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Positive solutions for a class of concave-convex semilinear elliptic systems with double critical exponents

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Abstract. In this paper, we consider the following concave-convex semilinear elliptic system with double critical exponents:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \frac{\alpha}{2^*}|u|^{\alpha-2}|v|^{\beta}u + \lambda|u|^{q-2}u, & \text{in } \Omega, \\ -\Delta v = |v|^{2^*-2}v + \frac{\beta}{2^*}|u|^{\alpha}|v|^{\beta-2}v + \mu|v|^{q-2}v, & \text{in } \Omega, \\ u,\ v > 0, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $\lambda,\ \mu > 0,\ 1 < q < 2,\ \alpha > 1,\ \beta > 1,\ \alpha + \beta = 2^* = \frac{2N}{N-2}$. By the Nehari manifold method and variational method, we obtain two positive solutions which improves the recent results in the literature.

Keywords: semilinear elliptic system, double critical exponents, positive solutions, Nehari manifold, variational method.

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1 Introduction and main result

In this paper, we mainly study the following concave-convex semilinear elliptic system with double critical exponents

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \frac{\alpha}{2^*}|u|^{\alpha-2}|v|^{\beta}u + \lambda|u|^{q-2}u, & \text{in } \Omega, \\ -\Delta v = |v|^{2^*-2}v + \frac{\beta}{2^*}|u|^{\alpha}|v|^{\beta-2}v + \mu|v|^{q-2}v, & \text{in } \Omega, \\ u,\ v > 0, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $\lambda,\ \mu > 0,\ 1 < q < 2,\ \alpha > 1,\ \beta > 1,\ \alpha + \beta = 2^* = \frac{2N}{N-2}$. System (1.1) is abstracted from some physical phenomenon, especially some description in nonlinear optics. As we all known, it is also a model in Hartree–Fock theory for a double condensate, i.e., a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$, which was first discovered and proposed by B.D. Esry et al in 1997. There is a lot of literatures about the origin and physical background of system (1.1), and we refer the readers to see [5,7,11,26].

It is well known that many results have been obtained in these years on critical semilinear elliptic equation and system. For example, in 1983, Brézis and Nirenberg in reference [3] studied the case of positive solutions of semilinear elliptic equations with critical exponent in different dimensions and got many important results. In 1994, Ambrosetti et al in [2] showed that some problems of critical elliptic equation with concave-convex nonlinearities. With the development of variational methods, people gradually shifted their focus from equation to system. In 2000, Alves et al first studied elliptic system involving subcritical or critical Sobolev exponent in [1] as following

$$\begin{cases} -\Delta u = au + bv + \frac{2\alpha}{\alpha+\beta} u|u|^{\alpha-2}|v|^{\beta}, & \text{in } \Omega, \\ -\Delta v = bu + cv + \frac{2\beta}{\alpha+\beta} |u|^{\alpha} v|v|^{\beta-2}, & \text{in } \Omega, \\ u, v > 0, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ with smooth boundary, and $a,\ b,\ c \in \mathbb{R},\ \alpha,\ \beta > 1,\ \alpha + \beta = 2^*$. They obtained some existence results and nonexistence results for the corresponding elliptic system with different dimensions and in different domain's shapes. In 2009, Hsu and Lin in [19] studied the following critical elliptic system

$$\begin{cases} -\Delta u = \lambda |u|^{q-2} u + \frac{2\alpha}{\alpha+\beta} |u|^{\alpha-2} u |v|^{\beta}, & \text{in } \Omega, \\ -\Delta v = \mu |v|^{q-2} v + \frac{2\beta}{\alpha+\beta} |u|^{\alpha} |v|^{\beta-2} v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$ with $N \geq 3$, and $\lambda,\ \mu > 0,\ \alpha,\ \beta > 1,\ \alpha + \beta = 2^*$. For $1 < q < 2$, they got two positive solutions when $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda$, where Λ is a positive constant. What's more, there are some other references on semilinear elliptic system with critical exponent, such as [5,6,8,9,12,13,16–18,20–22,24–26]. However, among the references mentioned above, the elliptic system involving double critical exponential terms with one strongly coupled and the other weakly coupled was studied only in [9] and [8]. Recently, Duan, Wei and Yang in [9], on an incompressible bounded domain, studied the following nonhomogeneous semilinear elliptic system with double critical exponents

$$\begin{cases} -\Delta u = |u|^{2^*-2} u + \frac{\alpha}{2^*} |u|^{\alpha-2} u |v|^{\beta} + \varepsilon f, & \text{in } \Omega, \\ -\Delta v = |v|^{2^*-2} v + \frac{\beta}{2^*} |u|^{\alpha} |v|^{\beta-2} v + \varepsilon g, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\alpha, \beta > 1,\ \alpha + \beta = 2^*,\ \varepsilon > 0$, for non-homogeneous terms $f,\ g$, which satisfy $0 \leq f(x),\ g(x) \in L^{\infty}(\Omega),\ f,\ g \not\equiv 0$, and for the incompressible bounded domains with smooth boundary Ω satisfies the following condition:

(V) $\Omega \subset \mathbb{R}^N (N \geq 3)$, and there exist two positive constants $0 < R_1 < R_2 < \infty$ such that

$$\{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \subset \Omega, \qquad \{x \in \mathbb{R}^N : |x| < R_1\} \not\subset \bar{\Omega}.$$

If condition (V) holds, by splitting Nehari manifold and the knowledge of topology, they got that there is a $\varepsilon' > 0$, for any $0 < \varepsilon < \varepsilon'$, such that the above system has at least three solutions in the incompressible domain Ω, one of which is a positive ground state solution. Furthermore, if R_1 in condition (V) is small enough, then there is a ε'' such that for any $0 < \varepsilon < \varepsilon''$ there are at least four solutions on the incompressible domain Ω.

Inspired by [9], we replace the abstract inhomogeneous terms with the concave-convex terms. In order to get a more general result, we extend the constraints of the "incompressible" domain to the general bounded domain. So, we study system (1.1).

We denote the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$ of $H_0^1(\Omega)$; and $E = H_0^1(\Omega) \times H_0^1(\Omega)$ with the norm:

$$\|(u,v)\|_E = \left[\int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx\right]^{\frac{1}{2}}.$$

Then, we use $|\cdot|_p$ to denote the $L^p(\Omega)$-norm, and denote S as the Sobolev optimal embedding constant, where S is defined as follows:

$$S = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}}} = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{\frac{2}{2^*}}} > 0. \tag{1.2}$$

From reference [27], we know that S is achieved by the function:

$$U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}, \qquad (x \in \mathbb{R}^N) \tag{1.3}$$

which is also a solution of the following equation:

$$\begin{cases} -\Delta u = u^{2^*-1}, & x \in \mathbb{R}^N, \\ u > 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

with $|\Delta U|_2^2 = |U|_{2^*}^{2^*} = S^{\frac{N}{2}}$. Let

$$S_{\alpha,\beta} = \inf_{(u,v) \in E\setminus\{(0,0)\}} \frac{\int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx}{\left[\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx\right]^{\frac{2}{2^*}}} = f(\tau_{min}) S. \tag{1.5}$$

According to [9, Lemma 1], we know that $S_{\alpha,\beta} = f(\tau_{min})S$, where

$$f(\tau) = \frac{1+\tau^2}{(1+\tau^{2^*}+\tau^\beta)^{\frac{2}{2^*}}}$$

and $f(\tau_{min}) \in [2^{-\frac{2}{2^*}}, 1]$ for any $\tau \geq 0$.

Based on (1.1), we know that the corresponding energy functional as follows:

$$I_{\lambda,\mu}(u,v) = \frac{1}{2}\|(u,v)\|_E^2 - \frac{1}{2^*}\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx - \frac{1}{q}\int_\Omega (\lambda |u|^q + \mu |v|^q) dx \tag{1.6}$$

and (u,v) is a weak solution of system (1.1) if for any $(\xi_1,\xi_2)\in E$ it satisfies

$$\begin{aligned}
&\langle I'_{\lambda,\mu}(u,v),(\xi_1,\xi_2)\rangle\\
&\quad=\int_\Omega(\nabla u\nabla\xi_1+\nabla v\nabla\xi_2)dx-\int_\Omega\left(|u|^{2^*-2}u\xi_1+|v|^{2^*-2}v\xi_2\right)dx\\
&\quad\quad-\int_\Omega\left(\frac{\alpha}{2^*}|u|^{\alpha-2}|v|^\beta u\xi_1+\frac{\beta}{2^*}|u|^\alpha|v|^{\beta-2}v\xi_2\right)dx-\int_\Omega\left(\lambda|u|^{q-2}u\xi_1+\mu|v|^{q-2}v\xi_2\right)dx=0.
\end{aligned}$$

When $(\xi_1,\xi_2)=(u,v)$, we can get:

$$\|(u,v)\|_E^2-\int_\Omega(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta)dx-\int_\Omega(\lambda|u|^q+\mu|v|^q)dx=0. \tag{1.7}$$

Define Nehari manifold as follows:

$$\mathcal{N}_{\lambda,\mu}=\{(u,v)\in E:\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle=0\}. \tag{1.8}$$

Set $z=(u,v)$, $\|z\|_E=\|(u,v)\|_E=\left(\|u\|^2+\|v\|^2\right)^{\frac{1}{2}}$. Define the function $\Psi(z)=\langle I'_{\lambda,\mu}(z),z\rangle$, such that

$$\begin{aligned}
\langle\Psi'(z),z\rangle&=2\|z\|_E^2-2^*\int_\Omega(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta)dx-q\int_\Omega(\lambda|u|^q+\mu|v|^q)dx\\
&=(2-q)\|z\|_E^2-(2^*-q)\int_\Omega(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta)dx\\
&=(2-2^*)\|z\|_E^2-(q-2^*)\int_\Omega(\lambda|u|^q+\mu|v|^q)dx
\end{aligned} \tag{1.9}$$

for any $z=(u,v)\in\mathcal{N}_{\lambda,\mu}$. To obtain the two positive solutions, we now split $\mathcal{N}_{\lambda,\mu}$ into three parts as follows:

$$\begin{aligned}
\mathcal{N}^+_{\lambda,\mu}&=\{z\in\mathcal{N}_{\lambda,\mu}:\langle\Psi'(z),z\rangle>0\},\\
\mathcal{N}^0_{\lambda,\mu}&=\{z\in\mathcal{N}_{\lambda,\mu}:\langle\Psi'(z),z\rangle=0\},\\
\mathcal{N}^-_{\lambda,\mu}&=\{z\in\mathcal{N}_{\lambda,\mu}:\langle\Psi'(z),z\rangle<0\},
\end{aligned} \tag{1.10}$$

where $\mathcal{N}_{\lambda,\mu}=\mathcal{N}^+_{\lambda,\mu}\cup\mathcal{N}^0_{\lambda,\mu}\cup\mathcal{N}^-_{\lambda,\mu}$. In addition, we will prove $\mathcal{N}^\pm_{\lambda,\mu}\neq\emptyset$ and $\mathcal{N}^0_{\lambda,\mu}=\{(0,0)\}$ for $0<\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}<T$, where

$$T=\left(\frac{2-q}{2^*-q}\right)^{\frac{2}{2^*-2}}\left(\frac{2^*-2}{2^*-q}\right)^{\frac{2}{2-q}}(S_{\alpha,\beta})^{\frac{2^*}{2^*-2}}S^{\frac{q}{2-q}}|\Omega|^{-\frac{2(2^*-q)}{2^*(2-q)}} \tag{1.11}$$

in Section 2.

