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = k^-_{\lambda,\mu} > 0$.

(ii) *$(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is a nontrivial solution of the system $(\mathcal{D}_{\lambda,\mu})$.*

Proof. By virtue of Lemma 5.3 (ii), for $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Upsilon_1$, there exists a $(PS)_{k^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$ in $\mathcal{H}$ for $I_{\lambda,\mu}$. Then, from Lemma 3.2, we find that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. Now, using Lemma 3.3 and Lemma 6.1, $I_{\lambda,\mu}$ satisfies the $(PS)_{k^-_{\lambda,\mu}}$-condition. Then, there exists $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{H}$ such that up to subsequence $(u_n, v_n) \to (u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ in $\mathcal{H}$. Moreover, $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = k^-_{\lambda,\mu} > 0$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. Using the argument as applied in Lemma 5.4, one can easily obtain that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is a nontrivial solution of system $(\mathcal{D}_{\lambda,\mu})$ for $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Upsilon_1$. □

Proof of Theorem 1.2. By Lemma 5.4 and Lemma 6.2, system $(\mathcal{D}_{\lambda,\mu})$ has one solution $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and another solution $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. Afterwards, we show that the solutions $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ are not semi-trivial. Using Lemma 5.4 (i) and Lemma 6.2 (i) respectively, we get

$$I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) < 0 \quad \text{and} \quad I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) > 0. \tag{6.4}$$

We observe that, if $(u, 0)$ (or $(0, v)$) is a semi-trivial solution of system $(\mathcal{D}_{\lambda,\mu})$, then we have

$$\begin{cases} \Delta^2 u = \lambda F(x)|u|^{r-2}u & \text{in } \Omega, \\ u = \nabla u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Now, the energy functional $I_{\lambda,\mu}(u, 0)$ corresponding to (6.5) is

$$I_{\lambda,\mu}(u, 0) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{r}\int_\Omega F(x)|u|^r dx = -\frac{2-r}{2r}\|u\|^2 < 0. \tag{6.6}$$

Thus (6.4) and (6.6), we conclude that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is not a semi-trivial solution. Next, we prove that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is also not a semi-trivial solution. Without loss of generality, we assume that $v^1_{\lambda,\mu} \equiv 0$. Then $u^1_{\lambda,\mu}$ is a non-trivial solution of (6.5) and

$$\|(u^1_{\lambda,\mu}, 0)\|^2 = \|u^1_{\lambda,\mu}\|^2 = \lambda\int_\Omega F(x)|u^1_{\lambda,\mu}|^r dx \geq 0.$$

Moreover, we choose $w \in H^2_0(\Omega) \setminus \{0\}$ such that

$$\|(0, w)\|^2 = \|w\|^2 = \mu\int_\Omega G(x)|w|^r dx > 0.$$

From Lemma 4.4, there exists a unique $0 < t_1 < t_{\max}(u^1_{\lambda,\mu}, w)$ such that $(t_1 u^1_{\lambda,\mu}, t_1 w) \in \mathcal{N}^+_{\lambda,\mu}$, where

$$t_{\max}(u^1_{\lambda,\mu}, w) = \left(\frac{(22^*_\alpha - r)\int_\Omega(\lambda F(x)|u^1_{\lambda,\mu}|^r + \mu G(x)|w|^r)dx}{(22^*_\alpha - 2)\|(u^1_{\lambda,\mu}, w)\|^2}\right)^{\frac{1}{2-r}} = \left(\frac{22^*_\alpha - r}{22^*_\alpha - 2}\right)^{\frac{1}{2-r}} > 1.$$

Furthermore,

$$I_{\lambda,\mu}(t_1 u^1_{\lambda,\mu}, t_1 w) = \inf_{0 \leq t \leq t_{\max}} I_{\lambda,\mu}(t u^1_{\lambda,\mu}, t w).$$

This together with the fact that $(u^1_{\lambda,\mu}, 0) \in \mathcal{N}^+_{\lambda,\mu}$ imply that

$$\mu^+_{\lambda,\mu} \leq I_{\lambda,\mu}(t_1 u^1_{\lambda,\mu}, t_1 w) \leq I_{\lambda,\mu}(u^1_{\lambda,\mu}, w) < I_{\lambda,\mu}(u^1_{\lambda,\mu}, 0) = \mu^+_{\lambda,\mu},$$

which is a contradiction. Hence, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is not a semi-trivial solution. □

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