Transformatio formarum quadraticarum Weierstrassiana.

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SUMMA

Transformatio Weierstrassiana quaestionem notam reductionis formae cuiuslibet quadraticae in summam solis variabilium quadratis constantem solvit, et quidem tali modo, ut transformatio hanc reductionem perficiens etiam seriem quandam parametrorum ad libitum electarum involvat.[1]) Itaque transformatio Weierstrassiana finitima transformationi generalissimae ab Ill$^{\text{mo}}$. DARBOUX detectae est habenda; satis enim constat transformationem a DARBOUX

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[1]) De rebus in hac commentatione pertractatis vide et cfr. K. WEIERSTRASS' *Math. Werke*, VII. Band: Vorlesungen über Variationsrechnung, bearb. von RUDOLF ROTHE (Leipzig, 1927); Zweites Kapitel: Hülfssätze aus der Theorie der quadratischen Formen. Etc., p. 11—20. In hoc opere considerationes reperiuntur, quibus disquisitiones subsequentes, — in particulis vero ulterius progredientes, — innituntur.

ortam iam ab origine quam maximum numerum parametrorum introducere.[2])

Per hanc occasionem animadvertimus, quod in constructione reductionis cognitae Kroneckerianae „alternantis“ *vertices* quidam speciales formarum coniunctarum dominantur,[3]) quare methodus haec alternans et *verticem-quaerens* nominari potest, quae itaque contrarium offert rationis Weierstrassianae nunc discernendae, cuius quidem quantitates fundamentales — scilicet illae proxime commemoratae parametri — vertices formarum in processu reductionis apparentium repraesentare nequeunt; quamobrem transformatio Weierstrassiana methodum *verticem-fugientem* constituit.

CAPUT PRIMUM.

Structura generalis reductionis Weierstrassianae.

Proposita forma quadratica generali

$$f(x_1, \ldots, x_n) \equiv \sum_{i,k}^{n} a_{ik} x_i x_k \not\equiv 0, \quad (a_{ik} = a_{ki}) \tag{1}$$

sint numeri

$$e_1, e_2, \ldots, e_n \tag{e}$$

ita sumpti, ut

$$f(e_1, e_2, \ldots, e_n) = \gamma \neq 0 \tag{2}$$

evadat; tales numeros reperiri posse, posito $f \not\equiv 0$, dubium non est. Ulterius certum est

$$E = (e_1, e_2, \ldots, e_n) \neq (0, 0, \ldots, 0)$$

cadere, quippe quum aliter et $\gamma = 0$ esset; punctum E secundum (2) vertex esse certe nequit.[4]) Argumentis adeo appositis

$$\frac{1}{2}\frac{\partial f}{\partial x_i} \equiv u_i(x_1, x_2, \ldots, x_n) \equiv a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \quad (i = 1, 2, \ldots, n) \tag{3}$$

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[2]) G. Darboux, Mémoire sur la théorie algébrique des formes quadratiques, *Journal de math. pures et appliquées,* (2) **19** (1874), pp. 347—396. — Cfr. et infra sub adnot. [19]).

[3]) Vide ex. gr. M. Bôcher, *Introduction to Higher Algebra* (New-York, 1907), Section 49.

[4]) Complexus $A = (a_1, a_2, \ldots, a_n)$, n valoribus $a_1, \ldots, a_n$ conflatus breviter *punctum* dicitur, duo huiusmodi puncta aequalia dicuntur, i. e.

$$A = (a_1, \ldots, a_n) = B = (b_1, \ldots, b_n),$$

si $a_1 = b_1, \ldots, a_n = b_n$.

scribentes, videlicet

$$(4)\qquad i\sum_1^n e_i u_i(e_1, \ldots, e_n) = f(e_1, \ldots, e_n) = \gamma$$

esse, et quum $\gamma \neq 0$, unusquisque valorum

$$(5)\qquad c_i = u_i(e_1, e_2, \ldots, e_n) \quad (i = 1, 2, \ldots, n)$$

non evanescet, nec, ratione (4), unumquodque ex productis

$$e_i c_i = e_i u_i(e_1, \ldots, e_n).$$

Et quidem ponamus[5])

$$(6)\qquad e_1 u_1(e_1, \ldots, e_n) = e_1 c_1 \neq 0,$$

unde patet etiam

$$(6')\qquad e_1 \neq 0,\ c_1 \neq 0$$

fore.[6])