Here is our main result.

Theorem 1.1. *Assume that $1<q<2$ and $\Omega\subset\mathbb{R}^N(N\geq 3)$ is a bounded domain with smooth boundary, $\lambda,\ \mu>0$, $\alpha>1$, $\beta>1$, $\alpha+\beta=\frac{2N}{N-2}$. Then,*

(i) for any $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,T)$, system (1.1) has a positive ground state solution;

(ii) for any $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,(\frac{q}{2})^{\frac{2}{2-q}}T)$, system (1.1) has two positive solutions, one of which is the positive ground state solution.

Remark 1.2. To the best of our knowledge, our result is up to date. On the one hand, we generalize [9] to system (1.1) on general bound domain and obtain two positive solutions. On the other hand, noting that [9, Claim 2], that is,

$$\int_\Omega (u_1 + tu_\delta^{\sigma,\rho})^{2^*} - u_1^{2^*} - (tu_\delta^{\sigma,\rho})^{2^*} - 2^* u_1^{2^*-1} tu_\delta^{\sigma,\rho} dx \geq O(\delta^{\frac{N-2}{2}})$$

and

$$\int_\Omega (v_1 + tv_\delta^{\sigma,\rho})^{2^*} - v_1^{2^*} - (tv_\delta^{\sigma,\rho})^{2^*} - 2^* v_1^{2^*-1} tv_\delta^{\sigma,\rho} dx \geq O(\delta^{\frac{N-2}{2}}).$$

From [9] we know that [15, (4.7)] is used in the proof of Claim 2. However, [15, (4.7)] has a restriction of $q \geq 3$ on the exponential q. For $2^* = \frac{2N}{N-2} \geq 3$, it implies that $N \leq 6$. Thus, when $N > 6$ the inequality in [9, Claim 2] may not hold, which may have some influence on the estimation of corresponding energy functional. So, for $N \geq 3$, we revalued [9, Claim 2], which is important for estimating the value of corresponding energy functional $I_{\lambda,\mu}$.

The content structure of this paper is organized as the following way. In Section 2, we will give some important lemmas for preparation to prove our main result. In Section 3, we will give the proof of the existence of positive ground state solutions for system (1.1). Finally, we will prove the existence of two positive solutions in Section 4.

2 Some preliminary results

In this section, we first give some important lemmas which are valuable preparation for the proof of our main result.

Lemma 2.1. *Assume that* $z = (u, v) \in E \backslash \{(0, 0)\}$ *with* $\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx > 0$, *then:*

(i) *there exist unique* t^+, t^- *with* $0 < t^+ < t_{max} < t^-$ *when* $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$, *such that* $t^+ z \in \mathcal{N}^+_{\lambda,\mu}$, $t^- z \in \mathcal{N}^-_{\lambda,\mu}$ *and*

$$I_{\lambda,\mu}(t^+ z) = \inf_{0 \leq t \leq t_{max}} I_{\lambda,\mu}(tz), \qquad I_{\lambda,\mu}(t^- z) = \sup_{t_{max} \leq t} I_{\lambda,\mu}(tz); \tag{2.1}$$

(ii) *for* $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$, $\mathcal{N}^0_{\lambda,\mu} = \{(0, 0)\}$ *and* $\mathcal{N}^-_{\lambda,\mu}$ *is a closed set.*

Proof. (i) For $t \geq 0$, $z = (u, v) \in E$ such that $\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx > 0$, we have

$$\langle I'_{\lambda,\mu}(tz), tz \rangle = t^2 \|z\|_E^2 - t^{2^*} \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx - t^q \int_\Omega (\lambda |u|^q + \mu |v|^q) dx.$$

Then, set $y_1, y_2 : \mathbb{R}^+ \to \mathbb{R}$,

$$y_1(t) = t^{2-q} \|z\|_E^2 - t^{2^*-q} \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx - \int_\Omega (\lambda |u|^q + \mu |v|^q) dx, \tag{2.2}$$

$$y_2(t) = t^{2-q} \|z\|_E^2 - t^{2^*-q} \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx. \tag{2.3}$$

Obviously, $y_1(t) = y_2(t) - \int_\Omega (\lambda |u|^q + \mu |v|^q)\, dx$. We now proceed with the analysis of $y_2(t)$,

$$\begin{aligned} y_2'(t) &= (2-q) t^{1-q} \|z\|_E^2 - (2^* - q) t^{2^*-1-q} \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx \\ &= t^{1-q} \left[(2-q) \|z\|_E^2 - (2^* - q) t^{2^*-2} \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx \right]. \end{aligned}$$

It is easy to figure out that $y_2'(t_{max}) = 0$ with

$$t_{max} = \left[\frac{(2-q)\|z\|_E^2}{(2^*-q)\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right]^{\frac{1}{2^*-2}} > 0.$$

Moreover, $y_2'(t) > 0$ for all $0 < t < t_{max}$, and $y_2'(t) < 0$ for all $t > t_{max}$. Through a simple analysis, we can get that

$$y_2(t_{max}) = \max y_2(t) = \left[\frac{(2-q)\|z\|_E^2}{(2^*-q)\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right]^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q} \|z\|_E^2.$$

According to the definition of $S_{\alpha,\beta}$, Hölder's inequality and (1.2), one has

$$\begin{aligned}
y_1(t_{max}) &= y_2(t_{max}) - \int_\Omega (\lambda |u|^q + \mu |v|^q)\, dx \\
&= \left[\frac{(2-q)\|z\|_E^2}{(2^*-q)\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right]^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q} \|z\|_E^2 \\
&\quad - \int_\Omega (\lambda |u|^q + \mu |v|^q)\, dx \\
&\geq \left(\frac{2-q}{2^*-q} \right)^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q} \|z\|_E^2 \left(\frac{\|z\|_E^2}{\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right)^{\frac{2-q}{2^*-2}} \\
&\quad - (\lambda \|u\|^q + \mu \|v\|^q)\, |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\
&\geq \left(\frac{2-q}{2^*-q} \right)^{\frac{2-q}{2^*-q}} \frac{2^*-2}{2^*-q} \|z\|_E^2 \left(\frac{\|z\|_E^2}{\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right)^{\frac{2-q}{2^*-2}} \\
&\quad - \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \right)^{\frac{2-q}{2}} (\|u\|^2 + \|v\|^2)^{\frac{q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\
&= \left(\frac{2-q}{2^*-q} \right)^{\frac{2-q}{2^*-q}} \frac{2^*-2}{2^*-q} \|z\|_E^2 \left(\frac{\|z\|_E^2}{\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right)^{\frac{2-q}{2^*-2}} \\
&\quad - \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \right)^{\frac{2-q}{2}} \|z\|_E^q |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\
&\geq \left(\frac{2-q}{2^*-q} \right)^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q} \|z\|_E^2 \left(\frac{\|z\|_E^2}{(S_{\alpha,\beta})^{-\frac{2^*}{2}} \|z\|_E^{2^*}} \right)^{\frac{2-q}{2^*-2}} \\
&\quad - \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \right)^{\frac{2-q}{2}} \|z\|_E^q |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\
&= \|z\|_E^q \left[\left(\frac{2-q}{2^*-q} \right)^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q} (S_{\alpha,\beta})^{\frac{2^*(2-q)}{2(2^*-2)}} - \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \right)^{\frac{2-q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \right] \\
&> 0,
\end{aligned} \tag{2.4}$$

for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$, where T is defined by (1.11). Because $y_1(t)$ is a continuous function, according to inequality preserving for continuous functions and (2.4), there exist unique t^+, t^- with $0 < t^+ < t_{max} < t^-$, which makes

$$y_1(t^+) = y_1(t^-) = 0.$$

So, for $t^+ < t_{max} < t^-$, since $y_2'(t^+) > 0$ and $y_2'(t^-) < 0$, we have $t^+z \in \mathcal{N}^+_{\lambda,\mu}$, $t^-z \in \mathcal{N}^-_{\lambda,\mu}$. Moreover, one has

$$I_{\lambda,\mu}(t^-z) \geq I_{\lambda,\mu}(tz) \geq I_{\lambda,\mu}(t^+z),$$

for each $t \in [t^+, t^-]$, and $I_{\lambda,\mu}(t^+z) < I_{\lambda,\mu}(tz)$ for each $t \in [0, t^+)$. Thus, one obtains

$$I_{\lambda,\mu}(t^+z) = \inf_{0 \leq t \leq t_{max}} I_{\lambda,\mu}(tz), \qquad I_{\lambda,\mu}(t^-z) = \sup_{t_{max} \leq t} I_{\lambda,\mu}(tz).$$

(*ii*) Set $z_0 = (u_0, v_0) \neq (0,0) \in \mathcal{N}^0_{\lambda,\mu}$, from (1.9) and (1.10), we know

$$\frac{2^*-2}{2^*-q}\|z_0\|_E^2 = \int_\Omega \left(\lambda|u_0|^q + \mu|v_0|^q\right), \tag{2.5}$$

$$\frac{2-q}{2^*-q}\|z_0\|_E^2 = \int_\Omega |u_0|^{2^*} + |v_0|^{2^*} + |u_0|^\alpha |v_0|^\beta dx. \tag{2.6}$$

We can deduce from (2.4), (2.5) and (2.6) that

$$\begin{aligned}
0 < &\left[\frac{(2-q)\|z_0\|_E^2}{(2^*-q)\int_\Omega |u_0|^{2^*} + |v_0|^{2^*} + |u_0|^\alpha|v_0|^\beta dx}\right]^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q}\|z_0\|_E^2 \\
&- \int_\Omega \left(\lambda|u_0|^q + \mu|v_0|^q\right) dx \\
= &\left[\frac{(2-q)\|z_0\|_E^2}{(2^*-q)\frac{2-q}{2^*-q}\|z_0\|_E^2}\right]^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q}\|z_0\|_E^2 - \frac{2^*-2}{2^*-q}\|z_0\|_E^2 \\
= &\frac{2^*-2}{2^*-q}\|z_0\|_E^2 - \frac{2^*-2}{2^*-q}\|z_0\|_E^2 \\
= &\,0,
\end{aligned}$$

which is a contradiction for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$. So, for $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$, we obtain $\mathcal{N}^0_{\lambda,\mu} = \{(0,0)\}$. Then, we will prove $\mathcal{N}^-_{\lambda,\mu}$ is a closed set when $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$. Assume $\{z_n\} \subset \mathcal{N}^-_{\lambda,\mu}$, $z_n \to z$, $z \in E$, and now we prove $z \in \mathcal{N}^-_{\lambda,\mu}$. From (1.10), we have

$$(2-q)\|z_n\|_E^2 - (2^*-q)\int_\Omega |u_n|^{2^*} + |v_n|^{2^*} + |u_n|^\alpha|v_n|^\beta dx < 0. \tag{2.7}$$

