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Existence and regularity of solutions for a singular anisotropic (p,q)-Laplacian with variable exponent

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Abstract. In this paper, we investigate the existence and regularity of positive solutions for certain singular problems that involve an anisotropic (p,q)-Laplacian-type operator and a singular term with a variable exponent, under zero Dirichlet boundary conditions on $\partial\Omega$. The main equation we analyze is

$$-\sum_{i=1}^{N}\partial_i\left(|\partial_i u(x)|^{p_i-2}\partial_i u(x)\right)-\sum_{i=1}^{N}\partial_i\left(|\partial_i u(x)|^{q_i-2}\partial_i u(x)\right)=\frac{f(x)}{u(x)^{\gamma(x)}}\quad\text{in }\Omega,$$

where Ω is a bounded, regular domain in $\mathbb{R}^N$, f is a positive function belonging to a specific Lebesgue space, and $\gamma(x)$ is a positive continuous function on $\overline{\Omega}$. In our study, we do not make comparisons between p_i and q_i, and as a result, we show that the solution belongs to either $W_0^{1,\vec{p}}(\Omega)\cap W_0^{1,\vec{q}}(\Omega)$ or $W_{loc}^{1,\vec{p}}(\Omega)\cap W_{loc}^{1,\vec{q}}(\Omega)$ depending on the summability of $f(x)$ and the values of $\gamma(x)$. The results are achieved using approximation techniques that include truncation, comparison, and variational methods.

Keywords: anisotropic singular (p,q)-equations, approximation, variational methods, existence and regularity results.

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1 Introduction and main results

This study investigates elliptic partial differential equations (PDEs) characterized by anisotropic (p,q)-Laplacian-type operators, focusing on their mathematical properties and physical relevance. Anisotropic (p,q)-Laplacians extend the classical Laplacian by incorporating directional dependence, making them instrumental in modeling complex phenomena in heterogeneous materials, such as heat conduction, fluid dynamics, and nonlinear elasticity. The research also highlights the importance of reaction-diffusion systems in plasma physics and

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chemical reaction design, which are driven by parabolic reaction-diffusion equations.

$$u_t - \operatorname{div}\left[|\nabla u|^{p-2}\nabla u\right] - \operatorname{div}\left[|\nabla u|^{q-2}\nabla u\right] = \mathfrak{G}(x,u).$$

These systems are fundamental for understanding pattern formation, wave propagation, and turbulence control in various scientific and engineering applications. For more technical details, we point out the following refs. [4,7,21,27,47]. By examining the theoretical framework surrounding anisotropic elliptic (PDEs) and referencing foundational works, this study underscores the critical role of these mathematical constructs in solving real-world problems in fields such as material science, image processing, and biological modeling. Can also refer to Refs. [32,34,46] and the references therein.

In this paper, we focus on the study of the following anisotropic singular problem:

$$\begin{cases} -\sum_{i=1}^{N}\partial_i\left(|\partial_i u|^{p_i-2}\partial_i u\right) - \sum_{i=1}^{N}\partial_i\left(|\partial_i u|^{q_i-2}\partial_i u\right) = \dfrac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2 boundary $\partial\Omega$ and $N \geq 3$. The exponent $\gamma(x)$ is a positive continuous function on $\overline{\Omega}$, and $f(x) \in L^r(\Omega)$ for some $r \geq 1$.

This study is inspired by the significant work of Giacomoni, Kumar, and Sreenadh [23], who investigate a singular quasilinear elliptic equation involving the (p,q)-Laplacian operator. In their work, the equation under consideration is

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) - \operatorname{div}(|\nabla u|^{q-2}\nabla u) = \frac{f(x)}{u^\delta} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < q < p < \infty$, $\delta > 0$, and $f(x)$ is a nonnegative function that exhibits singular behavior near the boundary of Ω. Their paper proves the existence of weak solutions in appropriate Sobolev spaces and establishes the solutions' boundary behavior for different values of δ. Specifically, they derive optimal Sobolev regularity, prove uniqueness under certain conditions, and provide non-existence results in other regimes. Moreover, they obtain Hölder regularity estimates for the gradient of weak solutions, offering significant insights into equations with singular nonlinearities. For further details, the reader is referred to [12,14] and related references. Arruda and Nascimento [1] generalize the results of Giacomoni, Kumar, and Sreenadh by considering a broader class of quasilinear operators beyond the standard p-Laplacian. They study a class of nonlinear, nonhomogeneous singular elliptic equations involving the (p,q)-Laplacian operator given by

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = \frac{h(x)}{u^\delta} + f(x,u) & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ (with $N \geq 3$), $2 \leq p < N$, and $\delta > 0$. By using the sub–supersolution method combined with variational techniques, they establish the existence of at least one weak solution and, under additional assumptions on f, prove the existence of two distinct solutions. Key contributions include addressing nonhomogeneous operators without truncation techniques and accommodating general singular terms for all $\gamma > 0$. These results apply to a broad class of quasilinear operators, including the p-Laplacian (when $a(t) \equiv 1$) and the (p,q)-Laplacian (when $a(t) = 1 + t^{q-p}/p$). Interested readers are referred to [37] and [45] for additional details.

Our work extends the (p,q)-Laplacian operators studied in (1.2) and (1.3) by introducing two anisotropic operators with distinct vector exponents $\{p_i\}_{i=1}^N$ and $\{q_i\}_{i=1}^N$. This generalization allows for direction-dependent nonlinearities, unlike the isotropic (p,q)-Laplacian. Additionally, the right-hand side of our problem (1.1) features a variable exponent $\gamma(x)$ in the singular term $\frac{f(x)}{u^{\gamma(x)}}$, which introduces spatial heterogeneity in the singularity strength. Moreover, we establish L^∞-regularity for solutions to (1.1), proving that $u \in L^\infty(\Omega)$ despite the anisotropic operators and the singular term.

The singular (q,p)-Laplacian denotes a class of singular differential equations that involve both the p-Laplacian and the q-Laplacian operators. These problems frequently arise in the study of various physical and mathematical phenomena, serving as models for steady-state solutions of reaction-diffusion problems encountered in biophysics, plasma physics, and chemical reaction studies (see, for example, [5,7]). In the context of (p,q)-Laplacian-driven reaction-diffusion equations, the (p,q)-Laplacian term represents the diffusion component of the equation, which is given by:

$$|\partial_i u|^{p-2}|\partial_i u| + |\partial_i u|^{q-2}|\partial_i u| \quad \text{in } \Omega,$$

where $\partial_i u$ denotes the partial derivative of u with respect to x_i, $2 \leq q < p$. The terms $|\partial_i u|^{p-2}|\partial_i u|$ and $|\partial_i u|^{q-2}|\partial_i u|$ represent the diffusion terms associated with the (p,q)-Laplacian operator.

Researchers have explored various aspects of the (p,q)-Laplacian, including the existence and uniqueness of solutions, regularity properties, and the behavior of solutions under different conditions. Recent results in this area can be found in [2,12,31,35,36,38]. Our research builds on these findings by focusing on the properties of solutions in anisotropic singular problems.

The study of elliptic equations with singular nonlinearities has garnered significant attention from researchers. A pioneering contribution was made in 1976 by Crandall, Rabinowitz, and Tartar [11], where the authors explored nonlinear elliptic boundary value problems characterized by singular terms. Their work established the existence of both classical and generalized solutions under various conditions for problems associated with $-\Delta$ and Dirichlet boundary conditions. Specifically, when $p_i = q_i = 2$, the problem (1.1) reduces to the well-known semilinear elliptic equation that has been extensively studied:

$$\begin{cases} -\operatorname{div}(\nabla u) = \frac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

with f is a non-negative function in a suitable Lebesgue space. In 1991, Lazer and McKenna published an important paper [28], focusing on the case where f is continuous and $\gamma(x) = \gamma$. They proved existence and regularity results concerning the behavior of solutions near the boundary. Boccardo and Orsina later extended this work in [3], where they used approximation methods by truncating the singular term to prove existence, regularity, and nonexistence results for problems modeled by (1.4) with $\gamma(x) = \gamma$. These results were further generalized by Chu, Y. Gao, and W. Gao [8] as well as by Carmona and Martínez-Aparicio [6], who studied the case where $\gamma(x)$ varies as a function of x. Their research examined how the summability of f and the values of $\gamma(x)$ affect the existence and regularity of solutions to problem (1.4).

For general p, the following p-Laplacian problem has been widely studied by numerous researchers:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where f is a positive function and $p > 1$. For example, in the paper [9], Chu and Gao applied the approximation method to analyze this problem involving the p-Laplacian operator with a constant exponent $\gamma(x) = \gamma$. Subsequently, Chu, Gao, and Sun extended these results in [10], where considering cases $\gamma(x)$ varies as a function of x. Their research explored various scenarios based on the behavior of $\gamma(x)$ and f, demonstrating that solutions to problem (1.5) exist in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

There has been increasing interest in singular problems involving anisotropic operators, which correspond to a specific case of our problem (1.1) when $p_i = q_i$. For instance, in [30], the following problem was studied:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = \frac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\gamma(x) = \gamma$ is a positive constant and f belongs to an appropriate Lebesgue space. Using perturbation techniques, Leggat and Miri demonstrated the existence of positive solutions for problem (1.6). Additionally, Miri extended this analysis in [33], considering $\gamma(x) > 0$ as a variable function and f as a positive function in $L^m(\Omega)$. They established the existence and regularity of solutions in $W_0^{1,\vec{p}}(\Omega)$ when $\gamma(x) \leq 1$ and $m = \bar{p}^{*\prime}$, and in $W_{loc}^{1,\vec{p}}(\Omega)$ when $\|\gamma\|_{L^\infty(\Omega)} \leq \gamma^*$, with $\gamma^* > 1$.

It is important to note that the results in [30] and [33] are largely based on the strong maximum principle. Additional results on singular anisotropic problems, particularly using the sub-supersolution method, can be found in works like [15] and [16].

In our previous work [25], we extended existing results on similar problems (1.6) by incorporating variable exponent singularities and critical growth within an anisotropic framework. The problem is formulated as follows:

$$\begin{cases} -\sum_{i=1}^{N} M\left(\int_\Omega |\partial_i u|^{p_i}\, dx \right) \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = \frac{f_1(x)}{u^{\beta(x)}} + \lambda\, f_2(x)\, u^{\bar{p}^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω is a bounded regular domain in $\mathbb{R}^N$ with $N > \bar{p}$. The critical Sobolev exponent $\bar{p}^*$ is given by $\bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}$, where $\bar{p} = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i}}$. The parameter $\lambda > 0$ is a positive constant, while the exponent function satisfies $0 < \beta(x) < 1$. The functions f_1 and f_2 have specific properties, and M represents the Kirchhoff coefficient.

