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Complex dynamics of the system of nonlinear difference equations in the Hilbert space

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Abstract. In the given article the necessary and sufficient conditions of the existence of solutions of boundary value problem for the nonlinear system in the Hilbert spaces are obtained. Examples of such systems like a Lotka–Volterra are considered. Bifurcation and branching conditions of solutions are obtained.

Keywords: Lotka–Volterra models, population dynamics, Moore–Penrose pseudo-inverse operators, Fibonacci numbers.

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1 Introduction

The system of difference equations is the subject of numerous publications, and it is impossible to analyse all of them in detail. In this article we develop constructive methods of analysis of linear and weakly nonlinear boundary-value problems for difference equations, which occupy a central place in the qualitative theory of dynamical systems. We consider such problems that the operator of the linear part of the equation does not have an inverse. Such problems include the so called critical (or resonance) problems (when the considered problem can have non unique solution and not for any right-hand sides). We use the well-known technique of generalised inverse operators [4] and the notion of a strong generalised solution of an operator equation developed in [20]. In such way, one can prove the existence of solutions of different types for the system of operator equations in the Hilbert spaces. There exist three possible types of solutions: classical solutions, strong generalised solutions, and strong pseudo solutions [32]. For the analysis of a weakly nonlinear system, we develop the well-known Lyapunov–Schmidt method. This approach gives possibility to investigate a lot of problems in difference equations and mathematical biology from a single point of view.

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2 Statement of the problem

Consider the following boundary-value problem

$$x(n+1,\varepsilon) = a(n)x(n,\varepsilon) + b(n)y(n,\varepsilon) + \varepsilon Z_1(x(n,\varepsilon), y(n,\varepsilon), n, \varepsilon) + f_1(n); \tag{2.1}$$

$$y(n+1,\varepsilon) = c(n)x(n,\varepsilon) + d(n)y(n,\varepsilon) + \varepsilon Z_2(x(n,\varepsilon), y(n,\varepsilon), n, \varepsilon) + f_2(n); \tag{2.2}$$

$$l\begin{pmatrix} x(\cdot,\varepsilon) \\ y(\cdot,\varepsilon) \end{pmatrix} = \alpha, \tag{2.3}$$

where operators $\{a(n), b(n), c(n), d(n) \in \mathcal{L}(\mathcal{H}), n \in J \subset \mathbb{Z}\}$, $\mathcal{L}(\mathcal{H})$ is the space of linear and bounded operators which acts from $\mathcal{H}$ into itself, vector-functions $f_1(n), f_2(n) \in l_\infty(J, \mathcal{H})$,

$$l_\infty(J,\mathcal{H}) = \left\{ f : J \to \mathcal{H}, \|f\|_{l_\infty} = \sup_{n\in J} \|f(n)\|_{\mathcal{H}} < \infty \right\},$$

Z_1, Z_2 are smooth nonlinearities; a linear and bounded operator l translates solutions of (2.1), (2.2) into the Hilbert space $\mathcal{H}_1$, α is an element of the space $\mathcal{H}_1$, $\alpha \in \mathcal{H}_1$ (instead of $l_\infty(J,\mathcal{H})$ we can consider another functional space $\mathcal{T}(J, \mathcal{L}(\mathcal{H}))$).

We find solutions of the boundary-value problem (2.1)–(2.3) which for $\varepsilon = 0$ turns in one of solutions of generating boundary-value problem

$$x_0(n+1) = a(n)x_0(n) + b(n)y_0(n) + f_1(n); \tag{2.4}$$

$$y_0(n+1) = c(n)x_0(n) + d(n)y_0(n) + f_2(n); \tag{2.5}$$

$$l\begin{pmatrix} x_0(\cdot) \\ y_0(\cdot) \end{pmatrix} = \alpha. \tag{2.6}$$

3 Results

3.1 Linear case

Consider the following vector $z_0(n) = (x_0(n), y_0(n))$, sequence of operator matrices

$$A_n = \begin{pmatrix} a(n) & b(n) \\ c(n) & d(n) \end{pmatrix},$$

and sequence of vector–functions $f(n) = (f_1(n), f_2(n))$. Then we can rewrite the generating boundary-value problem (2.4)–(2.6) in the following form