According to $z_n \to z$, $z \in E$ and (2.7), one has

$$(2-q)\|z\|_E^2 - (2^*-q)\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha|v|^\beta) dx \leq 0. \tag{2.8}$$

From (2.8), we can get $z \in \mathcal{N}^0_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$. We already know from the above proof that $\mathcal{N}^0_{\lambda,\mu} = \{(0,0)\}$ when $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$. So, if $z \in \mathcal{N}^0_{\lambda,\mu}$, then $z = (0,0)$. According to (1.5) and (2.7), we obtain

$$\|z_n\|_E \geq \left[\frac{(2-q)}{(2^*-q)}(S_{\alpha,\beta})^{\frac{2^*}{2}}\right]^{\frac{1}{2^*-2}} > 0,$$

which implies a contradiction with $z = (0,0)$. Thus, $z \in \mathcal{N}^-_{\lambda,\mu}$ for $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$. So, we can prove that $\mathcal{N}^-_{\lambda,\mu}$ is a closed set. The proof of Lemma 2.1 is complete. □

Lemma 2.2. *The energy functional $I_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Assume that $z=(u,v)\in\mathcal{N}_{\lambda,\mu}$. By the Hölder inequality, (1.2) and (1.7), one has

$$\begin{aligned} I_{\lambda,\mu}(z) &= \left(\frac{1}{2}-\frac{1}{2^*}\right)\|z\|_E^2-\left(\frac{1}{q}-\frac{1}{2^*}\right)\int_\Omega(\lambda|u|^q+\mu|v|^q)dx \\ &= \frac{1}{N}\|z\|_E^2-\left(\frac{1}{q}-\frac{1}{2^*}\right)\int_\Omega(\lambda|u|^q+\mu|v|^q)dx \\ &\geq \frac{1}{N}\|z\|_E^2-\left(\frac{1}{q}-\frac{1}{2^*}\right)(\lambda\|u\|^q+\mu\|v\|^q)|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{q}{2}} \\ &\geq \frac{1}{N}\|z\|_E^2-\left(\frac{1}{q}-\frac{1}{2^*}\right)\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}(\|u\|^2+\|v\|^2)^{\frac{q}{2}}|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{q}{2}} \\ &= \frac{1}{N}\|z\|_E^2-\left(\frac{1}{q}-\frac{1}{2^*}\right)\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\|z\|_E^q|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{q}{2}}. \end{aligned} \tag{2.9}$$

Because $1<q<2<2^*$, from (2.9) we know that $I_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$. The proof of Lemma 2.2 is completed. □

According to Lemma 2.1 and Lemma 2.2, we set $\mathcal{N}_{\lambda,\mu}=\mathcal{N}_{\lambda,\mu}^+\cup\mathcal{N}_{\lambda,\mu}^0\cup\mathcal{N}_{\lambda,\mu}^-$. And we define

$$m=\inf_{z\in\mathcal{N}_{\lambda,\mu}}I_{\lambda,\mu}(z),\qquad m^+=\inf_{z\in\mathcal{N}_{\lambda,\mu}^+}I_{\lambda,\mu}(z),\qquad m^-=\inf_{z\in\mathcal{N}_{\lambda,\mu}^-}I_{\lambda,\mu}(z). \tag{2.10}$$

Lemma 2.3.

(i) *We have $m\leq m^+<0$, for $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,T)$;*

(ii) *there exists a positive constant m_0 depending on λ,μ,S,N, such that $m^-\geq m_0>0$ for all $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in\left(0,(\frac{q}{2})^{\frac{2}{2-q}}T\right)$.*

Proof. (i) Assume that $z=(u,v)\in\mathcal{N}_{\lambda,\mu}^+$, by (1.7), (1.9) and (1.10), we can get

$$\frac{2-q}{2^*-q}\|z\|_E^2>\int_\Omega(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta)dx. \tag{2.11}$$

Then, by (2.11) we have

$$\begin{aligned} I_{\lambda,\mu}(z) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|z\|_E^2+\left(\frac{1}{q}-\frac{1}{2^*}\right)\int_\Omega(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta)dx \\ &< \left(\frac{1}{2}-\frac{1}{q}\right)\|z\|_E^2+\left(\frac{1}{q}-\frac{1}{2^*}\right)\left(\frac{2-q}{2^*-q}\right)\|z\|_E^2 \\ &= \frac{q-2}{Nq}\|z\|_E^2 \\ &< 0. \end{aligned}$$

So, $m=\inf_{z\in\mathcal{N}_{\lambda,\mu}}I_{\lambda,\mu}(z)\leq m^+=\inf_{z\in\mathcal{N}_{\lambda,\mu}^+}I_{\lambda,\mu}(z)\leq I_{\lambda,\mu}(z)<0$.

(ii) For $z=(u,v)\in\mathcal{N}_{\lambda,\mu}^-$, we can deduce that

$$\frac{2-q}{2^*-q}\|z\|_E^2<\int_\Omega(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta)dx\leq S_{\alpha,\beta}^{-\frac{2^*}{2}}\|z\|_E^{2^*}.$$

Consequently, from (1.5), (1.9) and (1.10), one has

$$\|z\|_E > \left(\frac{2-q}{2^*-q}\right)^{\frac{1}{2^*-2}} S_{\alpha,\beta}^{\frac{2^*}{2(2^*-2)}}. \tag{2.12}$$

By (2.9) and (2.12), for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in \left(0, (\frac{q}{2})^{\frac{2}{2-q}} T\right)$, we will get

$$\begin{aligned} I_{\lambda,\mu}(z) &\geq \left(\frac{1}{2}-\frac{1}{2^*}\right)\|z\|_E^2 - \left(\frac{1}{q}-\frac{1}{2^*}\right)\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|z\|_E^q |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\ &= \|z\|_E^q \Bigg[\left(\frac{1}{2}-\frac{1}{2^*}\right)\|z\|_E^{2-q} \\ &\quad - \left(\frac{1}{q}-\frac{1}{2^*}\right)\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}\Bigg] \\ &> \|z\|_E^q \Bigg[\left(\frac{1}{2}-\frac{1}{2^*}\right)\left(\frac{2-q}{2^*-q}\right)^{\frac{2-q}{2^*-2}} S_{\alpha,\beta}^{\frac{2^*(2-q)}{2(2^*-2)}} \\ &\quad - \left(\frac{1}{q}-\frac{1}{2^*}\right)\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}\Bigg] \\ &\geq m_0 \\ &> 0, \end{aligned} \tag{2.13}$$

where m_0 is a positive constant. So, $m^- = \inf_{z\in\mathcal{N}^-_{\lambda,\mu}} I_{\lambda,\mu}(z) \geq m_0 > 0$. Then, the proof of Lemma 2.3 is complete. □

Lemma 2.4. *Suppose $z_0 \in E$ is a local minimizer of $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, then we have $I'_{\lambda,\mu}(z_0) = 0$ in E^{-1}.*

Proof. Set $z_0 = (u_0, v_0) \in E$ is a local minimizer of $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$. Then, $I_{\lambda,\mu}(z_0) = \min_{z\in\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(z)$. According to the Lagrange multiplier theorem, there is a $\theta \in \mathbb{R}$ such that $I'_{\lambda,\mu}(z_0) = \theta\Psi'(z_0)$, where $\Psi(z) = \langle I'_{\lambda,\mu}(z), z\rangle$. Due to $z_0 \in \mathcal{N}_{\lambda,\mu}$, we have

$$0 = \langle I'_{\lambda,\mu}(z_0), z_0\rangle = \theta\langle\Psi'(z_0), z_0\rangle.$$

By Lemma 2.3, if $z_0 \notin \mathcal{N}^0_{\lambda,\mu}$, we can get $\langle\Psi'(z_0), z_0\rangle \neq 0$ for $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$. Thus, $\theta = 0$, $I'_{\lambda,\mu}(z_0) = 0$. The Lemma 2.4 is proved. □

3 The positive ground state solution

Lemma 3.1. *For any $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$, then there exists a $(PS)_m$-sequence $\{z_n\} = \{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ for $I_{\lambda,\mu}$ and T is defined as (1.11).*

Proof. The proof process is the same as [28, Proposition 9], which is omitted here. □

Lemma 3.2. *The energy functional $I_{\lambda,\mu}$ has a minimizer $z_* = (u_*, v_*) \in \mathcal{N}^+_{\lambda,\mu}$, for $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$. What's more z_* is a positive ground state solution of system (1.1), which makes $I_{\lambda,\mu}(z_*) = m = m^+ < 0$.*

Proof. According to Lemma 3.1, there is a $(PS)_m$-sequence, which is recorded as $\{z_n\} = \{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$. Then, we have

$$I_{\lambda,\mu}(z_n) = m + o_n(1), \qquad I'_{\lambda,\mu}(z_n) = o_n(1), \tag{3.1}$$

where $o_n(1) \to 0$ as $n \to \infty$. Combining with (2.9) and (3.1), we can get

$$\begin{aligned} m + o_n(1) &= I_{\lambda,\mu}(z_n) \\ &\geq \frac{1}{N}\|z_n\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|z_n\|_E^q |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}. \end{aligned}$$

Thus, $\{z_n\}$ is bounded in E. Then, $\{z_n\}$ has a subsequence (still denoted by $\{z_n\}$) which weakly converges to $z_* = (u_*, v_*) \in E$, and

$$\begin{cases} u_n \rightharpoonup u_*,\ v_n \rightharpoonup v_*, & \text{in } H_0^1(\Omega), \\ u_n \to u_*,\ v_n \to v_*, & \text{in } L^s(\Omega)(1 \leq s < 2^*), \\ u_n(x) \to u_*(x),\ v_n(x) \to v_*(x), & \text{a.e. in } \Omega. \end{cases} \tag{3.2}$$

According to (3.1), we have $\langle I'_{\lambda,\mu}(z_n), \xi\rangle \to 0$ as $n \to \infty$ for any $\xi \in E$. What's more, combining with (3.2), we have

$$\langle I'_{\lambda,\mu}(z_*), \xi\rangle = 0, \quad \text{for all } \xi \in E,$$

which implies that z_* is a solution of system (1.1) and $z_* \in \mathcal{N}_{\lambda,\mu}$.