Our main result establishes the existence of at least two weak solutions with opposite energy signs. The methodology is based on the fibering method using the Nehari manifold.

While our variational framework shares similarities with previous works, this study extends the analysis by considering a double anisotropic operator, singular reaction terms and regularity analysis. To handle these challenges, we employ techniques such as truncation methods and weighted Sobolev embeddings. In addition to establishing existence, we discuss various cases of the function $\gamma(x)$ to determine the corresponding solution in each scenario. Under further assumptions on the function f, it is proven that the obtained solutions belong to $L^\infty(\Omega)$, ensuring their boundedness and physical relevance. The incorporation of a double anisotropic operator enriches the analysis, providing deeper insights into the behavior of solutions in complex settings.

The techniques developed in this work not only generalize classical results but also pave the way for future research in problems involving variable exponent spaces, nonlocal operators, and anisotropic phenomena. The rigorous approach presented here contributes to the broader understanding of non-standard growth conditions in partial differential equations.

There is a substantial amount of literature and increasing interest in anisotropic (p,q)-Laplacian problems. Noteworthy recent contributions can be found in works such as [29, 39, 40, 43]. For example, in [39], Razani and Figueiredo studied the following problem:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{q_i-2} \partial_i u \right) = \mathfrak{H}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\mathfrak{H}(x,u) = \lambda u^{\gamma-1}$ with $\gamma > 1$. By applying a sub-supersolution approach along with minimization techniques in convex sets, the authors proved the existence of positive solutions for problem (1.7). A similar approach was used in [43], where Tavares considered problem (1.7) with $\mathfrak{H}(x,u) = k(x)u^{\alpha-1} + f(x,u)$, where $\alpha > 1$. Under a broad set of conditions, the existence and multiplicity of solutions were demonstrated. Additionally, Leggat and Miri [29], used the variational method to prove multiplicity results for the case where $\mathfrak{H}(x,u) = \lambda f(u)$.

We now introduce the key notations and assumptions relevant to problem (1.1). Specifically, we define the following parameters

$$\vec{p} = (p_1, \ldots, p_N), \qquad \bar{p} := \frac{N}{\sum_{i=1}^{N} 1/p_i}, \qquad \bar{p}^* := \frac{N\bar{p}}{N-\bar{p}}, \qquad p_\infty := \max\{\bar{p}^*, p_N\},$$

$$\vec{q} = (q_1, \ldots, q_N), \qquad \bar{q} := \frac{N}{\sum_{i=1}^{N} 1/q_i}, \qquad \bar{q}^* := \frac{N\bar{q}}{N-\bar{q}}, \qquad q_\infty := \max\{\bar{q}^*, q_N\}.$$

Throughout this paper, we impose the following assumptions

$$2 \le p_1 \le p_2 \le \cdots \le p_N < \bar{p}^*, \quad \text{and} \quad \sum_{i=1}^{N} \frac{1}{p_i} > 1,$$

$$2 \le q_1 \le q_2 \le \cdots \le q_N < \bar{q}^*, \quad \text{and} \quad \sum_{i=1}^{N} \frac{1}{q_i} > 1,$$

$$\gamma^+ = \sup_{x \in \overline{\Omega}} \gamma(x), \quad \text{and} \quad \gamma^- = \inf_{x \in \overline{\Omega}} \gamma(x).$$

We denote by X the anisotropic Sobolev space associated with problem (1.1), defined as

$$X := W_0^{1,\vec{p}}(\Omega) \cap W_0^{1,\vec{q}}(\Omega),$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded and regular domain with $N \geq 3$. Furthermore, the space X is given by the closure

$$X = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_X},$$

where the norm $\|\cdot\|_X$ is defined as

$$\|u\|_X := \|u\|_{W_0^{1,\vec{p}}(\Omega)} + \|u\|_{W_0^{1,\vec{q}}(\Omega)} = \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)} + \sum_{i=1}^{N} \|\partial_i u\|_{L^{q_i}(\Omega)}.$$

Moreover, $(X, \|\cdot\|_X)$ is a uniformly convex Banach space, and consequently, it is reflexive. For further details, see [19].

Additionally, we assume that f is a nontrivial measurable function satisfying the condition

$$(H_f) \qquad \operatorname*{ess\,inf}_{\Omega} f(x) > 0.$$

The primary objective of this article is to investigate the existence and regularity of positive weak solutions for the singular problem (1.1) in the space X by considering various cases of the function $\gamma(x)$. Our strategy is to establish the existence of critical points for the energy functional

$$\Psi(u) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i}\, dx + \sum_{i=1}^{N} \frac{1}{q_i} \int_\Omega |\partial_i u|^{q_i}\, dx - \int_\Omega f|u|^{1-\gamma(x)}\, dx.$$

The singular term $u^{-\gamma(x)}$ introduces significant challenges by causing a loss of Gateaux differentiability, despite Ψ being weakly lower semi-continuous (in fact, continuous). With this framework in place, we are now ready to rigorously define the notion of a solution for the singular problem.

Definition 1.1. A function $u \in X$ is called a weak solution of (1.1) if it satisfies the following conditions:

(i) $u > 0$ a.e. in Ω, $fu^{-\gamma(x)} \in L^1_{loc}(\Omega)$, and

(ii) $\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i \phi dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{q_i-2} \partial_i u \partial_i \phi dx = \int_\Omega \frac{f\phi}{u^{\gamma(x)}} dx,$

for all $\phi \in C_0^1(\Omega)$.

The principal theorems established in this work are as follows:

Theorem 1.2. *Suppose that f is a positive function in $L^1(\Omega)$, and (H_f) holds, with $0 < \gamma^- \leq \gamma(x) \leq \gamma^+ < 1$. Then, the problem (1.1) has a solution in X.*

Theorem 1.3. *Let f be a positive function in $L^1(\Omega)$ and assume that condition (H_f) holds, with $0 < \gamma^- < 1 < \gamma^+$. Then problem (1.1) admits a solution in the space $X_{loc} := W_{loc}^{1,\vec{p}}(\Omega) \cap W_{loc}^{1,\vec{q}}(\Omega)$. Moreover, the solution u belongs to $L^r(\Omega)$, where $r = \frac{N(\gamma^+ - 1 + \overline{m})}{(N - \overline{m})}$, with $\overline{m} = \overline{p}, \overline{q}$.*

Theorem 1.4. *Assume that f is a positive function in $L^1(\Omega)$ and condition (H_f) is satisfied, with $1 < \gamma^- \leq \gamma(x) \leq \gamma^+$. Then, problem (1.1) admits a solution in the space X_{loc}.*

Theorem 1.5. *Assume that (H_f) holds and $f \in L^r(\Omega)$ with $r \geq \frac{\min\{\overline{p}^*, \overline{q}^*\}}{\min\{\overline{p}^*, \overline{q}^*\} - \max\{\overline{p}, \overline{q}\}}$, and let $\gamma(x) > 0$. Then, the solution u of problem (1.1), as established by Theorems 1.2, 1.3, and 1.4, belongs to $L^\infty(\Omega)$.*

This paper is organized as follows. In Section 2 establishes several technical results that are essential for proving our main theorem. We begin by presenting a regularity result for the solution of the anisotropic (p,q)-Laplacian problem. Next, we introduce a comparison principle, and finally, we demonstrate that the solution remains strictly greater than a positive term. Section 3 applies the variational method and leverages the auxiliary results from Section 2 to prove that the approximate problem admits a solution in $X \cap L^\infty(\Omega)$. Section 4 is dedicated to the proof of the main results, while Section 5 explores possible generalizations of our existence result and discusses future research perspectives.

2 Preliminaries and technical results

This section is dedicated to establishing fundamental results essential for proving our main theorems. We begin by addressing the uniqueness and L^∞-regularity of solutions to the anisotropic (p,q)-Laplacian problem under consideration. Next, we introduce a comparison principle and conclude with a boundary behavior lemma, which ensures that a solution u remains bounded below by a positive quantity proportional to the distance to the boundary. Together, these results provide crucial insights into how the anisotropic structure influences the solution's regularity and its interaction with the domain's geometry. We start by examining the following anisotropic (p,q)-Laplacian problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{q_i-2} \partial_i u \right) = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Lemma 2.1 ([43]). *Let $h \in X'$. Then, the problem (2.1) has a unique solution in X.*

The proof follows from Minty–Browder's Theorem, as stated in Lemma 3.3 of [43].

Lemma 2.2. *Let $h \in L^r(\Omega)$ with $r > \frac{\max\{\bar{p}^*, \bar{q}^*\}}{\max\{\bar{p}^*, \bar{q}^*\} - \max\{p_N, q_N\}}$. The solution u to the problem (2.1) belongs to $L^\infty(\Omega)$.*

Proof. Let $\Omega_k := \{x \in \Omega : |u(x)| > k\}$ for $k > 0$. Define the truncated test function

$$\xi_k := \operatorname{sign}(u) \cdot (|u| - k)^+ ,$$

where $(\cdot)^+$ denotes the positive part. Let $\beta := \max\{\overline{p}^*, \overline{q}^*\}$, where $\overline{p}^*$ and $\overline{q}^*$ are the critical Sobolev exponents associated with the anisotropic exponents $\{p_i\}$ and $\{q_i\}$, respectively. Observe that $\xi_k \in X$ and $\partial_i \xi_k = \partial_i u$ in Ω_k, while $\partial_i \xi_k = 0$ outside Ω_k. Let $|\Omega_k|$ denote the Lebesgue measure of Ω_k.

