Electronic Journal of Qualitative Theory of Differential Equations
2025, No. **51**, 1–10; https://doi.org/10.14232/ejqtde.2025.1.51 www.math.u-szeged.hu/ejqtde/

Note on the oscillation of second-order hybrid type delay differential equations

Natarajan Prabaharan[1], **Radhakrishnan Srinivasan**[2], **John R. Graef**[✉3] and **Ethiraju Thandapani**[4]

[1]Department of Mathematics, R. M. D. Engineering College, Kavaraipettai 601206, Tamil Nadu, India
[2]Department of Mathematics, SRM Institute of Science and Technology, Ramapuram Campus, Chennai 600089, India
[3]Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA
[4]Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600005, India

Received 26 May 2025, appeared 12 October 2025
Communicated by Zuzana Došlá

Abstract. This note presents some new criteria for the oscillation of all solutions of the second-order hybrid type delay differential equation

$$(a(t)x'(t))' - p_1(t)x(t) + p_2(t)x^{\alpha}(\sigma(t)) = 0,$$

by making use of a positive solution of the associated linear differential equation

$$(a(t)u'(t))' - p_1(t)u(t) = 0.$$

The results obtained are new and are illustrated by three examples.

Keywords: second-order, hybrid type, delay differential equation, oscillation.

2020 Mathematics Subject Classification: 34C10, 34K11.

1 Introduction

Consider the second-order hybrid type delay differential equation of the form

$$(a(t)x'(t))' - q_1(t)x(t) + q_2(t)x^{\alpha}(\sigma(t)) = 0, \qquad t \geq t_0, \tag{E}$$

where we assume throughout that the following conditions hold:

(H_1) $a \in C^1([t_0,\infty),(0,\infty))$, $q_1,\ q_2 \in C([t_0,\infty),(0,\infty))$, and α is a ratio of odd positive integers;

(H_2) $\sigma \in C^1([t_0,\infty),\mathbb{R})$ with $\sigma'(t) \geq 0$, $\sigma(t) \leq t$, and $\sigma(t) \to \infty$ as $t \to \infty$.

✉Corresponding author. Email: johngraef9@gmail.com

By a *proper solution* of (E), we mean a function $x : [T_x, \infty) \to \mathbb{R}$ that satisfies (E) for all sufficiently large t and $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We always assume that equation (E) possesses proper solutions. A solution of (E) is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Note that if $q_1 = 0$, then (E) reduces to the second order delay differential equation

$$(a(t)x'(t))' + q_2(t)x^{\alpha}(\sigma(t)) = 0, \qquad t \geq t_0, \tag{E_1}$$

and if $q_2 = 0$, then we obtain the linear differential equation without a delay

$$(a(t)x'(t))' - q_1(t)x(t) = 0, \qquad t \geq t_0. \tag{E_2}$$

Therefore, we may refer to equation (E) as a "hybrid type" differential equation.

Oscillation properties of solutions of second-order nonlinear delay differential equations were first studied in [9]. Since then, there has been a large number of papers regarding the oscillation of delay differential equations of different forms; see, for example, the monographs [1,2,10], the recent papers [3,5,7,8,13,17], and the references cited therein. This is due to the fact that oscillation and delay/advanced phenomena appear in different models arising from real world applications.

Recently in [8], the authors studied oscillation properties of the trinomial differential equation

$$x''(t) - q_1(t)x(\tau(t)) - q_2(t)x(\sigma(t)) = 0, \qquad t \geq t_0, \tag{E_3}$$

where $\tau(t) \geq t$ and $\sigma(t) \leq t$ for all $t \geq t_0$. Moreover, in [5,17], the authors presented oscillation criteria for equations of the form

$$(a(t)x'(t))' + q_1(t)x(t) + q_2(t)x(\sigma(t)) = 0, \qquad t \geq t_0, \tag{E_4}$$

in case (E_4) is in either canonical or noncanonical form. Note that if $a(t) > 0$, $q_1(t) > 0$, and $q_2(t) > 0$ for all $t \geq t_0$, then for any positive solution $x(t)$ of (E_3) or (E_4), the sign of the second-order derivative becomes

$$x''(t) > 0 \quad \text{or} \quad (a(t)x'(t))' < 0, \quad \text{for } t \geq t_0,$$

and from this it can easily be seen that $x'(t) > 0$ or $x'(t) < 0$ eventually. However, equation (E) includes both of the binomial equations (E_1) and (E_2) as special cases. Properties of both of these equations have been studied by several authors; see, for example, the monographs [1,2,15] and the references contained therein for details.

