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On the kernel of the Stieltjes derivative and the space of bounded Stieltjes-differentiable functions

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Abstract. We investigate the existence and uniqueness of solutions to first-order Stieltjes differential problems, focusing on the role of the Stieltjes derivative and its kernel. Unlike the classical case, the kernel of the Stieltjes derivative operator is nontrivial, leading to non-uniqueness issues in Cauchy problems. We characterize this kernel by providing necessary and sufficient conditions for a function to have a zero Stieltjes derivative. To address the implications of this nontrivial kernel, we introduce a function space which serves as a suitable framework for studying Stieltjes differential problems. We explore its topological structure and propose a metric that facilitates the formulation of existence and uniqueness results. Our findings demonstrate that solutions to first-order Stieltjes differential equations are, in general, not unique, underscoring the need for a refined analytical approach to such problems.

Keywords: Stieltjes derivative, kernel of the derivative, Stieltjes differential equations, uniqueness, Cauchy problem.

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1 Introduction

The study of existence and uniqueness of solutions to Stieltjes differential problems has gained significant attention in recent years (see, for instance, [5, 9]). Regarding uniqueness, many results rely on classical techniques—such as Lipschitz conditions—and on an appropriate definition of the concept of solution. In the case of first-order problems, solutions are typically assumed to belong either to the space of Stieltjes absolutely continuous functions [9] or to the space of continuously Stieltjes differentiable functions [5], mirroring the classical setting.

These choices, however, are not merely natural consequences of the problem's structure. A key observation is that there exist Stieltjes differentiable functions with an everywhere zero

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Stieltjes derivative that are not constant. In other words, the kernel of the Stieltjes derivative operator is larger than expected. As a direct consequence, the first-order Cauchy problem does not generally admit a unique solution if we consider solutions that are everywhere Stieltjes differentiable, possibly with a Stieltjes continuous derivative, but not necessarily Stieltjes continuous. This distinction between differentiable functions that are continuous and those that are not has been highlighted in the literature—see, for example, [6]—and stems from the absence of a mean value theorem for Stieltjes derivatives with the same strength as in the classical case. Nevertheless, certain versions of this result do exist, and we will explore some of them in this article.

Given this setting, a fundamental step is to determine the kernel of the Stieltjes derivative operator, as it parameterizes the solutions of the Cauchy problem. This is a nontrivial task, but we will provide several characterizations of the kernel along with necessary and sufficient conditions for a function to have a zero Stieltjes derivative.

The fact that we are working with differentiable functions that are not necessarily continuous raises a crucial question: *in which space are we operating?* We will introduce a function space, denoted $\mathcal{BD}$, which contains the kernel of the derivative and serves as a suitable framework for studying first-order Cauchy problems. Equipping this space with an appropriate topology is challenging. Even in the classical case, it is uncommon to work with the space of everywhere differentiable functions; instead, continuity of the derivative is usually imposed to obtain a Banach space structure. We will investigate under which conditions this is possible in our setting and propose a metric topology for the general case.

This work is structured to build a coherent development of the theoretical framework and main results. We begin in Section 2 by introducing the fundamental notions that will serve as the foundation for our analysis. A key concept in our formulation is that of derivators, which provide the necessary structure for defining and studying Stieltjes differential problems. We outline their role and significance before turning to the Stieltjes derivative itself. This derivative, which generalizes the classical notion of differentiation, possesses distinctive properties that set it apart from its conventional counterpart. To further clarify its behavior, we also discuss the set of points where a given derivator remains constant, as these play a crucial role in describing the kernel of the Stieltjes derivative operator.

Building on these preliminaries, Section 3 is devoted to a detailed study of the kernel of the Stieltjes derivative. We establish precise conditions that characterize when a function has a zero Stieltjes derivative and explore the implications of these results for uniqueness in first-order differential problems. To provide further insight, we complement the theoretical discussion with explicit examples that illustrate how the structure of the kernel depends on the choice of derivator.

In Section 4, we introduce the function space $\mathcal{BD}$, which is specifically designed to accommodate solutions to Stieltjes differential problems. Our objective is to capture the essential features of differentiability in this setting while ensuring a well-defined analytical framework. To this end, we examine the topological properties of $\mathcal{BD}$ and introduce a suitable metric structure that facilitates the development of an existence and uniqueness theory.

Finally, in Section 5, we present our main results on the existence and uniqueness of solutions to first-order Stieltjes differential equations. We show that first order problems have in general several solutions, thereby highlighting the subtleties introduced by the Stieltjes setting.

2 Preliminaries

Let $[a,b] \subset \mathbb{R}$ be an interval, $\mathbb{F}$ the field $\mathbb{R}$ or $\mathbb{C}$ and $g : \mathbb{R} \to \mathbb{R}$ a left-continuous non-decreasing function. We will refer to such functions as *derivators*. For these functions, we define the set $D_g = \{d_n\}_{n\in\Lambda}$ (where $\Lambda \subset \mathbb{N}$) as the set of all discontinuity points of g, namely, $D_g = \{t \in \mathbb{R} : \Delta g(t) > 0\}$ where $\Delta g(t) := g(t^+) - g(t)$, $t \in \mathbb{R}$, and $g(t^+)$ denotes the right hand side limit of g at t. We also define

$$C_g := \{t \in \mathbb{R} : g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}.$$

Observe that C_g is open in the usual topology of $\mathbb{R}$, so we can write

$$C_g = \bigcup_{n\in\widetilde{\Lambda}} (a_n, b_n), \tag{2.1}$$

where $\widetilde{\Lambda} \subset \mathbb{N}$ and $(a_k, b_k) \cap (a_j, b_j) = \emptyset$ for $k \neq j$. With this notation, we denote $N_g^- := \{a_n\}_{n\in\widetilde{\Lambda}} \backslash D_g$, $N_g^+ := \{b_n\}_{n\in\widetilde{\Lambda}} \backslash D_g$ and $N_g := N_g^- \cup N_g^+$.

Definition 2.1 ([5, Definition 3.7])**.** We define the *Stieltjes derivative*, or *g-derivative*, of a map $f : [a,b] \to \mathbb{C}$ at a point $t \in [0,T]$ as

$$f'_g(t) = \begin{cases} \displaystyle\lim_{s\to t} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_g \cup C_g, \\ \displaystyle\lim_{s\to t^+} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \in D_g, \\ \displaystyle\lim_{s\to b_n^+} \frac{f(s)-f(b_n)}{g(s)-g(b_n)}, & t \in C_g,\ t \in (a_n, b_n), \end{cases}$$

where a_n, b_n are as in (2.1), provided the corresponding limits exist. In that case, we say that f is *g-differentiable at t*.

Remark 2.2. It is possible to further simplify the definition of the Stieltjes derivative at a point $t \in [a,b]$ by defining

$$t^* = \begin{cases} t, & t \notin C_g, \\ b_n, & t \in (a_n, b_n) \subset C_g, \end{cases}$$

with a_n, b_n as in (2.1). With this notation, we have that

$$f'_g(t) = \begin{cases} \displaystyle\lim_{s\to t} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_g \cup C_g, \\ \displaystyle\lim_{s\to t^{*+}} \frac{f(s)-f(t^*)}{g(s)-g(t^*)}, & t \in D_g \cup C_g, \end{cases}$$

provided the corresponding limit exists. We will define the function f^* as $f^*(t) = f(t^*)$ for $t \in [a,b]$.

The following result includes some basic properties of this derivative.

Proposition 2.3 ([5, Proposition 3.9])**.** *Let $t \in [a,b]$. If $f_1, f_2 : [a,b] \to \mathbb{F}$ are g-differentiable at t, then:*

- *The function* $\lambda_1 f_1 + \lambda_2 f_2$ *is g-differentiable at t for any* $\lambda_1, \lambda_2 \in \mathbb{F}$ *and*

$$(\lambda_1 f_1 + \lambda_2 f_2)'_g(t) = \lambda_1 (f_1)'_g(t) + \lambda_2 (f_2)'_g(t).$$

- (Product rule). *The product* $f_1 f_2$ *is g-differentiable at t and*

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t^*) + (f_2)'_g(t) f_1(t^*) + (f_1)'_g(t)\,(f_2)'_g(t)\Delta g(t^*).$$

- (Quotient rule). *If* $f_2(t^*)\,(f_2(t^*) + (f_2)'_g(t)\,\Delta g(t^*)) \neq 0$, *the quotient* f_1/f_2 *is g-differentiable at t and*

$$\left(\frac{f_1}{f_2}\right)'_g(t) = \frac{(f_1)'_g(t) f_2(t^*) - (f_2)'_g(t) f_1(t^*)}{f_2(t^*)\,(f_2(t^*) + (f_2)'_g(t)\,\Delta g(t^*))}.$$

Furthermore, we have the following chain rule.

Proposition 2.4 ([5, Proposition 4.1])**.** *Let* $x_0 \in [a,b]$, $f : [a,b] \to \mathbb{R}$ *and* $h : \mathbb{R} \to \mathbb{F}$. *Then, the following hold:*

1. *If* $x_0 \in [a,b]\backslash(C_g \cup D_g)$ *and there exist* $h'(f(x_0))$ *and* $f'_g(x_0)$, *then* $h \circ f$ *is g-differentiable at* x_0 *and*

$$(h \circ f)'_g(x_0) = h'(f(x_0)) f'_g(x_0). \tag{2.2}$$

2. *If* $x_0 \in C_g$ *and there exist* $h'(f(x_0))$ *and* $f'_g(x_0)$, *then* $h \circ f$ *is g-differentiable at* x_0 *and*

$$(h \circ f)'_g(x_0) = h'(f(x_0^*)) f'_g(x_0),$$

Observe that if f is g-continuous, $f(x_0) = f(x_0^*)$ *and we recover the formula* (2.2)*.*

3. *If* $x_0 \in D_g$ *and*

$$f(s) = f(x_0), \quad s \in (x_0, x_0 + \delta) \text{ for some } \delta > 0, \tag{2.3}$$

then $f'_g(x_0) = (h \circ f)'_g(x_0) = 0$. *In particular,* (2.2) *holds provided* $h'(f(x_0))$ *exists.*

4. *Suppose that* $x_0 \in D_g$ *and condition* (2.3) *does not hold. If* $f(x_0^+)$ *exists, h is continuous at* $f(x_0^+)$ *and the limit*

$$\lim_{s \to x_0^+} \frac{h(f(s)) - h(f(x_0))}{f(s) - f(x_0)}$$

exists, then there exist $f'_g(x_0)$ *and* $(h \circ f)'_g(x_0)$ *and*

$$(h \circ f)'_g(x_0) = \frac{h(f(x_0^+)) - h(f(x_0))}{f(x_0^+) - f(x_0)} f'_g(x_0).$$

Remark 2.5. It is clear from the proof of [5, Proposition 4.1], which is itself a continuation of the proof of [14, Proposition 2.56 and Proposition 3.15], that it is not necessary for h to be defined on $\mathbb{R}$. It is enough that $f([a,b])$ is contained in the domain of h and that the point at which h' is evaluated in each of the cases is an accumulation point of the domain.

We shall write as μ_g the Lebesgue–Stieltjes measure associated to g given by

$$\mu_g([c,d)) = g(d) - g(c), \quad c, d \in \mathbb{R},\ c < d.$$

We will use the term "g-measurable" for a set or function to refer to μ_g-measurability in the corresponding sense, and we denote by $\mathcal{L}^1_g(X,\mathbb{F})$ the set of Lebesgue–Stieltjes μ_g-integrable functions on a g-measurable set X with values in $\mathbb{F}$, whose integral we write as

$$\int_X f(s)\,\mathrm{d}\,\mu_g(s), \quad f\in\mathcal{L}^1_g(X,\mathbb{F}).$$

Similarly, we will talk about properties holding *g-almost everywhere* in a set X (shortened to *g-a.e.* in X), or holding for *g-almost all* (or, simply, *g-a.a.*) $x\in X$, as a simplified way to express that they hold μ_g-almost everywhere in X or for μ_g-almost all $x\in X$, respectively. $L^1_g(X,\mathbb{F})$ will be the Banach space associated to $\mathcal{L}^1_g(X,\mathbb{F})$ by taking equivalence classes of functions that are equal μ_g-a.e. The spaces $L^p_g(X,\mathbb{F})$ are defined as usual (they are L^p spaces with respect to the measure μ_g) and we will denote their respective norms by $\|\cdot\|_{L^p_g}$.

Definition 2.6 ([9, Definition 3.1]). A function $f:[a,b]\to\mathbb{F}$ is *g-continuous* at a point $t\in[a,b]$, or *continuous with respect to g* at t, if for every $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(t)-f(s)|<\varepsilon, \quad \text{for all } s\in[0,T],\ |g(t)-g(s)|<\delta.$$

If f is g-continuous at every point $t\in[a,b]$, we say that f is g-continuous on $[a,b]$. We denote by $\mathcal{C}_g([a,b];\mathbb{F})$ the *set of g-continuous functions* on $[a,b]$; and by $\mathcal{BC}_g([a,b];\mathbb{F})$ the *set of bounded g-continuous functions* on $[a,b]$.

Definition 2.7. Given $k\in\mathbb{N}$, we define $\mathcal{C}^0_g([a,b];\mathbb{F}):=\mathcal{C}_g([a,b];\mathbb{F})$ and $\mathcal{C}^k_g([a,b];\mathbb{F})$ recursively as

$$\mathcal{C}^k_g([a,b]):=\{f\in\mathcal{C}^{k-1}([a,b];\mathbb{F}):\ (f_g^{(k-1)})'_g\in\mathcal{C}_g([a,b];\mathbb{F})\},$$

where $f_g^{(0)}:=f$ and $f_g^{(k)}:=(f_g^{(k-1)})'_g$, $k\in\mathbb{N}$. Similarly, given $k\in\mathbb{N}$, we define $\mathcal{BC}^0_g([a,b];\mathbb{F}):=\mathcal{BC}_g([a,b];\mathbb{F})$ and $\mathcal{BC}^k_g([a,b];\mathbb{F})$ recursively as

$$\mathcal{BC}^k_g([a,b];\mathbb{F}):=\{f\in\mathcal{C}^k_g([a,b];\mathbb{F}):\ f_g^{(n)}\in\mathcal{BC}_g([a,b];\mathbb{F}),\ \forall n=0,\dots,k\}.$$

We also define $\mathcal{C}^\infty_g([a,b];\mathbb{F}):=\bigcap_{n\in\mathbb{N}}\mathcal{C}^k_g([a,b];\mathbb{F})$ and $\mathcal{BC}^\infty_g([a,b];\mathbb{F}):=\bigcap_{n\in\mathbb{N}}\mathcal{BC}^k_g([a,b];\mathbb{F})$.

Definition 2.8 ([12, Definition 5.1]). A function $F:[a,b]\to\mathbb{R}$ is *absolutely continuous with respect to g* (or *g-absolutely continuous*) if to each $\varepsilon>0$ there is some $\delta>0$ such that for any family $\{(a_n,b_n)\}_{n=1}^m$ of pairwise disjoint open subintervals of $[a,b]$ the inequality

$$\sum_{n=1}^m \left(g\left(b_n\right)-g\left(a_n\right)\right)<\delta$$

implies

$$\sum_{n=1}^m \left|F\left(b_n\right)-F\left(a_n\right)\right|<\varepsilon.$$

Theorem 2.9 ([12, Theorem 2.4]). *Assume that $f:[a,b)\longrightarrow\overline{\mathbb{R}}$ is integrable on $[a,b)$ with respect to μ_g and consider its indefinite Lebesgue–Stieltjes integral*

$$F(x)=\int_{[a,x)} f\,\mathrm{d}\,\mu_g \quad \textit{for all } x\in[a,b].$$

Then there is a g-measurable set $N\subset[a,b)$ such that $\mu_g(N)=0$ and

$$F'_g(x)=f(x) \quad \textit{for all } x\in[a,b]\backslash N.$$

Theorem 2.10 (Fundamental Theorem of Calculus for the Lebesgue–Stieltjes Integral [12, Theorem 5.4])**.** *A function $F : [a,b] \longrightarrow \mathbb{R}$ is g-absolutely continuous on $[a,b]$ if and only if the following three conditions are fulfilled:*

1. *There exists $F'_g(t)$ for g-almost all $t \in [a,b]$;*
2. *$F'_g \in \mathcal{L}^1_g([a,b))$; and*
3. *For each $t \in [a,b]$ we have*

$$F(t) = F(a) + \int_{[a,t)} F'_g \, \mathrm{d}\,\mu_g.$$

Definition 2.11. Given a function $p : [a,b] \to \mathbb{F}$, we say that it is *g-regressive* if

$$1 + p(t)\,\Delta g(t) \neq 0, \quad t \in [a,b] \cap D_g.$$

Given a g-regressive function $p \in \mathcal{L}^1_g([0,T);\mathbb{C})$, we define the *g-exponential map* associated to the map p as

$$\exp_g(p;t) := \exp\left(\int_{[0,t)} \widetilde{p}(s) \, \mathrm{d}\,\mu_g(s) \right), \quad t \in [a,b],$$

where, denoting by $\ln(z) := \ln|z| + i\,\mathrm{Arg}(z)$ the principal branch of the complex logarithm,

$$\widetilde{p}(s) := \begin{cases} p(s), & s \in [0,T) \setminus D_g, \\ \dfrac{\ln\left(1 + p(s)\,\Delta g(s)\right)}{\Delta g(s)}, & s \in [0,T) \cap D_g. \end{cases}$$

Observe that $\exp_g(p;t)$ can be written in the following way:

$$\exp_g(p;t) = \prod_{s \in [0,t) \cap D_g} \left(1 + p(s)\,\Delta g(s)\right) \exp\left(\int_{[0,t) \setminus D_g} p(s) \, \mathrm{d}\,\mu_g \right).$$

Remark 2.12. The g-exponential map belongs to $\mathcal{AC}_g([a,b];\mathbb{C})$, see [5, Theorem 4.2], and, furthermore, it is the only function in that space satisfying

$$\begin{cases} v'_g(t) = p(t)\,v(t), & g\text{-a.a.}\, t \in [a,b], \\ v(a) = 1. \end{cases}$$

In particular, if $p \in \mathcal{BC}_g([a,b];\mathbb{C})$, then $\exp_g(p;\cdot) \in \mathcal{BC}^1_g([a,b];\mathbb{C})$. Furthermore, if $p(t) = \lambda \in \mathbb{C}$, $t \in [a,b]$, then $\exp_g(p;\cdot) \in \mathcal{BC}^\infty_g([a,b];\mathbb{C})$.

