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Existence and asymptotic behavior of nontrivial solutions for the Klein–Gordon–Maxwell system with steep potential well

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Abstract. In this paper, we consider the following nonlinear Klein–Gordon–Maxwell system with a steep potential well

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u - \mu(2\omega + \phi)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ \Delta\phi = \mu(\omega + \phi)u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\omega > 0$ is a constant, μ and λ are positive parameters, $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and the nonlinearity f satisfies the Ambrosetti–Rabinowitz condition. We use parameter-dependent compactness lemma to prove the existence of nontrivial solution for μ small and λ large enough, then explore the asymptotic behavior as $\mu \to 0$ and $\lambda \to \infty$. Moreover, we also use truncation technique to study the existence and asymptotic behavior of positive solutions of the Klein–Gordon–Maxwell system when $f(u) := |u|^{q-2}u$ where $2 < q < 4$.

Keywords: Klein–Gordon–Maxwell system, asymptotic behavior, variational method.

2020 Mathematics Subject Classification: 35J60, 35J20.

1 Introduction

In recent years, the Klein–Gordon–Maxwell system has been widely studied. It is well known that this type of system has a strong physical meaning, and it arises in a very interesting physical context: as a model describing the nonlinear Klein–Gordon field interacting with the electromagnetic field. More specifically, the model represents standing waves $\psi = u(x)e^{i\omega t}$ in equilibrium with a purely electrostatic field $E = -\nabla\phi(x)$, where ϕ is the gauge potential. Using the variational method, Benci and Fortunato [4,5] first introduced the Klein–Gordon–Maxwell equations. In addition, they first studied the following special Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + [m_0^2 - (\omega + \phi)^2]u = |u|^{q-2}\, u, & \text{in } \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

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where $q \in (4,6)$, $m_0 > 0$ and $\omega > 0$ are constants, and establish the existence of infinitely many solitary wave solutions when $0 < \omega < m_0$ and $4 < q < 6$. D'Aprile and Mugnai in [12] also obtained the same conclusion for system (1.1) if one of the following assumptions holds:

(i) $0 < \omega < \sqrt{(q-2)/2}m_0$ and $q \in (2,4)$;

(ii) $q \in [4,6)$ and $0 < \omega < m_0$.

By a Pohozaev-type argument, D'Aprile and Mugnai in [13] showed that (1.1) only has a trivial solution when $0 < q \leq 2$ or $q \geq 6$. Inspired by [5, 12], Azzollini and Pomponio [1] proved that (1.1) admits a ground state solution if one of the following conditions holds:

(i) $4 \leq q < 6$ and $0 < \omega < m_0$;

(ii) $2 < q < 4$ and $0 < \omega < \sqrt{(q-2)/(6-q)}m_0$.

This range has been improved by authors in references [2] and [25]. We point out that the approaches used in [1, 2, 25] are heavily dependent on the form $f(u) := |u|^{q-2}u$. After that, many mathematicians focused on the more general system. For instance, Chen and Tang in[10] generalized the above results to the nonlinear term $f(u)$. They obtained a ground state solution with positive energy under some parameter limitations and f satisfied a superlinear condition.

It can be seen that many early articles are about Klein–Gordon–Maxwell with constant potential, and later more and more researchers concentrated on the non-constant potential. In recent years, there are a large number of articles concerning the existence, nonexistence and multiplicity of nontrivial solutions for the following problem (1.2).

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

In [16], He obtained infinitely many solutions of (1.2). Later, Li and Tang [17] improved the results of [16]. From these two references, we can see that $V(x)$ satisfies the following condition:

($\widehat{V}$) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$, $\inf_{\mathbb{R}^3} V(x) > 0$ and there exists $a_0 > 0$ such that

$$\lim_{|y|\to\infty} \text{meas}\left\{x \in \mathbb{R}^3 : |x-y| \leq a_0, V(x) \leq M\right\} = 0, \quad \forall M > 0.$$

Condition ($\widehat{V}$) plays a crucial role in guaranteeing the compactness of embedding of the weighted Sobolev space. If $V(x)$ is radially symmetric, we recall (see [6] or [23]) that, for $2 < s < 6$, the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is compact. Without this conditions we can see that the compactness is lost. It will make it more difficult for us to deal with the Klein–Gordon–Maxwell system. In this paper we consider the potential satisfied (a_1)–(a_3) below. The conditions (a_1)–(a_3) were first introduced in [3] and $\lambda a(x) + 1$ was called a steep potential well when λ was large. In [20], Liu, Kang and Tang studied the existence of positive solution for the Klein–Gordon–Maxwell system with steep potential well where $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ satisfied the following conditions:

(f_1') There exists a $\bar{C} > 0$ such that $|f(x,t)| \leq \bar{C}|t|$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$.

(f_2') There exists a $k \in \{1,2,\dots\}$ such that uniformly in $x \in \mathbb{R}^3$,

$$v_k < \liminf_{|t|\to 0} \frac{f(x,t)}{t} \leq \limsup_{|t|\to 0} \frac{f(x,t)}{t} < v_{k+1}.$$

(f_3') $\limsup_{|t|\to\infty} \frac{f(x,t)}{t} < v_1$.

Where $0 < v_1 < v_2 < v_3 < \cdots$ were the eigenvalues of the following eigenvalue problem (1.3) and can be written as $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$ counting their multiplicity.

$$\begin{cases} -\Delta u + u = \mu u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

For more articles about steep potential well, readers can refer to [7–9, 11, 17, 19, 21, 22, 24] and references therein for relevant conclusions.

In [27], Zhang and Du studied the existence and asymptotic behavior of positive solutions for Kirchhoff type problems with steep potential well by combining the truncation technique and the parameter-dependent compactness lemma. Motivated by the above works, one of the purposes of this paper is to investigate the existence and asymptotic behavior of nontrivial solution for the following Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u - \mu(2\omega + \phi)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ \Delta\phi = \mu(\omega + \phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\omega > 0$ is a constant, μ and λ are positive parameters, $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $a(x)$ satisfy the following conditions:

(a_1) $a(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $a(x) \geq 0$ on $\mathbb{R}^3$;

(a_2) there exists $c > 0$ such that $A_c := \{x \in \mathbb{R}^3 : a(x) < c\}$ is nonempty and bounded;

(a_3) $\Omega = \operatorname{int} a^{-1}(0)$ is non-empty and has smooth boundary with $\overline{\Omega} = a^{-1}(0)$;

(f_1) $\lim_{|s|\to 0} \frac{f(x,s)}{s} = 0$ uniformly for $x \in \mathbb{R}^3$;

(f_2) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist $c_1 > 0$, and $p \in (4,6)$ such that

$$|f(x,s)| \leq c_1(1 + |s|^{p-1});$$

(f_3) there exist $\alpha > 4$ such that $0 < \alpha F(x,s) \leq f(x,s)s$ uniformly $x \in \mathbb{R}^3$.

Remark 1.1. In [16], in order to show that the associated functional has a mountain pass geometry and obtain the boundedness of Cerami sequence, the authors used a global Ambrosetti–Rabinowitz condition (f_3). To the best of our knowledge, there are only a few articles about the asymptotic behavior of the solution of the Klein–Gordon–Maxwell system with a steep potential well and f satisfies the super-quartic condition.