It is clear that the solutions spaces of equations (E_1) and (E_2) are completely different. If we denote by S the set of all nonoscillatory solutions of a considered equation, then for (E_1), the set S becomes

$$S = S_1 \cup S_0$$

where positive solutions satisfy

$$x(t) \in S_1 \quad \text{if and only if} \quad x'(t) > 0 \text{ and } (a(t)x'(t))' < 0,$$

and

$$x(t) \in S_0 \quad \text{if and only if} \quad x'(t) < 0 \text{ and } (a(t)x'(t))' < 0.$$

On the other hand, for (E_2), the set S has the structure

$$S = S_0^* \cup S_2^*$$

where positive solutions satisfy

$$x(t) \in S_0^* \quad \text{if and only if} \quad x'(t) < 0 \text{ and } (a(t)x'(t))' > 0,$$

and

$$x(t) \in S_2^* \quad \text{if and only if} \quad x'(t) > 0 \text{ and } (a(t)x'(t))' > 0.$$

Consequently, the structure of the set of nonoscillatory solutions of the hybrid equation (E) with positive and negative parts is unclear.

A method often employed in the study of the oscillation of solutions of trinomial differential equations is to omit one term. If, we omit the negative part of (E), then we are led to the differential inequality

$$\{(a(t)x'(t))' + q_2(t)x^{\alpha}(\sigma(t))\} \operatorname{sgn} x(t) \geq 0. \tag{E$_5$}$$

However, it is well known that properties of the corresponding differential equation (E_1) are connected to solutions of a differential inequality in the opposite direction. Similarly, omitting the positive term in (E) gives the differential inequality

$$\{(a(t)x'(t))' - q_1(t)x(t)\} \operatorname{sgn} x(t) \leq 0, \tag{E$_6$}$$

which is again opposite to the one that we would need. There are only a limited number of papers for equations like (E) with positive and negative parts. In this paper, we use a different approach that overcomes those difficulties caused by the presence of both negative and positive terms in (E). We then use comparison methods to obtain criteria for the oscillation of all solutions of (E). Three examples are provided to illustrate the importance and novelty of our main results.

2 Main results

In this section, we first transform the hybrid equation (E) into the form of a binomial equation, and then use a positive solution of the related auxiliary second-order linear ordinary differential equation

$$(a(t)u'(t))' = q_1(t)u(t), \qquad t \geq t_0. \tag{2.1}$$

Lemma 2.1. *The differential equation* (2.1) *has a positive nonincreasing solution on* $[t_0, \infty)$.

Proof. The proof is similar to Theorem 2.46 of [15] and so the details are omitted. □

Next, using a positive solution $u(t)$ of (2.1), we can obtain the following result.

Lemma 2.2. *Let* $u(t)$ *be a positive decreasing solution of* (2.1) *on* $[t_0, \infty)$. *Then* (E) *can be written in the form*

$$\left(a(t)u^2(t)\left(\frac{x(t)}{u(t)}\right)'\right)' + u(t)q_2(t)x^{\alpha}(\sigma(t)) = 0, \qquad t \geq t_0. \tag{E$_7$}$$

