Electronic Journal of Qualitative Theory of Differential Equations
2025, No. **22**, 1–16; https://doi.org/10.14232/ejqtde.2025.1.22 www.math.u-szeged.hu/ejqtde/

Bifurcation in two parameters for a quasilinear Schrödinger equation

David Arcoya[1], **José Carmona**[2] and **Miguel Martínez-Teruel**[✉][1]

[1]Universidad de Granada, Avda. Fuentenueva s/n, Granada, 18071, Spain
[2]Universidad de Almería, Ctra. Sacramento s/n, Almería, 04120, Spain

Received 4 December 2024, appeared 20 May 2025
Communicated by Roberto Livrea

Abstract. This paper deals with existence and multiplicity of positive solutions for the quasilinear Schrödinger equation

$$\begin{cases} -\Delta u - \lambda m(x) u \Delta(u^2) = f(\mu, x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary and m is a bounded non negative continuous function. Under suitable assumptions on the asymptotically linear f, we use bifurcation theory to analyze the set of positive solutions.

Keywords: bifurcation theory, Schrödinger equation.

2020 Mathematics Subject Classification: 35J10, 35B32, 34K18.

1 Introduction

The quasilinear Schrödinger equation

$$i\partial_t z = -\Delta z + V_0(x)z - z(\Delta|z|^2) - \theta|z|^{p-1}z, \tag{1.1}$$

known as the superfluid film equation in plasma physics due to Kurihara ([9]), has been studied by Poppenberg, Schmitt and Wang in [10]. Here the potential V_0 is a function $V_0 \colon \mathbb{R}^N \to \mathbb{R}$, while $\theta > 0$, $p > 1$ are constants. Using the change of variables $z(t,x) = \exp\{-i\kappa t\}u(x)$ for $\kappa \in \mathbb{R}$ and putting $V = V_0 - \kappa$, they obtain the elliptic equation

$$-\Delta u - u(\Delta|u|^2) = \theta|u|^{p-1}u - V(x)u, \tag{1.2}$$

corresponding to the standing wave solutions of (1.1). In that work, variational techniques are applied to prove the existence of standing wave solutions. Also, readers can find more references in [10] about the physical meanings of the equation.

✉Corresponding author. Email: mmteruel@ugr.es

In this work we study positive solutions of

$$\begin{cases} -\Delta u - \lambda m(x) u \Delta(u^2) = f(\mu, x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_μ^λ}$$

where Ω is a bounded open domain with smooth boundary, λ and μ are real constants, $m(x)$ is a non negative bounded continuous function in $\overline{\Omega}$ and $f\colon \mathbb{R} \times \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ is a continuous function such that $f(\mu, x, 0) = 0$ for every $x \in \overline{\Omega}$ and $\mu \in \mathbb{R}$.

We emphasize that the equation in (P_μ^λ) is more general than (1.2). Observe also that when $\lambda = 0$ the problem (P_μ^0) is just a classical semilinear problem which has been extensively studied. For the case of asymptotically linear datum f at 0 see [2]. The case of the quasilinear problem (P_μ^λ) with $m(x) = 1$ has been studied for $\lambda > 0$ in several works made subsequently after [10], see for instance [6–8] and references therein. In all these works, the problem (with $m(x) = 1$) is variational and one of the main ideas is to reduce the problem to a semilinear one by using the change of variables $u = g(v)$ where g is the unique solution of the differential equation

$$g' = \frac{1}{\sqrt{1 + 2\lambda g^2}}, \qquad g(0) = 0. \tag{1.3}$$

Contrary to the cited previous works, here λ can be either positive or negative, implying an accelerating ($\lambda > 0$) or slowdown ($\lambda < 0$) process for the non linear diffusion term. In addition, since we are considering nonconstant density $m(x)$, this term also increases or decreases this accelerating-slowdown process depending on the point of the domain. Note that even in the simplest case $m(x) = 1$, the uniform ellipticity of the operator (which may be rewritten in divergence form as $-\mathrm{div}((1 + 2\lambda u^2)\nabla u) + 2\lambda u |\nabla u|^2$) can be lost when λ is negative.

Moreover, although the change of variables (1.3) is still available for negative λ when $m(x) = 1$, it leads to a semilinear problem of a completely different nature that in the case positive λ. Indeed, in that case, the nonlinear term has an asymptote and the main properties of this nonlinear term, as used in [6–8] and references therein, are no longer satisfied for negative λ. In the general case where $m(x)$ is not constant, no similar change of variable becomes the problem semilinear.

To overcome these difficulties, for every fixed $\lambda \in \mathbb{R}$ we see that problem (P_μ^λ) is equivalent to

$$\Phi_\mu^\lambda(u) = 0, \quad u \in \mathcal{U}_\lambda = \{u \in C_0^1(\overline{\Omega}) : 1 + 2\lambda m(x) u^2(x) > 0, x \in \Omega\};$$

i.e., to find the ceros u of an operator Φ_μ^λ defined in the open subset $\mathcal{U}_\lambda$ of the Banach space $C_0^1(\overline{\Omega})$ of the differentiable functions with continuous derivative in $\overline{\Omega}$ which vanishes on the boundary $\partial\Omega$. Notice that $\mathcal{U}_\lambda = C_0^1(\overline{\Omega})$ if $\lambda \geq 0$, while $\mathcal{U}_\lambda \subsetneq C_0^1(\overline{\Omega})$ when $\lambda < 0$. Because of this we prove in Theorem 3.1 an extension of the global bifurcation theorem of Rabinowitz ([11]) for operators (depending on the parameter μ) which are defined in open subsets of a Banach space.

Next, assuming that $f(\mu, x, s)$ is asymptotically linear at $s = 0$; i.e.,

$$\lim_{s \to 0^+} \frac{f(\mu, x, s)}{s} = \mu f'_+(x, 0) \in L^\infty(\Omega),$$

uniformly in (μ, x), with $f'_+(x, 0)$ nonnegative and not identically zero and denoting by $\mu_1(f'_+)$ the first eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta\varphi = \mu f'_+(x, 0)\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

we apply the cited extension theorem to obtain in Theorem 3.3 that the closure $\overline{\Sigma}$ of the nontrivial solution set of all $(\lambda, \mu, u) \in \mathbb{R} \times \mathbb{R} \times C_0^1(\overline{\Omega})$ such that $u \not\equiv 0$ solves (P_μ^λ); i.e.,

$$\overline{\Sigma} = \mathrm{cl}\{(\lambda, \mu, u) \in \mathbb{R} \times \mathbb{R} \times C_0^1(\overline{\Omega}) : u \text{ solves } (P_\mu^\lambda), u \not\equiv 0\}$$

intersects the plane $\pi_0 = \{(\lambda, \mu, u) \in \mathbb{R} \times \mathbb{R} \times C_0^1(\overline{\Omega}) : u \equiv 0\}$ in the line

$$\overline{\Sigma} \cap \pi_0 = r_0 \equiv \{(\lambda, \mu_1(f'_+), 0) : \lambda \in \mathbb{R}\}.$$

Moreover, there exists a continuum $C \subset \overline{\Sigma}$ such that $r_0 \subset C$ and, for every $\lambda \in \mathbb{R}$, $C \setminus \{(\lambda, \mu_1(f'_+), 0)\}$ contains a connected subset $\{\lambda\} \times C_\lambda$ of nontrivial solutions.