The following results holds:

Theorem 1.2. *Suppose that* (a_1)–(a_3) *and* (f_1)–(f_3) *are satisfied. Then there exist* λ_1^* *and* $\mu_0 > 0$ *such that for* $\lambda > \lambda_1^*$ *and* $\mu \in (0, \mu_0)$*, problem* (1.4) *has at least a nontrivial solution* $u_{\lambda,\mu} \in E_\lambda$*. Moreover, exist constants* $\tau_0, M > 0$ *(independent of* λ *and* μ*) such that*

$$\tau_0 \leq \|u_{\lambda,\mu}\|_\lambda \leq 2\sqrt{M} \quad \textit{for all } \lambda \textit{ and } \mu. \tag{1.5}$$

Then we show the asymptotic behavior of the nontrivial solution for system (1.4) as $\mu \to 0$ and $\lambda \to \infty$. By means of Theorem 1.2, we have the following results.

Theorem 1.3. *Let* $u_{\lambda,\mu}$ *be the nontrivial solution of* (1.4) *obtained by Theorem* 1.2*. Then for each* $\mu \in (0, \mu_0)$ *be fixed,* $u_{\lambda,\mu} \to u_\mu$ *in* $H^1(\Omega)$ *as* $\lambda \to \infty$*, where* u_μ *is a nontrivial solution of*

$$\begin{cases} -\Delta u + u - \mu(2\omega + \phi)\phi u = f(x,u), & \textit{in } \Omega, \\ \Delta\phi = \mu(\omega + \phi)u^2, & \textit{in } \Omega, \\ u = 0, & \textit{in } \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{1.6}$$

Theorem 1.4. *Let* $u_{\lambda,\mu}$ *be the nontrivial solution of* (1.4) *obtained by Theorem* 1.2*. Then for each* $\lambda \in (\lambda_1^*, \infty)$ *be fixed,* $u_{\lambda,\mu} \to u_\lambda$ *in* E_λ *as* $\mu \to 0$*, where* u_λ *is a nontrivial solution of*

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u = f(x,u), & \textit{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.7}$$

Theorem 1.5. *Let* $u_{\lambda,\mu}$ *be the nontrivial solution of* (1.4) *obtained by Theorem* 1.2*. Then* $u_{\lambda,\mu} \to u_0$ *in* $H^1(\Omega)$ *as* $\mu \to 0$ *and* $\lambda \to \infty$*, where* u_0 *is a nontrivial solution of*

$$\begin{cases} -\Delta u + u = f(x,u), & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases} \tag{1.8}$$

Remark 1.6. For Theorem 1.2, applying the Mountain Pass Theorem directly to the associated functional $I_{\lambda,\mu}$, we can get a Cerami sequence for $\mu > 0$ small enough. Then we will obtain the boundedness of this Cerami sequence.

Next, we consider the following Klein–Gordon–Maxwell system where $f(u) := |u|^{q-2}u$,

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u - \mu(2\omega + \phi)\phi u = |u|^{q-2}u, & \text{in } \mathbb{R}^3, \\ \Delta\phi = \mu(\omega + \phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.9}$$

where $\omega > 0$ is a constant, μ and λ are positive parameters, $2 < q < 4$ and $a(x)$ also satisfies $(a_1) - (a_3)$.

Remark 1.7. In particular, we note that the nonlinearity $u \mapsto f(u) := |u|^{q-2}u$ with $2 < q < 4$ does not satisfy the Ambrosetti–Rabinowitz type condition which would readily obtain a bounded Palais–Smale sequence or Cerami sequence.

Then we have the following results.

Theorem 1.8. *Suppose that* $(a_1) - (a_3)$ *and* $2 < q < 4$ *are satisfied. Then there exist* λ_2^* *and* $\mu_1, \mu_2 > 0$ *such that for* $\lambda > \lambda_2^*$ *and* $\mu \in (0, \min\{\mu_1, \mu_2\})$*, problem* (1.9) *has at least a positive solution* $\hat{u}_{\lambda,\mu} \in E_\lambda$*. Moreover, exist constants* $\tau_1, T > 0$ *(independent of* λ *and* μ*) such that*

$$\tau_1 \leq \|\hat{u}_{\lambda,\mu}\|_\lambda \leq T \quad \textit{for all } \lambda \textit{ and } \mu. \tag{1.10}$$

Theorem 1.9. *Let $\hat{u}_{\lambda,\mu}$ be the positive solution of* (1.9) *obtained by Theorem 1.8. Then for each $\mu \in (0, \min\{\mu_1, \mu_2\})$ be fixed, $\hat{u}_{\lambda,\mu} \to \hat{u}_\mu$ in $H^1(\Omega)$ as $\lambda \to \infty$, where $\hat{u}_\mu$ is a positive solution of*

$$\begin{cases} -\Delta u + u - \mu(2\omega + \phi)\phi u = |u|^{p-2}u, & \text{in } \Omega, \\ \Delta\phi = \mu(\omega + \phi)u^2, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{1.11}$$

Theorem 1.10. *Let $\hat{u}_{\lambda,\mu}$ be the positive solution of* (1.9) *obtained by Theorem 1.8. Then for each $\lambda \in (\lambda_2^*, \infty)$ be fixed, $\hat{u}_{\lambda,\mu} \to \hat{u}_\lambda$ in E_λ as $\mu \to 0$, where $\hat{u}_\lambda$ is a positive solution of*

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.12}$$

Theorem 1.11. *Let $\hat{u}_{\lambda,\mu}$ be the positive solution of* (1.9) *obtained by Theorem 1.8. Then $\hat{u}_{\lambda,\mu} \to \hat{u}_0$ in $H^1(\Omega)$ as $\mu \to 0$ and $\lambda \to \infty$, where $\hat{u}_0$ is a positive solution of*

$$\begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.13}$$

Remark 1.12. For Theorem 1.8, if we apply the Mountain Pass Theorem directly to the associated functional $\hat{I}_{\lambda,\mu}$, we also can get a Cerami sequence for $\mu > 0$ small enough. But it is difficult to obtain the boundedness of this Cerami sequence. Then we will use a new method (refer to [27]) called truncation technique to get over this difficulty.

The remainder of this paper is organized as follows. Next Section 2 we derive a variational setting for problems and give some preliminary lemmas. In Section 3 we will prove Theorem 1.2 to Theorem 1.5. Section 4 is devoted to the proof of Theorem 1.8 to Theorem 1.11.

2 Variational setting and preliminaries

Throughout this paper, we use the standard notations. We denote by $C, c_i, C_i, i = 1, 2, \ldots$ for various positive constants whose exact value may change from lines to lines but are not essential to the analysis of problem. We use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space respectively. We will write $o(1)$ to denote quantity that tends to 0 as $n \to \infty$. X' denotes the dual space of X.

$|\cdot|_q$ denotes the usual Lebesgue space with the norm $L^q(\mathbb{R}^3)$ for any $q \in [1, \infty]$. $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space with the standard scalar product and norm $\|\cdot\|_{H^1}$. $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{D^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}}$.

In the paper, we work in the following Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} a(x)u^2 dx < \infty \right\}$$

with the inner product and norm

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + (a(x) + 1)u^2 dx) \right)^{\frac{1}{2}}, \quad \|u\| = \langle u, u \rangle^{\frac{1}{2}}.$$

For $\lambda > 0$, we also need the following inner product and norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^3}(|\nabla u|^2 + (\lambda a(x)+1)u^2)dx\right)^{\frac{1}{2}}, \quad \|u\|_\lambda = \langle u,u\rangle_\lambda^{\frac{1}{2}}.$$

It is clear that $\|u\| \le \|u\|_\lambda$ for $\lambda \ge 1$. Set $E_\lambda = (E, \|\cdot\|_\lambda)$.