Proof. By a direct calculation and using (2.1), we see that

$$\begin{aligned}\frac{1}{u(t)}\left(a(t)u^2(t)\left(\frac{x(t)}{u(t)}\right)'\right)' &= \frac{1}{u(t)}\left(a(t)u(t)x'(t) - a(t)u'(t)x(t)\right)' \\ &= \frac{1}{u(t)}\left(u(t)(a(t)x'(t))' - (a(t)u'(t))'x(t)\right) \\ &= (a(t)x'(t))' - \frac{(a(t)u'(t))'x(t)}{u(t)} \\ &= (a(t)x'(t))' - q_1(t)x(t). \end{aligned} \tag{2.2}$$

Substituting (2.2) into (E) yields the desired result, and completes the proof. □

For the sake of convenience, we define the functions:

$$b(t) = a(t)u^2(t), \qquad B(t) = \int_{t_*}^{t} \frac{1}{a(s)u^2(s)} ds,$$

$$Q(t) = u(t)q_2(t)u^{\alpha}(\sigma(t)), \quad \text{and} \quad z(t) = \frac{x(t)}{u(t)}$$

for $t \geq t_* \geq t_0$. Employing the oscillation-preserving transformation $x(t) = u(t)z(t)$, where u is a positive solution of the differential equation (2.1), we can obtain the following consequence of Lemma 2.2.

Theorem 2.3. *Let $u(t)$ be a positive decreasing solution of (2.1). Then (E) is oscillatory if and only if the differential equation*

$$(b(t)z'(t))' + Q(t)z^{\alpha}(\sigma(t)) = 0, \qquad t \geq t_0, \tag{2.3}$$

is oscillatory.

For convenience in what follows, we will assume that (2.3) is in canonical form, that is,

$$B(t) \to \infty \quad \text{as } t \to \infty. \tag{2.4}$$

Remark 2.4. If $B(t) < \infty$ so that equation (2.3) is in noncanonical form, define

$$\Omega(t) = \int_t^{\infty} \frac{1}{b(s)} ds, \qquad b_1(t) = b(t)\Omega^2(t), \qquad \mu(t) = z(t)/\Omega(t),$$

and

$$Q_1(t) = \Omega(t)Q(t)\Omega^{\alpha}(\sigma(t)).$$

Then equation (2.3) can be written in the canonical form as

$$(b_1(t)\mu'(t))' + Q_1(t)\mu^{\alpha}(\sigma(t)) = 0, \qquad t \geq t_0. \tag{2.5}$$

The oscillation of (2.5) can then be deduced as for equation (2.3).

Theorem 2.5. *Let (2.4) hold and $u(t)$ be a positive decreasing solution of (2.1). Assume that $z(t)$ is an eventually positive (nonoscillatory) solution of (2.3). Then,*

$$z'(t) > 0 \quad \text{and} \quad b(t)z'(t))' \leq 0 \tag{2.6}$$

for all $t \geq t_1 \geq t_0$.

Proof. Let $z(t)$ be an eventually positive solution of (2.3), say $z(t) > 0$ and $z(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then from (2.3), we see that $(b(t)z'(t))' \leq 0$ and so $b(t)z'(t) > 0$ or $b(t)z'(t) < 0$ for all $t \geq t_1$.

If $b(t)z'(t) < 0$ for $t \geq t_1$, then

$$b(t)z'(t) \leq b(t_1)z'(t_1) < 0.$$

Dividing the last inequality by $b(t)$ and then integrating from t_1 to t gives

$$z(t) \leq z(t_1) + b(t_1)z'(t_1)\int_{t_1}^{t}\frac{1}{b(s)}ds \to -\infty$$

as $t \to \infty$ by (2.4). This contradicts the positivity of $z(t)$ and proves the theorem. □

Theorem 2.6. *Let (2.4) hold and u be a positive decreasing solution of (2.1). If*

$$\int_{t_1}^{\infty} Q(t)dt = \infty, \tag{2.7}$$

then equation (E) is oscillatory.