Praeterea facile probatur aequationem

$$(7)\qquad \sum_{i=1}^n y_i u_i(x_1, \ldots, x_n) \equiv \sum_{i=1}^n x_i u_i(y_1, \ldots, y_n)$$

pro unoquoque systemate valorum x et y locum habere; symmetrica enim est matrix $(f) = (a_{ik})$.[7])

Quibus omnibus ita praeparatis aspiciamus deinde identitatem cognitam[8]):

$$(\mathrm{I})\qquad \begin{aligned} f(x_1 + \lambda e_1, x_2 + \lambda e_2, \ldots, x_n + \lambda e_n) &\equiv f(x_1, \ldots, x_n) + \\ &+ 2\lambda . \sum_{i,k}^n a_{ik} x_i e_k + \lambda^2 . f(e_1, \ldots, e_n) \equiv \\ &\equiv f(x_1, \ldots, x_n) + 2\lambda . \sum_i^n u_i(e_1, \ldots, e_n) . x_i + \lambda^2 \gamma \equiv \\ &\equiv f(x_1, \ldots, x_n) + 2\lambda . \sum_i^n c_i x_i + \lambda^2 \gamma; \end{aligned}$$

ibidem iam formulis (2) et (5) usi sumus. Et si hic scribatur

$$(8)\qquad y_1 = \sum_i^n c_i x_i = c_1 x_1 + \ldots + c_n x_n,$$

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5) Propter (6) sententia generalis ne minimum laeditur, quia si initio fuisset $e_i c_i \neq 0$, transpositio $(1, i)$ hunc casum ad (6) reduceret.

6) Inaequalitate $c_1 \neq 0$ infra non utemur; at saltem cognitionem iuvat et huic condicioni satisfieri posse.

7) Expressio bilinearis (7) vulgo *forma polaris* ad f pertinens dici solet.

8) Bôcher: 3), p. 130, (4).

secundum (I) oritur

$$f(x_1+\lambda e_1,\ldots,x_n+\lambda e_n)\equiv\lambda^2\gamma+2\lambda y_1+f(x_1,\ldots,x_n),$$

aut quod idem est

$$f(x_1+\lambda e_1,\ldots)\equiv\frac{1}{\gamma}(\lambda\gamma+y_1)^2+f(x_1,\ldots)-\frac{y_1^2}{\gamma};$$

hinc autem, designante

$$\frac{1}{\gamma}=\frac{1}{f(e_1,\ldots,e_n)}=g(\neq 0),\text{[9]} \tag{9}$$

habebimus

$$f(x_1,\ldots,x_n)\equiv gy_1^2+f(x_1+\lambda e_1,\ldots)-g(\lambda\gamma+y_1)^2.$$

Deliberantes deinde ex (4), (5) et (8)

$$\lambda\gamma+y_1=\lambda\sum_i^n c_ie_i+\sum_i^n c_ix_i=\sum_i^n c_i(x_i+\lambda e_i)$$

fluere, aequationem antecedentem hunc aspectum induere perspicimus:

$$\begin{aligned} f(x_1,x_2,\ldots,x_n)\equiv gy_1^2+f(x_1+\lambda e_1,\,x_2+\lambda e_2,\ldots,x_n+\lambda e_n)-\\ -g\left(\sum_i^n c_i(x_i+\lambda e_i)\right)^2; \end{aligned} \tag{10}$$

hinc tandem *passus generator elementaris* reductionis Weierstrassianae prorsus accipitur. Si enim parametrus λ designata formulae (10) aequatione

$$\lambda=-\frac{x_1}{e_1} \tag{11}$$

determinetur, cui dispositioni, quia $e_1\neq 0$, nihil obstat, formula (10) in aequationem

$$\begin{aligned} f(x_1,\ldots,x_n)\equiv gy_1^2+f\left(0,\,x_2-\frac{e_2}{e_1}x_1,\ldots,x_n-\frac{e_n}{e_1}x_1\right)-\\ -g\left(\sum_{i=2}^n c_i\left(x_i-\frac{e_i}{e_1}x_2\right)\right)^2 \end{aligned} \tag{W}$$

mutabit; haec autem iam istam transformationem genetricem exhibet. — Stabilita aequatione identica (W) visamus substitutionem linearem