Referring to [28], it is well known that $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $s \in [2,6]$. Thus, combining Sobolev embedding, for each $s \in [2,6]$, there exists $d_s > 0$ (independent of $\lambda \ge 1$) such that

$$|u|_s \le d_s\|u\| \le d_s\|u\|_\lambda \quad \text{for } u \in E. \tag{2.1}$$

S is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$ and

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{|\nabla u|_2^2}{|u|_6^2}.$$

It is easy to see that the weak solutions $(u,\phi) \in E_\lambda \times D^{1,2}(\mathbb{R}^3)$ of system (1.4) are critical points of the functional given by

$$G_{\lambda,\mu}(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + (\lambda a(x)+1)u^2 - |\nabla\phi|^2 - \mu(2\omega+\phi)\phi u^2\right)dx - \int_{\mathbb{R}^3}F(x,u)dx. \tag{2.2}$$

The functional $G_{\lambda,\mu}(u,\phi)$ is strongly indefinite, i.e., unbounded from below and from above on infinite dimensional spaces. We need the following technical results to study of the functional in the only variable u.

Lemma 2.1 ([4, 12])**.** *For any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ which solves equation*

$$-\Delta\phi + u^2\phi = -\omega u^2. \tag{2.3}$$

Moreover, the map $\Phi : u \in H^1(\mathbb{R}^3) \mapsto \Phi[u] := \phi_u \in D^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and

(i) $-\omega \le \phi_u \le 0$ on the set $\{x \in \mathbb{R}^3 | u(x) \ne 0\}$;

(ii) $\|\phi_u\|_{D^{1,2}} \le C\|u\|^2$ and $\int_{\mathbb{R}^3}|\phi_u|u^2dx \le C\|u\|_{12/5}^4 \le C\|u\|^4$.

Lemma 2.2 ([1, Lemma 2.7])**.** *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$. As a consequence $I'(u_n) \to I'(u)$ in the sense of distributions.*

The following lemma is a stronger version of the Mountain Pass Theorem, so we can find a Cerami sequence.

Proposition 2.3 ([15])**.** *Let X be a real Banach space with its dual space X', and suppose that $J \in C^1(X,\mathbb{R})$ satisfies*

$$\max\{J(0), J(e)\} \le \mu < \eta \le \inf_{\|u\|_X=\rho} J(u)$$

for some $\mu < \eta$, $\rho > 0$ and $e \in X$ with $\|e\|_X > \rho$. Let $c \ge \eta$ be characterized by

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e. Then there exists a sequence $\{u_n\} \subset X$ such that

$$J(u_n) \to c \ge \eta \quad \text{and} \quad (1+\|u_n\|_X)\|J'(u_n)\|_{X'} \to 0 \quad \text{as } n \to \infty.$$

3 The existence and concentration phenomenon of solution of (1.4)

We proof Theorem 1.2 to Theorem 1.5 in this section. By (2.3), multiplying both sides by ϕ_u and integrating we obtain

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 dx = -\int_{\mathbb{R}^3} \omega \phi_u u^2 dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 dx. \tag{3.1}$$

Using (3.1), we can rewrite $G_{\lambda,\mu}$ as a C^1 functional $I_{\lambda,\mu} : E_\lambda \to \mathbb{R}$ given by

$$I_{\lambda,\mu}(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + (\lambda a(x)+1)u^2)dx - \frac{\mu}{2}\int_{\mathbb{R}^3} \omega \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx. \tag{3.2}$$

Moreover, for any $u, v \in E_\lambda$, we have

$$\langle I'_{\lambda,\mu}(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + (\lambda a(x)+1)uv)\, dx - \mu \int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u uv dx - \int_{\mathbb{R}^3} f(x,u) v dx. \tag{3.3}$$

To begin with, we show that the $I_{\lambda,\mu}$ has the mountain pass geometry.

Lemma 3.1. *Suppose (a_1)–(a_3) and (f_1)–(f_3) are satisfied. There exists $\mu_0 > 0$ for each $\mu \in (0, \mu_0)$ and $\lambda \geq 1$. Then there exist $\rho, \beta > 0$ and $e_0 \in E_\lambda$, $\|e_0\|_\lambda > \rho$, such that*

$$\inf_{\|u\|=\rho} I_{\lambda,\mu}(u) \geq \beta > 0 \geq \max\{I_{\lambda,\mu}(0), I_{\lambda,\mu}(e_0)\}.$$

Proof. From (f_1) and (f_2), for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$

$$|f(x,s)| \leq \varepsilon |s| + C_\varepsilon |s|^{p-1} \tag{3.4}$$

and

$$|F(x,s)| \leq \frac{\varepsilon}{2}|s|^2 + \frac{C_\varepsilon}{p}|s|^p. \tag{3.5}$$

We choose $\varepsilon = \frac{1}{2d_2^2}$, where $d_2 > 0$ is from (2.1). For each $u \in E_\lambda$, by Lemma 2.1, (2.1), (3.2) and (3.5) we have

$$\begin{aligned} I_{\lambda,\mu}(u) &\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{\varepsilon}{2}|u|_2^2 - \frac{C_\varepsilon}{p}|u|_p^p \\ &\geq \left(\frac{1}{2} - \frac{\varepsilon d_2^2}{2}\right)\|u\|_\lambda^2 - \frac{C_\varepsilon d_p^p}{p}\|u\|_\lambda^p \\ &= \left(\frac{1}{4} - \frac{C_\varepsilon d_p^p}{p}\|u\|_\lambda^{p-2}\right)\|u\|_\lambda^2, \end{aligned}$$

where the constants $d_p > 0$ and $C_\varepsilon > 0$ are independent of μ and λ. Then there exist $\rho > 0$ small enough and $\beta > 0$, such that $\inf_{\|u\|=\rho} I_{\lambda,\mu}(u) \geq \beta > 0$.

Then, we define the functional $J_\lambda : E_\lambda \to \mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + (\lambda a(x)+1)u^2)dx - \int_{\mathbb{R}^3} F(x,u)dx.$$

By (f_1) and (f_3), there exist $c_3, c_4 > 0$ such that

$$F(x,s) \geq c_3|s|^\alpha - c_4 s^2. \tag{3.6}$$

Let $e \in C_0^\infty(\Omega)$ be a positive smooth function, since $\alpha > 4$ then we have

$$\begin{aligned} J_\lambda(te) &= \frac{t^2}{2}\int_\Omega (|\nabla e|^2 + e^2)dx - \int_\Omega F(x,te)dx \\ &\le \frac{t^2}{2}\int_\Omega (|\nabla e|^2 + e^2)dx + c_4 t^2 \int_\Omega e^2 dx - c_3 t^\alpha \int_\Omega |e|^\alpha dx \\ &\to -\infty, \end{aligned}$$

as $t \to \infty$. Therefore, there exist $t_0 > 0$ large enough and let $e_0 := t_0 e$ such that $J_\lambda(e_0) \le -1$ with $\|e_0\|_\lambda > \rho$. From Lemma 2.1(i), then

$$\begin{aligned} I_{\lambda,\mu}(e_0) &= J_\lambda(e_0) - \frac{\mu}{2}\int_{\mathbb{R}^3} \omega \phi_{e_0} e_0^2 dx \\ &\le -1 + \frac{\mu\omega^2}{2}|e_0|_2^2, \end{aligned}$$

there exists $\mu_0 := \frac{2}{\omega^2 |e_0|_2^2} > 0$ (independent of λ) such that $I_{\lambda,\mu}(e_0) < 0$ for each $\lambda \ge 1$ and $\mu \in (0,\mu_0)$. The proof is completed. □

Then we consider the mountain pass value

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0,\ \gamma(1) = e_0\}$. From Proposition 2.3 and Lemma 3.1, we can obtain that for each $\lambda \ge 1$ and $\mu \in (0,\mu_0)$, there exists a Cerami sequence $\{u_n\} \subset E_\lambda$ such that

$$I_{\lambda,\mu}(u_n) \to c_{\lambda,\mu} > 0 \quad \text{and} \quad (1 + \|u_n\|_\lambda) \left\| I'_{\lambda,\mu}(u_n) \right\|_{E'_\lambda} \to 0 \quad \text{as } n \to \infty. \tag{3.7}$$

Next we prove that $c_{\lambda,\mu}$ has an upper bound.