Proof. Assume, to the contrary, that $x(t)$ is an eventually positive solution of (E), say $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, by our transformation, $z(t) = \frac{x(t)}{u(t)}$ is an eventually positive solution of (2.3), say $z(t) > 0$ and $z(\sigma(t)) > 0$ for all $t \geq t_1$. By Theorem 2.5, $z(t)$ is increasing, so by (H_1), we have $z(\sigma(t)) \geq z(\sigma(t_1))$ for $t \geq t_1$. Using this in (2.3) and then integrating the resulting inequality from t_1 to t yields

$$z^{\alpha}(\sigma(t_1))\int_{t_1}^{t} Q(s)ds \leq b(t_1)z'(t_1) - b(t)z'(t) \leq b(t_1)z'(t_1)$$

since $b(t)z'(t)$ is nonnegative. Letting $t \to \infty$ in the last inequality contradicts (2.7). This completes the proof. □

Remark 2.7. The above theorem is independent of α and the delay argument, so it holds for linear, super-linear, and sublinear equations as well as for delay or advanced type equations.

Next, we give two results for the oscillation of equation (E) without condition (2.7) holding.

Theorem 2.8. *Let (2.4) hold and u be a positive decreasing solution of (2.1). If $\alpha = 1$ and*

$$\liminf_{t\to\infty} B(\sigma(t))\int_{t}^{\infty} Q(s)ds > \frac{1}{4}, \tag{2.8}$$

then equation (E) is oscillatory.

Proof. Assume that $x(t)$ is an eventually positive solution of (E), say $x(t) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$ for some $t_1 \geq t_0$. Then, proceeding as in the proof of Theorem 2.6, we have $z(t) > 0$, $z(\sigma(t)) > 0$, and condition (2.6) holds for all $t \geq t_1$. Using Corollary 2 in [7], we see that condition (2.8) implies that $z(t)$ is oscillatory. This contradiction completes the proof. □

Theorem 2.9. *Let (2.4) hold and u be a positive decreasing solution of (2.1). If the first order delay differential equation*

$$w'(t) + Q(t)B^{\alpha}(\sigma(t))w^{\alpha}(\sigma(t)) = 0 \tag{2.9}$$

is oscillatory, then equation (E) is oscillatory.

Proof. Assume that $x(t)$ is an eventually positive solution of (E), say $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.6, we see that $z(t)$ is a positive solution of (2.3) and (2.6) holds for $t \geq t_1$. Using the monotonicity of $b(t)z'(t) > 0$, we have

$$z(t) \geq \int_{t_1}^{t} \frac{b(s)z'(s)}{b(s)} ds \geq B(t)b(t)z'(t).$$

Using this in (2.3) yields

$$(b(t)z'(t))' + Q(t)B^{\alpha}(\sigma(t))(b(\sigma(t))z'(\sigma(t)))^{\alpha} \leq 0.$$

Let $w(t) = b(t)z'(t) > 0$; then $w(t)$ is a positive solution of the inequality

$$w'(t) + Q(t)B^{\alpha}(\sigma(t))w^{\alpha}(\sigma(t)) \leq 0.$$

By [14, Theorem 1], equation (2.9) also has a positive solution, which is a contradiction. This completes the proof. □

Next, we present some explicit oscillation criteria for equation (E) with the help of the results given in [10] and [16] for $\alpha = 1$, $0 < \alpha < 1$, and $\alpha > 1$.

Corollary 2.10. *Let* (2.4) *hold and* u *be a positive decreasing solution of* (2.1)*. If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)B(\sigma(s))ds > \frac{1}{e} \quad \text{for } \alpha = 1, \tag{2.10}$$

or

$$\int_{t_0}^{\infty} Q(t)B^{\alpha}(\sigma(t))dt = \infty \quad \text{for } 0 < \alpha < 1, \tag{2.11}$$

then equation (E) *is oscillatory.*

Proof. Based on results in [10] (see [10, Theorem 2.1.1] and [10, Theorem 3.6.3], we see that conditions (2.10) and (2.11), respectively, imply equation (2.3) is oscillatory. The conclusion then follows from Theorem 2.3. □

Corollary 2.11. *Let* $\alpha > 1$, (2.4) *hold, and* u *be a positive decreasing solution of* (2.1)*. Then every solution of* (E) *is oscillatory if any of the following conditions hold:*