3 The Mean Value Theorem for real-valued functions and the kernel of the Stieltjes derivative

Inspired by the ideas in [1, Theorem 1.67], in this section we propose a series of results that can be described as different versions of the Mean Value Theorem for the Stieltjes derivative as a result of imposing different levels of regularity on the function. These results will be then used to study the kernel of the Stieltjes derivative under the corresponding hypotheses. It is clear that a constant function will have zero g-derivative but, as we will see, not every function with zero g-derivative has to be constant.

3.1 g-absolutely continuous functions

We first start for a theorem in the case of functions with weak derivatives, that is, the space of absolutely continuous functions $\mathcal{AC}_g([a,b],\mathbb{F})$. We will be assuming that g is not constant throughout this work.

Theorem 3.1. *Let be $f \in \mathcal{AC}_g([a,b],\mathbb{R})$. Then*

$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) \geq \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0,$$
$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) \leq \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

Proof. We divide the proof in different cases.

1. If $\operatorname{ess\,inf} f'_g$, $\operatorname{ess\,sup} f'_g \in \mathbb{R}$, we consider the following cases:

(a) If $\operatorname{ess\,inf} f'_g = \operatorname{ess\,sup} f'_g$, then f'_g takes the same value c μ_g-a.e. and, by Theorem 3.1, $f(b)-f(a) = \int_{[a,b)} f'_g \,\mathrm{d}\,\mu_g = c(g(b)-g(a))$, so $c = \frac{f(b)-f(a)}{g(b)-g(a)}$ and, therefore,

$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) = \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

(b) In the case where

$$\operatorname{ess\,inf} f'_g \leqslant \frac{f(b)-f(a)}{g(b)-g(a)} < \operatorname{ess\,sup} f'_g,$$

we have that

$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) \geqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

Now, given

$$\varepsilon \in \left(0, \operatorname{ess\,sup} f'_g - \frac{f(b)-f(a)}{g(b)-g(a)}\right),$$

we define the following sets:

$$A = \left\{x \in [a,b) : \ f'_g(x) \in \left[\frac{f(b)-f(a)}{g(b)-g(a)} + \varepsilon, \operatorname{ess\,sup} f'_g\right]\right\}, \quad B = [a,b) \setminus A.$$

We have $\mu_g(A) > 0$. Indeed, assume, on the contrary, that $\mu_g(A) = 0$,

$$\operatorname{ess\,sup} f'_g \leq \frac{f(b)-f(a)}{g(b)-g(a)} + \varepsilon < \frac{f(b)-f(a)}{g(b)-g(a)} + \operatorname{ess\,sup} f'_g - \frac{f(b)-f(a)}{g(b)-g(a)} = \operatorname{ess\,sup} f'_g,$$

which leads to a contradiction. Therefore, we have that $\mu_g(A) > 0$. Now, assuming that

$$\operatorname{ess\,inf} f'_g = \frac{f(b)-f(a)}{g(b)-g(a)},$$

we have that

$$\begin{aligned}
f(b)-f(a) &= \int_{[a,b]} f'_g \,\mathrm{d}\,\mu_g = \int_A f'_g \,\mathrm{d}\,\mu_g + \int_B f'_g \,\mathrm{d}\,\mu_g \\
&\geq \mu_g(A)\left(\frac{f(b)-f(a)}{g(b)-g(a)}+\varepsilon\right) + \mu_g(B)\operatorname{ess\,inf} f'_g \\
&= \mu_g(A)\left(\frac{f(b)-f(a)}{g(b)-g(a)}+\varepsilon\right) + \mu_g(B)\frac{f(b)-f(a)}{g(b)-g(a)} \\
&= (\mu_g(A)+\mu_g(B))\frac{f(b)-f(a)}{g(b)-g(a)} + \mu_g(A)\varepsilon \\
&= f(b)-f(a)+\mu_g(A)\varepsilon > f(b)-f(a),
\end{aligned}$$

which leads to a contradiction. Therefore:

$$\operatorname{ess\,inf} f'_g < \frac{f(b)-f(a)}{g(b)-g(a)},$$

and thus:

$$\mu_g\left(\left\{t\in[a,b)\ :\ f'_g(t)\leqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

(c) In the case where

$$\operatorname{ess\,inf} f'_g < \frac{f(b)-f(a)}{g(b)-g(a)} \leqslant \operatorname{ess\,sup} f'_g,$$

the procedure is analogous, and we will complete it for the sake of thoroughness. As a consequence of the preceding chain of inequalities,

$$\mu_g\left(\left\{t\in[a,b)\ :\ f'_g(t)\leqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

Now, let us take

$$\varepsilon \in \left(0, \frac{f(b)-f(a)}{g(b)-g(a)} - \operatorname{ess\,inf} f'_g\right)$$

and define the following sets:

$$A = \left\{x\in[a,b):\ f'_g(x)\in\left[\operatorname{ess\,inf} f'_g, \frac{f(b)-f(a)}{g(b)-g(a)}-\varepsilon\right]\right\},\quad B=[a,b)\setminus A.$$

We have $\mu_g(A)>0$. Indeed, if $\mu_g(A)=0$,

$$\operatorname{ess\,inf} f'_g \geq \frac{f(b)-f(a)}{g(b)-g(a)} - \varepsilon > \frac{f(b)-f(a)}{g(b)-g(a)} - \frac{f(b)-f(a)}{g(b)-g(a)} + \operatorname{ess\,inf} f'_g,$$

which leads to a contradiction. Therefore, we have that $\mu_g(A)>0$. Now, suppose that

$$\operatorname{ess\,sup} f'_g = \frac{f(b)-f(a)}{g(b)-g(a)},$$

then,

$$\begin{aligned}
f(b)-f(a) &= \int_{[a,b)} f'_g \, \mathrm{d}\,\mu_g = \int_A f'_g \, \mathrm{d}\,\mu_g + \int_B f'_g \, \mathrm{d}\,\mu_g \\
&\leq \mu_g(A)\left(\frac{f(b)-f(a)}{g(b)-g(a)} - \varepsilon\right) + \mu_g(B) \operatorname{ess\,sup} f'_g \\
&= \mu_g(A)\left(\frac{f(b)-f(a)}{g(b)-g(a)} - \varepsilon\right) + \mu_g(B)\frac{f(b)-f(a)}{g(b)-g(a)} \\
&= (\mu_g(A)+\mu_g(B))\frac{f(b)-f(a)}{g(b)-g(a)} - \mu_g(A)\varepsilon \\
&= f(b)-f(a)-\mu_g(A)\varepsilon < f(b)-f(a),
\end{aligned}$$

which leads to a contradiction. Therefore,

$$\operatorname{ess\,sup} f'_g > \frac{f(b)-f(a)}{g(b)-g(a)},$$

and, thus,

$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) \geqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

2. If $\operatorname{ess\,inf} f'_g = -\infty$ and $\operatorname{ess\,sup} f'_g \in \mathbb{R}$, then:

$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) \leqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

Let us see that the following also holds:

$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) \geqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

We are in the following situation:

$$-\infty = \operatorname{ess\,inf} f'_g < \frac{f(b)-f(a)}{g(b)-g(a)} \leq \operatorname{ess\,sup} f'_g.$$

Given $\varepsilon > 0$, we define the following sets:

$$A = \left\{x \in [a,b) : \ f'_g(x) \leq \frac{f(b)-f(a)}{g(b)-g(a)} - \varepsilon\right\}, \quad B = [a,b) \setminus A.$$

We have, since $\operatorname{ess\,inf} f'_g = -\infty$, that $\mu_g(A) > 0$. Therefore, following the reasoning applied in section (c) of the previous point,

$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) \geqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

3. If $\operatorname{ess\,sup} f'_g = +\infty$ and $\operatorname{ess\,inf} f'_g \in \mathbb{R}$, then:

$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) \geqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

Let us see that the following also holds:

$$\mu_g\left(\left\{t\in[a,b)\ :\ f'_g(t)\leqslant\frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right)>0.$$

We are in the following situation:

$$\operatorname{ess\,inf} f'_g\leqslant\frac{f(b)-f(a)}{g(b)-g(a)}<\operatorname{ess\,sup} f'_g=+\infty.$$

Given $\varepsilon>0$, we define the following sets:

$$A=\left\{x\in[a,b):\ f'_g(x)\geq\frac{f(b)-f(a)}{g(b)-g(a)}+\varepsilon\right\},\quad B=[a,b)\setminus A.$$

We have, since $\operatorname{ess\,sup} f'_g=+\infty$, that $\mu_g(A)>0$. Therefore, following the reasoning applied in section (b) of the previous point,

$$\mu_g\left(\left\{t\in[a,b)\ :\ f'_g(t)\leqslant\frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right)>0.$$ □

Corollary 3.2. *Let $f\in\mathcal{AC}_g([a,b],\mathbb{R})$. Then there exist $c,d\in[a,b]$ such that*

$$f'_g(c)\leqslant\frac{f(b)-f(a)}{g(b)-g(a)}\leqslant f'_g(d).$$

Proof. Taking into account that, by Theorem 2.10, the derivative of an g-absolutely continuous function exist g-a.e., and Theorem 3.1, we get the result. □

Remark 3.3. It is clear that, in the conditions of the previous corollary, given $c\in\mathbb{R}$,

$$\mu_g\left(\left\{t\in[a,b]\ :\ f'_g(t)\geqslant c\right\}\right)+\mu_g\left(\left\{t\in[a,b]\ :\ f'_g(t)\leqslant c\right\}\right)\geqslant g(b)-g(a)>0,$$

and, in particular, for the value $c=\frac{f(b)-f(a)}{g(b)-g(a)}$. Nevertheless, $\frac{f(b)-f(a)}{g(b)-g(a)}$ is the only value for which we can guarantee, a priori, that both sets have positive measure for every function f. Indeed, just take a $\mathcal{AC}_g$ function with constant derivative x. If $x=c$ both sets have measure $g(b)-g(a)$; otherwise, one has measure $g(b)-g(a)$ and the other measure zero.

Lemma 3.4. *Let $f\in\mathcal{AC}_g([a,b],\mathbb{R})$. Then f is constant if and only if f is g-differentiable on $[a,b]$ and $f'_g=0$ μ_g-a.e.*

Proof. If f is constant, by the definition of g-derivative, f'_g exists everywhere in $[a,b]$ and $f'_g=0$. Now assume $f'_g=0$ μ_g-a.e. Since $f\in\mathcal{AC}_g([a,b],\mathbb{R})$, by Theorem 2.10, $f(t)=\int_{[a,t)}f'_g(s)\,\mathrm{d}\,s+f(a)=f(a)$, so f is constant. □

3.2 Functions which are continuous with respect to g

One of the main tools in the proof of [1, Theorem 1.67] is the generalization of the Induction Principle for a given time scale, see [1, Theorem 1.7]. In our setting, we will use the following version of the Induction Principle on the real line that can be directly deduced from [2, Theorem 1].

Theorem 3.5 (Principle of real induction)**.** *Let $c, d \in \mathbb{R}$ be such that $c < d$ and $S \subset [c, d]$. Then, $S = [c, d]$ if and only if the following conditions are satisfied:*

1. *$c \in S$.*

2. *If $x \in [c, d)$ is such that $x \in S$, then there exists $\delta > 0$ such that $[x, x + \delta] \subset S$.*

3. *If $x \in (c, d]$ is such that $[c, x) \subset S$, then $x \in S$.*

In this first step, we prove a version of the Mean Value Theorem for functions which are g-differentiable and g-continuous on an interval. This is, to some extent, the generalization of the set of hypotheses required for the Mean Value Theorem for the usual derivative. Nevertheless, instead of establishing this result under the mentioned conditions, we propose a formulation based on some of the properties that such functions present. We do so as we believe that this makes it easy to understand the improvements in the subsequent sections.

Theorem 3.6. *Let $f, h : [a, b] \to \mathbb{R}$ be such that the following conditions hold:*

(i) *The maps f and h are left-continuous on $(a, b]$.*

(ii) *If g is constant on some $[\alpha, \beta] \subset [a, b]$, then so are f and h.*

(iii) *For all $t \in [a, b]$, f and h are g-differentiable at t and $|f'_g(t)| \leqslant h'_g(t)$.*

Then for any $s, t \in [a, b]$,

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|. \tag{3.1}$$

Proof. Let $t \in [a, b]$. We shall prove that

$$|f(s) - f(t)| \leqslant h(s) - h(t), \quad \text{for all } s \in [a, b],\, s \geqslant t. \tag{3.2}$$

Observe that, if $t = b$, this is trivial, so we shall assume that $t < b$.

Let $\varepsilon > 0$ and define

$$S_\varepsilon = \{s \in [t, b] : |f(s) - f(t)| \leqslant h(s) - h(t) + \varepsilon(g(s) - g(t))\}.$$

Note that, in order to prove (3.2), it is enough to show that $S_\varepsilon = [t, b]$ as $\varepsilon > 0$ has been arbitrarily chosen. We do this by means of Theorem 3.5. In particular, we only need to check that 2 and 3 in Theorem 3.5 are satisfied as, by definition, $t \in S_\varepsilon$, which shows 1.

In order to check 2 in Theorem 3.5, let $s \in [t, b)$ be such that $s \in S_\varepsilon$. We study two cases separately: $s \in N_g^- \cup C_g$ and $s \notin N_g^- \cup C_g$.

First, if $s \in N_g^- \cup C_g$, then we find $\delta > 0$ such that g is constant on $[s, s + \delta]$. Hence, (ii) guarantees that f and h are also constant on $[s, s + \delta]$ and thus, since $s \in S_\varepsilon$, it follows that $[s, s + \delta] \subset S_\varepsilon$.

Otherwise, we have that $s \notin N_g^- \cup C_g$ in which case, since f and h are g-differentiable at s, we know that

$$f'_g(s) = \lim_{r \to s^+} \frac{f(r) - f(s)}{g(r) - g(s)}, \quad h'_g(s) = \lim_{r \to s^+} \frac{h(r) - h(s)}{g(r) - g(s)}.$$

Hence, there exists $\rho > 0$ such that if $r \in (s, s + \rho)$, then

$$\left| \frac{f(r) - f(s)}{g(r) - g(s)} - f'_g(s) \right| < \frac{\varepsilon}{2}, \quad \left| \frac{h(r) - h(s)}{g(r) - g(s)} - h'_g(s) \right| < \frac{\varepsilon}{2},$$

or, equivalently,

$$|f(r)-f(s)-f'_g(s)(g(r)-g(s))|<\frac{\varepsilon}{2}(g(r)-g(s)),$$
$$|h(r)-h(s)-h'_g(s)(g(r)-g(s))|<\frac{\varepsilon}{2}(g(r)-g(s)).$$

In particular, for any $r\in(s,s+\rho)$, we have that

$$|f(r)-f(s)|\leqslant(g(r)-g(s))\left(|f'_g(s)|+\frac{\varepsilon}{2}\right),$$
$$h'_g(s)(g(r)-g(s))<h(r)-h(s)+\frac{\varepsilon}{2}(g(r)-g(s)).$$

Hence, taking $\delta\in(0,\rho)$, it follows that for any $r\in[s,s+\delta]$,

$$\begin{aligned}|f(r)-f(t)|&\leqslant|f(r)-f(s)|+|f(s)-f(t)|\\&\leqslant(g(r)-g(s))\left(|f'_g(s)|+\frac{\varepsilon}{2}\right)+h(s)-h(t)+\varepsilon(g(s)-g(t))\\&\leqslant(g(r)-g(s))\left(h'_g(s)+\frac{\varepsilon}{2}\right)+h(s)-h(t)+\varepsilon(g(s)-g(t))\\&=h'_g(s)(g(r)-g(s))+\frac{\varepsilon}{2}(g(r)-g(s))+h(s)-h(t)+\varepsilon(g(s)-g(t))\\&\leqslant h(r)-h(s)+\frac{\varepsilon}{2}(g(r)-g(s))+\frac{\varepsilon}{2}(g(r)-g(s))+h(s)-h(t)+\varepsilon(g(s)-g(t))\\&=h(r)-h(t)+\varepsilon(g(r)-g(t)),\end{aligned}$$

which shows that $[s,s+\delta]\subset S_\varepsilon$.

Finally, for 3 in Theorem 3.5, let $s\in(t,b]$ be such that $[t,s)\subset S_\varepsilon$. In that case, since f, g and h are left-continuous at s, we see that

$$\begin{aligned}|f(s)-f(t)|&=\left|\lim_{r\to s^-}f(r)-f(t)\right|=\lim_{r\to s^-}|f(r)-f(t)|\\&\leqslant\lim_{r\to s^-}(h(r)-h(t)+\varepsilon(g(r)-g(t)))=h(s)-h(t)+\varepsilon(g(s)-g(t)),\end{aligned}$$

as we needed to show.

Hence, we have proven that (3.2) holds from which (3.1) follows. □

As a direct consequence of Theorem 3.6 and [9, Proposition 3.2] we have the anticipated version of the Mean Value Theorem for functions which are continuous with respect to g.

Corollary 3.7 (Mean Value Theorem for g-continuous functions)**.** *Let $f,h:[a,b]\to\mathbb{R}$ be g-continuous on $[a,b]$. If f and h are g-differentiable on $[a,b]$ and $|f'_g(t)|\leqslant h'_g(t)$ for all $t\in[a,b]$, then,*

$$|f(s)-f(t)|\leqslant|h(s)-h(t)|,\quad s,t\in[a,b].$$

Finally, we can use Corollary 3.7 to fully describe the kernel of the Stieltjes derivative operator on the set of g-continuous functions.

Theorem 3.8 (Kernel of the Stieltjes derivative for g-continuous functions)**.** *Let $f:[a,b]\to\mathbb{R}$ be g-continuous on $[a,b]$. Then, $f'_g(t)=0$ for all $t\in[a,b]$ if and only if f is constant on $[a,b]$.*

Proof. It is clear from the definition of g-derivative that if f is constant, then $f'_g(t) = 0$ so we shall focus on the converse implication.

Assume $f'_g(t) = 0$ for all $t \in [a,b]$. Then, the map $h(t) = 0$, $t \in [a,b]$, is g-continuous and g-differentiable on $[a,b]$ with $h'_g(t) = 0$, $t \in [a,b]$. Furthermore, we have that $|f'_g(t)| \leqslant h'_g(t)$ for all $t \in [a,b]$, so Corollary 3.7 guarantees that $|f(s) - f(t)| = 0$ for all $s, t \in [a,b]$, which finishes the proof. □

To end this section we recall some interesting and already known results in this direction.