Using ξ_k as a test function in the weak formulation of problem (2.1) and applying Hölder's inequality, we can derive the following results

$$\begin{aligned} \sum_{i=1}^{N} \int_{\Omega_k} |\partial_i \xi_k|^{p_i} \, dx + \sum_{i=1}^{N} \int_{\Omega_k} |\partial_i \xi_k|^{q_i} \, dx &= \int_{\Omega} h \xi_k dx \\ &\leq \left(\int_{\Omega} |\xi_k|^{\beta} \, dx \right)^{\frac{1}{\beta}} \left(\int_{\Omega} |h|^r dx \right)^{\frac{1}{r}} |\Omega_k|^{1 - \frac{1}{\beta} - \frac{1}{r}}. \end{aligned} \tag{2.2}$$

Define the energy minimization problem

$$\mathfrak{B} := \inf_{\substack{u \in D^{1,\vec{p}}(\mathbb{R}^N) \cap D^{1,\vec{q}}(\mathbb{R}^N) \\ \|u\|_{L^\beta(\mathbb{R}^N)}=1}} \sum_{i=1}^{N} \frac{1}{p_i} \|\partial_i u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + \sum_{i=1}^{N} \frac{1}{q_i} \|\partial_i u\|^{q_i}_{L^{q_i}(\mathbb{R}^N)}, \tag{2.3}$$

where

$$\mathcal{D}^{1,\vec{p}}\left(\mathbb{R}^N\right) := \left\{ u \in L^{\bar{p}^*}\left(\mathbb{R}^N\right) : |\partial_i u| \in L^{p_i}\left(\mathbb{R}^N\right) \right\}.$$

Form Lemma 3 in [18], if $\beta = \bar{p}^*$, we have

$$\mathfrak{B} \geq \inf_{\substack{u \in D^{1,\vec{p}}(\mathbb{R}^N) \\ \|u\|_{L^{\bar{p}^*}(\mathbb{R}^N)}=1}} \sum_{i=1}^{N} \frac{1}{p_i} \|\partial_i u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} > 0.$$

Similarly, if $\beta = \bar{q}^*$, the same argument yields $\mathfrak{B} > 0$.

By direct computation, we deduce that for every $u \in X$

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i}\, dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{q_i}\, dx \\ &\quad \geq \|u\|^\theta_{L^\beta(\Omega)} \sum_{i=1}^{N} \int_\Omega \left|\partial_i \left(\frac{u}{\|u\|_{L^\beta(\Omega)}}\right)\right|^{p_i} dx + \|u\|^\theta_{L^\beta(\Omega)} \sum_{i=1}^{N} \int_\Omega \left|\partial_i \left(\frac{u}{\|u\|_{L^\beta(\Omega)}}\right)\right|^{q_i} dx \\ &\quad \geq \|u\|^\theta_{L^\beta(\Omega)} \left[\sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega \left|\partial_i \left(\frac{u}{\|u\|_{L^\beta(\Omega)}}\right)\right|^{p_i} dx + \sum_{i=1}^{N} \frac{1}{q_i} \int_\Omega \left|\partial_i \left(\frac{u}{\|u\|_{L^\beta(\Omega)}}\right)\right|^{q_i} dx \right], \end{aligned}$$

where

$$\theta = \begin{cases} \min\{p_1, q_1\} & \text{if } \|u\|_{L^\beta(\Omega)} > 1, \\ \max\{p_N, q_N\} & \text{if } \|u\|_{L^\beta(\Omega)} \leq 1. \end{cases}$$

Furthermore, from (2.3), we derive

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i}\, dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{q_i}\, dx \geq \mathfrak{B} \times \|u\|^\theta_{L^\beta(\Omega)}. \tag{2.4}$$

Combining (2.2) and (2.4), we obtain

$$\mathfrak{B} \left(\int_{\Omega_k} |\xi_k|^\beta\, dx \right)^{\frac{\theta-1}{\beta}} \leq \|h\|_{L^r(\Omega)} |\Omega_k|^{1-\frac{1}{\beta}-\frac{1}{r}}. \tag{2.5}$$

Additionally, if $0 < k < \ell$, we have $\Omega_\ell \subset \Omega_k$, and

$$|\Omega_\ell|^{\frac{1}{\beta}} (\ell - k) = \left(\int_{\Omega_\ell} (\ell - k)^\beta dx \right)^{\frac{1}{\beta}} \leq \left(\int_{\Omega_k} |\xi_k|^\beta\, dx \right)^{\frac{1}{\beta}}.$$

Given that $\theta \geq 2$, we conclude

$$\mathfrak{B} |\Omega_\ell|^{\frac{\theta-1}{\beta}} (\ell - k)^{\theta-1} \leq \mathfrak{B} \left(\int_{\Omega_k} |\xi_k|^\beta\, dx \right)^{\frac{\theta-1}{\beta}}, \tag{2.6}$$

and by combining (2.5) and (2.6), we infer

$$|\Omega_\ell| \leq \frac{1}{(\ell-k)^\beta \mathfrak{B}^{\frac{\beta}{\theta-1}}} \|h\|_{L^r(\Omega)}^{\frac{\beta}{\theta-1}} |\Omega_k|^{\frac{\beta}{\theta-1}\left(1-\frac{1}{\beta}-\frac{1}{r}\right)}.$$

Since $r > \frac{\max\{\bar{p}^*,\bar{q}^*\}}{\max\{\bar{p}^*,\bar{q}^*\}-\max\{p_N,q_N\}} \geq \frac{\beta}{\beta-\theta}$, we deduce that $\frac{\beta}{\theta-1}(1-\frac{1}{\beta}-\frac{1}{r}) > 1$. Thus, defining

$$\begin{cases} \psi(h) = |\Omega_\ell|, \\ \sigma = \frac{\beta}{\theta-1}\left(1-\frac{1}{\beta}-\frac{1}{r}\right), \\ k_0 = 0, \end{cases}$$

we see that ψ is a positive, non-increasing function. Moreover,

$$\psi(\ell) \leq \frac{C}{(\ell-k)^\beta}\psi(k)^\beta, \quad \text{for all } \ell > k > 0.$$

Using ((*i*), Lemma 4.1, in [42]), we conclude that $\psi(d) = 0$ with $d^\beta = \frac{C\|h\|_{L^r(\Omega)}^{\frac{\beta}{\theta-1}}|\Omega|^{\sigma-1}}{\mathfrak{B}^{\frac{\beta}{\theta-1}}}$. Thus, we obtain

$$\|u\|_{L^\infty(\Omega)} \leq \frac{C\|h\|_{L^r(\Omega)}^{\frac{1}{\theta-1}}|\Omega|^{\frac{\sigma-1}{\beta}}}{\mathfrak{B}^{\frac{1}{\theta-1}}}.$$

Therefore, we have established that $u \in L^\infty(\Omega)$. □

Lemma 2.3 ([43]). *Consider $u,v \in X$ satisfying the following in the weak sense*

$$\begin{cases} -\sum_{i=1}^N \partial_i\left(|\partial_i u|^{p_i-2}\partial_i u\right) - \sum_{i=1}^N \partial_i\left(|\partial_i u|^{q_i-2}\partial_i u\right) \\ \qquad \leq -\sum_{i=1}^N \partial_i\left(|\partial_i v|^{p_i-2}\partial_i v\right) - \sum_{i=1}^N \partial_i\left(|\partial_i v|^{q_i-2}\partial_i v\right) \quad \text{in } \Omega, \\ (u-v)^+ \in X. \end{cases}$$

Then, it follows that $u \leq v$ almost everywhere in Ω.

Regarding the proof, we refer the reader to Lemma 3.5 in [43].

Lemma 2.4. *Let μ be a positive constant and $u \in X$ be the unique solution to the following problem*

$$\begin{cases} -\sum_{i=1}^N \partial_i\left(|\partial_i u|^{p_i-2}\partial_i u\right) - \sum_{i=1}^N \partial_i\left(|\partial_i u|^{q_i-2}\partial_i u\right) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Then, there exist constants $\eta \in (0,1]$ and M, which are independent of μ and u, such that

$$u(x) \geq M \min\left\{\mu^{\max\left\{\frac{1}{p_1-1},\frac{1}{q_1-1}\right\}}, \mu^{\min\left\{\frac{1}{p_N-1},\frac{1}{q_N-1}\right\}}\right\} \min\{\eta, d(x)\},$$

with $d(x) := \operatorname{dist}(x,\partial\Omega), x \in \Omega$.

Proof. Let $\rho > 0$ define the set

$$A_\rho := \{x \in \Omega \mid d(x) < \rho\},$$

where $d(x) = \text{dist}(x, \partial\Omega)$ denotes the distance from x to the boundary $\partial\Omega$. Since the boundary $\partial\Omega$ is of class C^2, there exists a sufficiently small constant $\eta \in (0, 1]$ such that

$$d \in C^2\left(\overline{A_{3\eta}}\right) \quad \text{and} \quad |\nabla d(x)| \equiv 1,$$

as stated in Lemma 14.16 of Gilbarg and Trudinger [24].

For

$$\beta > \frac{1}{\min\{p_1, q_1\} - 1},$$

we define the function

$$\varsigma(x) = \begin{cases} \kappa d(x), & \text{if } d(x) < \eta, \\ \kappa\eta + \displaystyle\int_\eta^{d(x)} \kappa \left(\frac{2\eta - t}{\eta}\right)^\beta dt, & \text{if } \eta \leq d(x) < 2\eta, \\ \kappa\eta + \displaystyle\int_\eta^{2\eta} \kappa \left(\frac{2\eta - t}{\eta}\right)^\beta dt, & \text{if } d(x) \geq 2\eta, \end{cases}$$

where $\kappa > 0$ is a constant to be determined later. It is evident that $\varsigma \in C_0^1(\overline{\Omega})$.