Afterwards, for each fixed λ in $\mathbb{R}$ we study in Theorem 4.1 the laterality of the branch C_λ of positive solutions emanating from $(\mu_1(f'_+), 0)$. Following [3, 4], the behavior of $f(\mu, x, s)$ when (μ, s) is close to $(\mu_1(f'_+), 0)$ determines the laterality. Specifically, if we suppose the existence of the limit

$$\lim_{(\mu,s)\to(\mu_1(f'_+),0^+)} \frac{f(\mu, x, s) - \mu f'_+(x, 0)s}{s^3} = G(x) \quad \text{uniformly in } x \in \overline{\Omega},$$

then the bifurcation of positive solutions from $(\mu_1(f'_+), 0)$ is subcritical (i.e., to the left of $\mu = \mu_1(f'_+)$) provided that

- either $G(x) = +\infty$ a.e. $x \in \Omega$
- or $0 < \int_\Omega G(x)\varphi_1(x)^4 < \infty$, with φ_1 the positive eigenfunction of norm 1 for (1.4), and $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x, 0)\varphi_1(x)^2] = 0$.

A similar supercriticality laterality, i.e., to the right, is also shown when either $G(x) = -\infty$ a.e. $x \in \Omega$ or $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)\varphi_1(x)^2] = 0$ and $-\infty < \int_\Omega G(x)\varphi_1(x)^4 < 0$. Roughly speaking, in these cases the laterality is the same that the one of the semilinear problem corresponding to the case $m(x) = 0$ and does not depends on the value of λ.

The case $G\varphi_1^4 \in L^1(\Omega)$, $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x, 0)\varphi_1(x)^2] \neq 0$ is more delicate due to the key role played by the quasilinear term $\lambda m(x)u\Delta(u^2)$. Indeed, if

$$\lambda^* = -\frac{1}{2}\int_\Omega G(x)\varphi_1(x)^4 \left[\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x, 0)\varphi_1(x)^2]\right]^{-1}, \tag{1.5}$$

and $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x, 0)\varphi_1(x)^2] > 0$, then the bifurcation of positive solution at $\mu_1(f'_+)$ is

- subcritical for $\lambda > \lambda^*$ and
- supercritical for $\lambda < \lambda^*$.

A similar result is also shown when the reverse inequality is satisfied, i.e. in the case $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x, 0)\varphi_1(x)^2] < 0$. We observe that λ^* may be positive, negative or zero.

For our knowledge this new phenomenon on the dependence of the laterality according to the value of λ is new and may cause significant changes in the existence of solutions of (P_λ^μ). As an example of the consequences of this phenomenon we study the simpler case $m(x) = 1$ and $f(\mu, x, u) = \mu u - u^p$ with $p > 1$ (logistic case). Specifically, we recover the previous results obtained by [8] proving uniqueness of positive solution when $\lambda > 0$. In contrast, we handle also the new case $\lambda < 0$ and prove the multiplicity of positive solutions when $\lambda < \lambda^* < 0$ and close to λ^*.

The outline of the paper is as follows. In Section 2, we study some properties of the open sets $\mathcal{U}_\lambda$ and extend the function $f(\mu, x, s)$ for $s < 0$ to ensure that every solution of the problem (P_μ^λ) is nonnegative. In Section 3, we prove the extension of the Rabinowitz's global bifurcation theorem for operators defined in open subsets and we apply it to show the existence of the continuum of positive solutions. In Section 4, we study the laterality of the branches of positive solutions obtained in Section 3. Finally, in Section 5, we discuss the logistic case.

2 Preliminaries

Let Ω be a bounded open domain with smooth boundary. Assume that $m(x)$ is a non negative bounded continuous function in $\overline{\Omega}$ and

(f_0) $f\colon \mathbb{R} \times \overline{\Omega} \times [0, +\infty) \to \mathbb{R}$ is a continuous function such that

$$f(\mu, x, 0) = 0 \quad \text{for every } x \in \overline{\Omega} \text{ and } \mu \in \mathbb{R}.$$

Extending f to $\mathbb{R} \times \overline{\Omega} \times \mathbb{R}$ by taking

$$f(\mu, x, s) = f(\mu, x, 0) = 0, \quad \text{for every } \mu \in \mathbb{R},\ x \in \overline{\Omega} \text{ and } s < 0, \tag{2.1}$$

we deduce that every weak solution of (P_μ^λ) is non negative.

Lemma 2.1. *Let f be a continuous function in $\mathbb{R} \times \overline{\Omega} \times [0, +\infty)$ satisfying (f_0) and extended by zero as (2.1). Let $u \in H_0^1(\Omega)$ be a solution of (P_μ^λ) for certain $\lambda, \mu \in \mathbb{R}$. Then, the following holds.*

1. *If $u \in \mathcal{U}_\lambda$, then $u(x) \geq 0$ for every $x \in \Omega$.*
2. *Moreover, if $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $f(\mu, \cdot, \cdot)$ is C^1 in $\overline{\Omega} \times \mathbb{R}$ and u is a nonzero solution of (P_μ^λ) with $1 + 2\lambda m(x) u(x)^2 > 0$, $x \in \overline{\Omega}$, then*

$$u(x) > 0, \quad \forall x \in \Omega \quad \textit{and} \quad \frac{\partial u}{\partial n}(x) < 0, \quad \forall x \in \partial\Omega, \tag{2.2}$$

where $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of u at the point $x \in \partial\Omega$.

Proof. Let us prove part *(1)*. Observe first that (P_μ^λ) reduces to

$$\begin{cases} -\Delta u = \dfrac{f(\mu, x, u) + 2\lambda m(x) u |\nabla u|^2}{1 + 2\lambda m(x) u^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Since $u \in \mathcal{U}_\lambda$ we can take $\varphi = u^- = \min\{u, 0\} \in H_0^1(\Omega)$ as a test function in (2.3) and using that $f(\mu, x, s) = 0$ for every $\mu \in \mathbb{R}$, $x \in \Omega$ and $s \leq 0$ we have

$$\int_\Omega |\nabla u^-|^2 = \int_\Omega \frac{2\lambda m(x)(u^-)^2}{1 + 2\lambda m(x)(u^-)^2} |\nabla u^-|^2 = \int_\Omega |\nabla u^-|^2 - \int_\Omega \frac{|\nabla u^-|^2}{1 + 2\lambda m(x)(u^-)^2}.$$

Then, we obtain that $u^- = 0$; i.e., $u \geq 0$.