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[9]) Ex (9) statim sequitur:

$$g=\frac{\gamma}{\gamma^2}=f\left(\frac{e_1}{\gamma},\frac{e_2}{\gamma},\ldots,\frac{e_n}{\gamma}\right). \tag{2'}$$

$$(12)\qquad \begin{cases} y_1 = & c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n, \\ y_2 = -\dfrac{e_2}{e_1}x_1 + x_2, \\ y_3 = -\dfrac{e_3}{e_1}x_1 & \qquad + x_3, \\ \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \\ y_n = -\dfrac{e_n}{e_1}x_1 & \qquad\qquad + x_n, \end{cases}$$

cuius matrix, scilicet

$$X = \begin{pmatrix} c_1 & c_2 & c_3 & \ldots & c_n \\ -\dfrac{e_2}{e_1} & 1 & 0 & \ldots & 0 \\ -\dfrac{e_3}{e_1} & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -\dfrac{e_n}{e_1} & 0 & 0 & \ldots & 1 \end{pmatrix}$$

certo non est singularis, quoniam, uti sine negotio invenitur:

$$(13)\qquad \text{det. } X = c_1 + \frac{1}{e_1}(e_2c_2 + e_3c_3 + \ldots + e_nc_n) = \frac{\gamma}{e_1} \neq 0.$$

Ceterum generaliter habemus:

$$\begin{vmatrix} a & b & c & \ldots & m \\ \beta & 1 & 0 & \ldots & 0 \\ \gamma & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu & 0 & 0 & \ldots & 1 \end{vmatrix} = a - b\beta - c\gamma - \ldots - m\mu;$$

quae propositio facillime probatur subtrahendo elementa 2., 3., etc. lineae per b, per c, etc. ordinatim multiplicata elementis primae lineae respondentibus.

Itaque substitutio ad (12) *inversa,* ponatur

$$(14)\qquad [x] = Y[y],$$

formam f secundum (W) in formam

$$(15)\qquad \bar{f}(y_1, \ldots, y_n) \equiv gy_1^2 + f(0, y_2, \ldots, y_n) - g\left(\sum_{i=2}^{n} c_iy_i\right)^2$$

transfert; et determinans non evanescens

$$(14')\qquad \text{det. } Y = \frac{1}{\text{det. } X} = \frac{e_1}{\gamma} = ge_1$$

habet. Hoc modo igitur terminus quadratus gy_1^2 „abscinditur", forma vero „residua", scilicet

(16)
$$\varphi(y_2,\ldots,y_n)\equiv f(0,y_2,\ldots,y_n)-g\left(\sum_{i=2}^{n} c_i y_i\right)^2\equiv$$
$$\equiv\sum_{i=2}^{n}\sum_{k=2}^{n}\alpha_{ik}y_i y_k,$$

variabilem y_1 iam nequaquam continebit; hic autem

(17)
$$\alpha_{ik}=a_{ik}-gc_i c_k \qquad (i,k=2,3,\ldots,n)$$

invenitur.

Antequam porro progrederemur observari convenit, „ordine"[10]) formae f, id est $r(f)=r$ accepto, ordinem formae residuae

(17')
$$r(\varphi)=r-1$$

fore. Nullo quidem negotio demonstrabimus in matrice n^{ti} gradus et ordinis r, aspectum

(18)
$$\mathfrak{a}=\begin{vmatrix} a_1 & & & \\ & \ddots & & \\ & & a_k & \\ & & & \mathfrak{b} \end{vmatrix}$$

habente, ubi enim $a_1a_2\ldots a_k \neq 0$ et $\mathfrak{b}$ aliquam matricem $(n-k)^{\text{ti}}$ gradus esse supponitur, dum campi vacui solum elementis „0" compleantur, ordinem matricis $\mathfrak{b}$

(18')
$$r(\mathfrak{b})=r-k\,^{11})$$

fore.