Theorem 3.9 ([13, Theorem 2.6])**.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a left-continuous and nondecreasing function, continuous and increasing on an interval $[a,b] \subset \mathbb{R}$. Let $f : [a,b] \to \mathbb{R}$ be g-continuous on $[a,b]$ and g-differentiable on (a,b) satisfying $f(a) = f(b)$. Then, there exists $c \in (a,b)$ such that $f'_g(c) = 0$.*

As a consequence of this theorem, the authors prove the following corollary.

Corollary 3.10 ([13, Corollary 2.7])**.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a left-continuous and nondecreasing function, continuous and increasing on an interval $[a,b] \subset \mathbb{R}$. Let $f : [a,b] \to \mathbb{R}$ be g-continuous on $[a,b]$ and g-differentiable on (a,b). Then, there exists $c \in (a,b)$ such that $f'_g(c) = \frac{f(b)-f(a)}{g(b)-g(a)}$.*

We note that in Theorem 3.9 it is required that g be continuous and increasing on the interval $[a,b]$, a hypothesis that, in our case, is not necessary. In any case, if the derivator satisfies the hypothesis of continuity and monotonicity in the interval $[a,b]$, the proof of Theorem 3.8 is a consequence of the Corollary 3.10.

3.3 Left-continuous functions

In this next step, we aim to prove a version of the Mean Value Theorem for functions that are not necessarily g-continuous but share one basic properties with them: left-continuity.

To that end, we shall denote by $\mathcal{I}_1$ the family of connected components of $[a,b] \backslash C_g$. We have the following result conveying some properties of this family.

Lemma 3.11. *The family $\mathcal{I}_1$ is nonempty consisting of singletons or closed subintervals of $[a,b]$. Furthermore, if $I \in \mathcal{I}_1$ is an interval, then $\min I + \varepsilon \notin C_g \cup N_g^-$, for all $\varepsilon \in [0, \max I - \min I)$.*

Proof. The hypotheses under which we define the Stieltjes derivative guarantee that $b \notin C_g$, which guarantees that $[a,b] \backslash C_g \neq \emptyset$ and, thus, $\mathcal{I}_1 \neq \emptyset$. Now, by definition, the elements of $\mathcal{I}_1$ are closed and connected subsets of the real line. Hence, the first part of the result follows.

Now, let $I \in \mathcal{I}_1$ be an interval and consider $t_* = \min I$. Observe that t_* is well-defined as I is closed. Furthermore, by definition, we have that $t_* \notin C_g$. Now, reasoning by contradiction, suppose $t_* + \varepsilon \in N_g^-$. Then, by definition, there exists $r > 0$ such that g is constant on $[t_* + \varepsilon, t_* + \varepsilon + r]$, which means that $(t_* + \varepsilon, t_* + \varepsilon + r) \subset C_g$. This is a contradiction since $(t_* + \varepsilon, t_* + \varepsilon + r) \cap I \neq \emptyset$. Hence, $t_* + \varepsilon \notin N_g^-$. □

We are now in position to prove a version of the Mean Value Theorem for left-continuous functions, in which the family $\mathcal{I}_1$ plays an important role.

Theorem 3.12 (Mean Value Theorem for left-continuous functions)**.** *Let $f, h : [a,b] \to \mathbb{R}$ be left-continuous on $(a,b]$. If f and h are g-differentiable on $[a,b]$ and $|f'_g(t)| \leqslant h'_g(t)$ for all $t \in [a,b]$, then for each $I \in \mathcal{I}_1$,*

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in I. \tag{3.3}$$

Proof. Let $I \in \mathcal{I}_1$. If I is a singleton, then (3.3) is trivially satisfied so we shall assume that $I = [c, d]$ for some $c, d \in [a, b]$, $c < d$.

Let $t \in [c, d]$. Following the ideas in the proof of Theorem 3.6, we shall prove that

$$|f(s) - f(t)| \leqslant h(s) - h(t), \quad \text{for all } s \in [c, d],\ s \geqslant t, \tag{3.4}$$

from which the result follows. Once again, if $t = d$, (3.4) is trivially satisfied, so we shall assume that $t < d$.

Let $\varepsilon > 0$ and define

$$S_\varepsilon = \{s \in [t, d] : |f(s) - f(t)| \leqslant h(s) - h(t) + \varepsilon(g(s) - g(t))\}.$$

As in the proof of Theorem 3.6, (3.4) will be proved if we show that $S_\varepsilon = [t, d]$ using Theorem 3.5. Note that 1 is trivially satisfied and 3 can be checked in an analogous manner to the proof of Theorem 3.6, so we shall focus on 2.

Let $s \in [t, d)$ be such that $s \in S_\varepsilon$. Observe that the definition of $\mathcal{I}_1$ and Lemma 3.11 ensure that $s \notin C_g \cup N_g^-$, so, since f and h are g-differentiable at s, we know that

$$f'_g(s) = \lim_{r \to s^+} \frac{f(r) - f(s)}{g(r) - g(s)}, \quad h'_g(s) = \lim_{r \to s^+} \frac{h(r) - h(s)}{g(r) - g(s)}.$$

From here, checking 2 in Theorem 3.5 is analogous to reasoning used in the proof of Theorem 3.6 and we omit it. □

We can use this new version of the Mean Value Theorem to obtain a characterization of the left-continuous functions which have null Stieltjes derivative. The proof of the following result is analogous to that of Theorem 3.8 and we omit it.

Corollary 3.13. *Let $f : [a, b] \to \mathbb{R}$ be a left-continuous function on $(a, b]$ such that $f'_g(t) = 0$ for all $t \in [a, b]$. Then,*

$$f(t) = f(s), \quad \textit{for all } s, t \in I, \quad I \in \mathcal{I}_1.$$

The reciprocal implication in Corollary 3.13 does not hold, as the following example shows.

Example 3.14. Consider g to be the Cantor function, that is,

$$g(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{c_n}{2^{n+1}}, & x = \displaystyle\sum_{n=1}^{\infty} \frac{c_n}{3^n} \in C \text{ for } c_n \in \{0, 2\}; \\ \displaystyle\sup_{y \le x,\, y \in C} g(y), & x \in [0, 1] \backslash C, \end{cases}$$

where C is Cantor's set, that is,

$$C = \bigcap_{n=0}^{\infty} E_n,$$

where

$$
\begin{aligned}
E_0 &= [0,1]\,,\\
E_1 &= \left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right],\\
E_2 &= \left[0,\frac{1}{9}\right]\cup\left[\frac{2}{9},\frac{1}{3}\right]\cup\left[\frac{2}{3},\frac{7}{9}\right]\cup\left[\frac{8}{9},1\right],\\
&\ \vdots\\
E_m &= \left[0,\frac{1}{3^m}\right]\cup\left[\frac{2}{3^m},\frac{3}{3^m}\right]\cup\left[\frac{6}{3^m},\frac{7}{3^m}\right]\cup\left[\frac{8}{3^m},\frac{9}{3^m}\right]\cup\cdots\\
&\qquad\cdots\cup\left[\frac{3^m-3}{3^m},\frac{3^m-2}{3^m}\right]\cup\left[\frac{3^m-1}{3^m},1\right],\ m\in\mathbb{N}.
\end{aligned}
$$

Observe that g is continuous and increasing and, therefore, a derivator. Furthermore, $C_g = [0,1]\backslash C$. Since C is totally disconnected, $\mathcal{I}_1 = \{\{x\} : x \in C\}$ and the conclusion of Corollary 3.13 is vacuous. Still, we have that g is g-differentiable and $g'_g = 1$, not zero.

As a final note for this case, observe that, effectively, neither Theorem 3.12 nor Corollary 3.13 describe what happens to the functions involved on the set C_g. This is because the definition of Stieltjes derivative on the set C_g does not take into account the behavior of the functions in C_g. Without any extra hypotheses, it is not possible to describe the functions on a neighborhood of such points.

3.4 Stieltjes differentiable functions

In this final section, we aim to obtain a version of the Mean Value Theorem for g-differentiable functions, without imposing any other conditions. To that end, we need to refine the family $\mathcal{I}_1$. This refined family will be $\mathcal{I}_2$, the family of connected subsets of $[a,b]\backslash(C_g \cup N_g^+ \cup D_g)$, for which we have the following result.

Lemma 3.15. *The family $\mathcal{I}_2$ is nonempty and consists of either singletons or subintervals of $[a,b]$. Moreover, letting $t_* = \inf(I)$ for any $I \in \mathcal{I}_2$, we have that, for every $\varepsilon > 0$ such that $t_* + \varepsilon < \sup(I)$, it holds that $t_* + \varepsilon \notin C_g \cup N_g \cup D_g$. Furthermore, if $t_* \in I$, then $t_* \notin C_g \cup N_g \cup D_g$.*

Proof. Let $\varepsilon > 0$ be such that $t_* + \varepsilon < \sup(I)$. If $t_* + \varepsilon \in N_g^-$, there exists $r > 0$ such that g is constant on $[t_* + \varepsilon, t_* + \varepsilon + r]$. In particular, $(t_* + \varepsilon, t_* + \varepsilon + r) \subset C_g$. This is a contradiction since $(t_* + \varepsilon, t_* + \varepsilon + r) \cap I \neq \emptyset$. Hence, $t_* + \varepsilon \notin N_g^-$. The proof for the case $t_* \in I$ is analogous to the previous one. □

This new family of sets is enough to obtain a version of the Mean Value Theorem for g-differentiable functions in a similar fashion to Theorem 3.12.

Theorem 3.16 (Mean Value Theorem for Stieltjes differentiable functions)**.** *Let $f, h : [a,b] \to \mathbb{R}$ be g-differentiable functions on $[a,b]$. If $|f'_g(t)| \leqslant h'_g(t)$ for all $t \in [a,b]$ then, for each $I \in \mathcal{I}_2$,*

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s,t \in I. \tag{3.5}$$

Proof. Let $I \in \mathcal{I}_2$. If I is a singleton, then (3.5) is trivially satisfied, so we shall assume that $\overline{I} = [c,d]$ for some $c < d$. Let us examine each case individually.

- $I = (c, d]$. If we can prove that, given any $\varepsilon \in (0, d-c)$, it holds that

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in [c+\varepsilon, d],$$

we will be done, since for any two elements $t, s \in (c, d]$, it is always possible to find a value of $\varepsilon \in (0, d-c)$ such that $t, s \in [c+\varepsilon, d]$. Thus, (3.5) is satisfied.

Now take an arbitrary $\varepsilon \in (0, d-c)$, let $t \in [c+\varepsilon, d]$ and we shall show that

$$|f(s) - f(t)| \leqslant h(s) - h(t), \quad \text{for all } s \in [c+\varepsilon, d],\, s \geqslant t. \tag{3.6}$$

We will assume that $t < d$, since, in the case $t = d$, inequality (3.6) holds trivially. Now take $\widehat{\varepsilon} > 0$ and define:

$$S_{\widehat{\varepsilon}} = \{s \in [t, d] : |f(s) - f(t)| \leqslant h(s) - h(t) + \widehat{\varepsilon}(g(s) - g(t))\}.$$

As in the previous proofs, if we show that $S_{\widehat{\varepsilon}} = [t, d]$ for any $\widehat{\varepsilon} > 0$, (3.6) holds. Let us verify that the hypotheses of Theorem 3.5 are satisfied:

1. $t \in S_{\widehat{\varepsilon}}$ trivially (equality holds).
2. Let $s \in [t, d)$ such that $s \in S_{\widehat{\varepsilon}}$. Thanks to Lemma 3.15, $s \notin C_g \cup N_g \cup D_g$, so, since f and h are g-differentiable at s, we have

$$f'_g(s) = \lim_{r \to s} \frac{f(r) - f(s)}{g(r) - g(s)}, \quad h'_g(s) = \lim_{r \to s} \frac{h(r) - h(s)}{g(r) - g(s)}.$$

In particular,

$$f'_g(s) = \lim_{r \to s^+} \frac{f(r) - f(s)}{g(r) - g(s)}, \quad h'_g(s) = \lim_{r \to s^+} \frac{h(r) - h(s)}{g(r) - g(s)}$$

and we can proceed as in the proof of Theorem 3.6.

3. Let $s \in (t, d]$ such that $[t, s) \subset S_{\widehat{\varepsilon}}$. Since $s \in [c+\varepsilon, d] \subset (c, d] \in \mathcal{I}_2$, we have $s \notin C_g \cup N_g^+ \cup D_g$. In particular, we have that $g(r) < g(s)$ for all $r < s$. Hence, since f and h are g-differentiable at s, [12, Proposition 2.1] ensures that they are left-continuous at s, so it is enough to follow the arguments in Theorem 3.6 to finish the proof of the result.

- $I = [c, d)$. This case is similar to the previous one, and we must prove that, given any $\varepsilon \in (0, d-c)$, it holds that

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in [c, d-\varepsilon].$$

Thus, (3.5) is satisfied. The reasoning is analogous to the previous case: we take $t \in [c, d-\varepsilon]$, assuming that $t < d - \varepsilon$, since the case $t = d - \varepsilon$ is trivial. Given an arbitrary $\widehat{\varepsilon} > 0$, define

$$S_{\widehat{\varepsilon}} = \{s \in [t, d-\varepsilon] : |f(s) - f(t)| \leqslant h(s) - h(t) + \widehat{\varepsilon}(g(s) - g(t))\}.$$

and verify, using Theorem 3.5, that $S_{\widehat{\varepsilon}} = [t, d-\varepsilon]$. As a consequence, we obtain that

$$|f(s) - f(t)| \leqslant h(s) - h(t), \quad \text{for all } s \in [c, d-\varepsilon],\, s \geqslant t,$$

from which it follows that

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in [c, d-\varepsilon].$$

- $I = (c, d)$. Analogous to the previous cases, if we prove that, given $\varepsilon \in (0, (d-c)/3)$, it holds that
$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in [c+\varepsilon, d-\varepsilon],$$
we will be done. The proof is a combination of the arguments used in the previous two points.

- $I = [c, d]$. In this case, we directly prove that
$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in [c, d],$$
with the proof being analogous to the previous cases. □

Remark 3.17. Notice that in the interior of an interval such as I in Theorem 3.16, the restriction of g satisfies the hypothesis of Theorem 3.9 and Corollary 3.10 (that is, continuous and increasing).

As a direct consequence of Theorem 3.16, we can obtain the following result which can be described as a different version of the Mean Value Theorem under a boundedness assumption. Note that this formulation of the result is closer to the corresponding result for the usual derivative.

Corollary 3.18 (Mean Value Theorem for Stieltjes differentiable functions). *Let $f : [a, b] \to \mathbb{R}$ be a g-differentiable function on $[a, b]$. Then, for each $I \in \mathcal{I}_2$,*
$$|f(\sup I^-) - f(\inf I^+)| \leqslant \sup_{u \in I} |f'_g(u)| (g(\sup I^-) - g(\inf I^+)). \tag{3.7}$$

Proof. Let $I \in \mathcal{I}_2$. By the definition of $\mathcal{I}_2$, $g(\sup I^-) - g(\inf I^+) > 0$. If $\sup_{u \in I} |f'_g(u)| = \infty$, there is nothing to prove, so we will assume that $\sup_{u \in I} |f'_g(u)| < \infty$. We shall only prove (3.7) whenever $\overline{I} = [c, d]$ with $c < d$, as the case when I is a singleton is trivial.

Given $\varepsilon \in (0, (d-c)/3)$, we define the following map:
$$h_\varepsilon : t \in [c+\varepsilon, d-\varepsilon] \to h_\varepsilon(t) = \sup_{u \in I} |f'_g(u)| (g(t) - g(c+\varepsilon)).$$
Clearly h_ε is g-differentiable on $[c+\varepsilon, d-\varepsilon]$ with
$$(h_\varepsilon)'_g(t) = \sup_{u \in I} |f'_g(u)|, \ \forall\, t \in [c+\varepsilon, d-\varepsilon].$$
So, it follows that $|f'_g(t)| \leqslant (h_\varepsilon)'_g(t)$, for all $t \in [c+\varepsilon, d-\varepsilon]$. Hence, applying Theorem 3.16 on the interval $[c+\varepsilon, d-\varepsilon]$ and noting that $[c+\varepsilon, d-\varepsilon]$ is the only connected component of $[c+\varepsilon, d-\varepsilon] \backslash (C_g \cup N_g^+ \cup D_g)$, we have that
$$|f(s) - f(t)| \leqslant |h_\varepsilon(s) - h_\varepsilon(t)| = \sup_{u \in I} |f'_g(u)| |g(t) - g(s)|, \quad s, t \in [c+\varepsilon, d-\varepsilon].$$
In particular,
$$|f(d-\varepsilon) - f(c+\varepsilon)| \leqslant \sup_{u \in I} |f'_g(u)| (g(d-\varepsilon) - g(c+\varepsilon)),$$
so we can conclude the proof of the result by taking $\varepsilon \to 0^+$ if we show first that the limits $f(c^+)$ and $f(d^-)$ exist. Observe that $f(c^+)$ exists by the g-differentiability of f at $c \in D_g \cup N_g^+$. On the other hand, $f(d^-)$ exists because, given $\varepsilon \in \mathbb{R}^+$, and defining
$$M = \frac{\varepsilon}{2 \max\{1, \sup_{u \in I} |f'_g(u)|\}},$$

since g is left continuous at d, there exists $\delta \in \mathbb{R}^+$ such that $|g(t)-g(d)|, |g(s)-g(d)| < M$ if $t,s \in (c,d)$, $d-t, d-s < \delta$. Thus, given $t,s \in (c,d)$, $d-t, d-s < \delta$,

$$|f(s)-f(t)| \leqslant \sup_{u\in I} |f'_g(u)||g(t)-g(s)| \leq \sup_{u\in I} |f'_g(u)|(|g(t)-g(d)|+|g(d)-g(s)|) < \varepsilon,$$

which is enough to guarantee that $f(d^-)$ exists, as it implies that, for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset (c,d)$ converging to d, $\{f(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence—cf. [11, p. 245]. □

Theorem 3.16 can also be used to describe some properties of functions in the kernel of the Stieltjes derivative as presented in the next result. The proof of the following result is in the style of the one for Theorem 3.8 and we omit it.