For the proof of the part *(2)*, since f is C^1 in $\overline{\Omega}\times\mathbb{R}$, there exists $C>0$ such that $f(\mu,x,s)+Cs$ is increasing in $s\in[0,\|u\|_\infty]$. Denoting $h(x)=f(\mu,x,u(x))+Cu(x)$, we have that $0\lneq h(x)\in C^1_0(\overline{\Omega})$.

Now, we distinguish cases depending on the sign of λ. If $\lambda\geq 0$ then u satisfies

$$\begin{cases} -\Delta u+\dfrac{C}{1+2\lambda m(x)u^2}u=\dfrac{h(x)+2\lambda m(x)u|\nabla u|^2}{1+2\lambda m(x)u^2}\gneq 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Since $\frac{C}{1+2\lambda m(x)u^2}$ is positive and bounded by the regularity of u and m, we can apply the strong maximum principle to conclude that u belongs to the interior of the cone of nonnegative solutions

$$P=\left\{u\in C^1_0(\overline{\Omega})\ :\ u \text{ is a nonnegative solution of (2.3)}\right\};$$

i.e., u satisfies (2.2).

In case $\lambda<0$, then u satisfies

$$\begin{cases} -\Delta u+\left(\dfrac{C}{1+2\lambda m(x)u^2}+\dfrac{2(-\lambda)m(x)|\nabla u|^2}{1+2\lambda m(x)u^2}\right)u=\dfrac{h(x)}{1+2\lambda m(x)u^2}\gneq 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Since $\frac{C}{1+2\lambda m(x)u^2}+\frac{2(-\lambda)m(x)|\nabla u|^2}{1+2\lambda m(x)u^2}$ is positive and bounded by the regularity of u and m, we can apply the strong maximum principle to conclude that u belongs to the interior of P. □

3 Continuum of positive solutions

Let $(X,\|\cdot\|)$ be a real Banach space, let $\mathcal{U}$ be an open set of X with $0\in\mathcal{U}$ and let $K\colon\mathbb{R}\times\mathcal{U}\to\mathbb{R}$ be a compact operator. For $\mu\in\mathbb{R}$, we consider the equation

$$\Phi(\mu,u)=u-K(\mu,u)=0,\quad u\in\mathcal{U}. \tag{3.1}$$

The operator K can be viewed as a family of compact operators $\{K_\mu:\mu\in\mathbb{R}\}$ by taking

$$K_\mu(u)=K(\mu,u),\quad u\in\mathcal{U}.$$

Similarly, we denote $\Phi_\mu=I-K_\mu$.

We also define

$$\Sigma=\{(\mu,u)\in\mathbb{R}\times\mathcal{U}\colon \Phi(\mu,u)=0, u\neq 0\},$$

and denote by Σ_μ the μ-slide of Σ, that is, $\Sigma_\mu=\{u\in\mathcal{U}\colon (\mu,u)\in\Sigma\}$.

Before showing the next result, let us recall that μ^* is a bifurcation point of (3.1) if there exists a sequence $(\mu_n,u_n)\in\mathbb{R}\times\mathcal{U}$ with $u_n\neq 0$, such that $\mu_n\to\mu^*$ and $\Phi(\mu_n,u_n)=0$.

The following theorem is a slight improvement of the global bifurcation theorem of Rabinowitz of [11] (see Theorem 1.3 and Corollary 1.12). In fact, with our assumptions it will be easy to arrive at the same point in which the proof of Theorem 6.3.1 of [1] is applicable.

Theorem 3.1. *Let $\mu^*\in\mathbb{R}$ and $\varepsilon_0>0$ be such that the set $(\mu^*-\varepsilon_0,\mu^*+\varepsilon_0)\backslash\{\mu^*\}$ does not contain any bifurcation point from (3.1). Assume that the Leray–Schauder topological index verifies*

$$i(\Phi_{\underline{\mu}},0)\neq i(\Phi_{\overline{\mu}},0),\quad \textit{for some } \underline{\mu}\in(\mu^*-\varepsilon_0,\mu^*),\ \overline{\mu}\in(\mu^*,\mu^*+\varepsilon_0).$$

Then the connected component $\mathcal{C}$ of $\overline{\Sigma}$ that contains $(\mu^*,0)$ satisfies at least one of the following properties:

(a) $(\mathbb{R} \times \partial\mathcal{U}) \cap C \neq \emptyset$.

(b) There exists $\mu^\# \neq \mu^*$ *such that* $(\mu^\#, 0) \in C$*, i.e.* $\mu^\#$ *is another bifurcation point from zero.*

(c) C *is unbounded.*

Proof. First, it is well known that a change of the index in μ^* implies that μ^* is a bifurcation point. Let C be the connected component of $\overline{\Sigma}$ that contains $(\mu^*, 0)$. Proceeding by contradiction, let us assume that C verifies neither (a) nor (b) nor (c). Denying property (a) implies that there is a neighborhood U_δ of C such that $(\mathbb{R} \times \partial\mathcal{U}) \cap U_\delta = \emptyset$. Then, knowing that C is far from $\mathbb{R} \times \partial\mathcal{U}$ the proof follows same as Theorem 6.3.1 of [1]. □

For fixed $\lambda \in \mathbb{R}$, we define the set

$$\mathcal{U}_\lambda = \{u \in C_0^1(\overline{\Omega}) : 1 + 2\lambda m(x)u^2(x) \geq m_0, x \in \Omega, \text{ for some } m_0 > 0\}.$$

The following result summarizes the main properties of this set.

Lemma 3.2. *For every* $\lambda \in \mathbb{R}$ *the set* $\mathcal{U}_\lambda$ *is open in* $C_0^1(\overline{\Omega})$.

Proof. Observe that $\mathcal{U}_\lambda = C_0^1(\overline{\Omega})$ for $\lambda \geq 0$. Assume now that $\lambda < 0$, $u \in \mathcal{U}_\lambda$ and $\|u - v\|_1 \leq r$. Then

$$|v(x)| - |u(x)| \leq ||u(x)| - |v(x)|| \leq \|u - v\|_\infty \leq r, \quad x \in \overline{\Omega},$$

and thus

$$1 + 2\lambda m(x)v^2(x) \geq 1 + 2\lambda m(x)(r + u^2(x)) \geq m_0 + 2\lambda\|m\|_\infty r,$$

where $m_0 > 0$ is the minimum that attains the continuous function $1 + 2\lambda m(x)u^2(x)$ in $\overline{\Omega}$. Taking $r < \frac{m_0}{-2\lambda\|m\|_\infty}$ we have that $v \in \mathcal{U}_\lambda$ for every $v \in C_0^1(\overline{\Omega})$ with $\|u - v\|_1 \leq r$. □

Now, we consider the Nemitski operator $N^\lambda : \mathbb{R} \times \mathcal{U}_\lambda \to C_0(\overline{\Omega})$ given by

$$N^\lambda(\mu, u) = \frac{f(\mu, x, u) + 2\lambda m(x)u|\nabla u|^2}{1 + 2\lambda m(x)u^2}.$$

Since f and m are continuous and $u \in C_0^1(\overline{\Omega})$ we have that N^λ is a continuous operator. Also, we recall the compactness of the inverse of the Laplacian $(-\Delta)^{-1} : C_0(\overline{\Omega}) \to C_0^1(\overline{\Omega})$.