According to direct calculations, if $x \in \Omega$ satisfies $d(x) < \eta$ and $\partial_i d(x) \neq 0$, then we have

$$\begin{aligned}
&-\sum_{i=1}^N \left[\partial_i \left(|\partial_i(\kappa d)|^{p_i-2} \partial_i(\kappa d)\right) + \partial_i \left(|\partial_i(\kappa d)|^{q_i-2} \partial_i(\kappa d)\right)\right] \\
&\quad = -\sum_{i=1}^N \left[(\text{sgn}\,(\partial_i d))\, \kappa^{p_i-1} \partial_i \left(\text{sgn}\,(\partial_i d)^{p_i-1} (\partial_i d)^{p_i-1}\right)\right] \\
&\quad\quad -\sum_{i=1}^N \left[(\text{sgn}\,(\partial_i d))\, \kappa^{q_i-1} \partial_i \left(\text{sgn}\,(\partial_i d)^{q_i-1} (\partial_i d)^{q_i-1}\right)\right] \\
&\quad = -\sum_{i=1}^N \left[\kappa^{p_i-1} (p_i - 1) \left((\text{sgn}\,(\partial_i d))\, \partial_i d\right)^{p_i-2} \partial_i^2 d\right] \\
&\quad\quad -\sum_{i=1}^N \left[\kappa^{q_i-1} (q_i - 1) \left((\text{sgn}\,(\partial_i d))\, \partial_i d\right)^{q_i-2} \partial_i^2 d\right] \\
&\quad =: \Upsilon_1(x),
\end{aligned}$$

where $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$. In the case where $\eta < d(x) < 2\eta$ and

$\partial_i d(x) \neq 0$, we find that

$$
\begin{aligned}
&-\sum_{i=1}^{N}\partial_i\left(\left|\partial_i\left(\kappa\eta+\int_{\eta}^{d(x)}\kappa\left(\frac{2\eta-t}{\eta}\right)^{\beta}dt\right)\right|^{p_i-2}\partial_i\left(\kappa\eta+\int_{\eta}^{d(x)}\kappa\left(\frac{2\eta-t}{\eta}\right)^{\beta}dt\right)\right)\\
&\quad-\sum_{i=1}^{N}\partial_i\left(\left|\partial_i\left(\kappa\eta+\int_{\eta}^{d(x)}\kappa\left(\frac{2\eta-t}{\eta}\right)^{\beta}dt\right)\right|^{q_i-2}\partial_i\left(\kappa\eta+\int_{\eta}^{d(x)}\kappa\left(\frac{2\eta-t}{\eta}\right)^{\beta}dt\right)\right)\\
&=-\sum_{i=1}^{N}\partial_i\left(\left|\partial_i(d(x))\kappa\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta}\right|^{p_i-2}\partial_i(d(x))\kappa\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta}\right)\\
&\quad-\sum_{i=1}^{N}\partial_i\left(\left|\partial_i(d(x))\kappa\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta}\right|^{q_i-2}\partial_i(d(x))\kappa\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta}\right)\\
&=-\sum_{i=1}^{N}\kappa^{p_i-1}\beta\left(p_i-1\right)\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta(p_i-1)-1}\left(\frac{-1}{\eta}\right)\left(\left(\operatorname{sgn}\left(\partial_i d\right)\right)\partial_i d\right)^{p_i-2}\left(\partial_i d\right)^2\\
&\quad-\sum_{i=1}^{N}\kappa^{p_i-1}\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta(p_i-1)}\left(p_i-1\right)\left|\partial_i d\right|^{p_i-2}\partial_i^2(d(x))\\
&\quad-\sum_{i=1}^{N}\kappa^{q_i-1}\beta\left(q_i-1\right)\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta(q_i-1)-1}\left(\frac{-1}{\eta}\right)\left(\left(\operatorname{sgn}\left(\partial_i d\right)\right)\partial_i d\right)^{q_i-2}\left(\partial_i d\right)^2\\
&\quad-\sum_{i=1}^{N}\kappa^{q_i-1}\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta(q_i-1)}\left(q_i-1\right)\left|\partial_i d\right|^{q_i-2}\partial_i^2(d(x))\\
&=:\Upsilon_2(x).
\end{aligned}
$$

According to the previous cases, we have

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i(\varsigma)|^{p_i-2}\partial_i(\varsigma)\partial_i\phi dx+\sum_{i=1}^{N}\int_{\Omega}|\partial_i(\varsigma)|^{q_i-2}\partial_i(\varsigma)\partial_i\phi dx=\int_{\Omega}\Upsilon\phi dx\quad\forall\phi\in X,$$

where

$$\Upsilon(x)=\begin{cases}\Upsilon_1(x) & \text{if } d(x)<\eta,\ \partial_i d(x)\neq 0,\\ \Upsilon_2(x) & \text{if } \eta<d(x)<2\eta,\ \partial_i d(x)\neq 0,\\ 0 & \text{if } d(x)>2\eta \text{ or } \partial_i d(x)=0.\end{cases}$$

Given that $\beta > 1/(\min\{p_1,q_1\}-1)$ and $p_i, q_i \geq 2$ for all $i = 1,\ldots,N$, we conclude that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}|\partial_i(\varsigma)|^{p_i-2}\partial_i(\varsigma)\partial_i\phi dx+\sum_{i=1}^{N}\int_{\Omega}|\partial_i(\varsigma)|^{q_i-2}\partial_i(\varsigma)\partial_i\phi dx\\
&\quad\leq M_0\max\left\{\kappa^{p_1-1},\ldots,\kappa^{p_N-1}\right\}+M_1\max\left\{\kappa^{q_1-1},\ldots,\kappa^{q_N-1}\right\}\\
&\quad\leq M\max\left\{\kappa^{p_1-1},\ldots,\kappa^{p_N-1},\kappa^{q_1-1},\ldots,\kappa^{q_N-1}\right\},
\end{aligned}
$$

in the weak sense, where M_0 and M_1 are the constants from Lemma 2.4 and $M = 2\max\{M_0, M_1\}$.

Let u be the solution to problem (2.7) and $\mu > 0$. Now we distinguish two cases:
In the first case, if $\frac{\mu}{M} < 1$, we choose $\kappa \in (0,1)$ such that

$$\max\left\{\kappa^{p_1-1},\ldots,\kappa^{p_N-1},\kappa^{q_1-1},\ldots,\kappa^{q_N-1}\right\}=\kappa^{\min\{p_1-1,q_1-1\}}=\frac{\mu}{M}.$$

Applying Lemma 2.3 yields

$$u(x) \geq \varsigma(x) \geq \kappa \min\{\eta, d(x)\} = \left(\frac{\mu}{M}\right)^{\max\{\frac{1}{p_1-1}, \frac{1}{q_1-1}\}} \min\{\eta, d(x)\}.$$

In the second case, when $\frac{\mu}{M} \geq 1$ we can employ the same reasoning as in the first case to obtain

$$u(x) \geq \varsigma(x) \geq \kappa \min\{\eta, d(x)\} = \left(\frac{\mu}{M}\right)^{\min\{\frac{1}{p_N-1}, \frac{1}{q_N-1}\}} \min\{\eta, d(x)\}.$$

Consequently, we arrive at

$$u(x) \geq \min\left\{\left(\frac{\mu}{M}\right)^{\max\{\frac{1}{p_1-1}, \frac{1}{q_1-1}\}}, \left(\frac{\mu}{M}\right)^{\min\{\frac{1}{p_N-1}, \frac{1}{q_N-1}\}}\right\} \min\{\eta, d(x)\}.$$

Thus, the statement is proven. □

3 Approximation problems

To formulate a minimization problem, the energy functional Ψ must be bounded from below in X, which, unfortunately, is not the case here. Additionally, the presence of a singular term introduces non-differentiability in the energy functional, creating an additional challenge in defining the natural constrained set. As a result, we cannot directly apply the standard critical point theory to Ψ to solve problem (1.1). To overcome this difficulty and obtain our result, we study an associated approximating problem. Specifically, for any $n \in \mathbb{N}^*$, we consider the following perturbed problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{p_i-2} \partial_i u_n \right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{q_i-2} \partial_i u_n \right) = \dfrac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_n$}$$

where $f_n(x) = \min\{f(x)\ ,\ n\}$.

Lemma 3.1. *Suppose that $\gamma \in C^1(\overline{\Omega})$. Then the problem ($\mathcal{P}_n$) has a non-negative solution $u_n \in X \cap L^\infty(\Omega)$.*

Proof. While many studies rely on Schauder's fixed point theorem to establish the existence of solutions, this paper adopts a variational approach. We define the functions
$f : \Omega \times (0, \infty) \longrightarrow \mathbb{R}$ and $f_n : \Omega \times [0, \infty) \longrightarrow \mathbb{R}$ by

$$f(x,t) = \frac{f(x)}{t^{\gamma(x)}}, \quad \text{and} \quad f_n(x,t) = \frac{f_n(x)}{(t+1/n)^{\gamma(x)}}, \qquad \text{for all } n > 0.$$

It is important to note that f_n is a Carathéodory function, and its primitive is given by the function F_n, where

$$F_n(x,t) = \int_0^t f_n(x,s)ds.$$

Remark 3.2. For $t \geq 0$, the following inequalities hold

$$0 < f_n(x,t) \leq f_n(x) n^{\|\gamma\|_{L^\infty(\Omega)}}, \quad \text{and} \quad F_n(x,t) \leq f_n(x) n^{\|\gamma\|_{L^\infty(\Omega)}} t.$$

We now define the functional $\Psi_n : X \longrightarrow \mathbb{R}$ associated with the problem ($\mathcal{P}_n$) as follows

$$\Psi_n(u) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i} dx + \sum_{i=1}^{N} \frac{1}{q_i} \int_\Omega |\partial_i u|^{q_i} dx - \int_\Omega F_n(x,u)dx.$$

Since it holds that

$$W_0^{1,\vec{p}}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^r(\Omega) \quad \text{for all} \quad r \in [1, p_\infty),$$

and

$$W_0^{1,\vec{q}}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^r(\Omega) \quad \text{for all} \quad r \in [1, q_\infty).$$

as established in [20]. This implies that the functional Ψ_n is sequentially weakly lower semi-continuous and coercive. Indeed, we have

$$\begin{aligned}
\Psi_n(u) &= \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i} dx + \sum_{i=1}^{N} \frac{1}{q_i} \int_\Omega |\partial_i u|^{q_i} dx - \int_\Omega F_n(x,u)dx \\
&\geq \frac{1}{p_N N^{p_N-1}} \left(\sum_{i=1}^{N} |\partial_i u|_{L^{p_i}(\Omega)} \right)^{p_0} + \frac{1}{q_N N^{q_N-1}} \left(\sum_{i=1}^{N} |\partial_i u|_{L^{q_i}(\Omega)} \right)^{q_0} \\
&\quad - n^{\|\gamma\|_{L^\infty(\Omega)}} \int_\Omega f_n(x)u dx - 2N \\
&\geq \frac{1}{q_N p_N N^{p_N-1} N^{q_N-1}} \left(\|u\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)} + \|u\|^{q_0}_{W_0^{1,\vec{q}}(\Omega)} \right) - n^{\|\gamma\|_{L^\infty(\Omega)}} \int_\Omega f_n(x)u dx - 2N.
\end{aligned}$$

Here,

$$\begin{aligned}
p_0 = p_1 \quad &\text{if } \|u\|_{W_0^{1,\vec{p}}(\Omega)} > 1 \quad \text{and} \quad p_0 = p_N \quad \text{if } \|u\|_{W_0^{1,\vec{p}}(\Omega)} \leq 1, \\
q_0 = q_1 \quad &\text{if } \|u\|_{W_0^{1,\vec{q}}(\Omega)} > 1 \quad \text{and} \quad q_0 = q_N \quad \text{if } \|u\|_{W_0^{1,\vec{q}}(\Omega)} \leq 1.
\end{aligned}$$