(i) $\sigma(t) = t - \tau$, $\tau > 0$, *and there exists* $\lambda > \tau^{-1}\ln\alpha$ *such that*

$$\liminf_{t\to\infty} \left[Q(t)B^{\alpha}(\sigma(t))\exp(-e^{\lambda t})\right] > 0; \tag{2.12}$$

(ii) $\sigma(t) = \theta t$, $0 < \theta < 1$ *and there exists* $\lambda > -\ln\alpha/\ln\theta$ *such that*

$$\liminf_{t\to\infty} \left[Q(t)B^{\alpha}(\sigma(t))\exp(-t^{\lambda})\right] > 0; \tag{2.13}$$

(iii) $\sigma(t) = t^{\theta}$, $0 < \theta < 1$ *and there exists* $\lambda > -\ln\alpha/\ln\theta$ *such that*

$$\liminf_{t\to\infty} \left[Q(t)B^{\alpha}(\sigma(t))\exp(-(\ln t)^{\lambda})\right] > 0. \tag{2.14}$$

Proof. The proof follows from Theorem 3(i), Theorem 4(i), and Theorem 5(i) in [16], respectively, along with Theorem 2.9 above. This proves the corollary. □

Before presenting our final theorem, we define the constants

$$\beta_* = \lim_{t\to\infty} b(t)B(\sigma(t))B(t)Q(t) \quad \text{and} \quad \lambda_* = \lim_{t\to\infty} \frac{B(t)}{B(\sigma(t))}.$$

Theorem 2.12. *Let (2.4) hold and u be a positive decreasing solution of (2.1). If $\alpha = 1$ and*

$$\beta_* > \begin{cases} 0, & \text{if } \lambda_* = \infty, \\ \frac{d+2-\sqrt{(d+2)^2-4d}}{2d}, & \text{if } \lambda_* < \infty, \end{cases} \tag{2.15}$$

where $d = \ln \lambda_*$, then equation (E) is oscillatory.

Proof. From Corollary 2 and Theorem 4 in [11], we see that equation (2.3) is oscillatory. The conclusion then follows from Theorem 2.3. □

3 Examples

In this section, we present three examples to show the importance and novelty of the main results.

Example 3.1. Consider the second order hybrid differential equation

$$\left(t^{3/2}x'(t)\right)' - \frac{1}{2\sqrt{t}}x(t) + \frac{q_0}{\sqrt{t}}x(\lambda t) = 0, \qquad t \geq 1, \tag{3.1}$$

where $q_0 > 0$, and $\lambda \in (0,1)$. Here we have $a(t) = t^{3/2}$, $q_1(t) = \frac{1}{2\sqrt{t}}$, $q_2(t) = \frac{q_0}{\sqrt{t}}$, $\alpha = 1$, and $\sigma(t) = \lambda t$.

The auxiliary equation (2.1) becomes

$$(t^{3/2}u'(t))' = \frac{1}{2\sqrt{t}}u(t), \qquad t \geq 1$$

which has the positive decreasing solution $u(t) = 1/t$. Thus, $b(t) = t^{-1/2}$, $B(t) = \frac{2}{3}\left(t^{3/2} - 1\right)$, $Q(t) = q_0/(\lambda t^{5/2})$, and $z(t) = tx(t)$.

The transformed binomial form (2.3) of (3.1) is

$$\left(\frac{1}{\sqrt{t}}z'(t)\right)' + \frac{q_0}{\lambda t^{5/2}}z(\lambda t) = 0, \qquad t \geq 1. \tag{3.2}$$

Condition (2.4) clearly holds and condition (2.8) becomes

$$\liminf_{t\to\infty} \frac{2}{3}\left(\lambda^{3/2}t^{3/2} - 1\right) \int_t^\infty \frac{q_0}{\lambda s^{5/2}} ds > \frac{4}{9}\sqrt{\lambda} q_0,$$

so it holds if $q_0 > \frac{9}{16\sqrt{\lambda}}$. Hence, by Theorem 2.8, equation (3.1) is oscillatory if $q_0 > \frac{9}{16\sqrt{\lambda}}$.