Thus, the operator $K^\lambda : \mathbb{R} \times \mathcal{U}_\lambda \to C_0^1(\overline{\Omega})$ given by

$$K^\lambda(\mu, u) = (-\Delta)^{-1}\left[N^\lambda(\mu, u)\right], \quad \mu \in \mathbb{R}, u \in \mathcal{U}_\lambda,$$

is compact.

We define $\Phi_\mu^\lambda : \mathcal{U}_\lambda \to C_0^1(\overline{\Omega})$ given by

$$\Phi_\mu^\lambda(u) = u - K^\lambda(\mu, u), \quad \forall u \in \mathcal{U}_\lambda.$$

In this way, the solutions of (P_μ^λ) are zeros of Φ_μ^λ.

Finally, we denote by $\mu_1(f'_+)$ the first eigenvalue and φ_1 the positive eigenfunction of norm 1 of the eigenvalue problem (1.4).

Theorem 3.3. *Let* f *be a continuous function in* $\mathbb{R} \times \overline{\Omega} \times [0, +\infty)$ *satisfying* (f_0)*, and*

(f_1) *For each bounded set Γ in $\mathbb{R} \setminus \{0\}$ and $\mu \in \Gamma$, we have*

$$\lim_{s \to 0^+} \frac{f(\mu, x, s)}{s} = \mu f'_+(x, 0), \quad \textit{uniformly in } (\mu, x) \in \Gamma \times \Omega,$$

with $0 \leq f'_+(x, 0) \in L^\infty(\Omega)$ not identically zero.

Then, for every $\lambda \in \mathbb{R}$, $\mu_1(f'_+)$ is a bifurcation point from zero of the problem (P^λ_μ). Moreover, there exists a continuum C of positive solutions, i.e., a closed and connected subset of the closure $\overline{\Sigma}$ of the set

$$\Sigma = \{(\lambda, \mu, u) \in \mathbb{R} \times \mathbb{R} \times C^1_0(\overline{\Omega}) : \ u \in \mathcal{U}_\lambda, \ \Phi^\lambda_\mu(u) = 0, \ u \neq 0\}$$

with $(\lambda, \mu_1(f'_+), 0) \in C$ for every $\lambda \in \mathbb{R}$.

Remark 3.4. In addition, the continuum C provided by Theorem 3.3 satisfies that the slice

$$C_\lambda = \{(\mu, u) \in \mathbb{R} \times C^1_0(\overline{\Omega}) \ : \ (\lambda, \mu, u) \in C\}$$

satisfies at least one of the following conditions:

- C_λ is unbounded (which is always the case for $\lambda \geq 0$).
- $C_\lambda \cap \mathbb{R} \times \partial \mathcal{U}_\lambda \neq \emptyset$.

Proof. We divide the proof into the following steps.

- Step 1. For every compact set $\Gamma \subset \mathbb{R}^2 \backslash \{(\lambda, \mu_1(f'_+)) : \lambda \in \mathbb{R}\}$ there exists $\varepsilon > 0$ such that

  $$\Phi^\lambda_\mu(u) \neq 0, \quad \forall (\lambda, \mu) \in \Gamma, \ \forall u \in \mathcal{U}_\lambda \text{ with } 0 < \|u\|_1 < \varepsilon,$$

  where $\|u\|_1$ denotes the norm of u in $C^1(\Omega)$.

- Step 2. If $\mu < \mu_1(f'_+)$, it is satisfied

  $$i(\Phi^\lambda_\mu, 0) = 1, \quad \forall \lambda \in \mathbb{R}.$$

- Step 3. If $\mu > \mu_1(f'_+)$, it is satisfied

  $$i(\Phi^\lambda_\mu, 0) = 0, \quad \forall \lambda \in \mathbb{R}.$$

To prove Step 1, we proceed by contradiction. Let Γ be a compact subset of $\mathbb{R}^2 \setminus \{(\lambda, \mu_1(f'_+)) \ : \ \lambda \in \mathbb{R}\}$ and let us assume that there exists a sequence (λ_n, μ_n, u_n) with $(\lambda_n, \mu_n) \in \Gamma$ and $u_n \in \mathcal{U}_{\lambda_n}$ satisfying

$$\lambda_n \to \lambda, \quad \mu_n \to \mu \neq \mu_1(f'_+), \quad \|u_n\|_1 \to 0,$$

$$\Phi^{\lambda_n}_{\mu_n}(u_n) = 0, \quad u_n \geq 0.$$

Since $\Phi^{\lambda_n}_{\mu_n}(u_n) = 0$, we have that

$$u_n = (-\Delta)^{-1} \left[\frac{f(\mu_n, x, u_n) + 2\lambda_n m(x) u_n |\nabla u_n|^2}{1 + 2\lambda_n m(x) u_n^2} \right].$$

Dividing by $\|u_n\|_1$ we come to the fact that

$$\frac{u_n}{\|u_n\|_1} = (-\Delta)^{-1}\left[\frac{f(\mu_n, x, u_n) + 2\lambda_n m(x) u_n |\nabla u_n|^2}{\|u_n\|_1(1+2\lambda_n m(x)u_n^2)}\right]. \tag{3.2}$$

Since $\|u_n\|_1 \to 0$, we have that $u_n \to 0$. Then, there exists $n_0 > 0$ such that

$$\frac{1}{2} < 1 + 2\lambda_n m(x) u_n^2, \quad \forall n \geq n_0.$$

Thus, we conclude

$$\frac{|f(\mu_n, x, u_n) + 2\lambda_n m(x) u_n |\nabla u_n|^2|}{\|u_n\|_1(1+2\lambda_n m(x)u_n^2)} < 2\left(\frac{|f(\mu_n, x, u_n)|}{\|u_n\|_1} + \frac{|2\lambda_n m(x) u_n |\nabla u_n|^2|}{\|u_n\|_1}\right)$$

and each term of the right hand side is bounded.

- Using (f_1) and that $f'_+(x,0) \in L^\infty(\Omega)$, $\frac{f(\mu_n,x,u_n)}{\|u_n\|_1}$ is bounded.
- Since λ_n is convergent, m is bounded, $|\nabla u_n|^2$ converges to zero and $\frac{u_n(x)}{\|u_n\|_1}$ is bounded by 1, the term $\frac{2\lambda_n m(x) u_n |\nabla u_n|^2}{\|u_n\|_1}$ is bounded.

Then, since the operator $(-\Delta)^{-1}$ is compact, up to a subsequence denoted the same way, $\frac{u_n}{\|u_n\|_1}$ converges strongly to a non negative function $\varphi \in C_0^1(\overline{\Omega})$. Taking limit in (3.2), we arrive to

$$\varphi = (-\Delta)^{-1}[\mu f'_+(x,0)\varphi].$$

So, φ is a positive solution with norm 1 of the problem (1.4). Therefore $\varphi = \varphi_1$ and $\mu = \mu_1(f'_+)$, which is a contradiction.