Then, we will prove $z_n \to z_*$. By using the Lebesgue dominated convergence theorem, we can get

$$\lim_{n\to\infty} \int_\Omega (\lambda|u_n|^q + \mu|v_n|^q)dx = \int_\Omega (\lambda|u_*|^q + \mu|v_*|^q)dx. \tag{3.3}$$

Since $z_* \in \mathcal{N}_{\lambda,\mu}$, by Fatou's Lemma and (3.3), one has

$$\begin{aligned} m &\leq I_{\lambda,\mu}(z_*) \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\|z_*\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\int_\Omega (\lambda|u_*|^q + \mu|v_*|^q)dx \\ &= \frac{1}{N}\|z_*\|_E^2 - \frac{2^*-q}{2^*q}\int_\Omega (\lambda|u_*|^q + \mu|v_*|^q)dx \\ &\leq \liminf_{n\to\infty}\left(\frac{1}{N}\|z_n\|_E^2 - \frac{2^*-q}{2^*q}\int_\Omega (\lambda|u_n|^q + \mu|v_n|^q)dx\right) \\ &= \liminf_{n\to\infty} I_{\lambda,\mu}(z_n) \\ &= m, \end{aligned}$$

which implies $I_{\lambda,\mu}(z_*) = m$, $\|z_n\|_E^2 \to \|z_*\|_E^2$. By combining with (3.2), we can derive $z_n \to z_*$ in E. Thus, z_* is a solution of system (1.1) that means $z_* \in \mathcal{N}_{\lambda,\mu}$. Moreover, we are going to prove $z_* \in \mathcal{N}^+_{\lambda,\mu}$. Since $z_* \in \mathcal{N}_{\lambda,\mu}$, from (1.6) and (1.7), we have

$$\begin{aligned} \int_\Omega (\lambda|u_*|^q + \mu|v_*|^q)dx &= \frac{q(2^*-2)}{2(2^*-q)}\|z_*\|_E^2 - \frac{2^*q}{2^*-q}m \\ &\geq -\frac{2^*q}{2^*-q}m \\ &> 0. \end{aligned} \tag{3.4}$$

Then, $z_* \neq (0,0)$, which implies $z_* \in \mathcal{N}^+_{\lambda,\mu}$ or $z_* \in \mathcal{N}^-_{\lambda,\mu}$. If $z_* \in \mathcal{N}^-_{\lambda,\mu}$, by Lemma 2.1 there are unique t^+, t^- with $t^+ < t^- = 1$ such that $t^+ z_* \in \mathcal{N}^+_{\lambda,\mu}$, $t^- z_* \in \mathcal{N}^-_{\lambda,\mu}$. From (1.10) we know that

$$\frac{d}{dt} I_{\lambda,\mu}(t^+ z_*) = 0, \qquad \frac{d^2}{dt^2} I_{\lambda,\mu}(t^+ z_*) > 0.$$

Moreover, according to Lemma 2.1 , for any t with $t^+ < t < t^- = 1$, one gets

$$m^+ \leq I_{\lambda,\mu}(t^+ z_*) < I_{\lambda,\mu}(t z_*) \leq I_{\lambda,\mu}(t^- z_*) = I_{\lambda,\mu}(z_*) = m,$$

which implies a contradiction. Thus, $z_* \in \mathcal{N}^+_{\lambda,\mu}$ and $m = m^+$, and according to Lemma 2.3 (i), we have $m^+ = I_{\lambda,\mu}(z_*) < 0$.

Finally, we are going to prove z_* is a positive solution. We have $z_* \neq (0,0)$ from (3.4). Then, the main purpose now is to exclude semi-trivial solutions. Assume that $u_* \not\equiv 0, v_* \equiv 0$, then u_* is a nontrivial solution to the following equation:

$$\begin{cases} -\Delta u = |u|^{2^*-2} u + \lambda |u|^{q-2} u, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0. & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Because $(u_*, 0)$ is a solution of equation (3.5), we have

$$\|(u_*, 0)\|_E^2 = W_*(u_*, 0) > 0,$$

where $W_*(u_*, 0) = \int_\Omega |u_*|^{2^*} dx + \int_\Omega \lambda u_*^q dx$. And similarly, we could take $\phi \in H_0^1(\Omega) \backslash \{0\}$ such that

$$\|(0, \phi)\|_E^2 = W_*(0, \phi) > 0.$$

Now,

$$W_*(u_*, \phi) = \|(u_*, \phi)\|_E^2 = W_*(u_*, 0) + W_*(0, \phi).$$

According to Lemma 2.1, there exists a unique $0 < t^+ < t_{max}$ such that $(t^+ u_*, t^+ \phi) \in \mathcal{N}^+_{\lambda,\mu}$, where

$$t_{max} = \left[\frac{(2^* - q) W_*(u_*, \phi)}{(2^* - 2) \|(u_*, \phi)\|_E^2} \right]^{\frac{1}{2-q}} = \left(\frac{2^* - q}{2^* - 2} \right)^{\frac{1}{2-q}} > 1$$

and

$$I_{\lambda,\mu}(t^+ u_*, t^+ \phi) = \inf_{0 \leq t \leq t_{max}} I_{\lambda,\mu}(t u_*, t\phi).$$

Then, we can deduce the following result:

$$m^+ \leq I_{\lambda,\mu}(t^+ u_*, t^+ \phi) \leq I_{\lambda,\mu}(u_*, \phi) < I_{\lambda,\mu}(u_*, 0) = m^+.$$

It is impossible. Finally, we can know that $u_*, v_* > 0$ in Ω by using the strong maximum principle, and $z_* = (u_*, v_*)$ is a positive solution of system (1.1). The proof of Theorem 1.1 (i) is complete. □

4 Proof of Theorem 1.1

In this part, we will prove Theorem 1.1 (ii), and obtain the second positive solution of system (1.1). Before that, due to lacking of compactness condition for $I_{\lambda,\mu}$, we first give the local $(PS)_c$ condition which is satisfied for the corresponding energy function.

Lemma 4.1. *Let $\{z_n = (u_n, v_n)\}$ be a $(PS)_c$ sequence of $I_{\lambda,\mu}$ with*

$$c < m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}},$$

we can get that $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition in E.

Proof. Let $\{z_n\} = \{(u_n, v_n)\}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(z_n) = c + o_n(1), \qquad I'_{\lambda,\mu}(z_n) = o_n(1). \tag{4.1}$$

Combining with (2.9), we have

$$\begin{aligned} c + o_n(1) &= I_{\lambda,\mu}(z_n) \\ &\geq \frac{1}{N}\|z_n\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|z_n\|_E^q |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}. \end{aligned}$$

Since $1 < q < 2$, we know that $\{z_n\}$ is bounded in E. Passing to a subsequence (still denoted by $\{z_n\}$), there exists $z = (u, v) \in E$ such that $z_n \rightharpoonup z$ in E, and we have

$$\begin{cases} u_n \rightharpoonup u,\ v_n \rightharpoonup v, & \text{in } H_0^1(\Omega), \\ u_n \to u,\ v_n \to v, & \text{in } L^s(\Omega)(1 \leq s < 2^*), \\ u_n(x) \to u(x),\ v_n(x) \to v(x), & \text{a.e. in } \Omega. \end{cases} \tag{4.2}$$

Similar to [9, Proposition 1], as $n \to \infty$, from (4.1) and (4.2), one has

$$\begin{aligned} \lim_{n\to\infty} \langle I'_{\lambda,\mu}(z_n), \xi\rangle &= \langle I'_{\lambda,\mu}(z), \xi\rangle \\ &= \int_\Omega (\nabla u \nabla \xi_1 + \nabla v \nabla \xi_2) dx \\ &\quad - \int_\Omega \left(|u|^{2^*-2} u\xi_1 + |v|^{2^*-2} v\xi_2\right) dx \\ &\quad - \int_\Omega \left(\frac{\alpha}{2^*}|u|^{\alpha-2}|v|^\beta u\xi_1 + \frac{\beta}{2^*}|u|^\alpha|v|^{\beta-2} v\xi_2\right) dx \\ &\quad - \int_\Omega \left(\lambda|u|^{q-2}u\xi_1 + \mu|v|^{q-2}v\xi_2\right) dx \\ &= 0, \end{aligned}$$

for any $\xi = (\xi_1, \xi_2) \in E$. Particularly, choosing $\xi = z$, one obtains $\langle I'_{\lambda,\mu}(z), z\rangle$=0 and $z = (u, v) \in \mathcal{N}_{\lambda,\mu}$.

Set $\{(\eta_n, \mu_n)\} = \{(u_n - u, v_n - v)\}$ in E, then, $(\eta_n, \mu_n) \rightharpoonup (0, 0)$ in E. And next, we give the following version of Brézis–Lieb Lemma from [14, Lemma 3.4]

$$\int_\Omega |u_n|^\alpha |v_n|^\beta dx = \int_\Omega (|\eta_n|^\alpha |\mu_n|^\beta + |u|^\alpha |v|^\beta) dx + o_n(1), \tag{4.3}$$

and the Brézis–Lieb Lemma for the other terms,

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega (|\nabla \eta_n|^2 + |\nabla u|^2) dx + o_n(1), \tag{4.4}$$

$$\int_\Omega |u_n|^{2^*} dx = \int_\Omega (|\eta_n|^{2^*} + |u|^{2^*}) dx + o_n(1), \tag{4.5}$$

where (4.4) and (4.5) are equally applicable to v_n. Moreover, according to the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_\Omega (\lambda|u_n|^q + \mu|v_n|^q) dx = \int_\Omega (\lambda|u|^q + \mu|v|^q) dx. \tag{4.6}$$

Then from (4.3)–(4.5), we have

$$\begin{aligned} o_n(1) &= \langle I'_{\lambda,\mu}(u_n, v_n), (u_n, v_n)\rangle \\ &= \|(\eta_n, \mu_n)\|_E^2 - \int_\Omega (|\eta_n|^{2^*} + |\mu_n|^{2^*} + |\eta_n|^\alpha |\mu_n|^\beta) dx + o_n(1). \end{aligned} \tag{4.7}$$

Assume that there exists a constant l, which makes $\|(\eta_n, \mu_n)\|_E^2 \to l$. Then, from (4.7) we can get $\int_\Omega (|\eta_n|^{2^*} + |\mu_n|^{2^*} + |\eta_n|^\alpha |\mu_n|^\beta) dx \to l$. According to (1.5), one obtains

$$S_{\alpha,\beta} \left[\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx \right]^{\frac{2}{2^*}} \le \int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx.$$

Then, $l \ge S_{\alpha,\beta} l^{\frac{2}{2^*}}$, which implies that $l = 0$ or $l \ge S_{\alpha,\beta}^{\frac{N}{2}}$. On the one hand, if $l = 0$, the proof is complete. On the other hand, if $l \ge S_{\alpha,\beta}^{\frac{N}{2}}$, according to the definition of m and $(u, v) \in \mathcal{N}$, it follows from (4.3)–(4.7) that

$$\begin{aligned} c &= I_{\lambda,\mu}(u, v) + \frac{1}{2}\|(\eta_n, \mu_n)\|^2 - \frac{1}{2^*} \int_\Omega (|\eta_n|^{2^*} + |\mu_n|^{2^*} + |\eta_n|^\alpha |u_n|^\beta) dx + o_n(1) \\ &= m + \left(\frac{1}{2} - \frac{1}{2^*}\right) l \\ &\ge m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}, \end{aligned}$$

which is contrary to the given condition of $c < m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}$. So, $l = 0$, i.e. $(u_n, v_n) \to (u, v)$ in E. The proof of Lemma 4.1 is complete. □

Set $\psi \in C_0^\infty$ and satisfies $0 \le \psi \le 1$, $|\nabla \psi| \le C$. The definition of ψ as follows:

$$\psi(x) = \begin{cases} 1, & |x| \le \frac{\rho_0}{2} \\ 0, & |x| \ge \rho_0, \end{cases}$$

where $\varepsilon \in (0, 1)$. Moreover, setting

$$u_\varepsilon(x) = \psi(x) U_\varepsilon(x) \in H_0^1(\Omega), \qquad v_\varepsilon(x) = \tau_{min} \psi(x) U_\varepsilon(x) \in H_0^1(\Omega). \tag{4.8}$$

Then, we will have the following estimates.