Observe that $W_0^{1,\vec{p}}(\Omega) \overset{\text{continuous}}{\hookrightarrow} L^{p_1}(\Omega)$, implying that there exists a constant C such that, $\|u\|_{L^{p_1}(\Omega)} \leq C\|u\|_{W_0^{1,\vec{p}}(\Omega)}$. This leads to the following inequality

$$\Psi_n(u) \geq \frac{1}{q_N p_N N^{p_N-1} N^{q_N-1}} \left(\|u\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)} + \|u\|^{q_0}_{W_0^{1,\vec{q}}(\Omega)} \right) - C n^{\|\gamma\|_{L^\infty(\Omega)}} \|f\|_{L^{p_1'}(\Omega)} \|u\|_{W_0^{1,\vec{p}}(\Omega)} - 2N.$$

Since $p_0, q_0 \geq 2$, we have

$$\Psi_n(u) \longrightarrow +\infty \quad \text{as } \|u\|_X \longrightarrow +\infty.$$

Consequently, Ψ_n has a global minimum, denoted u_n. Moreover, $\Psi_n \in C^1(X)$ with derivative at u given by

$$\langle \Psi_n'(u), \phi \rangle \sum_{i=1}^{N} \int_\Omega |\partial_i(u)|^{p_i-2} \partial_i(u)\partial_i\phi dx + \sum_{i=1}^{N} \int_\Omega |\partial_i(u)|^{q_i-2} \partial_i(u)\partial_i\phi dx - \int_\Omega f_n(x,u)\phi dx.$$

Hence, this global minimum is a critical point and thus constitutes a weak solution to the problem ($\mathcal{P}_n$). Next, we consider the following problem

$$\begin{cases}
-\sum_{i=1}^{N} \partial_i \left(|\partial_i w_n|^{p_i-2} \partial_i w_n \right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i w_n|^{q_i-2} \partial_i w_n \right) = n^{\|\gamma\|_{L^\infty(\Omega)}} f_n & \text{in } \Omega, \\
w_n = 0 & \text{on } \partial\Omega.
\end{cases}$$

According to Lemmas 2.1 and 2.2 the aforementioned problem has a solution $w_n \in L^\infty(\Omega)$, whenever the right-hand side belongs to $L^s(\Omega)$ with $s > \max\{\bar{p}^*, \bar{q}^*\}'$. Here, $\max\{\bar{p}^*, \bar{q}^*\}'$ denotes the Hölder conjugate exponent, defined by the relation $\frac{1}{\max\{\bar{p}^*,\bar{q}^*\}} + \frac{1}{\max\{\bar{p}^*,\bar{q}^*\}'} = 1$.

Since $\frac{f_n}{(u_n+\frac{1}{n})^{\gamma(x)}} \geq 0$ a.e. in Ω, by applying Lemma 2.3, we show that $u_n \geq 0$ a.e. in Ω.

Furthermore, using Lemma 2.3 again, along with the inequality $f_n(x,u) \leq n^{\|\gamma\|_{L^\infty(\Omega)}} f_n(x)$ a.e. in Ω for all $n \in \mathbb{N}^*$, we conclude that $u_n \leq w_n$ a.e. in Ω, for all $n \in \mathbb{N}^*$. This leads us to deduce that $u_n \in L^\infty(\Omega)$. □

Remark 3.3. If u_n and v_n are two solutions of problem $(\mathcal{P}_n)$, then by a straightforward calculation using Lemma 2.3, we can demonstrate that $u_n \leq v_n$. Due to symmetry, this implies that the solution to problem $(\mathcal{P}_n)$ is unique.

Lemma 3.4. *The sequence $\{u_n\}$ is increasing with respect to $n, u_n \geq 0$ a.e in Ω, and there exists $K > 0$ (independent of n) such that*

$$u_n(x) \geq Kd(x) > 0 \quad \text{a.e. } x \in \Omega, \tag{3.1}$$

for every $n \in \mathbb{N}$, where $d(x) := \operatorname{dist}(x, \partial\Omega)$.

Proof. The function $f_n(x) = \min\{f(x)\ ,\ n\}$ provides a pointwise truncation of $f(x)$. Additionally, $\gamma(x) > 0$ in Ω, observing that

$$-\sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{p_i-2} \partial_i u_n \right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{q_i-2} \partial_i u_n \right) = \frac{f_n}{(u_n + \frac{1}{n})^{\gamma(x)}} \leq \frac{f_{n+1}}{(u_n + \frac{1}{n+1})^{\gamma(x)}}$$

and

$$-\sum_{i=1}^{N} \partial_i \left(|\partial_i u_{n+1}|^{p_i-2} \partial_i u_{n+1} \right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u_{n+1}|^{q_i-2} \partial_i u_{n+1} \right) = \frac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^{\gamma(x)}},$$

which implies

$$\begin{aligned} &-\sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{p_i-2} \partial_i u_n \right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{q_i-2} \partial_i u_n \right) \\ &\qquad + \sum_{i=1}^{N} \partial_i \left(|\partial_i u_{n+1}|^{p_i-2} \partial_i u_{n+1} \right) + \sum_{i=1}^{N} \partial_i \left(|\partial_i u_{n+1}|^{q_i-2} \partial_i u_{n+1} \right) \\ &\leq \frac{f_{n+1}}{(u_n + \frac{1}{n+1})^{\gamma(x)}} - \frac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^{\gamma(x)}} \\ &\leq \left(\frac{(u_{n+1} + \frac{1}{n+1})^{\gamma(x)} - (u_n + \frac{1}{n+1})^{\gamma(x)}}{(u_n + \frac{1}{n+1})^{\gamma(x)} (u_{n+1} + \frac{1}{n+1})^{\gamma(x)}} \right) f_{n+1}. \end{aligned} \tag{3.2}$$

In the last inequality, applying $(u_n - u_{n+1})^+$ as a test function, the right-hand side shows

$$\left(\frac{(u_{n+1} + \frac{1}{n+1})^{\gamma(x)} - (u_n + \frac{1}{n+1})^{\gamma(x)}}{(u_n + \frac{1}{n+1})^{\gamma(x)} (u_{n+1} + \frac{1}{n+1})^{\gamma(x)}} \right) (u_n - u_{n+1})^+ f_{n+1} \leq 0 \quad \text{a.e. in } \Omega. \tag{3.3}$$

Combining (3.3) and (3.2) yields

$$\sum_{i=1}^{N}\int_\Omega \left(|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i u_{n+1}|^{p_i-2}\partial_i u_{n+1}\right)\partial_i\left(u_n-u_{n+1}\right)^+ \\ +\sum_{i=1}^{N}\int_\Omega \left(|\partial_i u_n|^{q_i-2}\partial_i u_n - |\partial_i u_{n+1}|^{q_i-2}\partial_i u_{n+1}\right)\partial_i\left(u_n-u_{n+1}\right)^+ \le 0.$$

We recall the following algebraic inequality (see [41], page 210)

$$\left[|\xi_1|^{m-2}\xi_1 - |\xi_2|^{m-2}\xi_2\right]\cdot(\xi_1-\xi_2) \ge C\,|\xi_1-\xi_2|^m,$$

where $\xi_1,\xi_2 \in \mathbb{R}^N$, $m \ge 2$, and C is a positive constant.

Since $p_i \ge 2$ for all $i = 1,\dots,N$, we apply the above inequality with $\xi_1 = \partial_i u_n$ and $\xi_2 = \partial_i u_{n+1}$, which yields

$$0 \le \sum_{i=1}^{N}\int_\Omega |\partial_i\left(u_n-u_{n+1}\right)^+|^{p_i}dx + \sum_{i=1}^{N}\int_\Omega |\partial_i\left(u_n-u_{n+1}\right)^+|^{q_i}dx \le 0.$$

Consequently, we find that $(u_n-u_{n+1})^+ = 0$ a.e. in Ω, which implies $u_n \le u_{n+1}$ for all $n\in\mathbb{N}$.

Given that the sequence u_n is increasing with respect to n, it suffices to prove that inequality (3.1) holds for u_1. We know that u_1 is a solution of the problem $(\mathcal{P}_n)$ when $n=1$, which implies, in the weak sense

$$-\sum_{i=1}^{N}\partial_i\left(|\partial_i u_1|^{p_i-2}\partial_i u_1\right) - \sum_{i=1}^{N}\partial_i\left(|\partial_i u_1|^{q_i-2}\partial_i u_1\right) = \frac{f_1}{(u_1+1)^{\gamma(x)}}.$$

According to Lemma 3.1 and hypothesis (H_f), we have

$$\frac{f_1}{(u_1+1)^{\gamma(x)}} \ge \frac{\operatorname{ess\,inf}_\Omega f_1}{\left(\|u_1\|_{L^\infty(\Omega)}+1\right)^{\gamma^-}} > 0.$$

Let $\lambda > 0$ be such that

$$\frac{\operatorname{ess\,inf}_\Omega f_1}{\left(\|u_1\|_{L^\infty(\Omega)}+1\right)^{\gamma^-}} \ge \lambda. \tag{3.4}$$

By applying Lemma 2.4, there exists a unique solution $\underline{u}\in X$, to the following problem

$$\begin{cases} -\sum_{i=1}^{N}\partial_i\left(|\partial_i \underline{u}|^{p_i-2}\partial_i \underline{u}\right) - \sum_{i=1}^{N}\partial_i\left(|\partial_i \underline{u}|^{q_i-2}\partial_i \underline{u}\right) = \lambda & \text{in } \Omega,\\ \underline{u} = 0 & \text{on } \partial\Omega,\end{cases}$$

such that

$$\underline{u}(x) \ge Kd(x), \tag{3.5}$$

where $d(x) := \operatorname{dist}(x,\partial\Omega)$ for $x\in\Omega$, and K is a positive constant independent of $\underline{u}$.

From (3.4), we obtain

$$-\sum_{i=1}^{N}\partial_i\left(|\partial_i u_1|^{p_i-2}\partial_i u_1\right) - \sum_{i=1}^{N}\partial_i\left(|\partial_i u_1|^{q_i-2}\partial_i u_1\right) \ge -\sum_{i=1}^{N}\partial_i\left(|\partial_i \underline{u}|^{p_i-2}\partial_i \underline{u}\right) - \sum_{i=1}^{N}\partial_i\left(|\partial_i \underline{u}|^{q_i-2}\partial_i \underline{u}\right).$$

Using Lemma 2.3 together with inequality (3.5), we conclude

$$u_1 \geq \underline{u} \geq Kd(x), \quad \text{a.e. in } \Omega. \tag{3.6}$$

Due to the monotonicity of the sequence $\{u_n\}_n$ and the result in (3.6), it follows that for all $n \in \mathbb{N}^*$

$$u_n \geq Kd(x), \quad \text{a.e. in } \Omega.$$

□

4 Proof of the main results

In this section, contingent on the behavior of the function $\gamma(x)$, we establish the main results of our work by analyzing the convergence of solutions to the regularized problems $(\mathcal{P}_n)$ and investigating the regularity properties of these solutions.