As a consequence of step 1, we have that for every $\lambda \in \mathbb{R}$ the unique possible bifurcation point from zero of Φ_μ^λ is $\mu_1(f'_+)$.

Moreover, further consequences of step 1 are

$$i(\Phi_\mu^\lambda, 0) = c_1, \quad \forall \mu < \mu_1(f'_+), \quad \forall \lambda \in \mathbb{R}$$

and

$$i(\Phi_\mu^\lambda, 0) = c_2, \quad \forall \mu > \mu_1(f'_+), \quad \forall \lambda \in \mathbb{R},$$

where c_1 and c_2 are constants.

To prove step 2, for every $t \in [0,1]$ we claim that, for each $t \in [0,1]$, there exists $\varepsilon > 0$ satisfying

$$u - (-\Delta)^{-1}[tf(0,x,u)] \neq 0, \quad \forall 0 < \|u\|_1 < \varepsilon.$$

Arguing by contradiction, let us suppose that there exists a sequence $u_n \in C_0^1(\overline{\Omega})$ such that

$$\|u_n\|_1 \to 0, \quad u_n \geq 0$$

and

$$u_n - t(-\Delta)^{-1}[f(0,x,u_n)] = 0. \tag{3.3}$$

Dividing (3.3) by $\|u_n\|_1$ we have that

$$\frac{u_n}{\|u_n\|_1} = t(-\Delta)^{-1}\left[\frac{f(0,x,u_n)}{\|u_n\|_1}\right]$$

Now, using (f_1) as before, we have that, up to a subsequence noted the same way, $\frac{u_n}{\|u_n\|_1}$ converges strongly to a non negative function $v \in C_0^1(\overline{\Omega})$ with norm 1. Taking limits in (3.3), we conclude that

$$v = t(-\Delta)^{-1}[0],$$

but, since $(-\Delta)^{-1}[0] = 0$, then $v = 0$ and we have a contradiction.

As a consequence of this fact, defining $\Psi_t(u) = u - (-\Delta)^{-1}[tf(0,x,u)]$, we have proved that

$$i(\Psi_t, 0) = \text{constant}, \quad \forall t \in [0,1].$$

Using that $\Psi_1 = \Phi_0^0$ and $\Psi_0 = \mathrm{Id}$, we deduce that

$$i(\Phi_\mu^\lambda, 0) = i(\Phi_0^0, 0) = i(\mathrm{Id}, 0) = 1, \quad \forall \mu < \mu_1(f'_+),\ \forall \lambda \in \mathbb{R}.$$

Finally we prove step 3. To do so, we will prove that for fixed $\lambda \in \mathbb{R}$ and $\mu > \mu_1(f'_+)$, there exists $\varepsilon > 0$ such that

$$\Phi_\mu^\lambda(u) \neq \tau\varphi_1, \quad \forall \tau > 0,\ \forall 0 < \|u\|_1 < \varepsilon.$$

Let us assume by contradiction that there exists a sequence $u_n \in C_0^1(\overline{\Omega})$ and $\tau_n \geq 0$ satisfying $u_n > 0$ in Ω, $\|u_n\|_1 \to 0$ and

$$\Phi_\mu^\lambda(u_n) = \tau_n\varphi_1.$$

Equivalently,

$$u_n = (-\Delta)^{-1}\left[\frac{f(\mu,x,u_n) + 2\lambda m(x)u_n|\nabla u_n|^2}{1 + 2\lambda m(x)u_n^2}\right] + \tau_n\varphi_1.$$

Dividing by $\|u_n\|_1$ and using the compactness of $(-\Delta)^{-1}$ we have that

$$(-\Delta)^{-1}\left[\frac{f(\mu,x,u_n) + 2\lambda m(x)u_n|\nabla u_n|^2}{\|u_n\|_1(1 + 2\lambda m(x)u_n^2)}\right]$$

is convergent. Adding that $\frac{u_n}{\|u_n\|_1}$ is bounded by 1, the sequence $\frac{\tau_n}{\|u_n\|_1}$ is bounded. Thus, taking another subsequence if it's necessary, we have that $\frac{\tau_n}{\|u_n\|_1} \to \tau \geq 0$ and $\frac{u_n}{\|u_n\|_1} \to v$ with $v \in C_0^1(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta v = \mu f'_+(x,0)v + \tau\mu_1(f'_+)f'_+(x,0)\varphi_1 & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega, \\ \|v\|_1 = 1. \end{cases}$$

Using the weak formulation and taking φ_1 as test function, we have that

$$\int_\Omega \nabla\varphi_1\nabla v = \int_\Omega \mu f'_+(x,0)v\varphi_1 + \tau\mu_1(f'_+)f'_+(x,0)\varphi_1^2.$$

Now, using the definition of φ_1 we have that

$$\int_\Omega \mu_1(f'_+)f'_+(x,0)v\varphi_1 = \int_\Omega \mu f'_+(x,0)v\varphi_1 + \tau\mu_1(f'_+)f'_+(x,0)\varphi_1^2.$$

Equivalently, we arrive to

$$\int_\Omega (\mu_1(f'_+) - \mu)f'_+(x,0)v\varphi_1 = \int_\Omega \tau\mu_1(f'_+)f'_+(x,0)\varphi_1^2,$$

where the term in the right hand side is non-positive and the corresponding one in the left hand side is non-negative. So, each term is zero and conclude that $\tau = 0$ and $v = 0$, giving a contradiction.

Therefore, we have that

$$i(\Phi_\mu^\lambda, 0) = i(\Phi_\mu^\lambda - \tau\varphi_1, 0), \quad \forall \tau > 0.$$

However, the problem

$$\begin{cases} -\Delta w = \mu f'_+(x,0)w + \tau\mu_1(f'_+)f'_+(x,0)\varphi_1 & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega, \end{cases}$$

has no trivial solution and neither solutions with small norm. Hence,

$$i(\Phi_\mu^\lambda, 0) = i(\Phi_\mu^\lambda - \tau\varphi_1, 0) = 0.$$

Fixed λ^* in $\mathbb{R}$, there is a change of index in $\mu_1(f'_+)$, generating a continuum of solutions emanating from the point $(\lambda^*, \mu_1(f'_+), 0)$ and that it is contained in the plane $\lambda = \lambda^*$. Because of the arbitrariness of λ^*, a continuum of solutions emanates from each point on the line $\{(\lambda, \mu, u) : \mu = \mu_1(f'_+), u = 0\}$. So, since every branch is connected by the straight line $\{(\lambda, \mu, u) : \mu = \mu_1(f'_+), u = 0\}$ a general continuum is generated. □

Remark 3.5. For the case $f'_+(x,0) = +\infty$ Theorem 3.3 also holds considering $\mu_1(f'_+) = 0$, see Remark 4.2 of [3].