Lemma 4.2. *Under the assumptions of Theorem 1.1 (ii), for any* $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, (\frac{q}{2})^{\frac{2}{2-q}} T)$*, there exist* $\varepsilon_0 > 0$ *for any* $\varepsilon \in (0, \varepsilon_0)$ *such that*

$$\sup_{t \geq 0} I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) < m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}.$$

Proof. From [3], we can obtain the following classical conclusion:

$$|u_\varepsilon|_{2^*}^2 = |U_\varepsilon|_{2^*}^2 + O(\varepsilon^N); \tag{4.9}$$

$$\|u_\varepsilon\|^2 = \|U_\varepsilon\|^2 + O(\varepsilon^{N-2}). \tag{4.10}$$

Then, we have

$$\begin{aligned} I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) &= \frac{1}{2}\|(u_* + tu_\varepsilon, v_* + tv_\varepsilon)\|_E^2 \\ &\quad - \frac{1}{2^*}\int_\Omega (u_* + tu_\varepsilon)^{2^*} + (v_* + tv_\varepsilon)^{2^*} + (u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta dx \\ &\quad - \frac{1}{q}\int_\Omega \lambda(u_* + tu_\varepsilon)^q + \mu(v_* + tv_\varepsilon)^q dx \\ &= \frac{1}{2}\|(u_*, v_*)\|_E^2 + \frac{1}{2}\|(tu_\varepsilon, tv_\varepsilon)\|_E^2 \\ &\quad + t\int_\Omega u_*^{2^*-1}u_\varepsilon + \frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta u_\varepsilon + \lambda u_*^{q-1}u_\varepsilon dx \\ &\quad + t\int_\Omega v_*^{2^*-1}v_\varepsilon + \frac{\beta}{2^*}u_*^\alpha v_*^{\beta-1}v_\varepsilon + \mu v_*^{q-1}v_\varepsilon dx \\ &\quad - \frac{1}{2^*}\int_\Omega (u_* + tu_\varepsilon)^{2^*} + (v_* + tv_\varepsilon)^{2^*} + (u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta dx \\ &\quad - \frac{1}{q}\int_\Omega \lambda(u_* + tu_\varepsilon)^q + \mu(v_* + tv_\varepsilon)^q dx. \end{aligned}$$

According to [23, (4.11)]:

$$(a+b)^q \geq a^q + qa^{q-1}b,\ a, b > 0,\ 1 < q < 2. \tag{4.11}$$

Then, we have

$$\begin{aligned} &t\int_\Omega u_*^{2^*-1}u_\varepsilon + \frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta u_\varepsilon + \lambda u_*^{q-1}u_\varepsilon dx \\ &\qquad + t\int_\Omega v_*^{2^*-1}v_\varepsilon + \frac{\beta}{2^*}u_*^\alpha v_*^{\beta-1}v_\varepsilon + \mu v_*^{q-1}v_\varepsilon dx - \frac{1}{q}\int_\Omega \lambda(u_* + tu_\varepsilon)^q + \mu(v_* + tv_\varepsilon)^q dx \\ &\quad \leq t\int_\Omega u_*^{2^*-1}u_\varepsilon + v_*^{2^*-1}v_\varepsilon + \frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta u_\varepsilon + \frac{\beta}{2^*}u_*^\alpha v_*^{\beta-1}v_\varepsilon dx - \frac{1}{q}\int_\Omega (\lambda u_*^q + \mu v_*^q) dx. \end{aligned}$$

Thus,

$$\begin{aligned} I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) &= \frac{1}{2}\|(u_*, v_*)\|_E^2 + \frac{1}{2}\|(tu_\varepsilon, tv_\varepsilon)\|_E^2 \\ &\quad + t\int_\Omega u_*^{2^*-1}u_\varepsilon + \frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta u_\varepsilon + \lambda u_*^{q-1}u_\varepsilon dx \\ &\quad + t\int_\Omega v_*^{2^*-1}v_\varepsilon + \frac{\beta}{2^*}u_*^\alpha v_*^{\beta-1}v_\varepsilon + \mu v_*^{q-1}v_\varepsilon dx \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2^*}\int_\Omega (u_*+tu_\varepsilon)^{2^*}+(v_*+tv_\varepsilon)^{2^*}+(u_*+tu_\varepsilon)^\alpha(v_*+tv_\varepsilon)^\beta dx\\
&-\frac{1}{q}\int_\Omega \lambda(u_*+tu_\varepsilon)^q+\mu(v_*+tv_\varepsilon)^q dx\\
\le\ &\frac{1}{2}\|(u_*,v_*)\|_E^2+\frac{1}{2}\|(tu_\varepsilon,tv_\varepsilon)\|_E^2\\
&-\frac{1}{2^*}\int_\Omega (u_*+tu_\varepsilon)^{2^*}+(v_*+tv_\varepsilon)^{2^*}+(u_*+tu_\varepsilon)^\alpha(v_*+tv_\varepsilon)^\beta dx\\
&+\int_\Omega (u_*^{2^*-1}tu_\varepsilon+v_*^{2^*-1}tv_\varepsilon+\frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta tu_\varepsilon+\frac{\beta}{2^*}u_*^\alpha v_*^{\beta-1}tv_\varepsilon)dx\\
&-\frac{1}{q}\int_\Omega (\lambda u_*^q+\mu v_*^q)dx\\
=\ &I_{\lambda,\mu}(u_*,v_*)+\frac{1}{2}\|(tu_\varepsilon,tv_\varepsilon)\|_E^2\\
&-\frac{1}{2^*}\int_\Omega (tu_\varepsilon)^{2^*}+(tv_\varepsilon)^{2^*}+(tu_\varepsilon)^\alpha(tv_\varepsilon)^\beta dx\\
&-\frac{1}{2^*}\int_\Omega (u_*+tu_\varepsilon)^{2^*}-u_*^{2^*}-(tu_\varepsilon)^{2^*}-2^*u_*^{2^*-1}tu_\varepsilon dx\\
&-\frac{1}{2^*}\int_\Omega (v_*+tv_\varepsilon)^{2^*}-v_*^{2^*}-(tv_\varepsilon)^{2^*}-2^*v_*^{2^*-1}tv_\varepsilon dx\\
&-\frac{1}{2^*}\int_\Omega (u_*+tu_\varepsilon)^\alpha(v_*+tv_\varepsilon)^\beta-u_*^\alpha v_*^\beta-(tu_\varepsilon)^\alpha(tv_\varepsilon)^\beta\\
&-\alpha u_*^{\alpha-1}v_*^\beta tu_\varepsilon-\beta u_*^\alpha v_*^{\beta-1}tv_\varepsilon dx. \qquad (4.12)
\end{aligned}
$$

Let $\Phi_\varepsilon(t)=\Phi_{\varepsilon,1}(t)+\Phi_{\varepsilon,2}(t)+\Phi_{\varepsilon,3}(t)+\Phi_{\varepsilon,4}(t)$, where

$$\Phi_{\varepsilon,1}(t)=\frac{1}{2}\|(tu_\varepsilon,tv_\varepsilon)\|_E^2-\frac{1}{2^*}\int_\Omega (tu_\varepsilon)^{2^*}+(tv_\varepsilon)^{2^*}+(tu_\varepsilon)^\alpha(tv_\varepsilon)^\beta dx, \qquad (4.13)$$

$$\Phi_{\varepsilon,2}(t)=\frac{1}{2^*}\int_\Omega (u_*+tu_\varepsilon)^{2^*}-u_*^{2^*}-(tu_\varepsilon)^{2^*}-2^*u_*^{2^*-1}tu_\varepsilon dx, \qquad (4.14)$$

$$\Phi_{\varepsilon,3}(t)=\frac{1}{2^*}\int_\Omega (v_*+tv_\varepsilon)^{2^*}-v_*^{2^*}-(tv_\varepsilon)^{2^*}-2^*v_*^{2^*-1}tv_\varepsilon dx, \qquad (4.15)$$

$$
\begin{aligned}
\Phi_{\varepsilon,4}(t)=\ &\frac{1}{2^*}\int_\Omega (u_*+tu_\varepsilon)^\alpha(v_*+tv_\varepsilon)^\beta-u_*^\alpha v_*^\beta-(tu_\varepsilon)^\alpha(tv_\varepsilon)^\beta\\
&-\alpha u_*^{\alpha-1}v_*^\beta tu_\varepsilon-\beta u_*^\alpha v_*^{\beta-1}tv_\varepsilon dx. \qquad (4.16)
\end{aligned}
$$

Notice that $\Phi_\varepsilon(0)=0$, $\lim_{t\to+\infty}\Phi_\varepsilon(t)=-\infty$, and $\lim_{t\to 0^+}\Phi_\varepsilon(t)=0$ uniformly for all ε. On the one hand, when $\inf\sup_{t\ge 0}\Phi_\varepsilon(t)\le 0$, one has $I_{\lambda,\mu}(u_*+tu_\varepsilon,v_*+tv_\varepsilon)\le I_{\lambda,\mu}(u_*,v_*)=m<m+\frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}$. Conclusion naturally holds in this case. On the other hand, when $\inf\sup_{t\ge 0}\Phi_\varepsilon(t)>0$, then, $\sup_{t\ge 0}\Phi_\varepsilon(t)>0$ and it attains for some $t_\varepsilon>0$, that is, $\sup_{t\ge 0}\Phi_\varepsilon(t)=\Phi_\varepsilon(t_\varepsilon)$. According to the monotonicity of Φ_ε near $t=0$, we can find two positive constants $\overline{T_0},\underline{T_0}$, such that

$$|\Phi_\varepsilon(\underline{T_0})|=|\Phi_\varepsilon(\underline{T_0})-\Phi_\varepsilon(0)|<\xi=\frac{\Phi_\varepsilon(t_\varepsilon)}{4}.$$