Corollary 3.19. *Let $f : [a,b] \to \mathbb{R}$ be such that $f'_g(t) = 0$ for all $t \in [a,b]$. Then,*

$$f(t) = f(s), \quad \text{for all } s,t \in I, \quad I \in \mathcal{I}_2.$$

Remark 3.20. The converse of Corollary 3.19 is not generally true. For example, if we consider g as the Cantor function that we defined in Example 3.14, we have that $C_g \cup N_g^+ \cup D_g = [0,1] \setminus \widehat{C}$, where

$$\widehat{C} = \bigcap_{n=0}^{\infty} \widehat{E}_n,$$

with

$$\begin{aligned}
\widehat{E}_0 &= [0,1]\,,\\
\widehat{E}_1 &= \left[0,\frac{1}{3}\right] \cup \left(\frac{2}{3},1\right],\\
\widehat{E}_2 &= \left[0,\frac{1}{9}\right] \cup \left(\frac{2}{9},\frac{1}{3}\right] \cup \left(\frac{2}{3},\frac{7}{9}\right] \cup \left(\frac{8}{9},1\right],\\
&\vdots\\
\widehat{E}_m &= \left[0,\frac{1}{3^m}\right] \cup \left(\frac{2}{3^m},\frac{3}{3^m}\right] \cup \left(\frac{6}{3^m},\frac{7}{3^m}\right] \cup \left(\frac{8}{3^m},\frac{9}{3^m}\right] \cup \cdots\\
&\qquad \cdots \cup \left(\frac{3^m-3}{3^m},\frac{3^m-2}{3^m}\right] \cup \left(\frac{3^m-1}{3^m},1\right],\ m \in \mathbb{N}.
\end{aligned}$$

The set $\widehat{C}$ is totally disconnected, and therefore $\mathcal{I}_2 = \{\{x\} : x \in \widehat{C}\}$. Hence, $g(t) = g(s)$, for all $s,t \in I$, with $I \in \mathcal{I}_2$. However, $g'_g(t) = 1$ for all $t \in [0,1]$.

Now we present some characterizations of those g-differentiable functions with g-derivative 0 everywhere.

Lemma 3.21. *$f : [a,b] \to \mathbb{R}$ satisfies $f'_g(t) = 0$ for all $t \in [a,b)$ if and only if $(fh)'_g = f^* h'_g$ for every g-differentiable function $h : [a,b) \to \mathbb{R}$.*

Proof. If $f'_g(t) = 0$ for all $t \in [a,b)$, given g-differentiable function $h : [a,b) \to \mathbb{R}$, by the product rule,

$$(fh)'_g(t) = f'_g(t)h(t^*) + h'_g(t)f(t^*) + f'_g(t)h'_g(t)\Delta g(t^*) = f(t^*)h'_g(t).$$

Now assume $(fh)'_g = f^* h'_g$ for every g-differentiable function $h : [a,b] \to \mathbb{R}$. By taking $h = 1$, this equality implies that f is g-differentiable and $f'_g = 0$. □

We now aim to characterize the functions whose g-derivative is zero, using the following observation. Clearly, if φ is constant, then any function of the form $f = \varphi \circ g$ has g-derivative zero. Conversely, suppose φ is differentiable, g is continuous, and $f = \varphi \circ g$ has vanishing g-derivative. Then, by Proposition 2.4, for any t in the domain of g, we have

$$0 = f'_g(t) = \varphi'_g(g(t^*))g'_g(t) = \varphi'_g(g(t)),$$

so φ'_g has to vanish on the image of g. A similar characterization should hold for any g-differentiable function f; it suffices to express f as a composition $f = \varphi \circ g$ for a suitable function φ. This is precisely the content of Lemma 3.22.

For the next result we will denote by A'_+ the set of accumulation points of A from the right and by f'_+ the derivative of f from the right.

Lemma 3.22. *Let $f : \mathbb{R} \to \mathbb{R}$ be a g-differentiable function. f satisfies $f'_g(t) = 0$ for all $t \in [a, b)$ if and only if there exists a function $\varphi : I \to \mathbb{R}$, where I is the smallest interval containing $g(\mathbb{R})$, such that*

1. *φ is constant on $[g(t), g(t^+)]$ for $t \in \mathbb{R}$,*
2. *$\varphi(g(t^*)) = f(t^*)$ for $t \in \mathbb{R}$,*
3. *$\varphi|_{g(\mathbb{R})}$ is differentiable on $[g(\mathbb{R})\backslash g(D_g \cup N_g)] \cap g(\mathbb{R})'$, differentiable from the right on the set $[g(\mathbb{R})\backslash g(D_g)] \cap g(\mathbb{R})' \cap g(N_g^+)$ and differentiable from the left on $[g(\mathbb{R})\backslash g(D_g)] \cap g(\mathbb{R})' \cap g(N_g^-)$ and the derivative on those sets is zero, and*
4. *$\varphi|_{g(\mathbb{R})}$ is differentiable from the right on $g(D_g) \cap g(\mathbb{R})'_+$ and $(\varphi|_{g(\mathbb{R})})'_+ = 0$ on that set.*

Proof. Assume first that there exists a differentiable function $\varphi : I \to \mathbb{R}$ satisfying conditions 1-4. Let $h = \varphi \circ g = \varphi|_{g(\mathbb{R})} \circ g$. First, g is g-differentiable and $g'_g = 1$. Now we consider different cases for $t \in [a, b)$.

We start by making the observation that the points in $\varphi(\mathbb{R})\backslash\varphi(\mathbb{R})'$ are points $x = \varphi(t)$ where $t \in D_g$ and g is constant on $(t - \varepsilon, t)$ and $(t, t + \varepsilon)$ for some ε.

If $g(t) \notin \varphi(\mathbb{R})'$, $g(t) \in D_g$ and φ is constant on $[g(t), g(t^+)]$, so, given that φ is continuous,

$$h'_g(t) := \lim_{s \to t^+} \frac{\varphi(g(s)) - \varphi(g(t))}{g(s) - g(t)} = \frac{\varphi(g(t^+)) - \varphi(g(t))}{g(t^+) - g(t)} = 0.$$

If $g(t) \in \varphi(\mathbb{R})'$, taking into account Remark 2.5, we can apply points 1–3 of Lemma 2.4 to deduce that h is g-differentiable and $h'_g(t) = 0$ if $t \notin D_g$ or $t \in D_g$ and condition (2.3) holds. Suppose now that $t \in D_g$ and condition (2.3) does not hold. $g(t^+)$ exists, φ is continuous from the right at $g(t^+)$, as it is differentiable form the right, and we have that

$$\lim_{s \to t^+} \frac{\varphi(g(s)) - \varphi(g(t))}{g(s) - g(t)} = \frac{\varphi(g(t^+)) - \varphi(g(t))}{g(t^+) - g(t)} = 0.$$

Thus, by point 4 of Lemma 2.4, $h'_g = 0$.

Since f is g-differentiable, by [6, Proposition 4.1], f^* is g differentiable as well and $(f^*)'_g = f'_g$. Thus, $f'_g(t) = (f^*)_g(t)' = h'_g(t) = 0$ for $t \in [a, b)$.

To show the converse, we modify the proof of [8, Theorem 3.13]. Let $c = \sup g(\mathbb{R}) \in (-\infty, \infty]$. We start by defining the function $\sigma(t) = \sup g^{-1}(x)$ for every $x \in (-\infty, c) \cap g(\mathbb{R})$. If $c \in g(\mathbb{R})$, we define $\sigma(c) = t$ for some $t \in \mathbb{R}$ such that $g(t) = g(c)$. This way, we have defined σ on $g(\mathbb{R})$. Since g is increasing and left continuous, $\sigma(g(t)) = t^*$ for every $t \in \mathbb{R}$.

Let

$$\varphi(x) := \begin{cases} f(\sigma(x)), & x \in g(\mathbb{R}), \\ \varphi(g(t)), & x \in (g(t), g(t^+)]\backslash g(\mathbb{R}),\ t \in \mathbb{R}. \end{cases}$$

Observe that φ is well defined and, as g is nondecreasing, we have defined φ on all of I. Furthermore, φ is constant on $[g(t), g(t^+)]$ and $\varphi(g(t^*)) = f(\sigma(g(t^*))) = f(t^*)$ for $t \in \mathbb{R}$.

Now we check that $\varphi|_{g(\mathbb{R})}$ is differentiable on $[g(\mathbb{R})\backslash g(D_g \cup N_g)] \cap g(\mathbb{R})'$ (the other cases are similar) and $\varphi|'_{g(\mathbb{R})} = 0$. Given $t \in \mathbb{R}$, $g(t)$, is an accumulation point from the left in the set $g(\mathbb{R})$ because g is left continuous, so, given that $t^* \notin N_g^+$, we can consider

$$\lim_{\substack{y \to g(t)^- \\ y \in g(\mathbb{R})}} \frac{\varphi(y) - \varphi(g(t))}{y - g(t)} = \lim_{s \to t} \frac{\varphi(g(s)) - \varphi(g(t))}{g(s) - g(t)} = \lim_{s \to t^-} \frac{f(\sigma(g(s^*))) - f(\sigma(g(t^*)))}{g(s) - g(t)}$$
$$= \lim_{s \to t^-} \frac{f(s^*) - f(t^*)}{g(s) - g(t)} = (f^*)'_g(t) = f'_g(t) = 0.$$

If $g(t)$, is an accumulation point from the right in the set $g(\mathbb{R})$, and given that $t \notin N_g^-$ we conclude, in a similar manner, that

$$\lim_{\substack{y \to g(t)^+ \\ y \in g(\mathbb{R})}} \frac{\varphi(y) - \varphi(g(t))}{y - g(t)} = (f^*)'_g(t) = f'_g(t) = 0.$$

In any case, we have proven that the derivative is zero. Condition 4 is proven in a similar way. □

Remark 3.23. While Lemma 3.21 does not require f to be g-differentiable, Lemma 3.22 imposes this condition. The reason for this is that, although g-differentiability of f implies the same for f^*, with identical g-derivatives [6, Proposition 4.1], the reverse implication does not hold in general [6, Remark 4.3].

4 $\mathcal{BD}$-spaces

We now aim to define suitable function spaces, denoted by $\mathcal{BD}_g^k([a,b];\mathbb{F})$, for g-differentiable functions. To this end, we begin by establishing some basic properties of such functions, as these must be satisfied by any element of the proposed spaces. Moreover, we seek to ensure that these spaces provide a natural framework for the study of differential problems involving g-derivatives.

Remark 4.1. Taking into account Definition 2.1, given a derivator $g : [a,b] \to \mathbb{R}$ such that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$, the following conditions will be necessary for the existence of the g-derivative in all of the points of $[a,b] \setminus (C_g \cup D_g)$ (see [5, Remark 3.3]):

- Given $t \in [a,b] \setminus (C_g \cup D_g \cup N_g)$, f is g-continuous at t. In particular, f is continuous at t since g-continuous functions are, in particular, continuous at the points where g is continuous.

- Given $t \in N_g^-$, f is left g-continuous at t, in the sense that, given $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that, if $s < t$ with $g(t) - g(s) < \delta$ then, $|f(s) - f(t)| < \varepsilon$.

Observe that the function f might not be g-continuous at such points. Indeed, take for instance

$$g : x \in [0,3] \to g(s) = \begin{cases} x, & x \in [0,1], \\ 1, & x \in (1,2), \\ x-1, & x \in [2,3]. \end{cases} \tag{4.1}$$

Then,

$$f : x \in [0,3] \to f(x) = \begin{cases} x, & x \in [0,1], \\ x+1, & x \in (1,3], \end{cases}$$

is g-differentiable at $x = 1$ since

$$\lim_{s \to 1^-} \frac{f(s) - f(1)}{g(s) - g(1)} = \lim_{s \to 1^-} \frac{s-1}{s-1} = 1.$$

However, the function f is not g-continuous at $x = 1$ since g is continuous at that point.

- Given $t \in N_g^+$, f is right g-continuous at t, in the sense that, given $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that, if $t < s$ with $g(s) - g(t) < \delta$ then, $|f(s) - f(t)| < \varepsilon$.

  As in the previous case, the function does not have to be g-continuous at such points. Indeed, take for instance g as before and

$$f : x \in [0,3] \to f(x) = \begin{cases} x, & x \in [0,2), \\ 2x+1, & x \in [2,3], \end{cases} \tag{4.2}$$

  f is g-differentiable at $x = 2$, but f is not g-continuous at such point.

- Given $t \in D_g$, the right limit $f(t^+) = \lim_{s \to t^+} f(s)$ exists.

We conclude that, interestingly enough, the g-differentiability of a function at a point of N_g does not imply the g-continuity of the function at the point. The g-differentiability of a function only guarantees the g-continuity at the points of $[a,b] \backslash (C_g \cup D_g \cup N_g)$.

Remark 4.2. It is important to highlight that it may happen that the g-derivative of a function has more points at which it is g-continuous than the function itself. For example, let us consider the derivator (4.1) and the function (4.2). The function f is g-continuous on $[0,1) \cup (2,3]$, however, its g-derivative,

$$f_g' : x \in [0,3] \to f_g'(x) = \begin{cases} 1, & x \in [0,1], \\ 2, & x \in (1,3], \end{cases}$$

is g-continuous on $[0,1) \cup (1,3]$. That is, f_g' is g-continuous on $(1,2]$, however, f is not g-continuous on $(1,2)$ (in order for f to be continuous on the interval $(1,2)$, it would have to be constant on that interval, which is false).

Lemma 4.3. *Let $[a,b] \subset \mathbb{R}$ be a closed interval of the real line. Let us assume that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$. Given a function $f : [a,b] \to \mathbb{F}$ such that $f_g'(b_n)$ exists for some $n \in \widetilde{\Lambda}$, then there exists $f_g'(x)$ for all $x \in (a_n, b_n]$, and $f'_{g|_{(a_n,b_n]}}$ is g-continuous on $(a_n, b_n]$.*

Proof. The proof is straightforward considering that f_g' is constant on $(a_n, b_n]$ since $f_g'(x) = f_g'(b_n)$ for all $x \in (a_n, b_n]$. □

Building on the observations in Remark 4.1 regarding the properties of g-differentiable functions, we now introduce the space $\mathcal{BD}_g([a,b];\mathbb{F})$, consisting of functions that satisfy those same properties.

Definition 4.4. **($\mathcal{BD}_g([a,b];\mathbb{F})$ space)** Let $[a,b] \subset \mathbb{R}$ be a closed interval of the real line. Let us assume that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$. We say that $f : [a,b] \to \mathbb{F}$ belongs to $\mathcal{BD}_g([a,b];\mathbb{F})$ if it is bounded in $[a,b]$, g-continuous on $[a,b] \setminus (C_g \cup N_g \cup D_g)$, left g-continuous on N_g^- and right g-continuous on N_g^+.

In order to define the spaces $\mathcal{BD}_g^k([a,b];\mathbb{F})$, with $k \in \mathbb{N}$, we will denote by $\mathcal{BD}_g^0([a,b];\mathbb{F})$ the space $\mathcal{BD}_g([a,b];\mathbb{F})$ and by $f_g^{(0)}$ the function f.

Definition 4.5 **($\mathcal{BD}_g^k([a,b];\mathbb{F})$ space, $k \in \mathbb{N}$).** Let $[a,b] \subset \mathbb{R}$ be a closed interval of the real line. Let us assume that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$, we say that $f : [a,b] \to \mathbb{F}$ belongs to $\mathcal{BD}_g^k([a,b];\mathbb{F})$ if $f_g^{(k-1)} \in \mathcal{BD}_g^{k-1}([a,b];\mathbb{F})$ and there exists $(f^{(k-1)})'_g(t)$, for all $t \in [a,b]$ in terms of Definition 2.1, such that $(f^{(k-1)})'_g \in \mathcal{BD}_g^0([a,b];\mathbb{F})$. We write $\mathcal{BD}_g^\infty([a,b];\mathbb{F}) = \bigcap_{k\in\mathbb{N}} \mathcal{BD}_g^k([a,b];\mathbb{F})$.

In the next result we show that, in contrast with the case of the spaces $\mathcal{BC}_g^k([a,b];\mathbb{F})$, the spaces $\mathcal{BD}_g^k([a,b];\mathbb{F})$ are closed under the product of functions, showing that they have an algebra structure.

Lemma 4.6. *Given f_1, $f_2 \in \mathcal{BD}_g^k([a,b];\mathbb{F})$, with $k \in \mathbb{N}$, we have that $f_1 f_2 \in \mathcal{BD}_g^k([a,b];\mathbb{F})$.*

Proof. Let us see the case $k = 1$ (analogous for $k \in \mathbb{N}$, $k \geq 2$). Given f_1, $f_2 \in \mathcal{BD}_g^1([a,b];\mathbb{F})$, we have, by Proposition 2.3, that, the product $f_1 f_2$ is g-differentiable at $t \in [a,b]$ and

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t^*) + (f_2)'_g(t) f_1(t^*) + (f_1)'_g(t)\,(f_2)'_g(t)\Delta g(t^*).$$

In particular,

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t) + (f_2)'_g(t) f_1(t), \ \forall t \in [a,b] \setminus (D_g \cup C_g).$$

Hence, by the definition of space $\mathcal{BD}_g^1([a,b];\mathbb{F})$ we have that $(f_1 f_2)'_g$ is bounded in $[a,b]$, g-continuous on $[a,b] \setminus (C_g \cup N_g \cup D_g)$, left g-continuous on N_g^- and right g-continuous on N_g^+. □

Remark 4.7. In general we cannot ensure in general that $\mathcal{BD}_g^1([a,b];\mathbb{F}) \subset \mathcal{AC}_g([a,b];\mathbb{F})$ since the functions of $\mathcal{BD}_g^1([a,b];\mathbb{F})$ need not be left-continuous in D_g.

Remark 4.8. Under the hypotheses of Lemma 4.3, given $f \in \mathcal{BD}_g^k([a,b];\mathbb{F})$, we have that $f_g^{(l)}|_{(a_n,b_n]}$ is g-continuous on $(a_n,b_n]$, for all $n \in \widetilde{\Lambda}$ and for all $l = 1,\ldots,k$.

Observe that given $l \in \{1,\ldots,k\}$, we only have guaranteed left g-continuity of $f_g^{(l)}$ at $b_n \in D_g$, whereas if $b_n \in N_g^+$, the function $f_g^{(l)}$ is g-continuous at b_n. In both cases the function $f_g^{(l)}|_{(a_n,b_n]}$ is g-continuous on $(a_n,b_n]$.