4 Laterality

For a fixed $\lambda \in \mathbb{R}$, we have proved in Theorem 3.3, see also Remark 3.4, that a continuum $C_\lambda = \{(\mu, u) : (\lambda, \mu, u) \in C\}$ of positive solutions emanates from $(\mu_1(f'_+), 0)$. In this section, we study the behavior of this continuum. For λ fixed, let us recall that:

- the bifurcation at zero is subcritical if, when taking a sequence $\{(\mu_n, u_n)\}$ with $\mu_n \to \mu_1(f'_+)$, $u_n \to 0$ in $C_0^1(\overline{\Omega})$ and with $\Phi_{\mu_n}^\lambda(u_n) = 0$, we have that $\mu_n \leq \mu_1(f'_+)$ for n large enough.
- the bifurcation at zero is supercritical if, when taking a sequence $\{(\mu_n, u_n)\}$ with $\mu_n \to \mu_1(f'_+)$, $u_n \to 0$ in $C_0^1(\overline{\Omega})$ and with $\Phi_{\mu_n}^\lambda(u_n) = 0$, we have that $\mu_n \geq \mu_1(f'_+)$ for n large enough.

While the parameter λ did not play any role in the existence of positive solutions found in Theorem 3.3, we see here that the bifurcation is subcritical or supercritical depending on the value and sign of λ.

Theorem 4.1. *Let f be a continuous function in $\mathbb{R} \times \overline{\Omega} \times [0, +\infty)$ satisfying (f_0), (f_1) and*

$$\lim_{(\mu,s)\to(\mu_1(f'_+),0^+)} \frac{f(\mu,x,s) - \mu f'_+(x,0)s}{s^3} = G(x) \quad \text{uniformly in } x \in \overline{\Omega}. \tag{f_2}$$

1. *If $G(x) = +\infty$ a.e. $x \in \Omega$, then for every $\lambda \in \mathbb{R}$ the bifurcation of positive solutions is subcritical. Respectively, if $G(x) = -\infty$ a.e. $x \in \Omega$, then for every $\lambda \in \mathbb{R}$ the bifurcation of positive solutions is supercritical.*

2. *If* $G\varphi_1^4 \in L^1(\Omega)$ *and* $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2] = 0$*, then:*
   (a) *If* $\int_\Omega G(x)\varphi_1(x)^4 > 0$*, then for every* $\lambda \in \mathbb{R}$ *the bifurcation of positive solutions is subcritical.*
   (b) *If* $\int_\Omega G(x)\varphi_1(x)^4 < 0$*, then for every* $\lambda \in \mathbb{R}$ *the bifurcation of positive solutions is supercritical.*
3. *If* $G\varphi_1^4 \in L^1(\Omega)$ *and* $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2] \neq 0$*, then taking* λ^* *defined as* (1.5) *we have that:*
   (a) *If* $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2] > 0$*, then the bifurcation of positive solutions is subcritical for* $\lambda > \lambda^*$ *and supercritical for* $\lambda < \lambda^*$*.*
   (b) *If* $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2] < 0$*, then the bifurcation of positive solutions is supercritical for* $\lambda > \lambda^*$ *and subcritical for* $\lambda < \lambda^*$*.*

Proof. For fixed $\lambda \in \mathbb{R}$, using (f_0), (f_1) and (f_2) we can apply Theorem 3.3 to deduce that $\mu_1(f'_+)$ is a bifurcation point of positive solutions from zero of the problem (P_μ^λ). Thus, we can take a sequence (μ_n, u_n) verifying

$$u_n \neq 0, \quad \mu_n \to \mu_1(f'_+), \quad \|u_n\|_1 \to 0, \quad \Phi^\lambda_{\mu_n}(u_n) = 0.$$

Since u_n solves (2.3) with $\mu = \mu_n$, multiplying by φ_1 and integrating by parts, we have that

$$\int_\Omega \nabla u_n \nabla\varphi_1 = \int_\Omega \frac{f(\mu_n, x, u_n)}{1 + 2\lambda m(x)u_n^2}\varphi_1 + \int_\Omega \frac{2\lambda m(x)u_n|\nabla u_n|^2}{1 + 2\lambda m(x)u_n^2}\varphi_1. \tag{4.1}$$

Observe that, using u_n as test function in (1.4), we obtain that

$$\int_\Omega \nabla u_n \nabla\varphi_1 = \int_\Omega \mu_1(f'_+)f'_+(x,0)u_n\varphi_1.$$

Thus, by (4.1), we deduce

$$\begin{aligned}(\mu_1(f'_+) - \mu_n)\int_\Omega f'_+(x,0)u_n\varphi_1 &= \int_\Omega \nabla u_n\nabla\varphi_1 - \int_\Omega \mu_n f'_+(x,0)u_n\varphi_1 \\ &= \int_\Omega \frac{(f(\mu_n,x,u_n) - \mu_n f'_+(x,0)u_n)\varphi_1 + 2\lambda m(x)u_n\varphi_1[|\nabla u_n|^2 - \mu_n f'_+(x,0)u_n^2]}{1 + 2\lambda m(x)u_n^2}.\end{aligned}$$

If we multiply by $\|u_n\|_1^{-3}$ and denote $z_n = u_n/\|u_n\|_1$, we have that

$$\begin{aligned}(\mu_1(f'_+) - \mu_n)\int_\Omega \frac{f'_+(x,0)\varphi_1 z_n}{\|u_n\|_1^2} &= \int_\Omega \frac{(f(\mu_n,x,u_n) - \mu_n f'_+(x,0)u_n)\varphi_1 z_n^3}{u_n^3(1 + 2\lambda m(x)u_n^2)} \\ &\quad + \int_\Omega \frac{2\lambda m(x)\varphi_1 z_n}{1 + 2\lambda m(x)u_n^2}\left[|\nabla z_n|^2 - \mu_n f'_+(x,0)z_n^2\right].\end{aligned} \tag{4.2}$$

Now, taking into account that the sign of the left hand side coincides with the sign of $(\mu_1(f'_+) - \mu_n)$, the sign of the right hand side of (4.2) determines if the bifurcation is subcritical or supercritical.

In order to prove *(1)*, using that z_n converges to φ_1 and the dominated convergence theorem we obtain that the second term of the right hand side of (4.2) converges to

$$2\lambda\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2].$$

Applying Fatou's lemma to the first term of the right hand side, we deduce that

$$+\infty = \int_\Omega G(x)\varphi_1^4 \leq \liminf_{(\mu,s)\to(\mu_1(f'_+),0)} \int_\Omega \frac{(f(\mu_n,x,u_n) - \mu_n f'_+(x,0)u_n)\varphi_1 z_n^3}{u_n^3(1+2\lambda m(x)u_n^2)}.$$

Thus, the right hand side diverges to $+\infty$ (analogously diverges to $-\infty$ when $G(x) = -\infty$) and for every $\lambda \in \mathbb{R}$ the bifurcation is subcritical (respectively supercritical).