Supponamus quidem 1°: $\varrho=r(\mathfrak{b})>r-k$; si deinde determinans quoddam ϱ^{ti} gradus, puta b_ϱ, sub matrice $\mathfrak{b}$ contentum at $\neq 0$ eligatur,[12]) obvium est determinans $(\varrho+k)^{\text{ti}}$ gradus

$$\begin{vmatrix} a_1 & & & \\ & \ddots & & \\ & & a_k & \\ & & & b_\varrho \end{vmatrix}=a_1\ldots a_k b_\varrho$$

[10]) Hic et infra verbo „*ordo*" in eadem sententia utimur, quam ex. gr. lingua Germanica voce hodierna rite velut „*Rang* einer Matrix, resp. einer Form" designaret.

[11]) Liquet $r \geqq k$ esse. — Coniunctio ordinum supra exhibita in theoria matricum *dissolubilium* amplificationem respondentem nanciscitur; vide ex. gr. P. Muth, *Elementartheiler* (Leipzig, 1899), art. 22.

[12]) Praesente casu hoc perfici posse constat.

sub matrice $\mathfrak{a}$ contentum et non evanescens fore, quod autem suppositioni $\varrho + k > r$ opponeretur.

Si ex contrario 2°: $\varrho = r(\mathfrak{b}) < r - k$ fieri posset, ϱ etiam ordinem matricis cum $n-k$ lineis ultimis matricis $\mathfrak{a}$ formatae — et typum $(n-k) \times n$ habentis — significaret, quocirca omnia subdeterminantia r^{ti} gradus matricis $\mathfrak{a}$ — observando in his elementa linearum numero $\varrho + 1$ aut plurium certe ex illis $n-k$ lineis ultimis sumpta esse — secundum notum theorema Laplaceianum evanescerent; c. hyp. etc. Per haec formula (18′) et specialiter, respectu $g \neq 0$, etiam (17′) verificabantur.[13])

Et nunc ad thema principale, unde digressi sumus, revertentes, fiat adhuc

$$r(\varphi) = r - 1 > 0$$

et ideo

$$\varphi \mp 0;$$

tunc manifestum est formam quadraticam φ — iam solum $n-1$ variabiles involventem[14]) — processui, omni ratione illi simili, quo supra per occasionem formae nativae f usi sumus, subicere posse. Hac via inter variabiles $y_2, \ldots, y_n$[15]) et alias recentiores $z_2, \ldots, z_n$ substitutionem regularem typi ad (12) *inversae*

$$[y_2, \ldots, y_n] = Z_1 [z_2, \ldots, z_n] \tag{19}$$

nanciscemur, quae quidem formam φ in formam

$$\begin{aligned} \overline{\varphi}(z_2, \ldots, z_n) &\equiv g' z_2^2 + \varphi(0, z_3, \ldots, z_n) - g' \left(\sum_{i=3}^{n} c_i' z_i \right)^2 \equiv \\ &\equiv g' z_2^2 + \psi(z_3, \ldots, z_n) \end{aligned} \tag{20}$$

reducet.

Hic, postquam numeris idoneis

$$e_2', \ldots, e_n'$$

iam

$$\varphi(e_2', \ldots, e_n') \neq 0 \tag{20′}$$

et una $e_2' \neq 0$ adepti sumus,

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[13]) Apparet enim $r(\bar{f}) = r(f) = r$ fieri.

[14]) Fieri vero potest formam φ *de facto* pauciorem variabilium numerum quam $n-1$ continere; sed hoc eventum sequentia minime attinget.

[15]) Hoc loco signa $y_2, \ldots, y_n$ iam condicioni post transignationem antecedentem forte necessariam respondent; si transignatio facta sit, hanc in ultimis $n-1$ lineis substitutionis (12) pro futuris sine mora exsequimur.

$$g' = \frac{1}{\varphi(e_2', \ldots, e_n')} \neq 0$$

fiet; insuper posito

$$c_i' = \alpha_{i2} e_2' + \ldots + \alpha_{in} e_n', \quad (i = 2, 3, \ldots, n)$$

erit

$$z_2 = c_2' y_2 + \ldots + c_n' y_n,$$

et tandem:

$$z_3 = -\frac{e_3'}{e_2'} y_2 + y_3,$$
$$\cdots\cdots\cdots\cdots$$
$$z_n = -\frac{e_n'}{e_2'} y_2 \qquad + y_n.$$