Example 4.9. Let us consider the example we analyzed in Remark 4.2, in this case, $D_g = \emptyset$, $C_g = (1,2)$, $N_g^- = \{1\}$, and $N_g^+ = \{2\}$. Moreover,

- f is bounded, g-continuous on $[0,3] \setminus [1,2]$, g-continuous from the left at $x = 1$, and g-continuous from the right at $x = 2$.
- f'_g is bounded, g-continuous on $[0,3] \setminus [1,2]$, g-continuous from the left at $x = 1$, and g-continuous from the right at $x = 2$. In fact, f'_g is g-continuous on $[0,3] \setminus \{1\}$.
- $f_g^{(k)} = 0$ for all $k \geq 2$, in particular, it trivially holds that it is bounded, g-continuous on $[0,3] \setminus [1,2]$, g-continuous from the left at $x = 1$, and g-continuous from the right at $x = 2$.

Thus, $f \in \mathcal{BD}_g^k([a,b];\mathbb{F})$ for all $k \in \mathbb{N}$.

Remark 4.10. In the previous example, a very important property of the spaces $\mathcal{BD}_g^k([a,b];\mathbb{F})$ is revealed, namely that non-constant functions can exist whose g-derivative is zero.

We have that $\mathcal{BD}_g^k([a,b];\mathbb{F})$ is a normed vector space with the norm

$$\begin{aligned} \mathcal{BC}_g^k([a,b]) &\xrightarrow{\quad \|\cdot\|_k \quad} \mathbb{R} \\ f &\longmapsto \|f\|_k := \sum_{0 \leq i \leq k} \|f_g^{(i)}\|_\infty \end{aligned}$$

where $\|f\|_\infty := \sup_{x \in [a,b]} |f(x)|$ is the supremum norm, and $k \in \{0\} \cup \mathbb{N}$.

4.1 Sufficient conditions for a Banach space structure

In the following results we will assume that $[a,b] \subset \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is a derivator such that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$.

Remark 4.11. We have that $(\mathcal{BD}_g([a,b];\mathbb{F}), \|\cdot\|_\infty)$ is a Banach space. Indeed: the properties in the definition of $\mathcal{BD}_g([a,b];\mathbb{F})$, g-continuity on $[a,b] \setminus (C_g \cup N_g \cup D_g)$, left g-continuity on N_g^- and right g-continuity on N_g^+ are preserved by the convergence in the supremum norm. Let us see that under some hypotheses concerning the derivator g we have that $(\mathcal{BD}_g^k([a,b];\mathbb{F}), \|\cdot\|_k)$ is also a Banach space for all $k \in \mathbb{N}$.

Lemma 4.12. *The following families of statements are equivalent:*

1. $N'_g \setminus N_g \subset D_g$ *and* $D'_g \subset D_g$.
2. (a) *For all* $x \in [a,b] \setminus (D_g \cup N_g \cup C_g)$ *there exists* $\delta > 0$ *such that* $[x - \delta, x + \delta] \subset [a,b] \setminus (D_g \cup N_g \cup C_g)$,
   (b) *for all* $x \in N_g^+$ *there exists* $\delta > 0$ *such that* $(x, x + \delta] \subset [a,b] \setminus (D_g \cup N_g \cup C_g)$ *and*
   (c) *for all* $x \in N_g^-$ *there exists* $\delta > 0$ *such that* $[x - \delta, x) \subset [a,b] \setminus (D_g \cup N_g \cup C_g)$.

Proof. Let us prove each of the implications separately.

1⇒2. On the one hand, $\overline{D_g \cup N_g \cup C_g} = C_g \cup N_g \cup D_g \cup N'_g \cup D'_g = C_g \cup N_g \cup D_g$, from which we deduce that $C_g \cup N_g \cup D_g$ is closed. Therefore, given an element $x \in [a,b] \setminus (C_g \cup N_g \cup D_g)$, there exists $\delta > 0$ such that $[x - \delta, x + \delta] \subset [a,b] \setminus (C_g \cup N_g \cup D_g)$.

Now let $x \in N_g^+$ (the case N_g^- is analogous), since D_g is a closed set, there exists an element $\delta_1 > 0$ such that $[x - \delta_1, x + \delta_1] \subset [a,b] \setminus D_g$. Since $x \notin D_g$, in particular, $x \notin N'_g$, therefore, there exists an element $\delta_2 > 0$ such that $[x-\delta_2, x+\delta_2] \setminus \{x\} \subset [a,b] \setminus N_g$. Taking $\delta_3 = \min\{\delta_1, \delta_2\}$, we have $[x-\delta_3, x+\delta_3] \setminus \{x\} \subset [a,b] \setminus (N_g \cup D_g)$. Hence, $(x, x+\delta_3] \subset [a,b] \setminus (N_g \cup D_g)$ and, since the points of the set $N_g \cup D_g$ include the endpoints of the intervals that form the connected components of C_g, it follows that $(x, x+\delta_3] \subset [a,b] \setminus (C_g \cup N_g \cup D_g)$.

2⇒1. Let $x \in N_g$, and assume $x \in N_g^+$ (the case of N_g^- is analogous). By point 2. (b), we know that there exists an element $\delta > 0$ such that $(x, x+\delta] \subset [a,b] \setminus (C_g \cup N_g \cup D_g)$, in particular, $(x, x+\delta] \subset [a,b] \setminus (N_g \cup D_g)$. Additionally, there exists $\delta_2 > 0$ such that $[x-\delta_2, x) \subset C_g \subset [a,b] \setminus (N_g \cup D_g)$. Taking $\delta_3 = \min\{\delta_1, \delta_2\}$, we have $[x-\delta_3, x+\delta_3] \setminus \{x\} \subset [a,b] \setminus (N_g \cup D_g)$. Hence, $x \notin N_g' \cup D_g'$. Therefore, we have that $N_g \cap N_g' = N_g \cap D_g' = \emptyset$. Now, by point 2. (a), the set $C_g \cup N_g \cup D_g$ is closed in $[a,b]$, and since C_g is an open set, it follows that $N_g \cup D_g$ must be closed. Thus,

$$N_g \cup D_g = \overline{N_g \cup D_g} = N_g' \cup N_g \cup D_g \cup D_g'.$$

Finally, taking into account that $D_g \cap N_g = N_g \cap N_g' = N_g \cap D_g' = \emptyset$, we deduce that $N_g' \setminus N_g \subset D_g$ and $D_g' \subset D_g$. □

We recall the following result.

Lemma 4.13 ([5, Lemma 3.13]). *We have the continuous embedding $\mathcal{BC}_g^1([a,b]) \hookrightarrow \mathcal{AC}_g([a,b])$. Furthermore, for every $f \in \mathcal{BC}_g^1([a,b])$,*

$$f(x) = f(a) + \int_{[a,x)} f_g'(s) \, \mathrm{d}\,\mu_g(s), \quad \forall x \in [a,b].$$

The following result is a generalization of [5, Lemma 3.14]. The proof is essentially the same, but we include it here for completeness.

Lemma 4.14. *Let $h \in \mathcal{BD}_g^k([a,b];\mathbb{F})$ and consider the function*

$$H : x \in [a,b] \to H(x) = \int_{[a,x)} h(s) \, \mathrm{d}\,\mu_g(s).$$

We have that $H_g'(x) = h(x^*)$, for every $x \in [a,b]$, $h^* \in \mathcal{BD}_g^k([a,b];\mathbb{F})$, and, therefore, $H \in \mathcal{BD}_g^{k+1}([a,b];\mathbb{F})$.

Proof. On the one hand, since h and h^* have the same g-derivatives and $h \in \mathcal{BD}_g^k([a,b];\mathbb{F})$, we have that $h^* \in \mathcal{BD}_g^k([a,b];\mathbb{F})$. Furthermore, $H \in \mathcal{AC}_g([a,b];\mathbb{F})$ given that $h \in \mathcal{BD}_g([a,b];\mathbb{F}) \subset \mathcal{L}_g^1([a,b);\mathbb{F})$, so it is enough ot prove that $H_g'(x) = h(x^*)$ for every $x \in [a,b]$ to get the result. We study three different cases:

- If $x \in D_g$ it is clear that

$$\begin{aligned} H_g'(x) &= \lim_{s \to x^+} \frac{H(s) - H(x)}{g(s) - g(x)} \\ &= \lim_{s \to x^+} \frac{1}{g(s) - g(x)} \int_{[x,s)} h(s) \, \mathrm{d}\,\mu_g(s) \\ &= \lim_{s \to x^+} \frac{1}{g(s) - g(x)} \left(\int_{\{x\}} h(s) \, \mathrm{d}\,\mu_g(s) + \int_{(x,s)} h(s) \, \mathrm{d}\,\mu_g(s) \right) \\ &= \lim_{s \to x^+} \frac{h(x)\Delta^+ g(x)}{g(s) - g(x)} = h(x). \end{aligned}$$

- If $x \in [a,b] \backslash (C_g \cup D_g \cup N_g)$, let us compute the limit

$$\lim_{s \to x} \frac{H(s) - H(x)}{g(s) - g(x)},$$

on the domain where $g(s) \neq g(x)$. Fix $\varepsilon > 0$. Since h is g-continuous and g is continuous at x, there exists $\delta > 0$ such that $|h(u) - h(x)| < \varepsilon$ if $|u - x| < \delta$. Define $[\![x,s)\!) := [\min\{x,s\}, \max\{x,s\})$. Now, for $s \in [a,b]$, $|u - s| < \delta$, we have that

$$\begin{aligned} \left| \frac{H(s) - H(x)}{g(s) - g(x)} - h(x) \right| &= \left| \frac{\operatorname{sgn}(s-x)}{g(s) - g(x)} \int_{[\![x,s)\!)} h(u) \, \mathrm{d}\,\mu_g(u) - h(x) \right| \\ &= \frac{1}{|g(s) - g(x)|} \left| \int_{[\![x,s)\!)} (h(u) - h(x)) \, \mathrm{d}\,\mu_g(u) \right| \\ &\leq \frac{1}{|g(s) - g(x)|} \int_{[\![x,s)\!)} |h(u) - h(x)| \, \mathrm{d}\,\mu_g(u) \\ &\leq \frac{1}{|g(s) - g(x)|} \int_{[\![x,s)\!)} \varepsilon \, \mathrm{d}\,\mu_g(u) = \varepsilon. \end{aligned} \tag{4.3}$$

Thus,

$$\lim_{s \to x} \frac{H(s) - H(x)}{g(s) - g(x)} = h(x).$$

- If $x \in N_g^+ \backslash D_g$, let us compute the limit

$$\lim_{s \to x^+} \frac{H(s) - H(x)}{g(s) - g(x)},$$

on the domain where $g(s) \neq g(x)$. Fix $\varepsilon > 0$. Since h is g-continuous from the right and g is continuous at x, there exists $\delta > 0$ such that $|h(u) - h(x)| < \varepsilon$ if $0 < |u - s| < \delta$. Repeating the calculations at (4.3), we conclude that

$$H_g'(x) = \lim_{s \to x^+} \frac{H(s) - H(x)}{g(s) - g(x)} = h(x).$$

- In the case $x \in N_g^- \backslash D_g$, the reasoning is analogous.
- Finally, if $x \in (a_n, b_n) \subset C_g$, it holds that

$$H_g'(x) = H_g'(b_n) = h(b_n) = h(x^*),$$

where the first equality comes from the definition of the g-derivative at the points of C_g and the last is a consequence of the definition of x^*. □

Theorem 4.15. *Assume $N_g' \setminus N_g \subset D_g$ and $D_g' \subset D_g$. Then $(\mathcal{BD}_g^k([a,b];\mathbb{F}), \|\cdot\|_k)$ is a Banach space.*

Proof. We present a similar reasoning in the one in the proof of [5, Theorem 3.13]. We will check the case $k = 1$ (the case $k \geq 2$ is analogous). Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{BD}_g^1([a,b])$ be a Cauchy sequence. Then, $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{BD}_g([a,b])$ and $\{(f_n)_g'\}_{n\in\mathbb{N}} \subset \mathcal{BD}_g([a,b])$ are Cauchy sequences in the Banach space $\mathcal{BD}_g([a,b])$, so there exist $f, h \in \mathcal{BD}_g([a,b])$ such that $f_n \to f$ and $(f_n)_g' \to h$ in $\mathcal{BD}_g([a,b])$. Let us check that $f_g'(x)$ exists for every $x \in [a,b]$ for all possible cases and that, furthermore, $f_g' = h$.

• Given $x \in D_g$, we have that

$$(f_n)'_g(x) = \frac{f_n(x^+) - f_n(x)}{\Delta g(x)} \to \frac{f(x^+) - f(x)}{\Delta g(x)} = f'_g(x),$$

therefore $h(x) = f'_g(x)$.

• Thanks to Lemma 4.12, given $x \in [a,b]\backslash(D_g \cup C_g \cup N_g)$, there exists $\delta > 0$ such that $[x - \delta, x + \delta] \subset [a,b]\backslash(D_g \cup N_g \cup C_g)$. We have that f_n is bounded and g-continuous in $[x-\delta, x+\delta]$ and so is $(f_n)'_g$. Therefore, by Lemma 4.13, $f_n|_{[x-\delta,x+\delta]} \in \mathcal{AC}_g([x-\delta, x+\delta]; \mathbb{F})$ and

$$f_n(t) - f_n(x-\delta) = \int_{[x-\delta,t)} (f_n)'_g(s) \, \mathrm{d}\,\mu_g(s), \ \forall t \in [x-\delta, x+\delta].$$

On the other hand,

$$\begin{aligned}
&\left| \int_{[x-\delta,t)} (f_n)'_g(s) \, \mathrm{d}\,\mu_g(s) - \int_{[x-\delta,t)} h(s) \, \mathrm{d}\,\mu_g(s) \right| \\
&\quad \leq \int_{[x-\delta,t)} \left| (f_n)'_g(s) - h(s) \right| \, \mathrm{d}\,\mu_g(s) \\
&\quad \leq \varepsilon(g(x+\delta) - g(x-\delta)),
\end{aligned}$$

where the last inequality is valid for every $n \geq N$, where $N \in \mathbb{N}$ is such that $\|(f_n)'_g - h\|_\infty \leq \varepsilon$, for every $n \geq N$. Then, we have that

$$\lim_{n\to\infty} \int_{[x-\delta,t)} (f_n)'_g(s) \, \mathrm{d}\,\mu_g(s) = \int_{[x-\delta,t)} h(s) \, \mathrm{d}\,\mu_g(s)$$

uniformly on $[x-\delta, x+\delta]$. Thus,

$$\lim_{n\to\infty} (f_n(t) - f_n(x-\delta)) = \lim_{n\to\infty} \int_{[x-\delta,t)} (f_n)'_g(s) \, \mathrm{d}\,\mu_g(s) = \int_{[x-\delta,t)} h(s) \, \mathrm{d}\,\mu_g(s)$$

uniformly on $[x-\delta, x+\delta]$. Hence,

$$f(t) = f(x-\delta) + \int_{[x-\delta,t)} h(s) \, \mathrm{d}\,\mu_g(s).$$

Since $h|_{[x-\delta,x+\delta]} \in \mathcal{BC}_g([x-\delta, x+\delta])$, by Lemma 4.14, we get that

$$f'_g(t) = h(t), \ \forall x \in [x-\delta, x+\delta],$$

as we wanted to show. Thus, we have that $f'_g(t) = h(t)$, for all $t \in [x-\delta, x+\delta]$. In particular, $f'_g(x) = h(x)$.

• For $x \in N_g^-$ (analogous for N_g^+) the reasoning is similar. In this case, thanks to Lemma 4.12, given $x \in N_g^-$, we have that there exists $\delta > 0$ such that $[x-\delta, x) \subset [a,b]\backslash(D_g \cup N_g \cup C_g)$. Now since f_n and $(f_n)'_g$ are left g-continuous at x, we have that $f_n|_{[x-\delta,x]} \in \mathcal{AC}_g([x-\delta, x]; \mathbb{F})$ and we can proceed in a way resembling that of the previous point.

• For $x \in (a_n, b_n) \subset C_g$, we have, thanks to the previous results, that $f'_g(x) = f'_g(b_n) = h(b_n)$, where $b_n \in N_g^+ \cup D_g$. Now, thanks to Remark 4.8, $(f_n)'_g|_{(a_n,b_n]}$ is g-continuous on $(a_n, b_n]$, in particular, $(f_n)'_g(x) = (f_n)'_g(b_n)$ for all $x \in (a_n, b_n]$. Therefore, by taking the limit as n tends to infinity, $h(x) = h(b_n)$ for all $x \in (a_n, b_n]$, thus $f'_g(x) = h(x)$ for all $x \in (a_n, b_n]$. □

Remark 4.16. The previous Theorem is in fact a characterization. Indeed, suppose $N_g'\backslash N_g \not\subset D_g$ (the case $D_g' \not\subset D_g$ being similar) and let us see that $\mathcal{BD}_g^1([a,b];\mathbb{F})$ is not a Banach space. There exists $t \in N_g'\backslash(C_g \cup D_g \cup N_g)$, which we may assume to be an accumulation point of N_g from the right in the interior of $[a,b]$ (when $t \in \{a,b\}$ or t is an accumulation point from the left we can argue analogously). In that case there exist a family of connected components of C_g, $\{(c_n,d_n)\}_{n\in\mathbb{N}}$, such that $d_1 > c_1 > \cdots > d_n > c_n > \cdots > t$ and $c_n, d_n \to t$. We may take $g(t) = 0$ without loss of generality and define

$$f_n(x) = \begin{cases} 0, & x < d_n, \\ \frac{1}{2}g(c_{k-1}), & d_k \le x < d_{k-1},\ k = 2,\dots,n, \\ g(b), & x \ge d_1. \end{cases}$$

Notice that whenever $n \le m$ and $x \ge d_n$, $f_n(x) = f_m(x)$. It is clear that $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{BD}_g^1([a,b];\mathbb{F})$ and $(f_n)_g' = 0$ for every n. Furthermore, by continuity of g at t, $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{BD}_g^1([a,b];\mathbb{F})$; however, it cannot possibly converge in that space, since for any potential limit f we would have, for every n,

$$\frac{f(c_{n-1}) - f(t)}{g(c_{n-1}) - g(t)} = \frac{f_n(c_{n-1})}{g(c_{n-1})} = \frac{1}{2}\frac{g(c_{n-1})}{g(c_{n-1})} = \frac{1}{2} > 0 = (f_n)_g'(t).$$

Therefore $\mathcal{BD}_g^1([a,b];\mathbb{F})$ is not a Banach space.

The following example, in which $D_g = \emptyset$ and $N_g' \setminus N_g \not\subset \emptyset$, reinforces the previous remark.