Now we assume that $G\varphi_1^4 \in L^1(\Omega)$. In this case, using the dominated convergence theorem and that z_n converges to φ_1, the right hand side of (4.2) converges to

$$\int_\Omega G(x)\varphi_1^4 + 2\lambda \int_\Omega m(x)\varphi_1^2 \left[|\nabla\varphi_1|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1^2\right].$$

Therefore, the proof of item *(2)* follows since, in that case the sign of the right hand side of (4.2) coincides with that of $\int_\Omega G(x)\varphi_1(x)^4$ for every $\lambda \in \mathbb{R}$.

Finally, in the case of item *(3)*, we observe that

$$\begin{aligned}\int_\Omega G(x)\varphi_1^4 &+ 2\lambda \int_\Omega m(x)\varphi_1^2 \left[|\nabla\varphi_1|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1^2\right] \\ &= 2(\lambda-\lambda^*)\int_\Omega m(x)\varphi_1^2 \left[|\nabla\varphi_1|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1^2\right].\end{aligned}$$

Therefore, in the case of *(a)*, the right hand side of (4.2) is positive (respectively negative) and the bifurcation is subcritical (respectively supercritical) if, $\lambda > \lambda^*$ (respectively $\lambda < \lambda^*$). Analogously we conclude in the case of *(b)*. □

5 Multiplicity of solutions for the logistic datum

In this section, for $p > 1$, we study the problem (P_μ^λ) with $f(\mu,x,u) = \mu u - u^p$ and $m(x) = 1$; i.e.,

$$\begin{cases} -\Delta u - \lambda u\Delta(u^2) = \mu u - u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Condition (f_2) becomes

$$\lim_{s\to0^+} \frac{\mu s - s^p}{s} = \mu,$$

and $f'_+(x,0) = 1$. Thus, in this case (1.4) becomes

$$\begin{cases} -\Delta\varphi = \mu\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

First, we are able to apply Theorem 3.3, giving the existence of a continuum of positive solutions. For $\lambda > 0$, we can make the change of variables $u = g(v)$ with g given by (1.3). Thus, the problem turn into the semilinear equation

$$\begin{cases} -\Delta v = \mu g(v)g'(v) - g(v)^p g'(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

The datum satisfies that

$$\frac{\mu g(v)g'(v) - g(v)^p g'(v)}{v} \quad \text{is decreasing for } v \in (0,+\infty).$$

Then, by the classical result of Brezis and Oswald ([5]), the problem has a unique positive solution. Now, let us see that for the case $\lambda < 0$ we can find a range of values in which we have at least two solutions. To do this, observe that the condition (f_2) for the problem (5.1) becomes

$$\lim_{(\mu,u)\to(\mu_1,0)} \frac{\mu u - u^p - \mu u}{u^3} = \lim_{(\mu,u)\to(\mu_1,0)} -u^{p-3} = \begin{cases} -\infty & \text{if } p \in (1,3), \\ -1 & \text{if } p = 3, \\ 0 & \text{if } p \in (3,+\infty). \end{cases}$$

Consequently, (1.5) becomes

$$\lambda^* = \frac{\displaystyle\int_\Omega \varphi_1^4}{\displaystyle\int_\Omega 2\varphi_1^2[|\nabla\varphi_1(x)|^2 - \mu_1\varphi_1(x)^2]}.$$

if $p = 3$ and $\lambda^* = 0$ if $p > 3$.

Moreover, taking φ_1^3 as test function in the weak formulation of (5.2) we have that

$$\int_\Omega 3\varphi_1^2|\nabla\varphi_1|^2 = \int_\Omega \mu_1\varphi_1^4,$$

which implies that

$$\int_\Omega 2\varphi_1^2[|\nabla\varphi_1(x)|^2 - \mu_1\varphi_1(x)^2] < 0.$$

Therefore, $\lambda^* < 0$.

Corollary 5.1. *For a fixed $\lambda \in \mathbb{R}$, the bifurcation of positive solutions occurs according to the following cases:*

1. *If $p \in (1,3)$ then the bifurcation is supercritical for every $\lambda \in \mathbb{R}$.*
2. *If $p \geq 3$ then the bifurcation is subcritical if $\lambda < \lambda^*$ and supercritical if $\lambda > \lambda^*$.*

Now, we see that, in the case $p > 3$, there is a range of values of λ for which we can obtain two solutions. To do this, we first need the following result about a priori bounds on the solution.

Lemma 5.2. *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a positive solution of (5.1) with $\lambda < 0$. Then*

1. *if $1 + 2\lambda\|u\|_\infty^2 > 0$, then $\|u\|_\infty \leq \mu^{\frac{1}{p-1}}$.*
2. *if $\mu > \mu_1$ and $\lambda < 0$, then $\mu - \mu_1 \leq \|u\|_\infty^{p-1}$.*

Proof. First, since u is a solution of (P_μ^λ), we have that

$$\begin{aligned}(1 + 2\lambda u(x)^2)(-\Delta u(x)) &= \mu u(x) - u(x)^p + 2\lambda u(x)|\nabla u(x)|^2 \\ &= u(x)(\mu - u(x)^{p-1}) + 2\lambda u(x)|\nabla u(x)|^2, \quad \forall x \in \Omega.\end{aligned}$$

However, taking x_0 the point in which u attains the maximum, we obtain that

$$(1 + 2\lambda\|u\|_\infty^2)(-\Delta u(x_0)) = \|u\|_\infty(\mu - \|u\|_\infty^{p-1}).$$

with $\Delta u(x_0) \leq 0$. Hence, $\|u\|_\infty \leq \mu^{\frac{1}{p-1}}$ and *(1)* is proved.

To prove (2), we check that

$$-\mathrm{div}[(1+2\lambda u^2)\nabla u]+2\lambda u|\nabla u|^2=-2\lambda u|\nabla u|^2+(1+2\lambda u^2)(-\Delta u)=\mu u-u^p.$$

Thus $-\mathrm{div}[(1+2\lambda u^2)\nabla u]+2\lambda u|\nabla u|^2+u^p=\mu u$ and, defining the operator

$$-\mathrm{div}[(1+2\lambda u^2)\nabla(\cdot)]+2\lambda|\nabla u|^2+u^{p-1},$$

by the positivity of u, we have that

$$\sigma_1(-\mathrm{div}[(1+2\lambda u^2)\nabla(\cdot)]+2\lambda|\nabla u|^2+u^{p-1})=\mu,$$

where $\sigma_1(\cdot)$ denotes the first eigenvalue of the operator under zero Dirichlet boundary conditions. Using the monotonicity of the first eigenvalue and that $\lambda\le 0$, we have that

$$\mu\le\sigma_1\left[-\mathrm{div}[\nabla(\cdot)]+u^{p-1}\right]\le\sigma_1\left[-\mathrm{div}[\nabla(\cdot)]+\|u\|_\infty^{p-1}\right]=\mu_1+\|u\|_\infty^{p-1}.$$