Praeterea habemus observationis antecedentis memores:

$$r(\psi) = r(\overline{\varphi}) - 1 = r(\varphi) - 1 = r - 2.$$

Completo deinde systemate (19) aequatione

$$y_1 = z_1,$$

transformationem linearem n variabilium regularem

(21) $$[y] = Z[z]$$

nanciscimur, cuius composita cum transformatione (14), id est

(22) $$[x] = YZ[z],$$

formam $f(x_1, \ldots, x_n)$ directe in formam

(23) $$\overline{\overline{f}}(z_1, z_2, \ldots, z_n) \equiv g z_1^2 + g' z_2^2 + \psi(z_3, \ldots, z_n)$$

mutabit. Simili modo, ut in tractatibus forma f coniunctis invenitur

$$\text{det. } Z = \text{det. } Z_1 = g' e_2',$$

et ideo, respectu (14'),

(22') $$\text{det. } (YZ) = \text{det. } Y \cdot \text{det. } Z = g g' e_1 e_2'$$

fore.

Per haec revera via procedendi iam statis indicatur. Si enim

$$r(\psi) = r - 2 = 0, \text{ i. e. } r = 2,$$

quocirca $\psi \equiv 0$ sit, partem dexteram formulae (23) iam transformatam in summam quadratorum formae f suppeditare; si quidem contra adhuc

$$r(\psi) > 0$$

esset, formam ψ, quae tunc $n-2$ variabiles solum contineret,[16]) secundum praescripta supra stabilita ulterius reducere licebit; etc. etc. Inde perspicitur post r passus formam tandem χ residuam $n-r$ variabilium identice evanescentem nancisci; quo facto repraesentatio formae f sicut summa terminis quadratis solum conflata perfecta erit.[17]) Substitutiones denique passus intervenientes exhibentes connectendo, videmus successionem transformationum genetricum typi (W) substitutionem linearem et regularem

$$(24) \qquad [x] = H[\eta]$$

praebere, qua forma quadratica f ordinis r in summam quadratorum

$$(25) \qquad fH \equiv g_1\eta_1^2 + g_2\eta_2^2 + \ldots + g_r\eta_r^2$$

transfertur[18]); hic quidem patet

$$(25') \qquad g_1 g_2 \ldots g_r \neq 0$$

fieri.

Evolutiones praecedentes reductionem Weierstrassianam in summam quadratorum constituunt.

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Observationes ad modum pendendi matricis H a parametris e pertinentes in CAPITE TERTIO colligemus, nunc quidem praedicare solum volumus substitutionem (24) per varietatem parametrorum e totum examen transformationum typum (25) suppeditantium cohaerere.[19]) Ceterum videmus reductionem Weierstrassianam nec in ulla particula *alternantem* esse, sed magis per unumquemque passum, uniformi semper processu, terminum quendam *quadratum* recentem producere.

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[16]) Numerum variabilium semper sicut finem superiorem huius numeri intellegendo.

[17]) Visum est formam residuam k^{tam} ordinem $r-k$ habere, quocirca manifestum est condicionem identice evanescendi ante passum r^{tum} non apparere; post autem passum r^{tum} forma residua, ordinem $r-r=0$ ostendens, $\equiv 0$ evadet.

[18]) fH significat substitutionem (24) in forma $f(x_1, \ldots, x_n)$ exsecutam esse.

[19]) In capite subsequenti *omnes* reductiones non-singulares formae f in typum (26) per transformationes Weierstrassianas — casu quidem formae non-singularis — perfici posse videbimus. Patet igitur methodum Weierstrassianam pro hoc casu *generalitatem* processui a DARBOUX orto *aequalem* repraesentare.

Posito tandem $r=r(f)=n$, formula (25) habitum

(26) $$fH\equiv g_1\eta_1^2+\ldots+g_n\eta_n^2$$

induere videtur, ubi certo

(26') $$\text{det.}\,(fH)=g_1g_2\ldots g_n=|H|^2\,.\,\text{det.}\,f\neq 0$$

fore.

CAPUT SECUNDUM.