$$z_0(n+1) = A_n z_0(n) + f(n), \tag{3.1}$$

$$l z_0(\cdot) = \alpha. \tag{3.2}$$

Define an operator $\Phi(m,n) = A_{m+1}A_m \ldots A_{n+1}, m > n$, $\Phi(m,m) = I$. The operator $U(m) = \Phi(m,0)$ is an evolution operator [6]. General solution $z_0(n)$ of (3.1) can be represented in the following form

$$z_0(n) = \Phi(n,0)z_0 + g(n), \tag{3.3}$$

where

$$g(n) = \sum_{i=0}^{n} \Phi(n,i) f(i).$$

Remark 3.1. It should be noted that if the sequence of operator matrices A_n each has bounded inverse $A_n^{-1} \in \mathcal{L}(\mathcal{H})$, then general solution of (3.1) can be represented in the following form

$$z_0(n) = U(n)z_0 + \sum_{i=0}^{n} U(n)U^{-1}(i)f(i).$$

Substituting representation (3.3) in the boundary condition (3.2) we obtain the following operator equation

$$Qz_0 = h, \tag{3.4}$$

where the operator Q and the element h have the following form

$$Q = l\Phi(\cdot, 0), \qquad Q : \mathcal{H} \to \mathcal{H}_1, \qquad h = \alpha - lg(\cdot).$$

According to the theory of generalised solutions which was represented in [2] and theory of Moore–Penrose pseudo invertible operators [4] for the equation (3.4) we have the following variants:

1) Suppose that $R(Q) = \overline{R(Q)}$ ($R(Q)$ is the image of the operator Q). In this case we have that the equation (3.4) is solvable if and only if the following condition is hold [4]:

$$P_Y h = 0, \qquad \mathcal{H}_1 = R(Q) \oplus Y. \tag{3.5}$$

Here P_Y is an orthoprojector onto subspace Y. Under condition (3.5) the set of solutions of (3.4) has the following form:

$$z_0 = Q^+ h + P_{N(Q)} c, \qquad \forall c \in \mathcal{H},$$

where Q^+ is Moore–Penrose pseudo inverse [4,24,29] to the operator Q, $P_{N(Q)}$ is orthoprojector onto the kernel of the operator Q.

2) Consider the case when $R(Q) \neq \overline{R(Q)}$. In this case there is strong Moore–Penrose pseudo inverse $\overline{Q}^+$ [2] to the operator Q ($\overline{Q} : \overline{\mathcal{H}} \to \mathcal{H}_1$ is extension of the operator Q onto extended space $\mathcal{H} \subset \overline{\mathcal{H}}$ [2]). Condition of generalised solvability has the following form:

$$P_Y h = 0, \qquad \mathcal{H}_1 = \overline{R(Q)} \oplus Y. \tag{3.6}$$

Condition (3.6) guarantees only that $h \in \overline{R(Q)}$. Under condition (3.6) the set of strong generalised solutions of the equation (3.4) has the following form:

$$z_0 = \overline{Q}^+ h + P_{N(\overline{Q})} c, \qquad \forall c \in \mathcal{H}. \tag{3.7}$$

If $h \in R(Q)$ then strong generalised solutions are classical.

3) Suppose that $R(Q) \neq \overline{R(Q)}$ and $h \notin \overline{R(Q)}$. It means that the following condition is hold

$$P_Y h \neq 0. \tag{3.8}$$

Under condition (3.8) the set of strong generalised quasisolutions [2,4] has the following form:

$$z_0 = \overline{Q}^+ h + P_{N(\overline{Q})} c, \qquad \forall c \in \mathcal{H}.$$

Using the notion presented above, we obtain the following theorem.

Theorem 3.2. *Boundary value problem (3.1), (3.2) is solvable.*

a1) There are strong generalised solutions of (3.1), (3.2) if and only if

$$P_Y\left\{\alpha - l\sum_{i=0}^{\cdot}\Phi(\cdot,i)f(i)\right\} = 0, \tag{3.9}$$

if the element $(\alpha - l\sum_{i=0}^{\cdot}\Phi(\cdot,i)f(i)) \in R(Q)$ then solutions are classical;

b1) under condition (3.9) the set of generalised solutions of the boundary-value problem (3.1), (3.2) has the following form

$$z_0(n,c) = \overline{G[f,\alpha]}(n) + P_{N(\overline{Q})}c, \qquad \forall c \in \mathcal{H},$$