In view of condition (2.10), we see that if

$$q_0 > \frac{3}{2e\sqrt{\lambda}\ln\frac{1}{\lambda}},$$

then equation (3.1) is oscillatory.

For $\lambda = 1/4$, the condition (2.8) becomes $q_0 > 9/8$ and the condition (2.11) becomes $q_0 > 0.796107$. Thus, we see that condition (2.11) is better than (2.8).

Also note that equation (3.1) is of the noncanonical type (since $\int_{t_0}^\infty \frac{1}{a(s)} ds < \infty$) whereas the transformed equation (3.2) is in canonical form. Thus, our results apply to noncanonical equations as well as those in canonical form.

Example 3.2. Consider the second order sub-linear hybrid delay differential equation

$$x''(t) - x(t) + e^{\frac{2t}{3}} x^{\frac{1}{3}}(t-\pi) = 0, \qquad t \geq \pi. \tag{3.3}$$

We have $a(t) = 1$, $q_1(t) = 1$, $q_2(t) = e^{\frac{2t}{3}}$, $\alpha = \frac{1}{3}$, and $\sigma(t) = t - \pi$, so the auxiliary equation is

$$u''(t) - u(t) = 0$$

for which $u(t) = e^{-t}$ is a positive decreasing solution. The transformed equation is

$$\left(e^{-2t} z'(t)\right)' + e^{\frac{-2t}{3} + \frac{\pi}{3}} z^{\frac{1}{3}}(t-\pi) = 0, \qquad t \geq \pi.$$

Simple calculations show that $B(t) \approx \frac{e^{2t}}{2}$, $Q(t) = e^{\frac{-2t}{3} + \frac{\pi}{3}}$ and condition (2.11) becomes

$$\int_{\pi}^{\infty} \frac{e^{\frac{-2t}{3} + \frac{\pi}{3}} e^{\frac{2t}{3}}}{2^{\frac{1}{3}}} dt = \infty,$$

so it holds. Therefore, by Corollary 2.10, equation (3.3) is oscillatory.

Example 3.3. Consider the second order super-linear hybrid delay differential equation

$$x''(t) - x(t) + e^{t^2 + 2t} x^3(t/2) = 0, \qquad t \geq 1. \tag{3.4}$$

Now $a(t) = 1$, $q_1(t) = 1$, $q_2(t) = e^{t^2+2t}$, $\alpha = 3$, and $\sigma(t) = t/2$. The auxiliary equation is

$$u''(t) - u(t) = 0$$

and $u(t) = e^{-t}$ is a positive decreasing solution. The transformed equation is

$$\left(e^{-2t} z'(t)\right)' + e^{t^2 - 2t} z^3(t/2) = 0, \qquad t \geq 1.$$

Additional calculations give $b(t) = e^{-2t}$, $B(t) \approx \frac{e^{2t}}{2}$, and $Q(t) = e^{t^2 - t/2}$. Choosing $\lambda = 2$ we see that $2 > \ln 3 / \ln 2$. The condition (2.13) becomes

$$\liminf_{t \to \infty} \left[\frac{1}{8} e^{t^2 - t/2} e^{3t} e^{-t^2} \right] = \infty > 0,$$

so it holds. Therefore, by Corollary 2.11(ii), equation (3.4) is oscillatory.

4 Conclusion

In this paper, we studied the oscillatory properties of hybrid delay differential equations of second-order. This is achieved by transforming the equation under consideration to a binomial type equation. Then by applying the comparison method and integral averaging technique, we established conditions for the oscillation of all solutions. None of the results reported in [5, 8, 17] are applicable to Examples 3.1 or 3.3 since these equations contain both positive and negative parts. Thus, the oscillation results presented here are a significant new contribution to the theory of oscillation of trinomial delay differential equations. In a future work, we plan to extend the results in this paper to equations of the form

$$(a(t)x'(t))' + p_1(t)x(t) - p_2(t)x^{\alpha}(\sigma(t)) = 0,$$

where $\sigma(t) \geq t$.