Finally, we conclude that

$$\|u\|_\infty^{p-1}\ge\mu-\mu_1.$$

□

Theorem 5.3. *If $\lambda\in\left(\frac{-1}{2}\mu_1^{2/(1-p)},0\right)$ and $p>3$, then there is $\mu<\mu_1$ such that the problem (5.1) has at least two positive solutions.*

Proof. Let $\lambda\in\left(\frac{-1}{2}\mu_1^{2/(1-p)},0\right)$ be fixed, Theorem 3.3 implies that μ_1 is a bifurcation point from zero of positive solutions. Then, there exist a continuum C_λ that emanates from $(\mu_1,0)$, subset of $\overline{\Sigma}_\lambda$ where

$$\Sigma_\lambda=\{(\mu,u)\in\mathbb{R}\times C_0^1(\overline{\Omega})\,:\,u\in\mathcal{U}_\lambda,\Phi_\mu^\lambda(u)=0,u\ne 0\}.$$

By Theorem 3.1 there are three possibilities of behavior (not incompatibles) for C_λ. However, by the Step 1 of Theorem 3.3 there is no other bifurcation point from zero. Also, the condition $\lambda\in\left(\frac{-1}{2}\mu_1^{2/(1-p)},0\right)$ implies that $\mu_1^{\frac{1}{p-1}}<\frac{1}{\sqrt{-2\lambda}}$ and combining with Lemma (5.2) we deduce that

$$\|u\|_\infty\le\mu^{\frac{1}{p-1}},\quad\|u\|_\infty<\frac{1}{\sqrt{-2\lambda}},\quad\forall(\mu,u)\in C_\lambda. \tag{5.3}$$

$$(\mu-\mu_1)^{\frac{1}{p-1}}\le\|u\|_\infty,\quad\forall(\mu,u)\in C_\lambda\quad\text{with }\mu>\mu_1. \tag{5.4}$$

Combining the two inequalities of (5.3), see Figure 5.1, and that $(s-\mu_1)^{\frac{1}{p-1}}$ diverges positively when s tends to $+\infty$, we have that there is no solution for $\mu\ge\mu_1+(-2\lambda)^{\frac{1-p}{2}}$. Thus, we have that C_λ is bounded and $\overline{C}_\lambda\cap(\mathbb{R}\times\partial\mathcal{U}_\lambda)\ne\emptyset$. Again, by (5.3) the points $(\mu,u)\in\overline{C}_\lambda\cap(\mathbb{R}\times\partial\mathcal{U}_\lambda)$ verify that

$$(-2\lambda)^{\frac{1-p}{2}}\le\mu\le\mu_1+(-2\lambda)^{\frac{1-p}{2}},\quad\forall(\mu,u)\in\overline{C}_\lambda\cap(\mathbb{R}\times\partial\mathcal{U}_\lambda).$$

Now, notice that condition $\lambda\in\left(\frac{-1}{2}\mu_1^{2/(1-p)},0\right)$ implies that $\mu_1<(-2\lambda)^{\frac{1-p}{2}}$. Therefore, we have proved that there are positive solutions with $\mu>\mu_1$.

Nevertheless, since $p>3$ and $\lambda<0$, Corollary 5.1 implies that the bifurcation is subcritical. Then, the continuum C_λ emanates from zero at μ_1 with solutions with $\mu<\mu_1$ and turn around connecting with solutions with $\mu>\mu_1$. Consequently, there is at least one value of $\mu<\mu_1$ such that the problem (5.1) has two positive solutions.

□

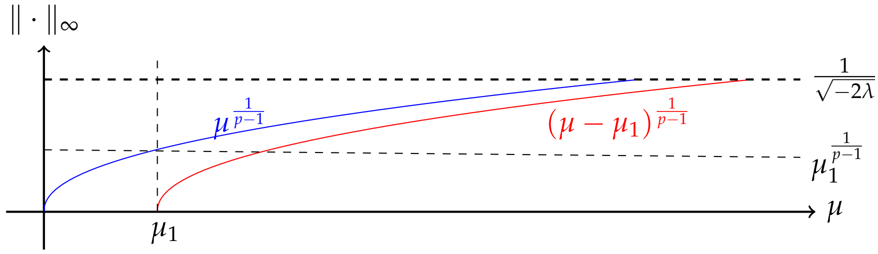

Figure 5.1: Bounds (5.3) of the continuum C_λ.

Acknowledgements

This research has been funded by Junta de Andalucía (grant FQM-116), by the Spanish Ministry of Science and Innovation, Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER) (grant PID2021-122122NB-I00) and by the FPU predoctoral fellowship of the Spanish Ministry of Universities (FPU21/05578).

References

[1] A. Ambrosetti, D. Arcoya, *An introduction to nonlinear functional analysis and elliptic problems*, Birkhäuser Boston, MA, 2011. https://doi.org/10.1007/978-0-8176-8114-2

[2] A. Ambrosetti, P. Hess, Positive solutions of asymptotically linear elliptic eigenvalue problems, *J. Math. Anal. Appl.* **73**(1980), No. 2, 411–422. https://doi.org/10.1016/0022-247X(80)90287-5

[3] D. Arcoya, J. Carmona, B. Pellacci, Bifurcation for some quasilinear operators, *Proc. Roy. Soc. Edinburgh Sect. A* **131**(2001), No. 4, 733–765. https://doi.org/10.1017/S0308210500001086

[4] D. Arcoya, J. L. Gámez, Bifurcation theory and related problems: anti-maximum principle and resonance, *Comm. Partial Differential Equations* **26**(2001), No. 9–10, 1879–1911. https://doi.org/10.1081/PDE-100107462

[5] H. Brezis, L. Oswald, Remarks on sublinear elliptic equations, *Nonlinear Anal.* **10**(1986), No. 1, 55–64. https://doi.org/10.1016/0362-546X(86)90011-8

[6] W. Cintra, E. Medeiros, U. Severo, On positive solutions for a class of quasilinear elliptic equations, *Z. Angew. Math. Phys.* **70**(2019), No. 79. https://doi.org/10.1007/s00033-019-1121-3

[7] M. Colin, L. Jeanjean, Solutions for a quasilinear Schrödinger equation: a dual approach, *Nonlinear Anal.* **56**(2004), No. 2, 213–226. https://doi.org/10.1016/j.na.2003.09.008

[8] G. M. Figueiredo, J. R. Santos Júnior, A. Suárez, Structure of the set of positive solutions of a non-linear Schrödinger equation, *Israel J. Math.* **227**(2018), No. 79, 485–505. https://doi.org/10.1007/s11856-018-1752-7

[9] S. Kurihara, Large-amplitude quasi-solitons in superfluid films, *J. Phys. Soc. Jpn* **50**(1981), No. 10, 3262–3267. https://doi.org/10.1143/JPSJ.50.3262

[10] M. Poppenberg, K. Schmitt, Z.-Q. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, *Calc. Var. Partial Differential Equations* **14**(2002), 329–344. https://doi.org/10.1007/s005260100105

[11] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Functional Analysis* **7**(1971), No. 3, 487–513. https://doi.org/10.1016/0022-1236(71)90030-9