where the generalised Green operator has the form

$$\overline{G[f,\alpha]}(n) = \Phi(n,0)\overline{Q}^{+}\left\{\alpha - l\sum_{i=0}^{\cdot}\Phi(\cdot,i)f(i)\right\};$$

a2) There are strong quasisolutions of (3.1), (3.2) if and only if the following condition is hold

$$P_Y\left\{\alpha - l\sum_{i=0}^{\cdot}\Phi(\cdot,i)f(i)\right\} \neq 0; \tag{3.10}$$

b2) Under condition (3.10) the set of strong quasisolutions of the boundary-value problem (3.1), (3.2) has the following form

$$z_0(n,c) = \overline{G[f,\alpha]}(n) + P_{N(\overline{Q})}c, \qquad \forall c \in \mathcal{H}.$$

3.2 Nonlinear case

Consider the nonlinear boundary-value problem (2.1)–(2.3). Using the introduced notations we can rewrite this problem in the following form

$$z(n+1,\varepsilon) = A_n z(n,\varepsilon) + \varepsilon Z(z(n,\varepsilon),n,\varepsilon), \tag{3.11}$$

$$lz(\cdot,\varepsilon) = \alpha. \tag{3.12}$$

Theorem 3.3 (Necessary condition)**.** *Suppose that the boundary value problem (3.11), (3.12) has solution $z(n,\varepsilon)$ which for $\varepsilon = 0$ turns in one of solutions $z_0(n,c)$ with element $c \in \mathcal{H}$ ($z(n,0) = z_0(n,c)$)). Then c satisfies the following operator equation for generating elements*

$$F(c) = P_Y l\sum_{i=0}^{\cdot}\Phi(\cdot,i)Z(z_0(i,c),i,0) \tag{3.13}$$

$$= P_Y l\sum_{i=0}^{\cdot}\Phi(\cdot,i)Z(\overline{G[f,\alpha]}(i) + P_{N(\overline{Q})}c,\cdot,0) = 0. \tag{3.14}$$

Proof. According to Theorem 3.3, the boundary value problem (3.11), (3.12) has solution if and only if the following condition is true:

$$P_Y\left\{\alpha - l\sum_{i=0}^{\cdot}\Phi(\cdot,i)(f(i) + \varepsilon Z(z(i,\varepsilon),i,\varepsilon))\right\} = 0. \tag{3.15}$$

From the condition (3.15) follows condition (3.13). □

Remark 3.4. It should be noted that theorem 3.2 is hold when the nonlinearities Z_1, Z_2 are continuous in the neighborhood of generating solution $z_0(n, c_0)$.

Now, we propose the following change of variables:

$$z(n,\varepsilon) = z_0(n, c_0) + u(n,\varepsilon),$$

where the element c_0 satisfies the operator equation (3.13). Then we can rewrite the boundary value problem (3.11), (3.12) in the following form

$$u(n+1,\varepsilon) = A_n u(n,\varepsilon) + \varepsilon\{Z(z_0(n,c^0),n,0) + Z'_u(z_0(n,c^0),n,0)u(n,\varepsilon) + \mathcal{R}(u(n,\varepsilon),n,\varepsilon)\}, \tag{3.16}$$

$$lu(\cdot,\varepsilon) = 0. \tag{3.17}$$

Here Z'_u is the Fréchet derivative,

$$\mathcal{R}(0,0,0) = \mathcal{R}'_u(0,0,0) = 0.$$

Boundary value problem (3.16), (3.17) has solutions if and only if the following condition is true:

$$P_{Y}l\sum_{i=0}^{\cdot}\Phi(\cdot,i)(Z(z_0(i,c^0),i,0) + Z'_u(z_0(i,c^0),i,0)u(i,\varepsilon) + \mathcal{R}(u(i,\varepsilon),i,\varepsilon)) = 0. \tag{3.18}$$

Under this condition the set of solutions of boundary value problem (3.16), (3.17) has the following form

$$u(n,\varepsilon) = P_{N(\overline{Q})}c + \overline{u}(n,\varepsilon), \tag{3.19}$$

where

$$\overline{u}(n,\varepsilon) = \varepsilon\overline{G[Z(z_0(\cdot,c^0),\cdot,0) + Z'_u(z_0(\cdot,c^0),\cdot,0)u(\cdot,\varepsilon) + \mathcal{R}(u(\cdot,\varepsilon),\cdot,\varepsilon),0]}(n). \tag{3.20}$$