Similarly, we can obtain $t_\varepsilon<\overline{T_0}$. So, $\underline{T_0}<t_\varepsilon<\overline{T_0}$ is bounded. Now, we evaluate the four parts separately. Let us evaluate (4.13) first, from (4.8) we can get:

$$\begin{aligned}\Phi_{\varepsilon,1}(t_\varepsilon) &= \frac{t_\varepsilon^2}{2}\|(u_\varepsilon, v_\varepsilon)\|_E^2 - \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega (|u_\varepsilon|^{2^*} + |v_\varepsilon|^{2^*} + |u_\varepsilon|^\alpha |v_\varepsilon|^\beta)dx \\ &= \frac{t_\varepsilon^2}{2}\int_\Omega (|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2)dx - \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega (|u_\varepsilon|^{2^*} + |v_\varepsilon|^{2^*} + |u_\varepsilon|^\alpha |v_\varepsilon|^\beta)dx \\ &= \frac{t_\varepsilon^2}{2}(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx - \frac{t_\varepsilon^{2^*}}{2^*}(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx.\end{aligned}$$

Then, define

$$J(t) = \frac{t^2}{2}\left[(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx\right] - \frac{t^{2^*}}{2^*}\left[(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx\right].$$

Obviously, $J'(t_{max}^\varepsilon) = 0$ with

$$t_{max}^\varepsilon = \left[\frac{(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx}{(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon^{2^*}|dx}\right]^{\frac{1}{2^*-2}} > 0.$$

By simple analysis, we get that $J(t)$ attains maximum at t_{max}^ε. Next, by using (1.5), (4.9), (4.10), we have the following result:

$$\begin{aligned}J(t_{max}^\varepsilon) &= \frac{1}{2}\left[\frac{(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx}{(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx}\right]^{\frac{2}{2^*-2}}\left[(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx\right] \\ &\quad - \frac{1}{2^*}\left[\frac{(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx}{(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx}\right]^{\frac{2^*}{2^*-2}}\left[(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx\right] \\ &= \frac{1}{N}\left\{\frac{(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left[(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx\right]^{\frac{2}{2^*}}}\right\}^{\frac{N}{2}} \\ &\le \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} + O(\varepsilon^{N-2}).\end{aligned}$$

Then, we can get

$$\Phi_{\varepsilon,1}(t_\varepsilon) \le J(t_{max}^\varepsilon) \le \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} + O(\varepsilon^{N-2}). \tag{4.17}$$

Next, let us analyze (4.14). According to [23, (4.12)]:

$$(a+b)^\gamma \ge a^\gamma + b^\gamma + \gamma a^{\gamma-1}b + C_1 ab^{\gamma-1}, \qquad 0 \le a \le M,\ b \ge 1,\ M > 0,\ \gamma > 2. \tag{4.18}$$

Then, according to (1.5) we can find a positive constant $C_2 > 1$, where C_2 satisfies:

$$\begin{aligned}S^{\frac{N}{2}} &= \left(\frac{1}{f(\tau_{min})}S_{\alpha,\beta}\right)^{\frac{N}{2}} \\ &\le C_2 S_{\alpha,\beta}^{\frac{N}{2}}.\end{aligned} \tag{4.19}$$

Moreover, by the standard elliptic estimates, we know that $u_*, v_* \in C(\overline{\Omega})$. Assume that $t_\varepsilon u_\varepsilon \geq 1$ for all $t_\varepsilon \geq \frac{1}{\sqrt{C_2 N(1+\tau_{min}^2)}}$, by using (4.18), we get

$$\begin{aligned}
\Phi_{\varepsilon,2}(t_\varepsilon) &= \frac{1}{2^*}\int_\Omega [(u_* + t_\varepsilon u_\varepsilon)^{2^*} - u_*^{2^*} - (t_\varepsilon u_\varepsilon)^{2^*} - 2^* u_*^{2^*-1} t_\varepsilon u_\varepsilon] dx \\
&\geq \frac{1}{2^*}\int_\Omega [u_*^{2^*} + (t_\varepsilon u_\varepsilon)^{2^*} + 2^* u_*^{2^*-1} t_\varepsilon u_\varepsilon \\
&\quad - u_*^{2^*} - (t_\varepsilon u_\varepsilon)^{2^*} - 2^* u_*^{2^*-1} t_\varepsilon u_\varepsilon + C_1 u_* t_\varepsilon^{2^*-1} u_\varepsilon^{2^*-1}] dx \\
&= \frac{t_\varepsilon^{2^*-1}}{2^*}\int_\Omega C_1 u_* u_\varepsilon^{2^*-1} dx \\
&\geq O(\varepsilon^{\frac{N-2}{2}}).
\end{aligned} \tag{4.20}$$

By using the same method as (4.20) to (4.15), we can get

$$\Phi_{\varepsilon,3}(t_\varepsilon) \geq O(\varepsilon^{\frac{N-2}{2}}). \tag{4.21}$$

At last, let's evaluate (4.16). First of all, we define a new function $f(x,y) : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$, and

$$f(x,y) = (1+x)^\alpha (1+y)^\beta - x^\alpha y^\beta - \alpha x - \beta y - 1.$$

Since

$$\begin{aligned}
\frac{\partial f(x,y)}{\partial x} &= \alpha(1+x)^{\alpha-1}(1+y)^\beta - \alpha x^{\alpha-1} y^\beta - \alpha \\
&\geq \alpha(1+x)^{\alpha-1}(1+y^\beta) - \alpha x^{\alpha-1} y^\beta - \alpha \\
&= \alpha(1+x)^{\alpha-1} - \alpha + \alpha(1+x)^{\alpha-1} y^\beta - \alpha x^{\alpha-1} y^\beta \\
&\geq 0.
\end{aligned}$$

We can also get $\dfrac{\partial f(x,y)}{\partial y} \geq 0$ in the same way. Obviously, $f(0,0) = 0$, so for any $x \geq 0, y \geq 0$, we have $f(x,y) \geq 0$. Because of $u_*, v_* > 0$, we have

$$\begin{aligned}
\Phi_{\varepsilon,4}(t_\varepsilon) &= \frac{1}{2^*}\int_\Omega (u_* + t_\varepsilon u_\varepsilon)^\alpha (v_* + t_\varepsilon v_\varepsilon)^\beta - u_*^\alpha v_*^\beta - (t_\varepsilon u_\varepsilon)^\alpha (t_\varepsilon v_\varepsilon)^\beta \\
&\quad - \alpha u_*^{\alpha-1} v_*^\beta t_\varepsilon u_\varepsilon - \beta u_*^\alpha v_*^{\beta-1} t_\varepsilon v_\varepsilon dx \\
&\geq 0.
\end{aligned} \tag{4.22}$$

Therefore, for $t = t_\varepsilon \geq \frac{1}{\sqrt{C_2 N(1+\tau_{min}^2)}}$, we know that there exists a $\varepsilon_1 > 0$ such for any $\varepsilon \in (0, \varepsilon_1)$ that

$$\begin{aligned}
I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) &\leq m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + O(\varepsilon^{N-2}) - O(\varepsilon^{\frac{N-2}{2}}) \\
&< m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}
\end{aligned} \tag{4.23}$$

by (4.17), (4.20), (4.21) and (4.22).

When $0 < t < \frac{1}{\sqrt{C_2 N(1+\tau_{min}^2)}}$, according to (4.8)–(4.11) and (4.19), there is a $\varepsilon_2 > 0$, when $\varepsilon \in (0, \varepsilon_2)$ we have the following estimates:

$$\begin{aligned}
I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) &= \frac{1}{2}\|(u_*, v_*)\|_E^2 + \frac{1}{2}\|(tu_\varepsilon, tv_\varepsilon)\|_E^2 \\
&\quad + t\int_\Omega (u_*^{2^*-1}u_\varepsilon + \frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta u_\varepsilon + \lambda u_*^{q-1}u_\varepsilon)dx \\
&\quad + t\int_\Omega (v_*^{2^*-1}v_\varepsilon + \frac{\beta}{2^*}u_*^{\alpha}v_*^{\beta-1} v_\varepsilon + \mu v_*^{q-1}v_\varepsilon)dx \\
&\quad - \frac{1}{2^*}\int_\Omega [(u_* + tu_\varepsilon)^{2^*} + (v_* + tv_\varepsilon)^{2^*} + (u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta]dx \\
&\quad - \frac{1}{q}\int_\Omega [\lambda(u_* + tu_\varepsilon)^q + \mu(v_* + tv_\varepsilon)^q]dx \\
&= I_{\lambda,\mu}(u_*, v_*) + \frac{t^2}{2}\|(u_\varepsilon, v_\varepsilon)\|_E^2 \\
&\quad + \frac{1}{2^*}\int_\Omega (u_*^{2^*} + v_*^{2^*} + 2^* u_*^{2^*-1}tu_\varepsilon + 2^* v_*^{2^*-1}tv_\varepsilon)dx \\
&\quad - \frac{1}{2^*}\int_\Omega [(u_* + tu_\varepsilon)^{2^*} + (v_* + tv_\varepsilon)^{2^*}]dx \\
&\quad + \frac{1}{2^*}\int_\Omega (u_*^\alpha v_*^\beta + \alpha u_*^{\alpha-1}v_*^\beta tu_\varepsilon + \beta u_*^\alpha v_*^{\beta-1}tv_\varepsilon)dx \\
&\quad - \frac{1}{2^*}\int_\Omega [(u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta]dx \\
&\quad + \frac{1}{q}\int_\Omega (\lambda u_*^q + \mu v_*^q)dx \\
&\quad - \frac{1}{q}\int_\Omega [\lambda(u_* + tu_\varepsilon)^q + \mu(v_* + tv_\varepsilon)^q]dx \\
&\le m + \frac{t^2}{2}\|(u_\varepsilon, v_\varepsilon)\|_E^2 \\
&= m + \frac{t^2}{2}\int_\Omega (|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2)dx \\
&= m + \frac{t^2}{2}(1 + \tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx \\
&= m + \frac{t^2}{2}(1 + \tau_{min}^2)\left[\|U_\varepsilon\|_E^2 + O(\varepsilon^{N-2})\right] \\
&\le m + t^2(1 + \tau_{min}^2)S^{\frac{N}{2}} \\
&\le m + t^2(1 + \tau_{min}^2)C_2 S_{\alpha,\beta}^{\frac{N}{2}} \\
&< m + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}.
\end{aligned} \tag{4.24}$$

Therefore, choosing $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$, for any $0 < \varepsilon < \varepsilon_0$, we can draw a conclusion

$$\sup_{t \ge 0} I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) < m + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}$$

from (4.23) and (4.24). The proof of Lemma 4.2 is finished. □

Lemma 4.3. *There is a $t_\varepsilon^-(u_\varepsilon) > 0$ such that $(u_* + t_\varepsilon^- u_\varepsilon, v_* + t_\varepsilon^- v_\varepsilon) \in \mathcal{N}_{\lambda,\mu}^-$, when $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, (\frac{q}{2})^{\frac{2}{2-q}}T)$. What is more, $0 < m^- < m + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}$.*