Example 4.17. Let us take as the generator g the Cantor function defined in Example 3.14, and consider the sequence of functions $\{F_n\}_{n=0}^{\infty}$ defined by $F_0(x) = 1$ and, for all $n \ge 0$,

$$F_{n+1} : x \in [0,1] \to F_{n+1}(x) = \begin{cases} \dfrac{1}{2}F_n(3x), & 0 \le x < \dfrac{1}{3}, \\ \dfrac{1}{2}, & \dfrac{1}{3} \le x < \dfrac{2}{3}, \\ \dfrac{1}{2} + \dfrac{1}{2}F_n(3x-2), & \dfrac{2}{3} \le x \le 1. \end{cases} \tag{4.4}$$

As proved in [4], the sequence of functions defined by the recurrence formula (4.4) converges uniformly to the Cantor function on $[0,1]$. Additionally, $\{F_n\}_{n=0}^{\infty} \subset \mathcal{BD}_g([0,1];\mathbb{R})$ and it holds that $(F_n)_g'(x) = 0$, for all $x \in [0,1]$ and for all $n \ge 0$.

For instance, in Figure 4.1 we can observe both the function F_3 (in blue) and an approximation of the Cantor function g (in black), which corresponds to the derivator. The aim of this representation is to provide a visual intuition of the convergence of the sequence $\{F_n\}_{n\in\mathbb{N}}$ towards the Cantor function, while also highlighting the fact that each function F_n has zero g-derivative. We note that this function F_3 is g-continuous on N_g^- (in particular, left g-continuous), right g-continuous on N_g^+, and g-continuous on $[0,1] \setminus N_g$. Since the function is piecewise constant and satisfies the g-continuity conditions described above, we have that $(F_3)_g' = 0$.

Taking this into account, $\{F_n\}_{n=0}^{\infty} \subset \mathcal{BD}_g^1([0,1];\mathbb{R})$ is a Cauchy sequence in $\mathcal{BD}_g^1([0,1];\mathbb{R})$. Finally, it is clear that the function $g \in \mathcal{BD}_g^1([0,1];\mathbb{R})$, moreover, $g_g'(x) = 1$ for all $x \in [0,1]$. Therefore, the sequence $\{(F_n)_g'\}_{n=0}^{\infty}$ does not converge uniformly to g_g' on $[0,1]$, and thus the sequence $\{F_n\}_{n=0}^{\infty}$ is not convergent in $\mathcal{BD}_g^1([0,1];\mathbb{R})$.

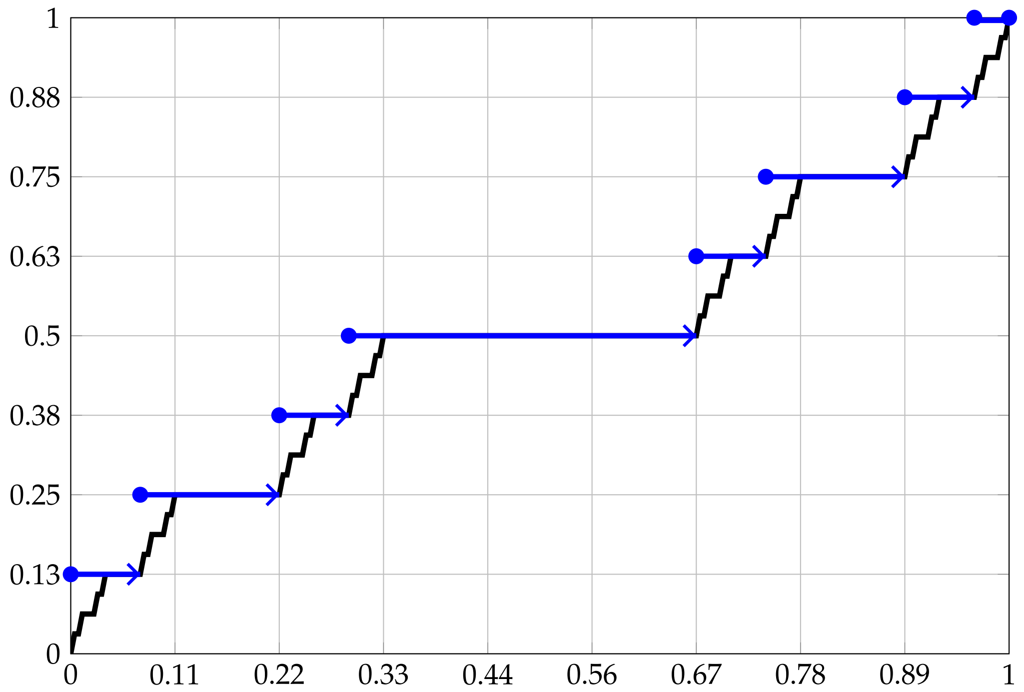

Figure 4.1: Graph of the function F_3 (in blue), along with an approximation of the Cantor function g (in black).

Proposition 4.18. *Assume $N_g' \setminus N_g \subset D_g$ and $D_g' \subset D_g$. Then $\mathcal{BC}_g^k([a,b];\mathbb{F})$ is a closed subspace of $\mathcal{BD}_g^k([a,b];\mathbb{F})$.*

Proof. It is clear that $\mathcal{BC}_g^k([a,b];\mathbb{F}) \subset \mathcal{BD}_g^k([a,b];\mathbb{F})$. Since the norm of $\mathcal{BC}_g^k([a,b];\mathbb{F})$ and that of $\mathcal{BD}_g^k([a,b];\mathbb{F})$ coincide on $\mathcal{BC}_g^k([a,b];\mathbb{F})$ and $\mathcal{BC}_g^k([a,b];\mathbb{F})$ is a Banach space, it is a closed subspace of $\mathcal{BD}_g^k([a,b];\mathbb{F})$. □

4.2 Complete metric space structure

Throughout this section, we will assume that $[a,b] \subset \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is a derivator such that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$. We will also denote by $\mathcal{BD}_g^k([a,b])$ the space $\mathcal{BD}_g^k([a,b];\mathbb{R})$, for $k \geq 0$.

Cases such as the one shown in Example 4.17 give rise to the following question: can the space $\mathcal{BD}_g^1([a,b];\mathbb{R})$ be given, without any further assumptions, a structure that makes differentiation continuous? This would imply that with such structure, the kernel of the g-derivative is a closed subspace of $\mathcal{BD}_g^1$.

A Banach or even Fréchet space structure would be desirable, but this has proven to be a rather difficult problem; one could point out that if, in Example 4.17, the supremum norm is strengthened, for instance to the total variation norm, the sequence is no longer Cauchy. If the new norm is still not strong enough, as is the case for the total variation, continuity of the g-derivative may fail, and therefore the Banach space structure; if it is too strong, however (e.g. a g-Lipschitz type norm), it may be necessary to exclude functions that we would like to have in the space (such as the ones in Example 4.17, none of which are g-Lipschitz).

In this section we provide a less ambitious (although still interesting) approach. A key tool is introduced in the following definition.

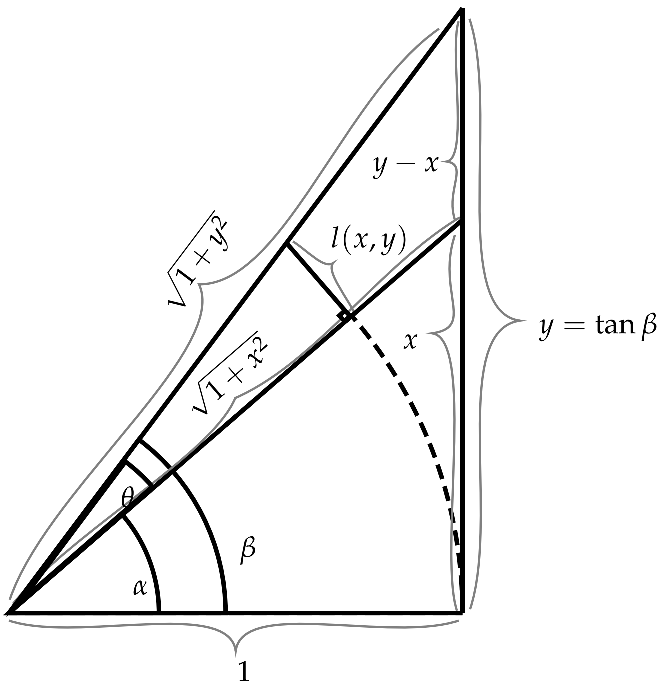

Figure 4.2: Geometric interpretation of the function l in two dimensions (without the cross product). In this figure we are assuming $y > x > 0$. Observe that $\alpha = \tan x$ and $y = \tan\beta$.

Definition 4.19. Given $x, y \in \mathbb{C}$, let

$$l(x,y) := \frac{|x-y|}{\sqrt{1+x^2}\sqrt{1+y^2}},$$

which is the *chordal distance* –see [3, 10].

Remark 4.20. The chordal distance between two complex numbers is the euclidean distance between their stereographic projections on the unit sphere. $(\mathbb{C}, l)$ is a metric space.

Remark 4.21. Another useful way of interpreting l in the case of $x, y \in \mathbb{R}$ is as $l(x,y) = \sin\theta$, where $\theta \in [0, \pi]$ is the angle between the vectors $(1, x)$ and $(1, y)$. To see this it is enough to note that

$$l(x,y) = \frac{\|(1,x,0) \times (1,y,0)\|_2}{\|(1,x,0)\|_2 \|(1,y,0)\|_2},$$

and remember that $\|\vec{a} \times \vec{b}\|_2 = \|\vec{a}\|_2 \|\vec{b}\|_2 \sin\theta$, with $\theta \in [0, \pi]$ the (small) angle between the vectors $\vec{a}$ and $\vec{b}$.

In Figure 4.2 we show a geometric interpretation that does not rely on the cross product.

Definition 4.22. For $f, h \in \mathcal{BD}^1_g([a,b])$, we define

$$\Gamma(f,h) := \sup_{\substack{s,t\in[a,b] \\ g(s)\neq g(t)}} l\left(\frac{f(s)-f(t)}{g(s)-g(t)}, \frac{h(s)-h(t)}{g(s)-g(t)}\right),$$

and from this,

$$d(f,h) := \|f-h\|_\infty + \|f'_g - h'_g\|_\infty + \Gamma(f,h).$$

Since l is a distance on $\mathbb{R}$, Γ is symmetric and satisfies the triangle inequality (on $\mathcal{BD}^1_g([a,b])$). Therefore, $(\mathcal{BD}^1_g([a,b]), d)$ is a metric space.

Theorem 4.23. *$(\mathcal{BD}^1_g([a,b]), d)$ is a complete metric space.*

Proof. It is clear that if $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{BD}^1_g([a,b]), d)$, there exist f and h such that $f_n \to f$ and $(f_n)'_g \to h$ in $(\mathcal{BD}([a,b]), \|\cdot\|_\infty)$. Let us see that $f'_g(t) = h(t)$ for every $t \in [a,b]$.

Let $t \in [a,b]$, and suppose that the g-derivative at that point is obtained by taking the right-hand limit (the remaining cases are analogous). Our goal is producing a bound in the following fashion

$$\begin{aligned}\left|\frac{f(s)-f(t)}{g(s)-g(t)} - h(t)\right| \le & \left|\frac{f(s)-f(t)}{g(s)-g(t)} - \frac{f_N(s)-f_N(t)}{g(s)-g(t)}\right| + \left|\frac{f_N(s)-f_N(t)}{g(s)-g(t)} - (f_N)'_g(t)\right| \\ & + \left|(f_N)'_g(t) - h(t)\right|,\end{aligned}$$

where N is big enough and $s \in [t, s_N)$. Recalling its sine interpretation, for the metric l to approach zero one of the following need to happen:

- x and y both tend to infinity (or minus infinity),
- x tends to infinity, and y to minus infinity,
- x and y are close to each other (and do not approach plus or minus infinity).

That said, for a big N and s_N near t, we can bound the value $\frac{f_N(s)-f_N(t)}{g(s)-g(t)}$ (close to $(f_N)'_g(t)$, which is uniformly bounded). Since $\Gamma(f_n, f_N)$ is small if $n \ge N$, the only possibility in $[t, s_N)$ (if indices are chosen accordingly) is that $\left|\frac{f_n(s)-f_n(t)}{g(s)-g(t)} - \frac{f_N(s)-f_N(t)}{g(s)-g(t)}\right|$ is close to zero for $n \ge N$, and consequently also

$$\left|\frac{f(s)-f(t)}{g(s)-g(t)} - \frac{f_N(s)-f_N(t)}{g(s)-g(t)}\right|.$$

Thanks to all this, and the definition of d, $f'_g(t) = h(t)$.

Since $f_n \xrightarrow{\|\cdot\|_\infty} f$ (in particular there is pointwise convergence), the continuity of l guarantees that $f_n \xrightarrow{d} f$. □

Remark 4.24. Observe that the sequence $\{F_n\}_{n\in\mathbb{N}}$ defined in Example 4.17 is not a Cauchy sequence. Indeed, given $n, m \in n \in \mathbb{N}$, $n > m$, there exists a point $t \in [a,b]$ such that F_n has a jump at t whereas F_m is constant on an open interval containing that point. This implies that

$$\lim_{s\to t} \frac{F_n(s)-F_n(t)}{g(s)-g(t)} = \infty, \quad \lim_{s\to t} \frac{F_n(s)-F_n(t)}{g(s)-g(t)} = 0$$

so, taking into account Remark 4.21,

$$\Gamma(F_n, F_m) := \sup_{\substack{s,t\in[a,b] \\ g(s)\ne g(t)}} l\left(\frac{F_n(s)-F_n(t)}{g(s)-g(t)}, \frac{F_m(s)-F_m(t)}{g(s)-g(t)}\right) = 1,$$

So $\{F_n\}_{n\in\mathbb{N}}$ cannot be a Cauchy sequence. Observe that this is a general behavior occurring when we have jumps appearing in constancy intervals.

As a consequence of the previous theorem we have the following corollary.

Corollary 4.25. *The differential operator* $\partial_g : (\mathcal{BD}^1_g([a,b],\mathbb{R}), d) \to (\mathcal{BD}([a,b],\mathbb{R}), \|\cdot\|_\infty)$, $\partial_g f = f'_g$ *is continuous.*

This structure has another desirable property.

Theorem 4.26. $(\mathcal{BC}^1_g([a,b]), d)$ *has the same topology* $(\mathcal{BC}^1_g([a,b]), \|\cdot\|_{\mathcal{BC}^1_g})$.

Proof. It is equivalent to check that for any $\{f_n\}_{n\in\mathbb{N}} \cup \{f\} \subset \mathcal{BC}^1_g([a,b])$ it holds that

$$f_n \xrightarrow{d} f \iff f_n \xrightarrow{\mathcal{BC}^1_g} f.$$

By definition of d, the first implication is immediate. For the second one, it is enough to note that for $f_n \xrightarrow{\mathcal{BC}^1_g} f$, we have the following bound for the difference quotients:

$$\begin{aligned}\sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} \left| \frac{f(s)-f(t)}{g(s)-g(t)} - \frac{f_n(s)-f_n(t)}{g(s)-g(t)} \right| &= \sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} \left| \frac{1}{g(s)-g(t)} \int_{[t,s)} (f'_g(u) - (f_n)'_g(u)) \mathrm{d}g(u) \right| \\ &\leq \|f'_g - (f_n)'_g\|_\infty.\end{aligned}$$

Hence, $\Gamma(f_n, f) \to 0$, and $d(f_n, f) \to 0$. □

Remark 4.27. Thanks to Theorem 4.26, the inclusion $(\mathcal{BC}^1_g([a,b]), \|\cdot\|_{\mathcal{BC}^1_g}) \hookrightarrow (\mathcal{BD}^1_g([a,b]), d)$ is an embedding.

The next example dispels the possibility of $(\mathcal{BD}^1_g([a,b]), d)$ being a topological vector space.

Example 4.28. Consider the functions

$$g(x) = \begin{cases} x, & x \in [-1,0], \\ x+1, & x \in (0,1]. \end{cases}, \qquad f(x) = \begin{cases} 0, & x \in [-1,0), \\ 1, & x \in [0,1], \end{cases}$$

and also $h = -f$, $f_k = (1-\frac{1}{k})f$, $h_k = -(1+\frac{1}{k})f$. We will check that $d(f, f_k) \to 0$, $d(h, h_k) \to 0$, but $d(f+h, f_k+h_k) \not\to 0$ (which implies that addition is not a continuous operation for the product topology of $\mathcal{BD}_g([-1,1]) \times \mathcal{BD}_g([-1,1])$).

On one hand, $\|f - f_k\|_\infty = \frac{1}{k}\|f\|_\infty$ and $\|f'_g - (f_k)'_g\|_\infty = \|0-0\|_\infty = 0$. On the other,

$$\begin{aligned}\Gamma(f, f_k) &:= \sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} l\left(\frac{f(s)-f(t)}{g(s)-g(t)}, \frac{f_k(s)-f_k(t)}{g(s)-g(t)} \right) \\ &= \sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} \frac{\left| -\frac{1}{k} \frac{f(s)-f(t)}{g(s)-g(t)} \right|}{\sqrt{1+\left(\frac{f(s)-f(t)}{g(s)-g(t)}\right)^2} \sqrt{1+\left(1+\frac{1}{k}\right)\left(\frac{f(s)-f(t)}{g(s)-g(t)}\right)^2}} \\ &\leq \frac{1}{k} \sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} \frac{\left| \frac{f(s)-f(t)}{g(s)-g(t)} \right|}{\sqrt{1+\left(\frac{f(s)-f(t)}{g(s)-g(t)}\right)^2}} \leq \frac{1}{k} \to 0,\end{aligned}$$

so we have that $f_k \xrightarrow{d} f$. The same reasoning is valid for h and h_k. Now,

$$\Gamma(f+h, f_k+h_k) = \Gamma\left(0, -\frac{2}{k}f\right) = \sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} \frac{\left|-\frac{2}{k}\frac{f(s)-f(t)}{g(s)-g(t)}\right|}{\sqrt{1+\left(\frac{2}{k}\frac{f(s)-f(t)}{g(s)-g(t)}\right)^2}}.$$

Since f is not g-Lipschitz (take $t=0$ and $s=-1/n$) and $\lim_{x\to\infty}\frac{|x|}{\sqrt{1+x^2}}=1$, we have that $\Gamma(f+h, f_k+h_k)=1$ for each k. Consequently, $f_k+h_k \overset{d}{\nrightarrow} f+h$.

Remark 4.29. In the previous example, any f with zero g-derivative that is not g-Lipschitz could have been chosen.

4.3 Relation of $\mathcal{BD}$ spaces to other spaces

In this section, we will assume that $[a,b]\subset\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ is a derivator such that $a\notin N_g^-$ and $b\notin N_g^+\cup D_g\cup C_g$. Let us examine the relationship between the space $\mathcal{BC}_g^k([a,b];\mathbb{F})$ and other more regular spaces.