References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations*, Kluwer, Dordrecht, 2002. https://doi.org/10.1007/978-94-017-2515-6; MR2091751; Zbl 1073.34002

[2] R. P. Agarwal, M. Bohner, W. T. Li, *Nonoscillation and oscillation: theory for functional differential equations*, Marcel Dekker, New York, 2004. https://doi.org/10.1201/9780203025741; MR2084730; Zbl 1068.34002

[3] B. Baculíková, J. Džurina, Oscillatory criteria via linearization of half-linear second order delay differential equations, *Opuscula Math.* **40**(2004), 523–536. https://doi.org/10.7494/opmath.2020.40.5.523; MR4302425; Zbl 1470.34172

[4] B. Baculíková, J. Džurina, Property A of differential equations with positive and negative term, *Electron. J. Qual. Theory Differ. Equ.* **2017**, No. 27, 1–7. https://doi.org/10.14232/ejqtde.2017.1.27; MR3650198

[5] Y. S. Chen, Existence of oscillatory solutions of second-order functional differential equations, *Ann. Differential Equations* **6**(1990), 389–394. MR1100103

[6] J. Džurina, B. Baculíková, Oscillation of trinomial differential equations with positive and negative terms, *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 43, 1–8. https://doi.org/10.14232/ejqtde.2014.1.43; MR3250034

[7] J. Džurina, Oscillation of second order delay differential equations, *Arch. Math. (Brno)* **33**(1997), 309–314. MR1601333

[8] J. Džurina, Oscillation of second-order trinomial differential equations with retarded and advanced arguments, *Appl. Math. Lett.* **153**(2024), No. 109058. https://doi.org/10.1016/j.aml.2024.109058; MR4715177; Zbl 1557.34068

[9] L. Erbe, Oscillation criteria for second order nonlinear delay equations, *Canadian Math. Bull.* **16**(1973), 49–56. https://doi.org/10.4153/CMB-1973-011-1; MR324173; Zbl 1125.34046

[10] L. H. Erbe, Q. Kong, B. G. Zhang, *Oscillation theory for functional-differential equations*, *Marcel Dekker,* New York, 1995. MR1309905; Zbl 0821.34067

[11] I. Jadlovská, J. Džurina, Kneser-type oscillation criteria for second-order half-linear delay differential equations, *Appl. Math. Comput.* **380**(2020), No. 125289. https://doi.org/10.1016/j.amc.2020.125289; MR4088426; Zbl 1451.34086

[12] Z. Opluštil, J. Šremr, Myshkis-type oscillation criteria for second-order linear delay differential equations, *Monatsh. Math.* **178**(2015), 143–161. https://doi.org/10.1007/s00605-014-0719-y; MR3384894; Zbl 1325.34077

[13] A. Özbekler, J. S. W. Wong, A. Zafer, Forced oscillation of second-order nonlinear differential equations with positive and negative coefficients, *Appl. Math. Lett.* **24**(2011), 1225–1230. https://doi.org/10.1016/j.aml.2011.02.013; MR2784187; Zbl 1223.34046

[14] Ch. G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, *Arch. Math. (Basel)* **36**(1981), 168–178. https://doi.org/10.1007/BF01223686; Zbl 0463.34050

[15] C. A. Swanson, *Comparison and oscillation theory of linear differential equations*, Academic Press, New York, 1968. MR34C10; Zbl 0191.09904

[16] X. H. Tang, Oscillation for first-order superlinear delay differential equations, *J. London Math. Soc. (2)* **65**(2002), 115–122. https://doi.org/10.1112/S0024610701002678; MR1875139; Zbl 1024.34058

[17] A. Zafer, T. Candan, Z. N. Guseinov, Oscillation of second-order canonical and non-canonical delay differential equations, *Math. Meth. Appl. Sci.* **28**(2005), 101–116. https://doi.org/10.1002/mma.10945; MR4922443; Zbl 08082021