Proof of Theorem 1.2.

When $0 < \gamma(x) < 1$, the existence of solutions in X for problem (1.1) is established by showing that the sequence of solutions to the approximate problem $(\mathcal{P}_n)$ remains bounded in X. Leveraging the reflexivity of X and applying Lemma 3.4, we then conclude the existence of a solution.

Lemma 4.1. *Suppose that (H_f) holds, $f \in L^1(\Omega)$, and let u_n be the solution of $(\mathcal{P}_n)$ with $0 < \gamma^- \leq \gamma(x) \leq \gamma^+ < 1$. Then the sequence $\{u_n\}_n$ is bounded in X.*

Proof. Begin by using u_n as a test function in $(\mathcal{P}_n)$. We have

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{q_i} dx &= \int_\Omega \frac{f_n u_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} dx \\
&\leq \int_\Omega f u_n^{1-\gamma(x)} dx \\
&\leq \int_{\Omega \cap \{u_n \leq 1\}} f dx + \int_{\Omega \cap \{u_n \geq 1\}} f u_n dx \\
&\leq \|f\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)} \|u_n\|_{L^\infty(\Omega)}.
\end{aligned} \tag{4.1}$$

Now, referencing the proof of Lemma 3.1, we can derive the following inequality:

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{q_i} dx \geq \frac{\|u_n\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)}}{N^{p_N - 1}} + \frac{\|u_n\|^{q_0}_{W_0^{1,\vec{q}}(\Omega)}}{N^{q_N - 1}} - 2N, \tag{4.2}$$

where p_0 and q_0 are identical in the proof of Lemma 3.1. Based on the previous inequalities and (4.1), we obtain

$$\begin{aligned}
\|u_n\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)} + \|u_n\|^{q_0}_{W_0^{1,\vec{q}}(\Omega)} &\leq \max\left\{N^{p_N - 1}, N^{q_N - 1}\right\} \left(\|f\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)} \|u_n\|_{L^\infty(\Omega)} + 2N\right) \\
&\leq \mathfrak{C}_1(f, \varphi).
\end{aligned}$$

The fact that $u_n \in L^\infty(\Omega)$ implies that $\{u_n\}$ is bounded in X. □

Since the embidding $X \hookrightarrow L^s(\Omega)$ is compact for all $s \in [1, \min\{\bar{p}^*, \bar{q}^*\})$, and given that $\{u_n\}$ is bounded in X by Lemma 4.1, we can conclude that, up to a subsequence, $u_n \rightharpoonup u$

weakly in X, where $u \in X$. Moreover, $u_n \to u$ strongly in $L^s(\Omega)$ for $1 \leq s < \min\{\bar{p}^*, \bar{q}^*\}$, and $u_n(x) \to u(x)$ almost everywhere in Ω. Thus, for every $\phi \in C_0^1(\Omega)$, we have

$$\lim_{n\longrightarrow\infty} \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i \phi dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{q_i-2}\partial_i u_n \partial_i \phi dx = \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \phi dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{q_i-2}\partial_i u \partial_i \phi dx. \quad (4.3)$$

According to Lemma 3.4, we have $u_n \geq kd(x)$ a.e. in Ω, which implies that

$$0 \leq \left| \frac{f_n \phi}{(u_n + \frac{1}{n})^{\gamma(x)}} \right| \leq \left(\|\phi\,(kd(\cdot))^{-\gamma^+}\|_{L^\infty(\Omega)} + \|\phi\,(kd(\cdot))^{-\gamma^-}\|_{L^\infty(\Omega)} \right) f,$$

for all $\phi \in C_0^1(\Omega)$. By using the Dominated Lebesgue's theorem, we have

$$\lim_{n\longrightarrow\infty} \int_\Omega \frac{f_n \phi}{(u_n + \frac{1}{n})^{\gamma(x)}} dx = \int_\Omega \frac{f\phi}{(u)^{\gamma(x)}} dx. \quad (4.4)$$

The fact that u_n is the solution of ($\mathcal{P}_n$) allows us to conclude, using (4.3) and (4.4), that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \phi dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{q_i-2}\partial_i u \partial_i \phi dx = \int_\Omega \frac{f\phi}{u^{\gamma(x)}} dx, \quad (4.5)$$

for all $\phi \in C_0^1(\Omega)$. This proves that (1.1) has a solution u in X. □

Proof of Theorem 1.3.

In the case $\gamma^- < 1 < \gamma^+$, we choose a test function $u_n^{\gamma^+}$ to demonstrate that u_n is bounded in X_{loc}, and also bounded in $L^r(\Omega)$ for some appropriate value of r. By utilizing the same arguments as in the case $0 < \gamma(x) < 1$ (Theorem 1.2), we can establish the existence and regularity of the solutions.

Lemma 4.2. *Assume that* (H_f) *is satisfied and that* f *is a nonnegative function in* $L^1(\Omega)$*. Let* u_n *denote the solution to* ($\mathcal{P}_n$) *for the parameters* $0 < \gamma^- < 1 < \gamma^+$*. Then, the sequence* $\{u_n\}_n$ *is bounded in both* X_{loc} *and* $L^r(\Omega)$*, where*

$$r = \frac{N(\gamma^+ - 1 + \overline{m})}{(N - \overline{m})},$$

with $\overline{m}$ *being either* $\overline{p}$ *or* $\overline{q}$*.*

Proof. Taking $u_n^{\gamma^+}$ as a test function, we obtain the following by using ($\mathcal{P}_n$) and Lemma 3.4. Given that $u_n \in L^\infty(\Omega)$ and $\gamma^+ - \gamma^- > 0$, we obtain

$$\begin{aligned}\sum_{i=1}^{N} \gamma^+ \int_\Omega |\partial_i u_n|^{p_i} u_n^{\gamma^+-1} dx + \sum_{i=1}^{N} \gamma^+ \int_\Omega |\partial_i u_n|^{q_i} u_n^{\gamma^+-1} dx &= \int_\Omega \frac{f_n u_n^{\gamma^+}}{(u_n + \frac{1}{n})^{\gamma(x)}} dx \\ &\leq \int_\Omega f u_n^{\gamma^+-\gamma^-} dx + \int_\Omega f dx \\ &\leq \|f\|_{L^1(\Omega)} \left(1 + \|u_n^{\gamma^+-\gamma^-}\|_{L^\infty(\Omega)}\right) \\ &\leq \mathfrak{C}_2(f,\gamma,\Omega). \end{aligned} \quad (4.6)$$

Furthermore, applying Lemma 3.4 for any compact set $K \subset \Omega$ and since $\gamma^+ - 1 > 0$, we conclude

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} u_n^{\gamma^+-1} dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{q_i} u_n^{\gamma^+-1} dx \geq \min_K (kd(x))^{\gamma^+-1} \left[\sum_{i=1}^{N} \int_K |\partial_i u_n|^{p_i} dx + \sum_{i=1}^{N} \int_K |\partial_i u_n|^{q_i} dx \right].$$

By the last inequalities and (4.6), we obtain

$$\sum_{i=1}^{N} \int_K |\partial_i u_n|^{p_i} dx + \sum_{i=1}^{N} \int_K |\partial_i u_n|^{q_i} dx \leq \frac{1}{\min_K (kd(x))^{\gamma^+-1}} \mathfrak{C}_2(f, \gamma, \Omega), \tag{4.7}$$

for any compact set $K \subset \Omega$. We conclude that u_n is bounded in X_{loc}.

Now from (4.6), we get for every $i = 1, \ldots, N$

$$\int_\Omega |\partial_i u_n|^{m_i} u_n^{\gamma^+-1} dx \leq \mathfrak{C}_2(f, \gamma, \Omega),$$

where m_i being either p_i or q_i, which implies

$$\left[\int_\Omega |\partial_i u_n|^{m_i} u_n^{\gamma^+-1} dx \right]^{\frac{1}{m_i}} \leq \mathfrak{C}_2(f, \gamma, \Omega)^{\frac{1}{m_i}}.$$

Hence,

$$\prod_{i=1}^{N} \left[\int_\Omega |\partial_i u_n|^{m_i} u_n^{\gamma^+-1} dx \right]^{\frac{1}{m_i}} \leq \mathfrak{C}_2(f, \gamma, \Omega)^{\frac{N}{\overline{m}}},$$

where $\frac{N}{\overline{m}} = \sum_{i=1}^{N} \frac{1}{m_i}$.

Now, we need to use the Sobolev inequality provided in [33, Theorem 1.2], with the following choice of exponents:

$$t_i m_i = \gamma^+ - 1, \quad r = \frac{N(\gamma^+ - 1 + \overline{m})}{(N - \overline{m})} \quad \text{and} \quad \frac{1}{r} = \frac{\gamma_i(N-1) - 1 + \frac{1}{m_i}}{t_i + 1},$$

where γ_i and t_i are defined as in [33, Theorem 1.2] . We deduce that

$$\left[\int_\Omega u_n^r dx \right]^{\frac{N}{\overline{m}} - 1} \leq \mathfrak{C}_2(f, \gamma, \Omega)^{\frac{N}{\overline{m}}}.$$

The fact that $N > \overline{m}$ allows us to conclude that u_n is bounded in $L^r(\Omega)$ with $r = \frac{N(\gamma^+-1+\overline{m})}{(N-\overline{m})}$.

□

Given that u_n is bounded in X_{loc}, we can prove the existence of a solution u in X_{loc} by following the proof of Theorem 1.2. Moreover, since u_n is bounded in $L^r(\Omega)$ with $r = \frac{N(\gamma^+-1+\overline{m})}{(N-\overline{m})}$, it follows that u is also in $L^r(\Omega)$.

Proof of Theorem 1.4.

Finally, when $\gamma(x) > 1$, we select a test function $u_n(\varphi^2 + \epsilon^{\gamma^+})$ to show that u_n is bounded in X_{loc}. By adapting the arguments from the case $0 < \gamma(x) < 1$, we can conclude the existence of solutions in X_{loc}.