We now define the Stieltjes–Sobolev spaces—see [8, Definition 5.1] and [15, Definition 3.2].

Definition 4.30. Let $p\in[1,\infty]$. We define the *Stieltjes–Sobolev spaces* as follows. $W_g^{0,1}([a,b),\mathbb{F}):=L_g^1([a,b),\mathbb{F})$, and, for $n\in\mathbb{N}$, $W_g^{n,1}([a,b),\mathbb{F}):=$

$$\left\{u\in L_g^1([a,b),\mathbb{F}) : \exists\, \widetilde{u}\in W_g^{n-1,1}([a,b),\mathbb{F}) \text{ s.t. } u(y)-u(x)=\int_x^y \widetilde{u}\,\mathrm{d}\mu_g,\ x,y\in[a,b)\right\}.$$

Observe that, due to Theorem 2.10, $W_g^{1,1}([a,b),\mathbb{F})=\mathcal{AC}([a,b),\mathbb{F})$.

Lemma 4.31. *Let $f\in\mathcal{BD}_g([a,b];\mathbb{F})$ be such that $f(t)=0$ μ_g-a.e. Then $f(t)=0$ for every $t\in[a,b]$.*

Proof. Let $A=\{t\in[a,b] : f(t)=0\}$. Since $f(t)=0$ μ_g-a.e., we have that $\mu_g(A)=\mu_g([a,b])$ and $\mu_g(X)=\mu_g(X\cap A)$ for any μ_g-measurable set $X\subset[a,b]$. Since $\mu_g(\{t\})>0$ for every $t\in D_g$, we conclude that $D_g\cap[a,b]\subset A$. Given $t\in(a,b)$, if there exists $\delta\in\mathbb{R}^+$, $\delta<\min\{t-a,b-t\}$, such that $[t-\delta,t+\delta)\cap A=\emptyset$, then

$$g(t+\delta)-g(t-\delta)=\mu_g([t-\delta,t+\delta))=\mu_g([t-\delta,t+\delta)\cap A)=0,$$

so $t\in C_g$. This implies that $A\cap(a,b)$ is dense in $(a,b)\backslash C_g$. Since $f\in\mathcal{BD}_g^0([a,b];\mathbb{F})$, f is g-continuous on $(a,b)\backslash(C_g\cup N_g\cup D_g)$ and, thus, continuous in that set. Given that $f|_A=0$ and $A\cap[(a,b)\backslash(C_g\cup N_g\cup D_g)]$ is dense in $(a,b)\backslash(C_g\cup N_g\cup D_g)$, we conclude that $f=0$ on $(a,b)\backslash(C_g\cup N_g\cup D_g)$.

We have proved that $D_g\cap[a,b]$, $(a,b)\backslash(C_g\cup N_g)\subset A$. It is left to see what happens on $N_g^+\backslash D_g$ and on $N_g^-\backslash D_g$. Given that $a\notin N_g^-$ and $b\notin N_g^+\cup D_g\cup C_g$, we do not have to consider a or b.

If $t\in N_g^+\backslash D_g$, since $t<b$, if there exists $\delta\in\mathbb{R}^+$, $\delta<\min\{b-t\}$, such that $[t,t+\delta)\cap A=\emptyset$, then

$$g(t+\delta)-g(t)=\mu_g([t,t+\delta))=\mu_g([t,t+\delta)\cap A=0,$$

so $t\notin N_g^+$, which is a contradiction. Therefore, we conclude that t is an accumulation point of $A\cap[a,b]$ from the right. Since f is right g-continuous and, thus, continuous from the right at $t\in N_g^+$, $f(t)=0$.

The argument is analogous for the case $t\in N_g^-\backslash D_g$, And we conclude that $f(t)=0$ for every $t\in[a,b]$. □

Remark 4.32. The condition $f \in \mathcal{BD}_g^1([a,b];\mathbb{F})$ in Lemma 4.31 is necessary as, in general, a function $f:\mathbb{R}\to\mathbb{R}$ can have g-derivative everywhere with $f_g' = 0$ μ_g-a.e., but $f_g' \neq 0$.

Indeed, let $g:\mathbb{R}\to\mathbb{R}$ be defined as

$$g(t) = t + \sum_{\substack{0<s<t \\ \frac{1}{s}\in\mathbb{Z}}} 2^{-s} + \sum_{\substack{t\leqslant s<0 \\ \frac{1}{s}\in\mathbb{Z}}} 2^{-s}.$$

Observe that $C_g = 0$, $D_g = \{\frac{1}{n}\}_{n\in\mathbb{Z}}$ and $\Delta g\left(\frac{1}{n}\right) = 2^{-|n|}$ for $n\in\mathbb{Z}$.

Define

$$f(t) := \begin{cases} \left\lfloor \dfrac{1}{x} \right\rfloor^{-1}, & t\in[-1,1]\backslash\{0\}, \\ 0 & t = 0. \end{cases}$$

f is right continuous. Furthermore, it is clear that if $t\in[-1,1]\backslash\{0\}$, $f_g' = 0$, so $f_g' = 0$ μ_g-a.e.

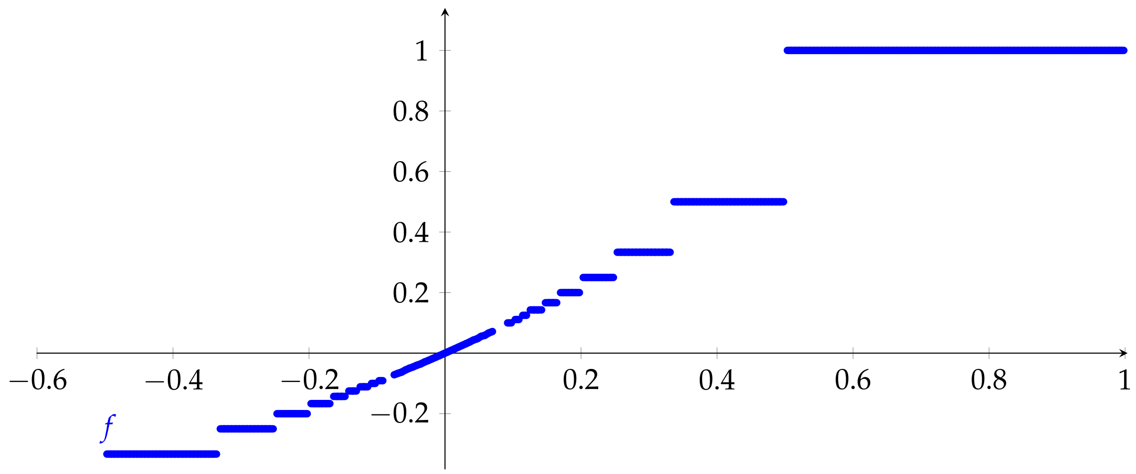

Figure 4.3: Representation of the function f.

—see Figure 4.3. Taking into account that, for $t\in[-1,1)\backslash\{0\}$,

$$t \leqslant \left\lfloor \frac{1}{t} \right\rfloor^{-1} < \frac{t}{1-t},$$

for $t\in(0,1)$,

$$\frac{f(t)-f(0)}{g(t)-g(0)} - 1 \leqslant \frac{\frac{t}{1-t}}{t + \sum\limits_{\substack{0<s<t \\ \frac{1}{s}\in\mathbb{Z}}} 2^{-s}} - 1 \leqslant \frac{\frac{t}{1-t}}{t} - 1 = \frac{1}{1-t} - 1 = \frac{t}{1-t},$$

and

$$\frac{f(t)-f(0)}{g(t)-g(0)} - 1 \geqslant \frac{t}{t + \sum\limits_{\substack{0<s<t \\ \frac{1}{s}\in\mathbb{Z}}} 2^{-s}} - 1 = \frac{t}{t} - 1 = 0,$$

whereas, for $t \in (-1/2, 0)$, given that $f(t) - f(0)$ and $g(t) - g(0)$ are negative,

$$\frac{f(t)-f(0)}{g(t)-g(0)} - 1 \leqslant \frac{t}{t + \sum_{\substack{t \leqslant s < 0 \\ \frac{1}{s} \in \mathbb{Z}}} 2^{-s} - 1} - 1 = \frac{-t}{1 - t - \sum_{\substack{t \leqslant s < 0 \\ \frac{1}{s} \in \mathbb{Z}}} 2^{-s}} - 1 \leqslant \frac{-t}{1-t},$$

and

$$\frac{f(t)-f(0)}{g(t)-g(0)} - 1 \geqslant \frac{\frac{t}{1-t}}{t + \sum_{\substack{t \leqslant s < 0 \\ \frac{1}{s} \in \mathbb{Z}}} 2^{-s} - 1} - 1 = \frac{-\frac{t}{1-t}}{1 - t - \sum_{\substack{t \leqslant s < 0 \\ \frac{1}{s} \in \mathbb{Z}}} 2^{-s}} - 1 \geqslant \frac{\frac{-t}{1-t}}{-t} - 1 = \frac{t}{t} - 1$$
$$= \frac{t}{1-t}.$$

In any case, we have that

$$f'_g(0) = \lim_{t \to 0} \frac{f(t)-f(0)}{g(t)-g(0)} = 1.$$

Thus, f is g-differentiable, $f'_g = 0$ μ_g-a.e. and

$$f'_g(x) := \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Theorem 4.33. *Let $n \in \mathbb{N}$. $f \in \mathcal{BD}^n_g([a,b],\mathbb{F})$ if and only if $f = h + \sum_{k=1}^n \rho_k$, where $h \in W^{n,1}_g([a,b],\mathbb{F}) \cap \mathcal{BD}^n_g([a,b];\mathbb{F})$ and $\rho_k \in W^{k-1,1}([a,b];\mathbb{F}) \cap \mathcal{BD}^n_g([a,b],\mathbb{F})$ is such that $\rho_g^{(k)} = 0$, for all $k = 1, \dots, n$.*

Proof. Given that $W^{n,1}_g([a,b],\mathbb{F}) \cap \mathcal{BD}^n_g([a,b];\mathbb{F}) \subset \mathcal{BD}^n_g([a,b];\mathbb{F})$, the sufficient condition is immediate. Let us now examine the necessary condition.

For $n = 1$, given an element $f \in \mathcal{BD}^1_g([a,b];\mathbb{F})$, we have that $f'_g \in \mathcal{BD}([a,b];\mathbb{F}) \subset \mathcal{L}^1_g([a,b];\mathbb{F})$. We can then consider $h(t) := \int_{[a,t)} f'_g \, \mathrm{d}\mu_g$, which is clearly $h \in W^{1,1}([a,b];\mathbb{F}) \cap \mathcal{BD}^1_g([a,b];\mathbb{F})$. Moreover, thanks to Lemma 4.14, we have $h'_g = (f'_g)^* = f'_g \in \mathcal{BD}_g([a,b];\mathbb{F})$. Now, let us define $\rho_1 := f - h \in \mathcal{BD}^1_g([a,b],\mathbb{F})$. By Lemmas 4.14 and 4.31, we obtain $\rho'_g = f'_g - h'_g = 0$.

Now, let $n \in \mathbb{N}$, $n \geq 2$, and assume the result holds for $n-1$. Since $f \in \mathcal{BD}^n_g([a,b];\mathbb{F})$, we have that $f'_g \in \mathcal{BD}^{n-1}_g([a,b];\mathbb{F})$. Therefore, there exist elements $\widetilde{h} \in W^{n-1,1}([a,b];\mathbb{F}) \cap \mathcal{BD}^{n-1}_g([a,b];\mathbb{F})$ and $\widetilde{\rho}_k \in W^{k-1,1}([a,b];\mathbb{F}) \cap \mathcal{BD}^{n-1}_g([a,b];\mathbb{F})$, with $(\widetilde{\rho}_k)^{k)}_g = 0$ for all $k = 1, \dots, n-1$, such that $f'_g = \widetilde{h} + \sum_{k=1}^{n-1} \widetilde{\rho}_k$. Define:

$$h(t) := \int_{[a,t)} \widetilde{h} \, \mathrm{d}\mu_g, \; t \in [a,b],$$
$$\rho_k(t) := \int_{[a,t)} \widetilde{\rho}_{k-1} \, \mathrm{d}\mu_g, \; t \in [a,b], \; k = 2, \dots, n,$$
$$\rho_1 := f - h - \sum_{k=2}^{n} \rho_k.$$

It follows that $h \in W^{n,1}([a,b];\mathbb{F}) \cap \mathcal{BD}^n_g([a,b];\mathbb{F})$, $\rho_k \in W^{k-1,1}([a,b];\mathbb{F}) \cap \mathcal{BD}^n_g([a,b];\mathbb{F})$ for $k = 1, \dots, n$. Furthermore, thanks to Lemmas 4.14 and 4.31 and [6, Proposition 4.1 and Remark 4.3] (that if φ is g-differentiable, φ^* is g-differentiable and the derivatives coincide), we obtain:

$$(\rho_k)_g^{k)} = (\widetilde{\rho}_{k-1}^*)_g^{k-1)} = (\widetilde{\rho}_{k-1})^{k-1)} = 0, \quad \forall k = 2, \dots, n,$$
$$(\rho_1)_g' = f_g' - h_g' - \sum_{k=2}^{n} (\rho_k)_g' = f_g' - \widetilde{h}^* - \sum_{k=1}^{n-1} (\widetilde{\rho}_k^*)_g = f_g' - (f_g')^* = 0. \qquad \square$$

Remark 4.34. The decomposition is unique if we impose $\rho_k(a) = 0$ for all $k = 1, \dots, n$.

Theorem 4.35. *Let $f : [a,b] \to \mathbb{F}$ a be such that $f_g'/f^* \in \mathcal{L}_g^1([a,b);\mathbb{F})$ is g-regressive. Then $f \in \mathcal{BD}_g^1([a,b],\mathbb{F})$ if and only if $f = \rho \cdot u$, where $u \in W_g^{1,1}([a,b],\mathbb{F}) \cap \mathcal{BD}_g^1([a,b];\mathbb{F})$ and $\rho \in \mathcal{BD}_g^1([a,b],\mathbb{F})$ is such that $\rho_g' = 0$.*

Furthermore, in that case, $u \in W_g^{1,1}([a,b],\mathbb{F}) \cap \mathcal{BD}_g^1([a,b];\mathbb{F})$ is a solution of equation

$$u_g'(t) = \frac{f_g'(t)}{f(t^*)} u(t), \quad \forall t \in [a,b]. \tag{4.5}$$

Proof. Since $W_g^{1,1}([a,b],\mathbb{F}) \cap \mathcal{BD}_g^1([a,b];\mathbb{F}) \subset \mathcal{BD}_g^1([a,b];\mathbb{F})$ and the product of two elements of $\mathcal{BD}_g^1([a,b];\mathbb{F})$ is in $\mathcal{BD}_g^1([a,b];\mathbb{F})$ (see Lemma 4.6), the sufficient condition is immediate. Let us now examine the necessary condition.

Let us consider the following initial value problem:

$$\begin{cases} u_g'(t) = \dfrac{f_g'(t)}{f(t^*)} u(t), \ g\text{-a.a. } t \in [a,b), \\ u(a) = 1. \end{cases}$$

Thanks to [5, Theorem 4.2 and Remark 4.4], we know that the above problem admits a unique solution in the space $\mathcal{AC}_g([a,b];\mathbb{F})$, which is also nonzero in $[a,b]$. Now, since $\frac{f_g'(t)}{f(t^*)} u(t) \in \mathcal{BD}_g([a,b];\mathbb{F})$, thanks to Lemma 4.14 and the fact that $(f_g')^* = f_g'$ and $u = u^*$ (since u is g-continuous), we can ensure that (4.5) holds at every point of the interval $[a,b]$. Consequently, we have $u \in \mathcal{AC}_g([a,b];\mathbb{F}) \cap \mathcal{BD}_g^1([a,b];\mathbb{F})$.

Now, let us define $\rho = f/u \in \mathcal{BD}_g^1([a,b];\mathbb{F})$. Clearly, we have $f = \rho \cdot u$. Moreover, given any element $t \in [a,b]$, we obtain:

$$\begin{aligned} \rho_g'(t) &= \left(\frac{f}{u}\right)_g'(t) \\ &= \frac{f_g'(t)u(t) - u_g'(t)f(t^*)}{u(t^*)\,(u(t^*) + u_g'(t)\,\Delta g(t^*))} = 0, \end{aligned}$$

since (4.5) holds for all elements in the interval $[a,b]$. $\square$

5 Applications to Stieltjes differential equations

Although we have so far considered an arbitrary interval $[a,b]$, in this section, we will focus on the interval $[0,T]$ as it is a more natural choice for the study of initial value problems. In any case, the generalization to generic intervals of the form $[a,b]$ is possible. Thus, throughout this section we will assume that $[0,T] \subset \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is a derivator such that $0 \notin N_g^-$ and $T \notin N_g^+ \cup D_g \cup C_g$.

Definition 5.1 (Kernel of the operator ∂_g). We define $\ker(\partial_g) \subset \mathcal{BD}_g^1([0,T];\mathbb{F})$ as the kernel of the following operator:

$$\begin{array}{rcl} \partial_g : \mathcal{BD}_g^1([0,T];\mathbb{F}) & \to & \mathcal{BD}_g([0,T];\mathbb{F}) \\ h & \to & \partial_g(h) = h'_g. \end{array}$$

In the following proposition, we will show that $\ker(\partial_g)$ is a non-empty set.

Proposition 5.2. *Assume $D'_g \subset D_g$ and let $h : [0,T] \to \mathbb{F}$ be bounded, right continuous at $[0,T] \cap D_g$ and constant on each connected component of $[0,T] \setminus D_g$. Then $h \in \ker(\partial_g)$.*

Proof. Given an element $t \in D_g$, we have that $h(t^+) = h(t)$, so $h'_g(t) = 0$. Now, given $t \in [0,T] \setminus D_g$, since D_g is closed, there exists $\delta > 0$ such that h is constant at $(t-\delta, t+\delta)$, therefore $h'_g(t) = 0$. □

Let us now examine the relationship between $\ker(\partial_g) \subset \mathcal{BD}_g^1([0,T];\mathbb{F})$ and the solutions of the homogeneous Stieltjes linear differential equation in the space $\mathcal{BD}_g^1([0,T];\mathbb{F})$.