Substituting (3.19) in (3.18) we obtain the following operator equation

$$B_0 c = r, \tag{3.21}$$

where the operator

$$B_0 = -P_{Y}l\sum_{i=0}^{\cdot}\Phi(\cdot,i)Z'_u(z_0(i,c^0),i,0)P_{N(\overline{Q})},$$

$$r = P_{Y}l\sum_{i=0}^{\cdot}\Phi(\cdot,i)(Z'_u(z_0(i,c^0),i,0)\overline{u}(i,\varepsilon) + \mathcal{R}(u(i,\varepsilon),i,\varepsilon)).$$

Condition $P_{N(B_0^*)}P_Y = 0$ guarantees that equation (3.21) is solvable and has at least one generalized solution in the following form $c = \overline{B}_0^+ r$. For a small enough ε considered operator system (3.19)–(3.21) has a contracting operator in the right-hand side and using contraction mapping principle [2] we have the following assertion.

Theorem 3.5 (Sufficient condition)**.** *Suppose that the following condition is true: $P_{N(\overline{B}_0^*)}P_Y = 0$. ($P_{N(\overline{B}_0^*)}$ is an orthoprojector onto the kernel of adjoint to the operator B_0). Then the boundary value problem* (3.11), (3.12) *has generalised solutions which can be found with using of iterative processes:*

$$u_{k+1}(n,\varepsilon) = P_{N(\overline{Q})}c_k + \overline{u}_k(n,\varepsilon),$$

$$c_{k+1} = \overline{B}_0^+ P_Y l \sum_{i=0}^{\cdot} \Phi(\cdot, i)(Z'_u(z_0(i, c^0), i, 0)\overline{u}_k(i, \varepsilon) + \mathcal{R}(u_k(i, \varepsilon), i, \varepsilon)),$$

$$\overline{u}_{k+1}(n, \varepsilon) = \varepsilon \overline{G[Z(z_0(\cdot, c^0), \cdot, 0) + Z'_u(z_0(\cdot, c^0), \cdot, 0)\overline{u}_k(\cdot, \varepsilon) + \mathcal{R}(u_k(\cdot, \varepsilon), \cdot, \varepsilon), 0]}(n),$$

where

$$\mathcal{R}(u_k(n, \varepsilon), n, \varepsilon) = Z(z_0(n, c^0) + u_k(n, \varepsilon), n, \varepsilon) - Z(z_0(n, c^0), n, 0) - Z'_u(z_0(n, c^0), n, 0)u_k(n, \varepsilon),$$

$$u_0 = c_0 = \overline{y}_0 = 0.$$

4 Applications

It is well-known that systems like a Lotka–Volterra [34, 35] plays an important role in the dynamics of population [26, 27] (mathematical biology). There exist many papers which are dedicated to investigation of such problems in continuous and discrete cases (see for example the recent works [1,5,7,9–19,21–23,25,28,30,31,33]). As a rule such problems are regular. We consider some examples of systems with different type of boundary conditions in the critical case. We show that the operator which generates considering problem can be Fredholm. We find bifurcation conditions of solutions with using of the equation for generating constants [3]. It should be noted that the proposed method also works in the case of boundary-value problems with fractional derivative [8].

4.1 Examples

4.1.1 Example 1

Consider the following periodic boundary-value problem in the finite dimensional case:

$$x_i(n+1, \varepsilon) = a_i(n)x_i(n, \varepsilon) + b_i(n)y_i(n, \varepsilon) + \varepsilon g_i^1(n)x_i(n, \varepsilon)\left(1 - \sum_{j=1}^{t} a_{ij}(n)y_j(n, \varepsilon)\right) + f_1^i(n), \tag{4.1}$$

$$y_i(n+1, \varepsilon) = c_i(n)x_i(n, \varepsilon) + d_i(n)y_i(n, \varepsilon) + \varepsilon g_i^2(n)y_i(n, \varepsilon)\left(1 - \sum_{j=1}^{t} b_{ij}(n)x_j(n, \varepsilon)\right) + f_2^i(n), \tag{4.2}$$

$$x_i(0, \varepsilon) = x_i(m, \varepsilon), \tag{4.3}$$

$$y_i(0, \varepsilon) = y_i(m, \varepsilon), \qquad i = \overline{1, p}. \tag{4.4}$$

Here $x_i(n, \varepsilon), y_i(n, \varepsilon), a_i(n), b_i(n), c_i(n), d_i(n), g_i^1(n), g_i^2(n), a_{ij}(n), b_{ij}(n) \in \mathbb{R},\ i = \overline{1, p},\ j = \overline{1, t}$.