Lemma 3.2. *Suppose (a_1)–(a_3) and $(f_1),(f_3)$ hold. Then for each $\lambda \ge 1$ and $\mu \in (0,\mu_0)$, there exists $M > 0$ (independent of μ and λ) such that $c_{\lambda,\mu} \le M$.*

Proof. By $(f_1),(f_3)$, we have (3.6). Since $e_0 \in C_0^\infty(\Omega)$ and Lemma 3.1, we obtain that

$$\begin{aligned} I_{\lambda,\mu}(te_0) &= \frac{t^2}{2}\int_\Omega (|\nabla e_0|^2 + e_0^2)dx - \frac{\mu}{2}\int_\Omega \omega\phi_{te_0}(te_0)^2 dx - \int_\Omega F(x,te_0)dx \\ &\le \frac{t^2}{2}\int_\Omega (|\nabla e_0|^2 + e_0^2)dx + \frac{\mu_0\omega^2}{2}t^2\int_\Omega e_0^2 dx + c_4 t^2 \int_\Omega e_0^2 dx - c_3 t^\alpha \int_\Omega |e_0|^\alpha dx, \end{aligned}$$

where $\alpha > 4$. Therefore, there exists $M > 0$ (independent of μ and λ) such that

$$c_{\lambda,\mu} \le \max_{t\in[0,1]} I_{\lambda,\mu}(te_0) \le M.$$

This completes the proof. □

Lemma 3.3. *Assume (a_1)–(a_3) and (f_1)–(f_3) hold, for each $\lambda > 1$, $\mu \in (0,\mu_0)$, if $\{u_n\} \subset E_\lambda$ is a sequence satisfying (3.7), then we have, up to a subsequence, $\{u_n\}$ is bounded in E_λ.*

Proof. From (3.2), (3.3), (f_3), Lemma 2.1(i) and Lemma 3.2, $\alpha > 4$, for $n \to \infty$ we have

$$\begin{aligned} M + o(1) \geq c_{\lambda,\mu} + o(1) =& I_{\lambda,\mu}(u_n) - \frac{1}{\alpha}\langle I'_{\lambda,\mu}(u_n), u_n\rangle \\ =& \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|_\lambda^2 + \left(\frac{2}{\alpha} - \frac{1}{2}\right)\mu\int_{\mathbb{R}^3}\omega\phi_{u_n}u_n^2dx \\ &+ \frac{\mu}{\alpha}\int_{\mathbb{R}^3}\phi_{u_n}^2u_n^2dx + \int_{\mathbb{R}^3}\left(\frac{1}{\alpha}f(x,u_n)u_n - F(x,u_n)\right)dx \\ \geq& \frac{1}{4}\|u_n\|_\lambda^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in E_λ and $\|u_n\|_\lambda \leq 2\sqrt{M}$ as $n \to \infty$, where M is given by Lemma 3.2. □

Then we will give the compactness conditions for $I_{\lambda,\mu}$. Before that, we introduce a lemma to deal with nonlinear term.

Lemma 3.4 ([14])**.** *Assume that (f_1) and (f_2) hold. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then along a subsequence of $\{u_n\}$,*

$$\lim_{n\to\infty}\sup_{\varphi\in H^1(\mathbb{R}^3),\|\varphi\|_{H^1}\leq 1}\left|\int_{\mathbb{R}^3}[f(x,u_n) - f(x,u_n-u) - f(x,u)]\varphi dx\right| = 0.$$

Lemma 3.5. *Suppose that (a_1)–(a_3) and (f_1)–(f_3) hold. If $\{u_n\} \subset E_\lambda$ is a sequence satisfying (3.7), up to a subsequence, there exists $\lambda_1^* \geq 1$ such that for each $\mu \in (0,\mu_0)$ and $\lambda \in (\lambda_1^*,\infty)$, $\{u_n\} \subset E_\lambda$ contains a convergent subsequence.*

Proof. By Lemma 3.3, we know that $\{u_n\}$ is bounded. We may assume that there exists $u \in E_\lambda$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } E_\lambda, \\ u_n &\to u \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^3), 2 \leq s < 6, \\ u_n &\to u \quad \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{3.8}$$

From Lemma 2.2 and (3.7), we have $\langle I'_{\lambda,\mu}(u), u\rangle = 0$, i.e.,

$$\|u\|_\lambda^2 - 2\mu\int_{\mathbb{R}^3}\omega\phi_u u^2dx - \mu\int_{\mathbb{R}^3}\phi_u^2u^2dx - \int_{\mathbb{R}^3}f(x,u)udx = 0. \tag{3.9}$$

Next, we prove that $u_n \to u$ in E_λ. Let $v_n = u_n - u$, by (a_2), then

$$|v_n|_2^2 = \int_{\mathbb{R}^3\setminus A_c}v_n^2dx + \int_{A_c}v_n^2dx \leq \frac{1}{c\lambda+1}\|v_n\|_\lambda^2 + o(1) \leq \frac{1}{c\lambda}\|v_n\|_\lambda^2 + o(1). \tag{3.10}$$

It follows from Brézis–Lieb Lemma the ([26, Lemma 1.32]) that

$$\|u_n\|_\lambda^2 - \|u\|_\lambda^2 = \|v_n\|_\lambda^2 + o(1). \tag{3.11}$$

Then, by (3.10), Hölder and Sobolev inequalities, we have

$$|v_n|_p \leq |v_n|_2^\theta|v_n|_6^{1-\theta} \leq S^{\frac{\theta-1}{2}}|v_n|_2^\theta|\nabla v_n|_2^{1-\theta} \leq S^{\frac{\theta-1}{2}}(c\lambda)^{-\frac{\theta}{2}}\|v_n\|_\lambda + o(1), \tag{3.12}$$

where $\theta = \frac{6-p}{2p} > 0$. Employing Lemma 3.4, we have

$$\left|\frac{1}{\|u_n\|_\lambda}\int_{\mathbb{R}^3}[f(x,u_n) - f(x,v_n) - f(x,u)]u_ndx\right| = o(1).$$

From (3.8), $v_n \rightharpoonup 0$ in E_λ, $v_n \to 0$ in $L^s_{loc}(\mathbb{R}^3)$ for $2 \le s < 6$ and $v_n \to 0$ a.e. on $\mathbb{R}^3$, there have

$$\begin{aligned}\frac{1}{\|u_n\|_\lambda}\int_{\mathbb{R}^3} f(x,u_n)u_n dx \le& \frac{1}{\|u_n\|_\lambda}\left[\int_{\mathbb{R}^3} f(x,u)u + f(x,v_n)v_n + f(x,v_n)u + f(x,u)v_n dx\right]\\ &+\left|\frac{1}{\|u_n\|_\lambda}\int_{\mathbb{R}^3}[f(x,u_n)-f(x,v_n)-f(x,u)]u_n dx\right| \\ =& \frac{1}{\|u_n\|_\lambda}\left[\int_{\mathbb{R}^3} f(x,u)udx + \int_{\mathbb{R}^3} f(x,v_n)v_n dx\right] + o(1).\end{aligned} \tag{3.13}$$