Lemma 4.3. *Assume that (H_f) is satisfied and that $f \in L^1(\Omega)$ is a nonnegative function. Let u_n represent the solution to $(\mathcal{P}_n)$ under the condition $1 < \gamma^- \le \gamma(x) \le \gamma^+$. Then, the sequence $\{u_n\}_n$ is bounded in X_{loc}.*

Proof. By using the test function $u_n(\varphi^2 + \epsilon^{\gamma^+})$ with $\varphi \in C^1(\overline{\Omega})$ and $\operatorname{supp}\varphi \subset \Omega$, and under the condition $0 < \epsilon < \frac{1}{n}$ for a fixed n, we can apply the properties from $(\mathcal{P}_n)$ and Lemma 3.4. We obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\left[\int_\Omega \left(|\partial_i u_n|^{p_i} + |\partial_i u_n|^{q_i}\right)(\varphi^2 + \epsilon^{\gamma^+})dx + 2\int_\Omega u_n\varphi\left(|\partial_i u_n|^{p_i-2} + |\partial_i u_n|^{q_i-2}\right)\partial_i u_n \partial_i \varphi dx\right]\\
&\quad = \int_\Omega \frac{f_n u_n(\varphi^2 + \epsilon^{\gamma^+})}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}}dx\\
&\quad \le \int_\Omega \frac{f_n u_n \varphi^2}{(kd(x))^{\gamma(x)}}dx + \int_\Omega \frac{f_n u_n \epsilon^{\gamma^+}}{\left(\frac{1}{n}\right)^{\gamma^+}}dx\\
&\quad \le \int_{\operatorname{supp}\varphi} \frac{f u_n \varphi^2}{\left(k \min\limits_{\operatorname{supp}\varphi} d(x)\right)^{\gamma(x)}}dx + \int_\Omega f_n u_n dx,
\end{aligned}$$

where $k > 0$ is a constant. Using Hölder's inequality,

$$\begin{aligned}
&\sum_{i=1}^{N}\left[\int_\Omega \left(|\partial_i u_n|^{p_i} + |\partial_i u_n|^{q_i}\right)(\varphi^2 + \epsilon^{\gamma^+})dx + 2\int_\Omega u_n\varphi\left(|\partial_i u_n|^{p_i-2} + |\partial_i u_n|^{q_i-2}\right)\partial_i u_n \partial_i \varphi dx\right]\\
&\quad \le \left[\|\varphi^2\|_{L^\infty(\Omega)} \min\left\{\left(k \min_{\operatorname{supp}\varphi} d(x)\right)^{-\gamma^-}, \left(k \min_{\operatorname{supp}\varphi} d(x)\right)^{-\gamma^+}\right\} + 1\right] \|f\|_{L^1(\Omega)} \|u_n\|_{L^\infty(\Omega)}.
\end{aligned} \tag{4.8}$$

Let $0 < \lambda < \frac{\epsilon^{\gamma^+}}{\left(\|\varphi\|_{L^\infty(\Omega)}+1\right)^{\frac{\max\{q_N,p_N\}}{\min\{q_1,p_1\}-1}}}$. By applying Young's inequality, we get

$$\begin{aligned}
&\sum_{i=1}^{N} 2\int_{\operatorname{supp}\varphi} |\partial_i u_n|^{m_i-1}|\partial_i\varphi||u_n\varphi|dx\\
&\quad \le \sum_{i=1}^{N}\left\{\frac{m_i-1}{m_i}\int_{\operatorname{supp}\varphi}\lambda|\partial_i u_n|^{m_i}|\varphi|^{\frac{m_i}{m_i-1}}dx + \frac{2^{m_i}}{m_i}\int_{\operatorname{supp}\varphi}\frac{1}{\lambda^{m_i-1}}|\partial_i\varphi|^{m_i}|u_n|^{m_i}dx\right\},
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{i=1}^{N} 2\int_\Omega u_n\varphi|\partial_i u_n|^{m_i-2}\partial_i u_n\partial_i\varphi dx\\
&\quad \ge -\sum_{i=1}^{N} 2\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{m_i-1}|\partial_i\varphi||u_n\varphi|dx\\
&\quad \ge -\sum_{i=1}^{N}\left\{\int_{\operatorname{supp}\varphi}\lambda|\partial_i u_n|^{m_i}|\varphi|^{\frac{m_i}{m_i-1}}dx + \frac{2^{m_i}}{m_i\lambda^{m_i-1}}\int_{\operatorname{supp}\varphi}|\partial_i\varphi|^{m_i}|u_n|^{m_i}dx\right\},
\end{aligned}$$

where $m_i = p_i$ or $m_i = q_i$. By combining the last inequalities and (4.8) yields

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{p_i}\left(\varphi^2+\epsilon^{\gamma^+}-\lambda|\varphi|^{\frac{p_i}{p_i-1}}\right)dx+\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{q_i}\left(\varphi^2+\epsilon^{\gamma^+}-\lambda|\varphi|^{\frac{q_i}{q_i-1}}\right)dx\\
&\quad\le\left[\|\varphi^2\|_{L^\infty(\Omega)}\min\left\{\left(k\min_{\operatorname{supp}\varphi}d(x)\right)^{-\gamma^-},\left(k\min_{\operatorname{supp}\varphi}d(x)\right)^{-\gamma^+}\right\}+1\right]\|f\|_{L^1(\Omega)}\|u_n\|_{L^\infty(\Omega)}\\
&\qquad+\sum_{i=1}^{N}\frac{2^{p_i}}{p_i\lambda^{p_i-1}}\|\partial_i\varphi\|^{p_i}_{L^\infty(\Omega)}\|u_n\|^{p_i}_{L^{p_i}(\Omega)}+\sum_{i=1}^{N}\frac{2^{q_i}}{q_i\lambda^{q_i-1}}\|\partial_i\varphi\|^{q_i}_{L^\infty(\Omega)}\|u_n\|^{q_i}_{L^{q_i}(\Omega)}\\
&\quad\le\mathfrak{C}_3(f,\varphi),
\end{aligned}\tag{4.9}$$

where the boundedness of the sequences $\|u_n\|_{L^\infty(\Omega)}, \|u_n\|_{L^{p_i}(\Omega)}$, and $\|u_n\|_{L^{q_i}(\Omega)}$ for each $i = 1,\dots,N$ ensures the validity of the inequality.

At this point, noting that

$$0<\lambda<\frac{\epsilon^{\gamma^+}}{\left(\|\varphi\|_{L^\infty(\Omega)}+1\right)^{\frac{\max\{q_N,p_N\}}{\min\{q_1,p_1\}-1}}},$$

we can deduce that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{p_i}\left(\varphi^2+\epsilon^{\gamma^+}-\lambda|\varphi|^{\frac{p_i}{p_i-1}}\right)dx+\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{q_i}\left(\varphi^2+\epsilon^{\gamma^+}-\lambda|\varphi|^{\frac{q_i}{q_i-1}}\right)dx\\
&\quad\ge\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{p_i}\left(\varphi^2+\epsilon^{\gamma^+}-\epsilon^{\gamma^+}\frac{|\varphi|^{\frac{p_i}{p_i-1}}}{\left(\|\varphi\|_{L^\infty(\Omega)}+1\right)^{\frac{\max\{q_N,p_N\}}{\min\{q_1,p_1\}-1}}}\right)dx\\
&\qquad+\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{q_i}\left(\varphi^2+\epsilon^{\gamma^+}-\epsilon^{\gamma^+}\frac{|\varphi|^{\frac{q_i}{q_i-1}}}{\left(\|\varphi\|_{L^\infty(\Omega)}+1\right)^{\frac{\max\{q_N,p_N\}}{\min\{q_1,p_1\}-1}}}\right)dx\\
&\quad\ge\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{p_i}\varphi^2dx+\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{q_i}\varphi^2dx,
\end{aligned}\tag{4.10}$$

since the inequality

$$\frac{|\varphi|^{\frac{m_i}{m_i-1}}}{\left(\|\varphi\|_{L^\infty(\Omega)}+1\right)^{\frac{\max\{q_N,p_N\}}{\min\{q_1,p_1\}-1}}}<1$$

holds for $m_i = q_i$ or $m_i = p_i$ for every $i = 1,\dots,N$, it follows that, in view of (4.9) and (4.10), we obtain

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i u_n|^{p_i}\varphi^2dx+\sum_{i=1}^{N}\int_{\Omega}|\partial_i u_n|^{q_i}\varphi^2dx\le\mathfrak{C}_3(f,\varphi).$$

Thus, the sequence $\{u_n\}_n$ is bounded in X_{loc} as desired. □

Minor adjustments in the proof of Theorem 1.2 allow us to prove the existence of u in X_{loc}.

Proof of Theorem 1.5.

By using $(u-\delta)^+$ as a test function in problem (1.1), we demonstrate that for all $\gamma(x) > 0$ and $f \in L^r(\Omega)$ for some suitable r, the solution to problem (1.1) belongs to $L^\infty(\Omega)$.