For $\varepsilon = 0$ we obtain the following generating boundary-value problem

$$x_i^0(n+1) = a_i(n)x_i^0(n) + b_i(n)y_i^0(n) + f_1^i(n), \tag{4.5}$$

$$y_i^0(n+1) = c_i(n)x_i^0(n) + d_i(n)y_i^0(n) + f_2^i(n), \tag{4.6}$$

$$x_i^0(0) = x_i^0(m), \tag{4.7}$$

$$y_i^0(0) = y_i^0(m). \tag{4.8}$$

$$l\begin{pmatrix} x_0(\cdot) \\ y_0(\cdot) \end{pmatrix} = \begin{pmatrix} x_i^0(m) - x_i^0(0) \\ y_i^0(m) - y_i^0(0) \end{pmatrix}_{i=\overline{1,p}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For the vector $z_i^0(n) = (x_i^0(n), y_i^0(n))$ we can write the following assertion.

Corollary 4.1. *The boundary value problem (4.5)–(4.8) has periodic solutions if and only if*

$$P_{Y_d} \sum_{k=0}^{m} \Phi(m,k) f(k) = 0, \tag{4.9}$$

where $Q = \Phi(m,0) - I$, d is a number of linearly independent columns of Q; under condition (4.9) the set of solutions has the form

$$z_i^0(n, c_r) = (G[f,0])(n) + P_{Q_r} c_r, \quad c_r \in \mathbb{R}^r, \tag{4.10}$$

where the generalised Green's operator $(G[f,0])(n)$ has the following form

$$(G[f,0])(n) = -\Phi(n,0) Q^+ \sum_{k=0}^{m} \Phi(m,k) f(k),$$

r is a number of linearly independent rows of Q (P_{Q_r} is an orthoprojector onto the kernel of matrix Q).

Remark 4.2. It should be noted that in the considered above case index of an operator $\mathcal{S}$ can be calculated in the following way

$$\operatorname{ind} \mathcal{S} = r - d,$$

where the operator $\mathcal{S}$ with boundary conditions has the following form

$$\mathcal{S} \begin{pmatrix} x_i^0(n) \\ y_i^0(n) \end{pmatrix} := \begin{pmatrix} x_i^0(n+1) - a_i(n) x_i^0(n) - b_i(n) y_i^0(n) \\ y_i^0(n+1) - c_i(n) x_i^0(n) - d_i(n) y_i^0(n) \end{pmatrix}.$$

It means that the operator $\mathcal{S}$ is Fredholm [4].

For the nonlinear boundary value problem (4.1)–(4.4) we obtain the following assertions.

Corollary 4.3 (Necessary condition)**.** *If the boundary value problem (4.1)–(4.4) has solution, then the element $c_r = c_r^0$ satisfies the following equation for generating constants:*

$$F(c_r) = P_{Y_d} \sum_{i=0}^{m} \Phi(m,i) Z(z_0(i, c_r), i, 0) = 0,$$

where

$$Z(z_0(n, c_r), n, 0) = \begin{pmatrix} g_i^1(n) x_i^0(n, c_r)(1 - \sum_{j=1}^{t} a_{ij}(n) y_j^0(n, c_r)) \\ g_i^2(n) y_i^0(n, c_r)(1 - \sum_{j=1}^{t} b_{ij}(n) x_j^0(n, c_r)) \end{pmatrix}.$$