From (3.5), (3.10)–(3.13), Lemma 3.3 and Fatou's Lemma, choose $\varepsilon = \frac{1}{2d_2^2}$, then

$$\begin{aligned} o(1) &= \langle I'_{\lambda,\mu}(u_n), u_n\rangle - \langle I'_{\lambda,\mu}(u), u\rangle \\ &= \|u_n\|^2_\lambda - \mu\int_{\mathbb{R}^3}(2\omega+\phi_{u_n})\phi_{u_n}u_n^2 dx - \int_{\mathbb{R}^3} f(x,u_n)u_n dx \\ &\quad -\|u\|^2_\lambda + \mu\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u u^2 dx + \int_{\mathbb{R}^3} f(x,u)udx \\ &\ge \|v_n\|^2_\lambda - \int_{\mathbb{R}^3} f(x,v_n)v_n dx + o(1) \\ &\ge \|v_n\|^2_\lambda - \varepsilon|v_n|_2^2 - C_\varepsilon|v_n|_p^p + o(1) \\ &\ge [\frac{1}{2} - C_\varepsilon(4d_p\sqrt{M})^{p-2}S^{\theta-1}(c\lambda)^{-\theta}]\|v_n\|^2_\lambda + o(1).\end{aligned}$$

Hence, there exists $\lambda_1 = [2C_\varepsilon(4d_p\sqrt{M})^{p-2}S^{\theta-1}c^{-\theta}]^{\frac{1}{\theta}}$ such that the previous coefficient of $\|v_n\|^2_\lambda$ is greater than 0 when $\lambda > \lambda_1$, where M is given by Lemma 3.2. Then choose $\lambda_1^* = \max\{\lambda_1, 1\}$ such that $v_n \to 0$ in E_λ for all $\lambda > \lambda_1^*$. □

Proof of Theorem 1.2. Assume (a_1)–(a_3) and (f_1)–(f_3) are satisfied. By Lemma 3.1, there exists $\mu_0 > 0$ such that for every $\lambda \ge 1$ and $\mu \in (0,\mu_0)$, $I_{\lambda,\mu}$ possesses a Cerami sequence $\{u_n\}$ at the mountain pass level $c_{\lambda,\mu}$ and satisfied

$$I_{\lambda,\mu}(u_n) \to c_{\lambda,\mu} > 0 \quad \text{and} \quad (1+\|u_n\|_\lambda)\|I'_{\lambda,\mu}(u_n)\|_{E'_\lambda} \to 0, \text{ as } n\to\infty.$$

From Lemmas 3.2 and 3.3, we thus deduce that for every $\lambda \ge 1$ and $\mu \in (0,\mu_0)$, after passing to a subsequence, $\{u_n\}$ is bounded in E_λ and $\|u_n\|_\lambda \le 2\sqrt{M}$ as $n \to \infty$. It follows from Lemma 3.5 that exists $\lambda_1^* \ge 1$ such that for each $\mu \in (0,\mu_0)$ and $\lambda \in (\lambda_1^*, \infty)$, the sequence $\{u_n\}$ has a convergent subsequence in E_λ. Then there exists $u_{\lambda,\mu} \in E_\lambda$, such that $u_n \to u_{\lambda,\mu}$ as $n \to \infty$, and thus

$$\|u_{\lambda,\mu}\|_\lambda \le 2\sqrt{M}, \quad I_{\lambda,\mu}(u_{\lambda,\mu}) = c_{\lambda,\mu} \quad \text{and} \quad I'_{\lambda,\mu}(u_{\lambda,\mu}) = 0.$$

Now we claim that $u_{\lambda,\mu} \ne 0$. Otherwise, $I_{\lambda,\mu}(u_{\lambda,\mu}) = 0 = c_{\lambda,\mu}$, which is a contradiction to $c_{\lambda,\mu} > 0$. Moreover, by the Hölder inequality, Lemma 2.1(ii) and Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^3}\phi^2_{u_{\lambda,\mu}}u^2_{\lambda,\mu}dx &\le \left(\int_{\mathbb{R}^3}(\phi^2_{u_{\lambda,\mu}})^3 dx\right)^{\frac{1}{3}}\left(\int_{\mathbb{R}^3}(u^2_{\lambda,\mu})^{\frac{3}{2}}dx\right)^{\frac{2}{3}} \\ &\le Cd_3^2\|\phi_{u_{\lambda,\mu}}\|^2_{D^{1,2}}\|u_{\lambda,\mu}\|^2_\lambda \\ &\le \widetilde{C}\|u_{\lambda,\mu}\|^4_\lambda.\end{aligned}$$

Then, since $\langle I'_{\lambda,\mu}(u_{\lambda,\mu}), u_{\lambda,\mu}\rangle = 0$, Lemma 2.1(i) and (3.4), there exists $\varepsilon = \frac{1}{2d_2^2}$, we have

$$\begin{aligned}\|u_{\lambda,\mu}\|_\lambda^2 &= 2\mu\int_{\mathbb{R}^3}\omega\phi_{u_{\lambda,\mu}}u_{\lambda,\mu}^2dx + \mu\int_{\mathbb{R}^3}\phi_{u_{\lambda,\mu}}^2u_{\lambda,\mu}^2dx + \int_{\mathbb{R}^3}f(x,u_{\lambda,\mu})u_{\lambda,\mu}dx \\ &\le \mu_0\widetilde{C}_1\|u_{\lambda,\mu}\|_\lambda^4 + \varepsilon d_2^2\|u_{\lambda,\mu}\|_\lambda^2 + C_\varepsilon d_p^p\|u_{\lambda,\mu}\|_\lambda^p.\end{aligned}$$

Hence, there exists $\tau_0 > 0$ (independent of μ and λ) such that $\|u_{\lambda,\mu}\|_\lambda \ge \tau_0$ for all $\mu \in (0,\mu_0)$ and $\lambda \in (\lambda_1^*, \infty)$. This finishes the proof. □

Proof of Theorem 1.3. Let $\mu \in (0,\mu_0)$ be fixed, then for any sequence $\lambda_n \to +\infty$. Let $u_n := u_{\lambda_n,\mu}$ be the nontrivial solution of (1.4) obtained by Theorem 1.2. From Theorem 1.2 we have

$$0 < \tau_0 \le \|u_n\|_{\lambda_n} \le 2\sqrt{M} \quad \text{for } n \to \infty. \tag{3.14}$$

Thus, up to a subsequence, we may assume that

$$\begin{aligned} u_n &\rightharpoonup u_\mu \quad ;\text{in } E,\\ u_n &\to u_\mu \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^3),\ 2 \le s < 6,\\ u_n &\to u_\mu \quad \text{a.e. in } \mathbb{R}^3.\end{aligned}$$

It follows from (3.14), Fatou's Lemma and (a_1) that

$$0 \le \int_{\mathbb{R}^3}(a(x)+1)u_\mu^2dx \le \liminf_{n\to\infty}\int_{\mathbb{R}^3}(a(x)+1)u_n^2dx \le \liminf_{n\to\infty}\frac{\|u_n\|_{\lambda_n}^2}{\lambda_n} = 0.$$

Hence, $u_\mu = 0$ a.e. in $\mathbb{R}^3 \setminus a^{-1}(0)$, and $u_\mu \in H^1(\Omega)$ by the condition (a_3).