Let u be the solution for problem (1.1). Taking $(u-\delta)^+$ as a test function in (1.1) with $\delta > 0$, we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i (u-\delta)^+ dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{q_i-2}\partial_i u \partial_i (u-\delta)^+ dx = \int_\Omega \frac{f(u-\delta)^+}{(u)^{\gamma(x)}} dx,$$

from which it follows that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i (u-\delta)^+|^{p_i} dx + \sum_{i=1}^{N} \int_\Omega |\partial_i (u-\delta)^+|^{q_i} dx \leq \int_\Omega f(u-\delta)^+ \left(kd(x)\right)^{-\gamma(x)} dx,$$

thus, we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i (u-\delta)^+|^{p_i} dx \leq \int_\Omega f(u-\delta)^+ \left(kd(x)\right)^{-\gamma(x)} dx,$$

which leads to the following inequality

$$\left(\int_\Omega |\partial_i (u-\delta)^+|^{p_i} dx \right)^{\frac{1}{p_i}} \leq \left(\int_\Omega f(u-\delta)^+ \left(kd(x)\right)^{-\gamma(x)} dx \right)^{\frac{1}{p_i}}. \tag{4.11}$$

Additionally, we employ the following Sobolev-type inequality, as shown in [13], there exists a constant $C > 0$, dependent only on the domain Ω, such that for any $s \in [1, \bar{p}^*]$:

$$\|\xi\|_{L^s(\Omega)} \leq C \prod_{i=1}^{N} \|\partial_i \xi\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}, \tag{4.12}$$

where $\xi \in W_0^{1,\vec{p}}(\Omega)$, and $p_i \geq 2$ for each $i = 1, 2, \dots, N$. From (4.11)–(4.12) and noting that $\min\{\bar{p}^*, \bar{q}^*\} \leq \bar{p}^*$, we obtain

$$\begin{aligned} \|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} &\leq C_0 \prod_{i=1}^{N} \left(\int_\Omega f(u-\delta)^+ \left(kd(x)\right)^{-\gamma(x)} dx \right)^{\frac{1}{Np_i}}, \\ &\leq C_0 \left(\int_\Omega f(u-\delta)^+ \left(kd(x)\right)^{-\gamma(x)} dx \right)^{\frac{1}{\bar{p}}}. \end{aligned}$$

Applying Hölder's inequality, we have

$$\begin{aligned} &\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \\ &\leq C_0 \left(\left[\int_{\Lambda_\delta} |f|^{\min\{\bar{p}^*,\bar{q}^*\}'} \left(kd(x)\right)^{-\gamma(x)} dx \right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}} \|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \right)^{\frac{1}{\bar{p}}} \\ &\leq C_0 \left(\left[\int_{\Lambda_\delta} |f|^{\min\{\bar{p}^*,\bar{q}^*\}'} dx \right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}} \| \left(kd(\cdot)\right)^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}} \|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \right)^{\frac{1}{\bar{p}}}, \end{aligned}$$

with $\Lambda_\delta = \{x \in \Omega : \ u(x) > \delta\}$.

Now, applying Young's inequality to the right-hand side of the last inequality, let $\epsilon > 0$, we have

$$\begin{aligned}&\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)}\\ &\leq C_\epsilon \| (kd(\cdot))^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \left[\int_{\Lambda_\delta} |f|^{\min\{\bar{p}^*,\bar{q}^*\}'} dx\right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{\bar{p}'}{\bar{p}}} + \frac{\epsilon}{\bar{p}}\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)},\end{aligned}$$

which implies

$$\left(1-\frac{\epsilon}{\bar{p}}\right)\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \leq C_1 \| (kd(\cdot))^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \left[\int_{\Lambda_\delta} |f|^{\min\{\bar{p}^*,\bar{q}^*\}'} dx\right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{\bar{p}'}{\bar{p}}}.$$

By choosing ϵ such that $1-\frac{\epsilon}{\bar{p}} > 0$, we can infer that

$$\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \leq C_2 \| (kd(\cdot))^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \left[\int_{\Lambda_\delta} |f|^{\min\{\bar{p}^*,\bar{q}^*\}'} dx\right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}.$$

Applying Hölder's inequality with exponents $\frac{r}{\min\{\bar{p}^*,\bar{q}^*\}'}$ and $\left(\frac{r}{\min\{\bar{p}^*,\bar{q}^*\}'}\right)'$ yields

$$\begin{aligned}&\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)}\\ &\leq C_3 \| (kd(\cdot))^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \left[\|f\|_{L^r(\Omega)}^{\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}} \operatorname{meas}(\Lambda_\delta)^{1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}}\right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}.\end{aligned}$$

Utilizing the fact that $f \in L^r(\Omega)$, where $r \geq \frac{\min\{\bar{p}^*,\bar{q}^*\}}{\min\{\bar{p}^*,\bar{q}^*\}-\max\{\bar{p},\bar{q}\}}$, we have

$$\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \leq C_4 \| (kd(\cdot))^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \operatorname{meas}(\Lambda_\delta)^{\left[1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}\right]\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}. \tag{4.13}$$

Again, by Hölder's inequality, the following inequality holds

$$\int_\Omega (u-\delta)^+ dx \leq \|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \operatorname{meas}(\Lambda_\delta)^{1-\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}}}. \tag{4.14}$$

Thus, from (4.13) and (4.14), we get

$$\begin{aligned}&\int_\Omega (u-\delta)^+ dx\\ &\leq C_5 \| (kd(\cdot))^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \operatorname{meas}(\Lambda_\delta)^{\left[1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}\right]\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}} \operatorname{meas}(\Lambda_\delta)^{1-\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}}},\\ &\leq C_5 \| (kd(\cdot))^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \operatorname{meas}(\Lambda_\delta)^{1-\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}}+\left[1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}\right]\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}.\end{aligned} \tag{4.15}$$

Defining $\tau(\delta) = \int_\Omega (u-\delta)^+ dx$, it follows straightforwardly that $\tau'(\delta) = -\operatorname{meas}(\Lambda_\delta)$, where $\Lambda_\delta = \{x \in \Omega : \ u(x) > \delta\}$ (see [26]). Thus, the previous inequality becomes

$$\tau(\delta)^{\frac{1}{1-\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}}+\left[1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}\right]\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}} \leq -C_6\tau'(\delta). \tag{4.16}$$

Let us define the parameter α as follows

$$\alpha = \frac{1}{1-\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}}+\left[1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}\right]\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}.$$

Using this expression in combination with equation (4.16), we can further deduce

$$1 \leq -C_6 \tau'(\delta)\tau(\delta)^{-\alpha} = \frac{-C_6}{1-\alpha}\left(\tau(\delta)^{1-\alpha}\right)'.$$

Observe that assumption $r \geq \frac{\min\{\bar{p}^*,\bar{q}^*\}}{\min\{\bar{p}^*,\bar{q}^*\}-\max\{\bar{p},\bar{q}\}}$ implies $1-\alpha > 0$. Integrating the last inequality from 0 to δ, we obtain

$$\delta \leq -C_6\left(\tau(\delta)^{1-\alpha} - \tau(0)^{1-\alpha}\right),$$

which leads to

$$C_6\tau(\delta)^{1-\alpha} \leq -\delta + C_6\|u\|^{1-\alpha}_{L^1(\Omega)}.$$

From this inequality and the fact that $\tau(\delta)$ is a non-negative and decreasing, there exists a δ_0 such that $\tau(\delta_0) = 0$. Therefore, $\|u\|_{L^\infty(\Omega)} \leq C_{\delta_0}$, which implies $u \in L^\infty(\Omega)$. □

5 Concluding remarks and future directions

As kindly suggested by one of the referees of this paper, we intend to further develop and extend the analysis presented here to singular double anisotropic variable exponent problems of the form:

$$\begin{cases} -\sum_{i=1}^{N}\partial_i\left(|\partial_i u|^{p_i(x)-2}\partial_i u\right) - \sum_{i=1}^{N}\partial_i\left(|\partial_i u|^{q_i(x)-2}\partial_i u\right) = \dfrac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^N$ is a rectangular domain with dimension $N \geq 3$, $\gamma(x)$ is a positive function in $C(\overline{\Omega})$, and $f(x)$ belongs to $L^r(\Omega)$ for some $r \geq 1$.

The vector functions $\vec{p},\vec{q} : \bar{\Omega} \to \mathbb{R}^N$ are defined as

$$\vec{p}(x) = (p_1(x),\ldots,p_N(x)), \qquad \vec{q}(x) = (q_1(x),\ldots,q_N(x)),$$

where p_i and q_i belong to $C_+(\overline{\Omega})$ and satisfy certain conditions.

The energy functional associated with this model contains the unbalanced variational integral

$$u \longmapsto \sum_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}|\partial_i u|^{p_i(x)}\,dx + \sum_{i=1}^{N}\int_\Omega \frac{1}{q_i(x)}|\partial_i u|^{q_i(x)}\,dx. \tag{5.2}$$

This functional provides a sharper version of the classical energy

$$u \longmapsto \sum_{i=1}^{N}\frac{1}{p_i}\int_\Omega |\partial_i u|^{p_i}\,dx + \sum_{i=1}^{N}\frac{1}{q_i}\int_\Omega |\partial_i u|^{q_i}\,dx.$$

These problems allow for a more precise description of heterogeneous and anisotropic physical phenomena in fields such as nonlinear elasticity, material science, and homogenization. They are particularly useful for modeling composite materials with spatially varying properties, where the material behavior depends on both location and direction.

In nonlinear elasticity and material science, composite materials with locally varying hardening exponents $\vec{p}(x)$ and $\vec{q}(x)$ can be effectively characterized using the energy functional in (5.2). Furthermore, such problems have important applications in elasticity, homogenization, the modeling of strongly anisotropic materials, the Lavrentiev phenomenon, and other related areas.

Inspired by the seminal work of Chems Eddine and Repovš [17], we observe that the main results in this study can be extended to more general anisotropic problems involving variable exponents as in (5.1). Specifically, we extend the regularity of solutions to anisotropic Hölder continuous function spaces $C^{0,\overrightarrow{\beta}(\cdot)}(\overline{\Omega})$ over rectangular domains. In their work, Chems Eddine and Repovš introduced a novel framework for embedding anisotropic variable exponent Sobolev spaces into anisotropic variable exponent Hölder continuous function spaces in rectangular domains. Their results generalize classical Sobolev embedding theorems by incorporating anisotropic settings with variable exponents, providing deeper insights into function regularity in different spatial directions. Under log-Hölder continuity conditions on the variable exponents $\overrightarrow{p}(\cdot)$.

A promising new research direction focuses on anisotropic operators with variable exponents and singular reactions. Within this framework, we aim to extend the qualitative analysis conducted in this paper to singular nonlinear boundary value problems with variable exponents, modeled by the following system:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i(x)-2} \partial_i u \right) = \dfrac{f(x)}{u^{\gamma(x)}} + \mathfrak{F}_u(x,u,v) & \text{in } \Omega, \\ -\sum_{i=1}^{N} \partial_i \left(|\partial_i v|^{q_i(x)-2} \partial_i v \right) = \dfrac{g(x)}{v^{\beta(x)}} + \mathfrak{F}_v(x,u,v) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

This anisotropic system, which features unbalanced growth, was introduced by Figueiredo and Silva [22], where the vectors $\vec{p}$ and $\vec{q}$ are independent of x.

We conclude by noting that a key characteristic of nonlinear problems with variable exponents is the possibility of a subcritical-critical-supercritical multiple regime. A particularly intriguing open problem involves analyzing the singular case of the anisotropic system described in problem (5.3) within this multiple regime. A closely related and highly intriguing research direction involves anisotropic systems with variable exponent operators and singular reactions. These systems exhibit complex behaviors, particularly in the presence of unbalanced growth conditions and multiple regime transitions. Exploring their qualitative properties, existence results, and regularity aspects presents significant mathematical challenges and potential applications in nonlinear elasticity, material science, and phase transition models.

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