Corollary 4.4 (Sufficient condition)**.** *Suppose that the following condition is true:*

$$P_{N(B_0^*)} P_{Q_d^*} = 0.$$

Then the boundary value problem (4.1)–(4.4) has generalized solutions which can be found using of iterative processes:

$$u_{k+1}(n, \varepsilon) = P_{N(Q)_r} c_k + \overline{u}_k(n, \varepsilon),$$

$$c_{k+1} = B_0^+ P_{N(Q^*)} \sum_{i=0}^{m} \Phi(m,i) (Z_u'(z_0(i, c^0), i, 0) \overline{u}_k(i, \varepsilon) + \mathcal{R}(u_k(i, \varepsilon), i, \varepsilon)),$$

$$\overline{u}_{k+1}(n, \varepsilon) = \varepsilon G[Z(z_0(\cdot, c^0), \cdot, 0) + Z_u'(z_0(\cdot, c^0), \cdot, 0) \overline{u}_k(\cdot, \varepsilon) + \mathcal{R}(u_k(\cdot, \varepsilon), \cdot, \varepsilon), 0](n),$$

where

$$\mathcal{R}(u_k(n,\varepsilon),n,\varepsilon) = Z(z_0(n,c^0)+u_k(n,\varepsilon),n,\varepsilon) - Z(z_0(n,c^0),n,0) - Z'_u(z_0(n,c^0),n,0)u_k(n,\varepsilon),$$

$$u_0 = c_0 = \overline{y}_0 = 0,$$

$$Z'_u(z_0(n,c^0),n,0)u_k(n,\varepsilon) = \begin{pmatrix} g_i^1(n)x_i^0(n,c_r^0)(1-\sum_{j=1}^t a_{ij}(n)u_{jk}^2(n)) + g_i^1(n)u_{ik}^1(n)(1-\sum_{j=1}^t a_{ij}(n)y_j^0(n,c_r^0)) \\ g_i^2(n)y_i^0(n,c_r^0)(1-\sum_{j=1}^t b_{ij}(n)u_{jk}^1(n)) + g_i^2(n)u_{ik}^2(n)(1-\sum_{j=1}^t b_{ij}(n)x_j^0(n,c_r^0)) \end{pmatrix}.$$

4.1.2 Example 2

Suppose that $a_i(n) = b_i(n) = c_i(n) = g_i^1(n) = g_i^2(n) = a_{ij}(n) = b_{ij}(n) = 1, d_i(n) = 0$. In this case

$$A_n = A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad n \in \mathbb{N}.$$

Then for the linear boundary value problem (4.5)–(4.8) we obtain that the evolution operator $\Phi(m,n)$ has the following form

$$\Phi(m,n) = A^{m-n+1} = \begin{pmatrix} F_{m-n+2} & F_{m-n+1} \\ F_{m-n+1} & F_{m-n} \end{pmatrix}.$$

Here $F_0 = 1, F_1 = 1$, $F_{n+2} = F_n + F_{n+1}, n \geq 0$ are Fibonacci numbers. In this case the matrix Q is nondegenerate ($Q^+ = Q^{-1}$, $P_{N(Q)} = I, P_Y = I$, I is an identity matrix) and we obtain the following corollary.

Corollary 4.5. *The boundary value problem* (4.1)–(4.4) *has periodic solution if and only if*

$$\sum_{k=0}^m A^{m-k+1}f(k) = \sum_{k=0}^m \begin{pmatrix} F_{m-k+2} & F_{m-k+1} \\ F_{m-k+1} & F_{m-k} \end{pmatrix} \begin{pmatrix} f_1^i(k) \\ f_2^i(k) \end{pmatrix} = 0; \tag{4.11}$$

under condition (4.11) *the solution of the boundary value problem* (4.1)–(4.4) *has the form*

$$\begin{aligned} z_i^0(n) &= (G[f,0])(n) = -A^{n+1}Q^{-1}\sum_{k=0}^m A^{m-k+1}f(k) \\ &= -\frac{1}{\Delta(m)}\sum_{k=0}^m \begin{pmatrix} a_{11}(n,m,k)f_1^i(k) + a_{12}(n,m,k)f_2^i(k) \\ a_{21}(n,m,k)f_1^i(k) + a_{22}(n,m,k)f_2^i(k) \end{pmatrix}, \end{aligned}$$

where

$$\Delta(m) = (F_{m+2}-1)(F_m-1) - F_{m+1}^2;$$

$$\begin{aligned} a_{11}(n,m,k) &= F_{n+2}(F_mF_{m-k+2} - F_{m+1}F_{m-k+1}) - (F_{n+2}F_{m-k+2} + F_{n+1}F_{m-k+1}) \\ &\quad + F_{n+1}(F_{m+2}F_{m-k+1} - F_{m+1}F_{m-k+2}); \end{aligned}$$