Now we show that $u_n \to u_\mu$ in $L^s(\mathbb{R}^3)$ for all $s \in (2,6)$. Otherwise, by Lions' vanishing Lemma ([18,26]) there exist $\delta, r > 0$ and $x_n \in \mathbb{R}^3$ such that

$$\int_{B_r(x_n)}(u_n - u_\mu)^2dx \ge \delta.$$

This implies that $|x_n| \to \infty$ as $n \to \infty$, and so $|B_r(x_n) \cap A_c| \to 0$. By the Hölder inequality, we then conclude that

$$\int_{B_r(x_n)\cap A_c}(u_n - u_\mu)^2dx \to 0 \quad \text{as } n \to \infty.$$

Consequently, we get

$$\begin{aligned}\|u_n\|_{\lambda_n}^2 &\ge (c\lambda_n + 1)\int_{B_r(x_n)\cap\{a(x)\ge c\}}u_n^2dx = (c\lambda_n+1)\int_{B_r(x_n)\cap\{a(x)\ge c\}}(u_n-u_\mu)^2dx \\ &= (c\lambda_n+1)\left(\int_{B_r(x_n)}(u_n-u_\mu)^2dx - \int_{B_r(x_n)\cap A_c}(u_n-u_\mu)^2dx\right) \to \infty,\end{aligned}$$

as $n \to \infty$, which contradicts (3.14).

We then prove that $u_n \to u_\mu$ in E. Since

$$\langle I'_{\lambda_n,\mu}(u_n), u_n\rangle = \langle I'_{\lambda_n,\mu}(u_n), u_\mu\rangle = 0,$$

we have

$$\|u_n\|_{\lambda_n}^2 - 2\mu\int_{\mathbb{R}^3}\omega\phi_{u_n}u_n^2dx - \mu\int_{\mathbb{R}^3}\phi_{u_n}^2u_n^2dx - \int_{\mathbb{R}^3}f(x,u_n)u_ndx = 0,$$

$$\int_{\mathbb{R}^3} (\nabla u_n \nabla u_\mu + (\lambda_n a(x)+1)u_n u_\mu)\, dx - 2\mu \int_{\mathbb{R}^3} \omega \phi_{u_n} u_n u_\mu dx \\ - \mu \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n u_\mu dx - \int_{\mathbb{R}^3} f(x,u_n) u_\mu dx = 0.$$

Since $u_\mu = 0$ a.e. in $\mathbb{R}^3 \setminus a^{-1}(0)$ and by Lemma 2.1(ii), (3.14), we deduce that

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n (u_n - u_\mu) dx \le |\phi_{u_n}|_6 |u_n|_2 |u_n - u_\mu|_3 \le C \|\phi_{u_n}\|_{D^{1,2}} \|u_n\|_\lambda |u_n - u_\mu|_3 \to 0,$$

$$\int_{\mathbb{R}^3} \phi_{u_n}^2 u_n (u_n - u_\mu) dx \le |\phi_{u_n}|_6^2 |u_n|_3 |u_n - u_\mu|_3 \le C \|\phi_{u_n}\|_{D^{1,2}}^2 \|u_n\|_\lambda |u_n - u_\mu|_3 \to 0,$$

$$\int_{\mathbb{R}^3} f(x,u_n)(u_n - u_\mu) dx \to 0.$$

Thus, $\lim_{n\to\infty} \|u_n\|_{\lambda_n}^2 = \|u_\mu\|^2$. Then from the weakly lower semi-continuity of norm, we have

$$\|u_\mu\|^2 \le \liminf_{n\to\infty} \|u_n\|^2 \le \limsup_{n\to\infty} \|u_n\|^2 \le \limsup_{n\to\infty} \|u_n\|_{\lambda_n}^2 = \|u_\mu\|^2. \tag{3.15}$$

Consequently, we yield that $u_n \to u_\mu$ in E.

Finally, we only need to show that u_μ is a weak solution of (1.6). Now for any $v \in C_0^\infty(\Omega)$, since $\langle I'_{\lambda_n,\mu}(u_n), v\rangle = 0$, it is easy to check that

$$\int_\Omega (\nabla u_\mu \nabla v + u_\mu v) dx - \mu \int_\Omega (2\omega + \phi_{u_\mu}) \phi_{u_\mu} u_\mu v dx - \int_\Omega f(x,u_\mu) v dx = 0.$$

i.e., u_μ is a weak solution of (1.6) by the density of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. From (3.14) and (3.15), we can see that

$$\|u_\mu\| = \lim_{n\to\infty} \|u_n\|_{\lambda_n} \ge \tau_0 > 0,$$

so $u_\mu \neq 0$. Thus, u_μ is a nontrivial weak solution of (1.6). □

Proof of Theorem 1.4. Let $\lambda \in (\lambda_1^*, \infty)$ be fixed, then for any sequence $\mu_n \to 0$. Let $u_n := u_{\lambda,\mu_n}$ be the nontrivial solution of (1.4) obtained by Theorem 1.2. From Theorem 1.2, we have

$$0 < \tau_0 \le \|u_n\|_{\lambda_n} \le 2\sqrt{M} \quad \text{for } n \to \infty. \tag{3.16}$$

Passing to a subsequence , we may assume that $u_n \rightharpoonup u_\lambda$ in E_λ. Note that $I'_{\lambda,\mu_n}(u_n) = 0$, we can obtain that $u_n \to u_\lambda$ in E_λ as the proof of Lemma 3.5.

To complete the proof, we will show that u_λ is a nontrivial solution of (1.7). Now for any $v \in E_\lambda$, since $\langle I'_{\lambda,\mu_n}(u_n), v\rangle = 0$, it is easy to check that

$$\int_{\mathbb{R}^3} \nabla u_\lambda \nabla v + (\lambda a(x)+1) u_\lambda v dx = \int_{\mathbb{R}^3} f(x,u_\lambda) v dx,$$

i.e., u_λ is a weak solution of (1.7). Then, by (3.16) we see that $u_\lambda \neq 0$. Therefore, u_λ is a nontrivial weak solution of (1.7). This completes the proof. □

Proof of Theorem 1.5. Following the same argument as in the proof of Theorems 1.3 and 1.4, we get the conclusion. □

4 The existence and concentration phenomenon of solution of (1.9)

In this section, we will give the asymptotic behavior of positive solution of (1.9), and devote to prove Theorem 1.8 to Theorem 1.11. We will use truncation technique to obtain the boundedness of Cerami sequence. Before that, we write the functional corresponding to (1.9). $\hat{I}_{\lambda,\mu} : E_\lambda \to \mathbb{R}$ given by

$$\hat{I}_{\lambda,\mu}(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + (\lambda a(x)+1)u^2)dx - \frac{\mu}{2}\int_{\mathbb{R}^3}\omega\phi_u u^2 dx - \frac{1}{q}\int_{\mathbb{R}^3}|u^+|^q dx \tag{4.1}$$

and $\hat{I}_{\lambda,\mu} \in C^1$. Moreover, for any $u, v \in E_\lambda$, we have

$$\langle \hat{I}'_{\lambda,\mu}(u), v\rangle = \int_{\mathbb{R}^3}(\nabla u\cdot\nabla v + (\lambda a(x)+1)uv)\,dx - \mu\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u uv dx - \int_{\mathbb{R}^3}|u^+|^{q-2}u^+ v dx. \tag{4.2}$$