Proof. According to Lemma 2.1, there is a $t^-(z) > 0$ for any $z = (u,v) \in E\backslash\{(0,0)\}$ such that $t^-(z)z \in \mathcal{N}^-_{\lambda,\mu}$. Let

$$E_1 = \left\{z \in E : u = 0 \text{ } or \text{ } \|z\| < t^-\left(\frac{z}{\|z\|_E}\right)\right\},$$
$$E_2 = \left\{z \in E : \|z\| > t^-\left(\frac{z}{\|z\|_E}\right)\right\}.$$

Then, we have $\mathcal{N}^-_{\lambda,\mu} = \left\{z \in E : \|z\| = t^-\left(\frac{z}{\|z\|_E}\right)\right\}$. So, $E = E_1 \cup E_2 \cup \mathcal{N}^-_{\lambda,\mu}$. We have $\mathcal{N}^+_{\lambda,\mu} \subset E_1$, since $t^+ < t^-$. Now, there is a positive constant M_1 such that $0 < t^-(z) < M_1$ for $\|z\|_E = 1$. When $t_0 = \frac{\left|M_1 - \|(u_*,v_*)\|^2_E\right|^{\frac{1}{2}}}{\|(u_\varepsilon,v_\varepsilon)\|^2_E} + 1$, we claim that

$$\omega_\varepsilon = (u_* + t_0 u_\varepsilon, v_* + t_0 v_\varepsilon) \in E_2,$$

for $\varepsilon > 0$ small enough. By (4.10), we can deduce that

$$\begin{aligned}\|(u_* + t_0 u_\varepsilon, v_* + t_0 v_\varepsilon)\|^2_E &= \|(u_*,v_*)\|^2_E + \|(t_0 u_\varepsilon, t_0 v_\varepsilon)\|^2_E \\ &\quad + 2t_0 \int_\Omega (u_*^{2^*-1}u_\varepsilon + \frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta u_\varepsilon + \lambda u_*^{q-1}u_\varepsilon)dx \\ &\quad + 2t_0 \int_\Omega (v_*^{2^*-1}v_\varepsilon + \frac{\beta}{2^*}u_*^{\alpha}v_*^{\beta-1} v_\varepsilon + \mu v_*^{q-1}v_\varepsilon)dx \\ &\geq \|(u_*,v_*)\|^2_E + t_0^2\|(u_\varepsilon,v_\varepsilon)\|^2_E + o_n(1) \\ &> M_1^2 \\ &\geq \left[t^-\left(\frac{\omega_\varepsilon}{\|\omega_\varepsilon\|_E}\right)\right]^2.\end{aligned}$$

We denote $h : [0,1] \to E$ by $h(t) = u_* + tt_0 v_\varepsilon$, then there exists $0 < (t_\varepsilon)^- < t_0$, which makes $(u_* + (t_\varepsilon)^- u_\varepsilon, v_* + (t_\varepsilon)^- v_\varepsilon) \in \mathcal{N}^-_{\lambda,\mu}$. Moreover, from Lemma 4.2 and Lemma 2.3 (ii), one has

$$0 < m^- \leq I_{\lambda,\mu}(u_* + (t_\varepsilon)^- u_\varepsilon, v_* + (t_\varepsilon)^- v_\varepsilon) \leq \sup_{t\geq 0} I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) < m + \frac{1}{N}S^{\frac{N}{2}}_{\alpha,\beta}.$$

Thus, the proof of Lemma 4.3 is complete. □

Next, for $z = (u,v)$, $\varphi = (\varphi_1,\varphi_2) \in E$, we define

$$\begin{aligned} z - \varphi &= (u - \varphi_1, v - \varphi_2), \\ \langle z, \varphi\rangle &= \int_\Omega \nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 dx, \\ G_{\lambda,\mu}(z,\varphi) &= \int_\Omega \left(\lambda |u|^{q-2}u\varphi_1 + \mu |v|^{q-2}v\varphi_2\right) dx, \\ H(z,\varphi) &= \int_\Omega \left(|u|^{2^*-2}u\varphi_1 + |v|^{2^*-2}v\varphi_2\right) dx + \int_\Omega \left(\frac{\alpha}{2^*}|u|^{\alpha-2}|v|^\beta u\varphi_1 + \frac{\beta}{2^*}|u|^\alpha |v|^{\beta-2}v\varphi_2\right) dx.\end{aligned}$$

Then, we have the following conclusion.

Lemma 4.4. *When $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, (\frac{q}{2})^{\frac{2}{2-q}}T)$, there exist $\eta > 0$ and a differentiable function $\zeta : B_\eta(0) \subset E \to \mathbb{R}^+$, for $z = (u,v) \in \mathcal{N}^-_{\lambda,\mu}$ such that $\zeta(0) = 1$, $\zeta(\varphi)(z - \varphi) \in \mathcal{N}^-_{\lambda,\mu}$ for any $\varphi = (\varphi_1,\varphi_2) \in B_\eta(0)$, and*

$$\langle \zeta'(0), \phi\rangle = \frac{2\langle z,\phi\rangle - 2^* H(z,\phi) - qG_{\lambda,\mu}(z,\phi)}{(2-q)\|z\|^2_E - (2^* - q)H(z,z)}$$

for all $\phi = (\phi_1,\phi_2) \in E$.

Proof. For $z \in \mathcal{N}^-_{\lambda,\mu}$, we define a function $F_z : \mathbb{R} \times E \to \mathbb{R}$ and

$$\begin{aligned}F_z(\zeta,\phi) &= \Big\langle I'_{\lambda,\mu}(\zeta(z-\phi)), \zeta(z-\phi)\Big\rangle \\ &= \zeta^2\|z-\phi\|_E^2 - \zeta^{2^*}H(z-\phi,z-\phi) - \zeta^q G_{\lambda,\mu}(z-\phi,z-\phi).\end{aligned}$$

Then, we have $F_z(1,0) = \langle I'_{\lambda,\mu}(z), z\rangle = 0$, moreover, from (1.8) and (1.9) we have

$$\begin{aligned}\frac{d}{d\zeta}F_z(1,0) &= 2\|z\|_E^2 - 2^*H(z,z) - qG_{\lambda,\mu}(z,z) \\ &= (2-q)\|z\|_E^2 - (2^*-q)H(z,z) \\ &< 0.\end{aligned}$$

According to the implicit function theorem, there is a $\eta > 0$ and a differential function $\zeta : B_\eta(0) \subset E \to \mathbb{R}$, which makes $\zeta(0) = 1$, then, $F_z(\zeta(0), 0) = F_z(1,0) = 0$, one has

$$\langle \zeta'(0), \phi\rangle = \frac{2\langle z,\phi\rangle - 2^*H(z,\phi) - qG_{\lambda,\mu}(z,\phi)}{(2-q)\|z\|_E^2 - (2^*-q)H(z,z)}$$

and

$$F_z(\zeta(\varphi), \varphi) = 0, \quad \text{for all } \varphi \in B_\eta(0)$$

which is equivalent to

$$\Big\langle I'_{\lambda,\mu}(\zeta(\varphi)(z-\varphi)), \zeta(\varphi)(z-\varphi)\Big\rangle = 0, \quad \text{for all } \varphi \in B_\eta(0).$$

This means that for all $\varphi \in B_\eta(0)$, we have $\zeta(\varphi)(z-\varphi) \in \mathcal{N}_{\lambda,\mu}$. The proof of Lemma 4.4 is complete. □

4.1 The proof of Theorem 1.1

Proof. For $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, (\frac{q}{2})^{\frac{2}{2-q}}T)$, there exists a $z \in \mathcal{N}^-_{\lambda,\mu}$ such that $m^- = \inf I_{\lambda,\mu}(z) > 0$, by Lemma 2.3. Setting $\{(u_n, v_n)\} \subset E$, which is a minimizing sequence of $I_{\lambda,\mu}$ at m^-. Now, we are going to prove that $\{(u_n, v_n)\}$ is a $(PS)_{m^-}$-sequence of $I_{\lambda,\mu}$. According to Ekeland's variational principle (see [10]), there exists a sequence (we still denote it as $\{(u_n, v_n)\}$) that satisfies

(i) $I_{\lambda,\mu}(u_n, v_n) < m^- + \dfrac{1}{n}$;

(ii) $I_{\lambda,\mu}(u_n, v_n) \le I_{\lambda,\mu}(w_1, w_2) + \dfrac{\|(w_1,w_2) - (u_n,v_n)\|_E}{n}, \qquad (w_1, w_2) \in \mathcal{N}^-_{\lambda,\mu}$.

So, we only need to prove $I'_{\lambda,\mu}(u_n, v_n) \to 0$ as $n \to \infty$ in E^{-1} to get that $\{(u_n, v_n)\}$ is a $(PS)_{m^-}$-sequence of $I_{\lambda,\mu}$. According to Lemma 4.4, there exist a $\eta_n > 0$ and differentiable function $\zeta_n : B((0,0); \eta_n) \subset E \to \mathbb{R}^+$ such that $\zeta_n(0,0) = 1$, $\zeta_n(w_1, w_2)((u_n, v_n) - (w_1, w_2)) \in \mathcal{N}^-_{\lambda,\mu}$, for any $(w_1, w_2) \in B((0,0); \eta_n)$. Let $(\varphi_1, \varphi_2) \in E$, $\|(\varphi_1, \varphi_2)\|_E = 1$, and $0 < \sigma < \eta_n$. Then we choose $(w_1, w_2) = \sigma(\varphi_1, \varphi_2)$, which makes $(w_1, w_2) = \sigma(\varphi_1, \varphi_2) \in B((0,0); \eta_n)$ and $\omega_{\sigma,n} = \zeta_n(\sigma(\varphi_1, \varphi_2))((u_n, v_n) - \sigma(\phi_1, \varphi_2)) \in \mathcal{N}^-_{\lambda,\mu}$, for $0 < \sigma < \eta_n$. From (ii) and the mean value theorem, let $\sigma \to 0^+$, we have

$$\begin{aligned}
\frac{\|\omega_{\sigma,n} - (u_n, v_n)\|_E}{n} &\geq I_{\lambda,\mu}(u_n, v_n) - I_{\lambda,\mu}(\omega_{\sigma,n}) \\
&= \left\langle I'_{\lambda,\mu}(t_0(u_n, v_n) + (1-t_0)\omega_{\sigma,n}), (u_n, v_n) - \omega_{\sigma,n} \right\rangle \\
&= \left\langle I'_{\lambda,\mu}(u_n, v_n), (u_n, v_n) - \omega_{\sigma,n} \right\rangle + o(\|(u_n, v_n) - \omega_{\sigma,n}\|_E) \\
&= \sigma\zeta_n(\sigma(\varphi_1, \varphi_2)) \left\langle I'_{\lambda,\mu}(u_n, v_n), (\varphi_1, \varphi_2) \right\rangle \\
&\quad + (1 - \zeta_n(\sigma(\varphi_1, \varphi_2))) \left\langle I'_{\lambda,\mu}(u_n, v_n), (u_n, v_n) \right\rangle + o(\|(u_n, v_n) - \omega_{\sigma,n}\|_E) \\
&= \sigma\zeta_n(\sigma(\varphi_1, \varphi_2)) \left\langle I'_{\lambda,\mu}(u_n, v_n), (\varphi_1, \varphi_2) \right\rangle + o(\|(u_n, v_n) - \omega_{\sigma,n}\|_E),
\end{aligned}$$

where $t_0 \in (0, 1)$. Next, let $\sigma \to 0^+$, we have

$$\begin{aligned}
&\left\langle I'_{\lambda,\mu}(u_n, v_n), (\varphi_1, \varphi_2) \right\rangle \\
&\leq \frac{\|\omega_{\sigma,n} - (u_n, v_n)\|_E(\frac{1}{n} + o(1))}{\sigma|\zeta_n(\sigma(\varphi_1, \varphi_2))|} \\
&\leq \frac{\|(u_n, v_n)(\zeta_n(\sigma(\varphi_1, \varphi_2)) - \zeta_n(0,0)) - \sigma(\varphi_1, \varphi_2)\zeta_n(\sigma(\varphi_1, \varphi_2))\|_E(\frac{1}{n} + |o(1)|)}{\sigma|\zeta_n(\sigma(\varphi_1, \varphi_2))|} \\
&\leq \frac{\|(u_n, v_n)\|_E|\zeta_n(\sigma(\varphi_1, \varphi_2)) - \zeta_n(0,0)| + \sigma\|(\varphi_1, \varphi_2)\|_E|\zeta_n(\sigma(\varphi_1, \varphi_2))|}{\sigma|\zeta_n(\sigma(\varphi_1, \varphi_2))|} \left(\frac{1}{n} + |o(1)|\right) \\
&\leq C(1 + \|\zeta'_n(0,0)\|) \left(\frac{1}{n} + |o(1)|\right).
\end{aligned}$$