$$\begin{aligned} a_{12}(n,m,k) &= F_{n+2}(F_mF_{m-k+1} - F_{m+1}F_{m-k}) - (F_{n+2}F_{m-k+1} + F_{n+1}F_{m-k}) \\ &\quad + F_{n+1}(F_{m+2}F_{m-k} - F_{m+1}F_{m-k+1}); \end{aligned}$$

$$\begin{aligned} a_{21}(n,m,k) &= F_{n+1}(F_mF_{m-k+2} - F_{m+1}F_{m-k+1}) - (F_{n+1}F_{m-k+2} + F_{n+1}F_{m-k+1}) \\ &\quad + F_n(F_{m+2}F_{m-k+1} - F_{m+1}F_{m-k+2}); \end{aligned}$$

$$\begin{aligned} a_{22}(n,m,k) &= F_{n+1}(F_mF_{m-k+1} - F_{m+1}F_{m-k}) - (F_{n+2}F_{m-k+1} + F_{n+1}F_{m-k}) \\ &\quad + F_{n+1}(F_{m+2}F_{m-k} - F_{m+1}F_{m-k+1}). \end{aligned}$$

In this case the necessary condition of solvability for the nonlinear boundary-value problem (4.1)–(4.4) has the form

$$\begin{pmatrix} \sum_{i=0}^{m} F_{m-i+2}x_i^0(n)(1-\sum_{j=1}^{n} y_j^0(n)) + F_{m-i+1}y_i^0(n)(1-\sum_{j=1}^{n} x_j^0(n)) \\ \sum_{i=0}^{m} F_{m-i+1}x_i^0(n)(1-\sum_{j=1}^{n} y_j^0(n)) + F_{m-i}y_i^0(n)(1-\sum_{j=1}^{n} x_j^0(n)) \end{pmatrix} = 0.$$

Fréchet derivate Z'_u has the following form

$$Z'_u(z_0(n), n, 0)u_k(n, \varepsilon) = \begin{pmatrix} x_i^0(n)(1-\sum_{j=1}^{n} u_{jk}^2(n,\varepsilon)) + u_{ik}^1(n,\varepsilon)(1-\sum_{j=1}^{n} y_j^0(n)) \\ y_i^0(n)(1-\sum_{j=1}^{n} u_{jk}^1(n,\varepsilon)) + u_{ik}^2(n,\varepsilon)(1-\sum_{j=1}^{n} x_j^0(n)) \end{pmatrix}.$$

4.1.3 Example 3

Consider the following boundary value problem

$$\begin{aligned} x_i(n+1,\varepsilon) &= a_i(n)x_i(n,\varepsilon) + b_i(n)y_i(n,\varepsilon) \\ &\quad + \varepsilon g_i^1(n)x_i(n,\varepsilon)\left(1-\sum_{j=1}^{n} a_{ij}(n)y_j(n,\varepsilon)\right) + f_1^i(n), \qquad (4.12) \\ y_i(n+1,\varepsilon) &= c_i(n)x_i(n,\varepsilon) + d_i(n)y_i(n,\varepsilon) \\ &\quad + \varepsilon g_i^2(n)y_i(n,\varepsilon)\left(1-\sum_{j=1}^{n} b_{ij}(n)x_j(n,\varepsilon)\right) + f_2^i(n), \qquad (4.13) \end{aligned}$$

with the following boundary conditions

$$l\begin{pmatrix} x_i(\cdot,\varepsilon) \\ y_i(\cdot,\varepsilon) \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{p_1} x_i(n_k,\varepsilon) \\ \sum_{l=0}^{p_2} y_i(n_l,\varepsilon) \end{pmatrix}_{i=\overline{1,p}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}. \qquad (4.14)$$

Here $n_k, k = \overline{0, p_1}$, $n_l, l = \overline{0, p_2}$ are finite sequences of integer numbers. In this case we obtain the multi-point boundary-value problem.

4.1.4 Example 4

Suppose that $x_i(n), y_i(n) \geq 0$ and boundary condition has the following form

$$l\begin{pmatrix} x_i(\cdot,\varepsilon) \\ y_i(\cdot,\varepsilon) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{p} x_i(0,\varepsilon) \\ \sum_{i=0}^{p} y_i(0,\varepsilon) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \qquad (4.15)$$

Such condition has practical meaning. It means the population distribution at the initial time (the proportion of the population in species).

5 Conclusion

Proposed in the given article approach gives possibility to investigate a lot of biological problems.

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