Then we define a cut-off function $\eta \in C^1([0,\infty),\mathbb{R})$ satisfying $0 \le \eta \le 1$, $\eta(t) = 1$ if $0 \le t \le 1$, $\eta(t) = 0$ if $t \ge 2$, $\max\limits_{t>0}|\eta'(t)| \le 2$ and $\eta'(t) \le 0$ for each $t > 0$. Using η, for every $T > 0$ we then consider the truncated functional $\hat{I}^T_{\lambda,\mu} : E_\lambda \to \mathbb{R}$ defined by

$$\hat{I}^T_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2_\lambda - \frac{\mu}{2}\eta\left(\frac{\|u\|^2_\lambda}{T^2}\right)\int_{\mathbb{R}^3}\omega\phi_u u^2 dx - \frac{1}{q}\int_{\mathbb{R}^3}|u^+|^q dx. \tag{4.3}$$

We can see that $\hat{I}^T_{\lambda,\mu}$ is of class C^1, then for each $u, v \in E_\lambda$,

$$\begin{aligned}\langle(\hat{I}^T_{\lambda,\mu})'(u), v\rangle =&\langle u,v\rangle_\lambda - \frac{\mu}{T^2}\eta'\left(\frac{\|u\|^2_\lambda}{T^2}4.1\right)\langle u,v\rangle_\lambda\int_{\mathbb{R}^3}\omega\phi_u u^2 dx \\ &- \mu\eta\left(\frac{\|u\|^2_\lambda}{T^2}\right)\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u uv dx - \int_{\mathbb{R}^3}|u^+|^{q-2}u^+ v dx.\end{aligned} \tag{4.4}$$

It is easy to see that every nontrivial critical point of $\hat{I}_{\lambda,\mu}$ is a positive solution of (1.9), and we will prove it in the following lemma.

Lemma 4.1. *Suppose that $2 < q < 4$ and (a_1)–(a_3) are satisfied. Then every nontrivial critical point of $\hat{I}_{\lambda,\mu}$ is a positive solution of (1.9).*

Proof. Let $u \in E_\lambda$ be a nontrivial critical point of $\hat{I}_{\lambda,\mu}$, then $\langle \hat{I}'_{\lambda,\mu}(u), v\rangle = 0$ for all $v \in E_\lambda$. We have

$$\int_{\mathbb{R}^3}(\nabla u\cdot\nabla v + (\lambda a(x)+1)uv)\,dx - \mu\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u uv dx - \int_{\mathbb{R}^3}|u^+|^{p-2}u^+ v dx = 0. \tag{4.5}$$

Taking $v = u^- = -\min\{u, 0\}$ in (4.5), by Lemma 2.1(i) we obtain that

$$\|u^-\|^2_\lambda \le \|u^-\|^2_\lambda - \mu\int_{\mathbb{R}^3}(2\omega+\phi_{u^-})\phi_{u^-}|u^-|^2 dx = 0$$

which is a contradiction. Then we can see $u \ge 0$ in $\mathbb{R}^3$. Hence, the strong maximum principle and the fact $u \ne 0$ imply that $u > 0$ in $\mathbb{R}^3$, and the proof is ready. □

Lemma 4.2. *Suppose $2 < q < 4$ and (a_1)–(a_3) are satisfied. There exists $\mu_1 > 0$ for each $T > 0$, $\mu \in (0, \mu_1)$ and $\lambda \ge 1$. Then there exist $\bar{\rho}, \bar{\beta} > 0$ and $\bar{e}_0 \in E_\lambda$, $\|\bar{e}_0\|_\lambda > \bar{\rho}$, such that*

$$\inf_{\|u\|=\bar{\rho}} \hat{I}^T_{\lambda,\mu}(u) \ge \bar{\beta} > 0 \ge \max\left\{\hat{I}^T_{\lambda,\mu}(0), \hat{I}^T_{\lambda,\mu}(\bar{e}_0)\right\}.$$

Proof. From Lemma 2.1, (2.1) and (4.3), for each $u \in E_\lambda$, we have

$$\hat{I}^T_{\lambda,\mu}(u) \geq \frac{1}{2}\|u\|^2_\lambda - \frac{1}{q}d^q_q\|u\|^q_\lambda = \|u\|^2_\lambda\left(\frac{1}{2} - \frac{1}{q}d^q_q\|u\|^{q-2}_\lambda\right),$$

where the constant $d_q > 0$ is independent of T, μ and λ. Since $q > 2$, there exist $\bar{\rho} > 0$ small enough and $\bar{\beta} > 0$, such that $\inf_{\|u\|=\bar{\rho}} \hat{I}^T_{\lambda,\mu}(u) \geq \bar{\beta} > 0$.

Then, we define the functional $\hat{J}_\lambda : E_\lambda \to \mathbb{R}$ by

$$\hat{J}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + (\lambda a(x) + 1)u^2)dx - \frac{1}{q}\int_{\mathbb{R}^3}|u^+|^q dx.$$

Since $2 < q < 4$, similar to Lemma 3.1, there also exist $\bar{t}_0 > 0$ large enough and let $\bar{e}_0 := \bar{t}_0 e$ such that $\hat{J}_\lambda(\bar{e}_0) \leq -1$ with $\|\bar{e}_0\|_\lambda > \bar{\rho}$. From Lemma 2.1(i), then

$$\begin{aligned}\hat{I}^T_{\lambda,\mu}(\bar{e}_0) &= \hat{J}_\lambda(\bar{e}_0) - \frac{\mu}{2}\eta\left(\frac{\|\bar{e}_0\|^2_\lambda}{T^2}\right)\int_{\mathbb{R}^3}\omega\phi_{\bar{e}_0}\bar{e}^2_0 dx \\ &\leq -1 + \frac{\mu\omega^2}{2}|\bar{e}_0|^2_2,\end{aligned}$$

there exists $\mu_1 := \frac{2}{\omega^2|\bar{e}_0|^2_2} > 0$ (independent of λ and T) such that $\hat{I}^T_{\lambda,\mu}(\bar{e}_0) < 0$ for each T, $\lambda \geq 1$ and $\mu \in (0, \mu_1)$. The proof is completed. □

Then we also can consider the mountain pass value

$$\hat{c}^T_{\lambda,\mu} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \hat{I}^T_{\lambda,\mu}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = \bar{e}_0\}$. From Proposition 2.3 and Lemma 4.2, we obtain that for each $T > 0$, $\lambda \geq 1$ and $\mu \in (0, \mu_1)$, there exists a Cerami sequence $\{\hat{u}_n\} \subset E_\lambda$ such that

$$\hat{I}^T_{\lambda,\mu}(\hat{u}_n) \to \hat{c}^T_{\lambda,\mu} > 0 \quad \text{and} \quad (1 + \|\hat{u}_n\|_\lambda)\left\|(\hat{I}^T_{\lambda,\mu})'(\hat{u}_n)\right\|_{E'_\lambda} \to 0 \quad \text{as } n \to \infty. \tag{4.6}$$

Since $2 < q < 4$ and the definition of $\eta(t)$, similar proof to Lemma 3.2, we can find a $\bar{M} > 0$ such that $\hat{c}^T_{\lambda,\mu}$ has an upper bound, i.e.,

$$\hat{c}^T_{\lambda,\mu} \leq \bar{M}. \tag{4.7}$$

Lemma 4.3. *Assume $2 < q < 4$ and (a_1)–(a_3) hold, let $T = \sqrt{\frac{2q(\bar{M}+1)}{q-2}}$. Then there exists $\mu_2 > 0$ small enough, for each $\lambda \geq 1$, $\mu \in (0, \min\{\mu_1, \mu_2\})$, if $\{\hat{u}_n\} \subset E_\lambda$ is a sequence satisfying (4.6), then we have, up to a subsequence,*

$$\sup_{n\in\mathbb{N}} \|\hat{u}_n\|_\lambda \leq T.$$

Proof. Otherwise, there exists a subsequence of $\{\hat{u}_n\}$, still denoted by $\{\hat{u}_n\}$ such that $\|\hat{u}_n\|_\lambda > T$. It can be divided into two situations:

(i) $T < \|\hat{u}_n\|_\lambda < \sqrt{2}T$; (ii) $\|\hat{u}_n\|_\lambda \geq \sqrt{2}T$.