Due to $\{(u_n, v_n)\}$ and $\zeta'_n(0,0)$ are bounded, we could learn that $I'_{\lambda,\mu}(u_n, v_n) \to 0$ in E^{-1} as $n \to \infty$. Thus, $\{(u_n, v_n)\}$ is a $(PS)_{m^-}$-sequence of $I_{\lambda,\mu}$.

In accordance with Lemma 4.1, Lemma 4.2 and Lemma 4.3, there is a list of convergent subsequences $\{(u_n, v_n)\}$, such that $(u_n, v_n) \to (u_{**}, v_{**})$, where $(u_{**}, v_{**}) \in \mathcal{N}^-_{\lambda,\mu}$. What's more, when $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in \left(0, (\frac{q}{2})^{\frac{2}{2-q}} T\right)$, we can get $I_{\lambda,\mu}(u_{**}, v_{**}) = m^- > 0$. Since $I_{\lambda,\mu}(u_{**}, v_{**}) = I_{\lambda,\mu}(|u_{**}|, |v_{**}|)$ and $(u_{**}, v_{**}) \in \mathcal{N}^-_{\lambda,\mu}$, we can deduce that

$$\int_\Omega |u_{**}|^{2^*} + |v_{**}|^{2^*} + |u_{**}|^\alpha |v_{**}|^\beta dx > \frac{2-q}{2^*-q}\|(u_{**}, v_{**})\|^2_E > 0 \tag{4.25}$$

from (1.9) and (1.10). So, $(u_{**}, v_{**}) \neq 0$. Applying the strong maximum principle, we could get that (u_{**}, v_{**}) is a positive solution of system (1.1). Finally, due to $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$, which implies that (u_*, v_*) and (u_{**}, v_{**}) are entirely different. The proof of Theorem 1.1 is complete. □

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References

[1] C. O. Alves, D. C. Morais Filho, M. A. S. Souto, On systems of elliptic equations involving subcritical or critical Sobolev exponents, *Nonlinear Anal.* **42**(2000), No. 5, 771–787. https://doi.org/10.1016/S0362-546X(99)00121-2; MR1776922; Zbl 0958.35037

[2] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122**(1994), No. 2, 519–543. https://doi.org/10.1006/jfan.1994.1078; MR1276168; Zbl 0805.35028

[3] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36**(1983), No. 4, 437–477. https://doi.org/10.1002/cpa.3160360405; MR709644; Zbl 0541.35029

[4] D. M. Cao, H. S. Zhou, Multiple positive solutions of nonhomogeneous semilinear elliptic equations in $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect.* **126**(1996), No. 2, 443–463. https://doi.org/10.1017/S0308210500022836; MR1386873; Zbl 0846.35042

[5] Z. J. Chen, W. M. Zou, Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent, *Arch. Ration. Mech. Anal.* **205**(2012), No. 2, 515–551. https://doi.org/10.1007/s00205-012-0513-8; MR2947540; Zbl 1256.35132

[6] Z. J. Chen, W. M. Zou, Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent: higher dimensional case, *Calc. Var. Partial Differential Equations* **52**(2015), No. 1–2, 423–467. https://doi.org/10.1007/s00526-014-0717-x; MR3299187; Zbl 1312.35158

[7] Z. J. Chen, W. M. Zou, Existence and symmetry of positive ground states for a doubly critical Schrödinger system, *Trans. Amer. Math. Soc.* **367**(2015), No. 5, 3599–3646. https://doi.org/10.1090/S0002-9947-2014-06237-5; MR3314818; Zbl 1315.35091

[8] M. Clapp, J. Faya, Multiple solutions to a weakly coupled purely critical elliptic system in bounded domains, *Discrete Contin. Dyn. Syst.* **39**(2019), No. 6, 3265–3289. https://doi.org/10.3934/dcds.2019135; MR3959429; Zbl 1415.35113

[9] X. L. Duan, G. M. Wei, H. T. Yang, Multiple solutions for critical nonhomogeneous elliptic systems in noncontractible domain, *Math. Methods Appl. Sci.* **44**(2021), No. 11, 8615–8637. https://doi.org/10.1002/mma.7289; MR4279812; Zbl 1472.35149

[10] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47**(1974), No. 2, 324–353. https://doi.org/10.1016/0022-247X(74)90025-0; MR346619; Zbl 0286.49015

[11] B. D. Esry, C. H. Greene, J. Burke, J. Bohn, Hartree–Fock theory for double condensates, *Phys. Rev. Lett.* 78(1997), No. 19, 3594-3597. https://doi.org/10.1103/PhysRevLett.78.3594

[12] H. N. Fan, Multiple positive solutions for semi-linear elliptic systems with sign-changing weight, *J. Math. Anal. Appl.* **409**(2014), No. 1, 399–408. https://doi.org/10.1016/j.jmaa.2013.07.014; MR3095049; Zbl 1310.35100

[13] Z. Y. Guo, W. M. Zou, Positive ground state solutions for an elliptic system with Hardy–Sobolev critical exponent growth, *J. Math. Phys.* **60**(2019), No. 10, 1–20. https://doi.org/10.1063/1.5081794; MR4015132; Zbl 1427.35069

[14] P. G. Han, Multiple positive solutions of nonhomogeneous elliptic systems involving critical Sobolev exponents, *Nonlinear Anal.* **64**(2006), No. 4, 869–886. https://doi.org/10.1016/j.na.2005.04.053; MR2197099; Zbl 1208.35046

[15] H. Y. He, J. F. Yang, Positive solutions for critical inhomogeneous elliptic problems in non-contractible domains, *Nonlinear Anal.* **70**(2009), No. 2, 952–973. https://doi.org/10.1016/j.na.2008.01.024; MR2468432; Zbl 1155.35373

[16] T. S. Hsu, Multiplicity of positive solutions for critical singular elliptic systems with concave-convex nonlinearities, *Adv. Nonlinear Stud.* **9**(2009), No. 2, 295–311. https://doi.org/10.1515/ans-2009-0205; MR2503831; Zbl 1181.35075

[17] T. S. Hsu, Multiple positive solutions for a critical quasilinear elliptic system with concave-convex nonlinearities, *Nonlinear Anal.* **71**(2009), No. 7-8, 2688–2698. https://doi.org/10.1016/j.na.2009.01.110; MR2532793; Zbl 1167.35356

[18] T. S. Hsu, Existence and multiplicity of positive solutions to a perturbed singular elliptic system deriving from a strongly coupled critical potential, *Bound. Value Probl.* **2012**(2012), No. 116, 1–14. https://doi.org/10.1186/1687-2770-2012-116; MR3016674; Zbl 1280.35041

[19] T. S. Hsu, H. L. Lin, Multiple positive solutions for a critical elliptic system with concave–convex nonlinearities, *Proc. Roy. Soc. Edinburgh Sect.* **139**(2009), No. 6, 1163–1177. https://doi.org/10.1017/S0308210508000875; MR2557317; Zbl 1183.35117

[20] D. S. Kang, On the quasilinear elliptic problems with critical Sobolev–Hardy exponents and Hardy terms, *Nonlinear Anal.* **68**(2008), No. 7, 1973–1985. https://doi.org/10.1016/j.na.2007.01.024; MR2388757; Zbl 1135.35042

[21] D. S. Kang, M. R. Liu, L. S. Xu, Critical elliptic systems involving multiple strongly-coupled Hardy-type terms, *Adv. Nonlinear Anal.* **9**(2020), No. 1, 866–881. https://doi.org/10.1515/anona-2020-0029; MR3990627; Zbl 1466.35134

[22] D. S. Kang, L. S. Xu, A critical surface for the solutions to singular elliptic systems, *J. Math. Anal. Appl.* **472**(2019), No. 2, 2017–2033. https://doi.org/10.1016/j.jmaa.2018.12.042; MR3906462; Zbl 1412.35100

[23] J. F. Liao, Y. Pu, C. L. Tang, Multiplicity of positive solutions for a class of concave–convex elliptic equations with critical growth, *Acta Math. Sci. Ser.* **38**(2018), No. 2, 497–518. https://doi.org/10.1016/S0252-9602(18)30763-X; MR3758287; Zbl 1399.35044

[24] J. F. Liao, X. Zhou, Existence of the second positive solution for a class of semilinear elliptic system with critical exponent (in Chinese), *J. China West Normal University (Natural Sciences)* **44**(2023), No. 2, 131–140. https://doi.org/10.16246/j.issn.1673-5072.2023.02.004

[25] H. L. Lin, Multiple positive solutions for semilinear elliptic systems, *J. Math. Anal. Appl.* **391**(2012), No. 1, 107–118. https://doi.org/10.1016/j.jmaa.2012.02.028; MR2899841; Zbl 1241.35080

[26] Z. X. Qi, Z. T. Zhang, Existence of multiple solutions to a class of nonlinear Schrödinger system with external sources terms, *J. Math. Anal. Appl.* **420**(2014), No. 2, 972–986. https://doi.org/10.1016/j.jmaa.2014.06.038; MR3240061; Zbl 1300.35133

[27] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, Vol. 24, Birkhäuser, Boston, 1977. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 0856.49001

[28] T. F. Wu, On semilinear elliptic equations involving concave–convex nonlinearities and sign-changing weight function, *J. Math. Anal. Appl.* **318**(2006), No. 1, 253–270. https://doi.org/10.1016/j.jmaa.2005.05.057; MR2210886; Zbl 1153.35036