Proposition 5.3. *Let $\beta \in \mathcal{BD}_g([0,T];\mathbb{F})$ be g-regressive. The solutions in $\mathcal{BD}_g^1([0,T];\mathbb{F})$ of problem*

$$\begin{cases} v'_g(t) - \beta(t)\, v(t) = 0, \ t \in [0,T), \\ v(0) = 1, \end{cases} \tag{5.1}$$

are the functions of the form $h \cdot v$ where $h \in \ker(\partial_g)$ with $h(0) = 1$ and $v \in \mathcal{BD}_g^1([0,T];\mathbb{F})$ is the unique g-absolutely continuous solution of problem (5.1).

Proof. We start by observing that the uniqueness of the g-absolutely continuous solution v of problem (5.1) was proven in [5, Theorem 4.2 and Remark 4.4]. Moreover, since v is g-absolutely continuous, it is bounded and g-continuous by [9, Proposition 5.5] and, therefore, by definition of $\mathcal{BD}_g([0,T];\mathbb{F})$, we have that $v \in \mathcal{BD}_g([0,T];\mathbb{F})$. Given that $\beta \in \mathcal{BD}_g([0,T];\mathbb{F})$, we have, by Lemma 4.6, that $\beta \cdot v \in \mathcal{BD}_g([0,T];\mathbb{F})$. Furthermore, since $v'_g = \beta \cdot v$ μ_g-a.e. and v is g-absolutely continuous, by Theorem 2.10 we have that $v(t) = v(0) + \int_{[0,t)} \beta \cdot v \, \mathrm{d}\mu_g$ for every $t \in [0,T]$. Thanks to Lemma 4.14, we have that $v'_g = \beta^* v \in \mathcal{BD}_g([0,T];\mathbb{F})$ and $v'_g(t) - \beta(t^*)\, v(t) = 0$ for all $t \in [0,T)$. Hence, $v \in \mathcal{AC}_g([0,T];\mathbb{F}) \cap \mathcal{BD}_g^1([0,T];\mathbb{F})$.

We now show that a function of the form $\widetilde{v} = h\, v$ is a solution of problem (5.1) in $\mathcal{BD}_g^1([0,T];\mathbb{F})$. Indeed, observe that, thanks to Lemma 4.6 and the fact that $h \in \mathcal{BD}_g^1([0,T];\mathbb{F})$, we have that $\widetilde{v} = h\, v \in \mathcal{BD}_g^1([0,T];\mathbb{F})$. Moreover, $\widetilde{v}(0) = h(0)\, v(0) = 1$.

Furthermore, for a given element $t \in [0,T) \setminus C_g$, $t^* = t$. If t is such that $v'_g(t) = \beta(t)\, v(t)$, it follows, thanks to Lemma 3.21, that $\widetilde{v}'_g(t) = h(t)\, v'_g(t) = h(t)\, \beta(t)\, v(t) = \beta(t)\, \widetilde{v}(t)$. We conclude that $\widetilde{v}'_g(t) = \beta(t)\, \widetilde{v}(t)$, g-a.e. $t \in [0,T)$ and, thus, $\widetilde{v}$ is a solution of problem (5.1) in $\mathcal{BD}_g^1([0,T];\mathbb{F})$.

Finally we show that if $w \in \mathcal{BD}_g^1([0,T];\mathbb{F})$ is a solution of problem (5.1), it has to be of the form $h \cdot v$ where $h \in \ker(\partial_g)$ with $h(0) = 1$. Indeed, given that $\beta = w'/w^* \in \mathcal{L}_g^1([0,T);\mathbb{F})$ is g-regressive, we can apply the construction in Theorem 4.35 to conclude that $w = h \cdot u$, for some $h \in \mathcal{BD}_g^1([0,T],\mathbb{F})$ such that $h'_g = 0$ and $u \in \mathcal{AC}_g([0,T];\mathbb{F}) \cap \mathcal{BC}_g^1([0,T];\mathbb{F})$ satisfying

$$\begin{cases} u'_g(t) = \dfrac{w'(t)}{w(t^*)}\, u(t) = \beta(t)\, u(t), \forall t \in [0,T], \\ u(0) = 1. \end{cases}$$

Hence, u is a g-absolutely continuous function that is a solution of problem (5.1). Since the g-absolutely continuous solution is unique, $u = v$, so $w = h \cdot v$, as we wanted to show. □

Proposition 5.4. *Let $\beta \in \mathcal{BD}_g([0,T];\mathbb{F})$ be g-regressive. Given $v \in \mathcal{AC}_g([0,T];\mathbb{F}) \cap \mathcal{BC}^1_g([0,T];\mathbb{F})$ the solution of (5.1) we have that a function $h \in \mathcal{BD}^1_g([0,T];\mathbb{F})$, with $h(0) = 1$, belongs to $\ker(\partial_g)$ if and only if $\widetilde{v} = h\,v \in \mathcal{BD}^1_g([0,T];\mathbb{F})$ is a solution of problem (5.1) and it satisfies that $\widetilde{v}'_g(t) - \beta(t)\,\widetilde{v}(t) = 0$, $\forall t \in [0,T] \setminus C_g$.*

Proof. Before starting the proof, it is worth remembering that the unique solution v of (5.1) in the space $\mathcal{AC}_g([0,T];\mathbb{F}) \cap \mathcal{BD}^1_g([0,T];\mathbb{F})$ is such that $v(t) \neq 0$, for all $t \in [0,T]$ —see [5, Theorem 4.2 and Remark 4.4]. Moreover, $v'_g(t) - \beta(t)\,v(t) = 0$, for all $t \in [0,T] \setminus C_g$.

Now, on the one hand, given an element $h \in \ker(\partial_g)$, by Proposition 5.3, it follows that $\widetilde{v} = h\,v \in \mathcal{BD}^1_g([0,T];\mathbb{F})$ is also a solution of (5.1). Moreover, $\widetilde{v}'_g(t) = h(t)\,v'_g(t) = h(t)\,\beta(t)\,v(t) = \beta(t)\,\widetilde{v}(t)$, for all $t \in [0,T] \setminus C_g$.

On the other hand, assume that $\widetilde{v} \in \mathcal{BD}^1_g([0,T];\mathbb{F})$ is a solution of (5.1) which also satisfies $\widetilde{v}'_g(t) - \beta(t)\,\widetilde{v}(t) = 0$, for all $t \in [0,T] \setminus C_g$. We have that:

$$\begin{aligned}\beta(t)\,\widetilde{v}(t) &= \widetilde{v}'_g(t)\\ &= (h\,v)'_g(t)\\ &= h'_g(t)\,v(t) + h(t)\,v'_g(t) + h'_g(t)\,v'_g(t)\,\Delta g(t), \ \forall t \in [0,T] \setminus C_g.\end{aligned}$$

Taking into account that $\widetilde{v} = hv$ and that $v'_g = \beta v$,

$$\beta(t)\,h(t)\,v(t) = h'_g(t)\,v(t) + \beta(t)\,h(t)\,v(t) + \beta(t)\,h'_g(t)\,v(t)\,\Delta g(t), \ \forall t \in [0,T] \setminus C_g.$$

Therefore,

$$h'_g(t)\,v(t)\,(1 + \beta(t)\,\Delta g(t)) = 0, \ \forall t \in [0,T] \setminus C_g.$$

Since $1 + \beta(t)\,\Delta g(t) \neq 0$ and $v(t) \neq 0$ for all $t \in [0,T]$, we conclude that $h'_g(t) = 0$ for all $t \in [0,T] \setminus C_g$, and thus, $h'_g(t) = 0$ for all $t \in [0,T]$. □

Remark 5.5. From Proposition 5.4 it follows that, given an element $\beta \in \mathcal{BD}_g([0,T];\mathbb{F})$ such that $1 + \beta(t)\Delta g(t) \neq 0$ for all $t \in [0,T] \cap D_g$, the dimension of the space of solutions in $\mathcal{BD}^1_g([0,T];\mathbb{F})$ of problem (5.1) is equal to the dimension of $\ker(\partial_g)$.

Example 5.6. Let be $\beta \in \mathcal{BD}_g([0,T];\mathbb{F})$ be g-regressive. Let $v = \exp_g(\beta,\cdot)$, see Definition 2.11.

Now, let us define the function $h : t \in [0,T] \to h(t) \in \mathbb{F}$, where

$$\begin{aligned}h(t) &= \exp\left(- \int_{[0,t]\cap D_g} \widetilde{\beta}(s)\, \mathrm{d}\,\mu_g(s) \right)\\ &= \exp\left(- \sum_{s\in[0,t]\cap D_g} \ln\left(1 + \beta(s)\Delta g(s)\right) \right)\\ &= \left[\prod_{s\in[0,t]\cap D_g} (1 + \beta(s)\,\Delta g(s)) \right]^{-1}.\end{aligned}$$

The above function is well-defined since $\beta \in \mathcal{L}^1_g([0,T);\mathbb{F})$ and $T \notin D_g$. Furthermore, h is bounded, right-continuous in $[0,T] \cap D_g$, constant in the connected components of $[0,T] \setminus D_g$, and satisfies $h(0) = 1$. Therefore, thanks to Proposition 5.2, $h \in \ker(\partial_g)$, and, by Proposition 5.3, the function $\widetilde{v} : t \in [0,T] \to \mathbb{F}$, defined by

$$\begin{aligned}\widetilde{v}(t) &= h(t)\, v(t) \\ &= \frac{\left[\prod_{s\in[0,t)\cap D_g} (1+\beta(s)\,\Delta g(s))\right]}{\left[\prod_{s\in[0,t]\cap D_g} (1+\beta(s)\,\Delta g(s))\right]} \exp\left(\int_{[0,t)\setminus D_g} \beta(s)\; \mathrm{d}\,\mu_g(s)\right) \\ &= \frac{1}{(1+\beta(t)\,\Delta g(t))} \exp\left(\int_{[0,t)\setminus D_g} \beta(s)\; \mathrm{d}\,\mu_g(s)\right)\end{aligned}$$

is another solution in the space $\mathcal{BD}^1_g([0,T];\mathbb{F})$ of the initial value problem (5.1).

For example, let $T > 2$ and consider the following generator:

$$g : t \in \mathbb{R} \to g(t) := \begin{cases} t, & t \le 1, \\ t+\delta_1, & t \in (1,2], \\ t+\delta_1+\delta_2, & t > 2, \end{cases}$$

where $\delta_k > 0$, for $k = 1,2$. In this case, $D_g = \{1,2\}$ and $N_g = \emptyset$, thus it is clear that $N_g' \setminus N_g \subset D_g$ and $D_g' \subset D_g$. Now, let $\beta : t \in [0,T] \to \beta(t) = 1$. It follows that β is g-regressive on $[0,T]$ for all $\delta_k > 0$, with $k = 1,2$. Therefore,

$$\begin{aligned} v(t) &= \exp_g(\beta,t) \\ &= \left[\prod_{s\in[0,t)\cap D_g} (1+\Delta g(s))\right] \exp\left(\mu_g([0,t)\setminus D_g\right) = \begin{cases} \exp(t), & t\le 1, \\ (1+\delta_1)\exp(t), & t \in (1,2], \\ (1+\delta_1)(1+\delta_2)\exp(t), & t>2 \end{cases}\end{aligned}$$

and

$$\begin{aligned}\widetilde{v}(t) &= h(t)\, v(t) \\ &= \frac{\prod_{s\in[0,t)\cap D_g} (1+\Delta g(s))}{\prod_{s\in[0,t]\cap D_g} (1+\Delta g(s))} \exp\left(\mu_g([0,t)\setminus D_g\right) = \begin{cases} \exp(t), & t<1, \\ (1+\delta_1)^{-1}\exp(t), & t=1, \\ \exp(t), & t\in(1,2), \\ (1+\delta_2)^{-1}\exp(t), & t=2, \\ \exp(t), & t>2. \end{cases}\end{aligned}$$

are two solutions in the space $\mathcal{BD}^1_g([0,T];\mathbb{R})$ of the initial value problem (5.1). Observe that both solutions converge to $\exp(t)$ as δ_1 and δ_2 tends to zero.

In Figure 5.1, we can observe the behavior of $v(t) = \exp_g(\beta,t)$, while in Figure 5.2, we see that of $\widetilde{v}(t) = h(t)\, v(t)$. Observe that, in both cases, the solutions exhibit discontinuities at the points of D_g. However, while v is left-continuous, $\widetilde{v}$ is not.

The significance of this example is crucial in the sense that the g-exponential function associated to β under the given conditions is unique in the space $\mathcal{AC}_g([0,T];\mathbb{F})$, while if we consider the space $\mathcal{BD}^1_g([0,T];\mathbb{F})$, the g-exponential function is not uniquely determined.

We end this work with the following corollaries of Theorem 5.3.

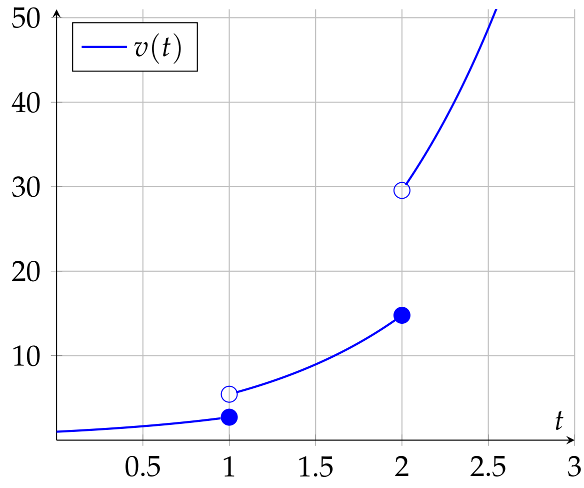

Figure 5.1: $v(t) = \exp_g(\beta, t)$.

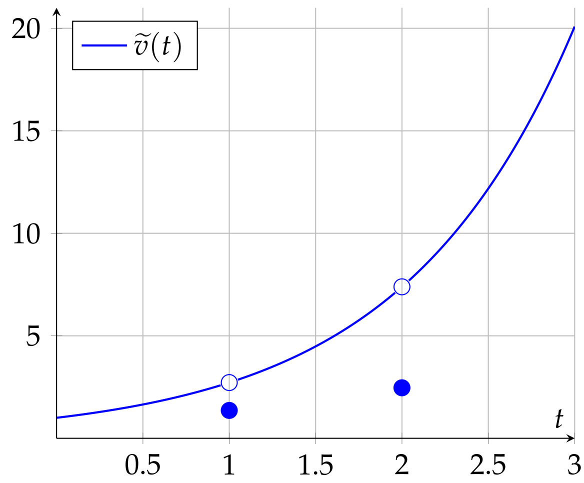

Figure 5.2: $\widetilde{v}(t) = h(t)\, v(t)$.

Corollary 5.7. *Let $n \in \mathbb{N}$, $\beta \in \mathcal{BD}_g^n([0,T];\mathbb{F})$ g-regressive. The solutions in $\mathcal{BD}_g^{n+1}([0,T];\mathbb{F})$ of problem*

$$\begin{cases} v_g'(t) - \beta(t)\, v(t) = 0, \ t \in [0,T), \\ v(0) = 1, \end{cases} \tag{5.2}$$

are the functions of the form $h \cdot v$ where $h \in \ker(\partial_g)$ with $h(0) = 1$ and $v \in \mathcal{BD}_g^{n+1}([0,T];\mathbb{F})$ is the unique g-absolutely continuous solution of problem (5.2).

Proof. We will assume $n \geqslant 2$, as the case $n = 1$ is covered by Proposition 5.3. We start by observing that $v \in \mathcal{BD}_g^n([0,T];\mathbb{F})$, where v is the unique g-absolutely continuous solution of problem (5.2). Indeed, as we reasoned in Proposition 5.3, $v \in \mathcal{BD}_g^1([0,T];\mathbb{F})$ and, given that $\beta \in \mathcal{BD}_g^n([0,T];\mathbb{F})$, we have, by Lemma 4.6, that $\beta \cdot v \in \mathcal{BD}_g^1([0,T];\mathbb{F})$. Furthermore, since $v_g' = \beta \cdot v$ μ_g-a.e. and v is g-absolutely continuous, by Theorem 2.10 we have that $v(t) = v(0) + \int_{[0,t)} \beta\, v \,\mathrm{d}\,\mu_g$ for every $t \in [0,T]$. Thanks to Lemma 4.14, we have that $v_g' = \beta\, v \in \mathcal{BD}_g^2([0,T];\mathbb{F})$ and $v_g'(t) - \beta(t^*)\, v(t) = 0$ for all $t \in [0,T]$. Hence, $v \in \mathcal{BD}_g^2([0,T];\mathbb{F})$. Repeating this argument inductively, we conclude that $v \in \mathcal{BD}_g^{n+1}([0,T];\mathbb{F})$.

Now take $\widetilde{v} = h \cdot v$ where $h \in \ker(\partial_g)$ with $h(0) = 1$ and we will show that $\widetilde{v}$ is a solution in $\mathcal{BD}_g^n([0,T];\mathbb{F})$ of problem (5.2). We already know from Proposition 5.3 that $\widetilde{v}$ is a solution in $\mathcal{BD}_g^1([0,T];\mathbb{F})$ of problem (5.2), so it is enough to show that $\widetilde{v} \in \mathcal{BD}_g^n([0,T];\mathbb{F})$, but this is evident given that, since $h'_g = 0$, and $v \in \mathcal{BD}_g^n([0,T];\mathbb{F})$, we have that $\widetilde{v}_g^{k)} = hv^{k)}$ for $k = 1,\dots,n$ and, thus, $\widetilde{v} \in \mathcal{BD}_g^n([0,T];\mathbb{F})$.

On the other hand, if $u \in \mathcal{BD}_g^n([0,T];\mathbb{F})$ is a solution of problem (5.2), by Proposition 5.3, $u = h\,v$ for some $h \in \ker \partial_g$ with $h(0) = 1$. □

The next corollary is proven following the same steps as in Proposition 5.3 and Corollary 5.7.

Corollary 5.8. *Let $n \in \mathbb{N}$, $\beta, f \in \mathcal{BD}_g^n([0,T];\mathbb{F})$ g-regressive. The solutions in $\mathcal{BD}_g^{n+1}([0,T];\mathbb{F})$ of problem*

$$\begin{cases} v'_g(t) - \beta(t)\,v(t) = f(t), \ t \in [0,T), \\ v(0) = 1, \end{cases} \tag{5.3}$$

are the functions of the form $h \cdot v$ where $h \in \ker(\partial_g)$ with $h(0) = 1$ and $v \in \mathcal{BD}_g^{n+1}([0,T];\mathbb{F})$ is the unique g-absolutely continuous solution of problem (5.3).

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