Firstly, for the case (i), due to (4.3), (4.4) and Lemma 2.1 we have

$$\begin{aligned}\bar{M}+o(1) \geq \hat{c}^T_{\lambda,\mu}+o(1) &= \hat{I}^T_{\lambda,\mu}(\hat{u}_n) - \frac{1}{q}\langle (\hat{I}^T_{\lambda,\mu})'(\hat{u}_n), \hat{u}_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{q}\right)\|\hat{u}_n\|^2_\lambda + \frac{\mu}{qT^2}\eta'\left(\frac{\|\hat{u}_n\|^2_\lambda}{T^2}\right)\|\hat{u}_n\|^2_\lambda \int_{\mathbb{R}^3}\omega\phi_{\hat{u}_n}\hat{u}_n^2dx \\ &\quad + \left(\frac{2}{q}-\frac{1}{2}\right)\mu\eta\left(\frac{\|\hat{u}_n\|^2_\lambda}{T^2}\right)\int_{\mathbb{R}^3}\omega\phi_{\hat{u}_n}\hat{u}_n^2dx + \frac{\mu}{q}\eta\left(\frac{\|\hat{u}_n\|^2_\lambda}{T^2}\right)\int_{\mathbb{R}^3}\phi^2_{\hat{u}_n}\hat{u}_n^2dx \\ &\geq \left(\frac{1}{2}-\frac{1}{q}\right)\|\hat{u}_n\|^2_\lambda - \left(\frac{2}{q}-\frac{1}{2}\right)\mu\omega^2 d_2^2\|\hat{u}_n\|^2_\lambda \\ &\geq (\bar{M}+1) - \frac{2(4-q)\mu\omega^2 d_2^2}{q-2}(\bar{M}+1),\end{aligned}$$

which is a contradiction when we choose $\mu_2 := \frac{q-2}{2(4-q)\omega^2 d_2^2(\bar{M}+1)} > 0$ such that $\mu \in (0, \min\{\mu_1,\mu_2\})$. Then we deduce that $\|\hat{u}_n\|_\lambda \geq \sqrt{2}T$ for n large enough. With the definition of $\eta(t)$, we conclude that

$$\begin{aligned}\bar{M}+o(1) \geq \hat{c}^T_{\lambda,\mu}+o(1) =& \hat{I}^T_{\lambda,\mu}(\hat{u}_n) - \frac{1}{q}\langle (\hat{I}^T_{\lambda,\mu})'(\hat{u}_n), \hat{u}_n\rangle \\ =& \left(\frac{1}{2}-\frac{1}{q}\right)\|\hat{u}_n\|^2_\lambda \\ \geq& 2(\bar{M}+1),\end{aligned}$$

this is obviously a contradiction. The proof of this lemma ends. □

Up to now, we have proved that the sequence $\{\hat{u}_n\}$ given by (4.6) satisfies $\|\hat{u}_n\|_\lambda \leq T$. In particular, this sequence $\{\hat{u}_n\}$ is also a Cerami sequence at level $\hat{c}^T_{\lambda,\mu}$ for $\hat{I}_{\lambda,\mu}$, i.e.,

$$\hat{I}_{\lambda,\mu}(\hat{u}_n) \to \hat{c}^T_{\lambda,\mu} > 0 \quad \text{and} \quad (1+\|\hat{u}_n\|_\lambda)\left\|\hat{I}'_{\lambda,\mu}(\hat{u}_n)\right\|_{E'_\lambda} \to 0 \quad \text{as } n\to\infty.$$

Then we will give the compactness conditions for $\hat{I}_{\lambda,\mu}$.

Lemma 4.4. *Suppose that $2<q<4$ and (a_1)–(a_3) hold. If $\{\hat{u}_n\} \subset E_\lambda$ is a sequence satisfying (4.6), up to a subsequence, there exists $\lambda_2^* \geq 1$ such that for each $\mu \in (0,\min\{\mu_1,\mu_2\})$ and $\lambda \in (\lambda_2^*,\infty)$, $\{\hat{u}_n\} \subset E_\lambda$ contains a convergent subsequence.*

Proof. Proof is similar to Lemma 3.5, there exists $\lambda_2 = [(2d_qT)^{q-2}S^{\theta-1}c^{-\theta}]^{\frac{1}{\theta}}$ where $\theta = \frac{6-q}{2q} > 0$ and choose $\lambda_2^* = \max\{\lambda_2, 1\}$ such that $(\hat{u}_n - \hat{u}) \to 0$ in E_λ for all $\lambda > \lambda_2^*$. □

Proof of Theorem 1.8. Assume $2<q<4$ and (a_1)–(a_3) are satisfied. By Lemma 4.2, there exists $\mu_1 > 0$ such that for every $\lambda \geq 1$ and $\mu \in (0,\mu_1)$, $\hat{I}^T_{\lambda,\mu}$ possesses a Cerami sequence $\{\hat{u}_n\}$ at the mountain pass level $\hat{c}^T_{\lambda,\mu}$. From (4.7) and Lemma 4.3, we thus deduce that there exist $\mu_2 > 0$ such that for every $\lambda \geq 1$ and $\mu \in (0,\min\{\mu_1,\mu_2\})$, after passing to a subsequence, $\{\hat{u}_n\}$ is a Cerami sequence of $\hat{I}_{\lambda,\mu}$ satisfying $\|\hat{u}_n\|_\lambda \leq T$, i.e.,

$$\sup_{n\in\mathbb{N}}\|\hat{u}_n\|_\lambda \leq T, \quad \hat{I}_{\lambda,\mu}(\hat{u}_n) \to \hat{c}^T_{\lambda,\mu} \quad \text{and} \quad (1+\|\hat{u}_n\|_\lambda)\|\hat{I}'_{\lambda,\mu}(\hat{u}_n)\|_{E'_\lambda} \to 0, \text{ as } n\to\infty.$$

It follows from Lemma 4.4 that exists $\lambda_2^* \geq 1$ such that for each $\mu \in (0, \min\{\mu_1, \mu_2\})$ and $\lambda \in (\lambda_2^*, \infty)$, the sequence $\{\hat{u}_n\}$ has a convergent subsequence in E_λ. Then there exists $\hat{u}_{\lambda,\mu} \in E_\lambda$, such that $\hat{u}_n \to \hat{u}_{\lambda,\mu}$ as $n \to \infty$, and thus

$$\|\hat{u}_{\lambda,\mu}\|_\lambda \leq T, \quad \hat{I}_{\lambda,\mu}(\hat{u}_{\lambda,\mu}) = \hat{c}^T_{\lambda,\mu} \quad \text{and} \quad \hat{I}'_{\lambda,\mu}(\hat{u}_{\lambda,\mu}) = 0.$$

Similarly, we can prove that $\hat{u}_{\lambda,\mu} \neq 0$ and there exists $\tau_1 > 0$ (independent of μ and λ) such that $\|\hat{u}_{\lambda,\mu}\|_\lambda \geq \tau_1$ for all $\mu \in (0, \min\{\mu_1, \mu_2\})$ and $\lambda \in (\lambda_2^*, \infty)$. □

Proof of Theorem 1.9 to Theorem 1.11. Please refer to the proofs of Theorems 1.3 to 1.5. The detailed proofs are omitted here.

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