Transformationes Weierstrassianae omnes reductiones formae in summam quadratorum complectuntur.

Pro insigni complemento explanationum antecedentium et quadam ratione sicut conversio earum est habendum

Theorema I. *Si forma $f(x_1,\ldots,x_n)$ non-singularis substitutione quadam regulari*

(24^{bis}) $$[x]=H[\eta]$$

in formam

(26^{bis}) $$fH\equiv g_1\eta_1^2+g_2\eta_2^2+\ldots+g_n\eta_n^2$$

reducitur, ubi itaque

(26'') $$g_1g_2\ldots g_n\neq 0;$$

matrix H matrice W transformationis cuiusdam Weierstrassianae — ad formam f pertinenti — permutari potest.[20]

Demonstratio.[21] Substitutio ad (24^{bis}) inversa systema formarum linearium, variabilium $x_1, x_2, \ldots, x_n$ constituit, puta

(η) $$\begin{cases}\eta_1=\eta_1(x_1,\,x_2,\ldots,\,x_n),\\ \eta_2=\eta_2(x_1,\,x_2,\ldots,\,x_n),\\ \ldots\ldots\ldots\ldots\ldots\\ \eta_n=\eta_n(x_1,\,x_2,\ldots,\,x_n),\end{cases}$$

cuius determinans:

$$\frac{1}{\text{det.}\,H}\neq 0.$$

Introductio autem harum formarum (η) in partem dextram relationis (26^{bis}) evidenter reddet formam $f(x_1,\ldots,x_n)$; unde quidem oritur

[20]) Quapropter $H=\Theta W$ statuere licet, ubi enim Θ matricem transformationis cuiusdam *automorphae* formae f significat.

[21]) Cfr. WEIERSTRASS: [1]), pp. 15–16.

$$(27)\qquad \begin{aligned} f_1(x_1,\ldots,x_n) &\equiv f(x_1,\ldots,x_n)-g_1(\eta_1(x_1,\ldots,x_n))^2\equiv\\ &\equiv g_2(\eta_2(x_1,\ldots,x_n))^2+\ldots+g_n(\eta_n(x_1,\ldots,x_n))^2, \end{aligned}$$

et porro, adhibente transformatione (24^{bis}) formae f_1, forma

$$(28)\qquad f_1H\equiv g_2\eta_2^2+\ldots+g_n\eta_n^2$$

iam solum $n-1$ variabiles η continens. Itaque perspicimus formam f_1 certo *singularem* evadere et ideo valores

$$\mu_1,\mu_2,\ldots,\mu_n$$

non omnes evanescentes identitatique

$$\sum_{i=1}^{n}\frac{1}{2}\frac{\partial f_1}{\partial x_i}\cdot\mu_i\equiv 0$$

satisfacientes reperiri posse.[22])

Licet igitur relationem scribere:

$$(29)\qquad \begin{aligned} \sum_{i=1}^{n}\frac{1}{2}\frac{\partial f_1}{\partial x_i}\cdot\mu_i &\equiv \sum_{i=1}^{n}\left(\frac{1}{2}\frac{\partial f}{\partial x_i}\cdot\mu_i-g_1\eta_1\frac{\partial\eta_1}{\partial x_i}\mu_i\right)\equiv\\ &\equiv\sum_{i=1}^{n}\mu_i u_i-g_1\eta_1\cdot\sum_{i=1}^{n}c_i\mu_i\equiv 0, \end{aligned}$$

ubi quidem numeri

$$c_i\equiv\frac{\partial\eta_1}{\partial x_i}$$
$$(i=1,2,\ldots,n)$$

coefficientibus formae linearis

$$(30)\qquad \eta_1(x_1,\ldots,x_n)\equiv c_1x_1+\ldots+c_nx_n$$

determinantur.

Ex (29) patet

$$(31)\qquad C=\sum_{i=1}^{n}c_i\mu_i\neq 0$$

fore, quia aliter iterum ex (29)

$$\sum_i\mu_i u_i\equiv 0$$

flueret, quod autem — non omnibus μ evanescentibus — suppositae *regularitati* formae f contradiceret. Ex (31) deinde obvium est non omnes c evanescere[23]) et proinde $\eta_1\not\equiv 0$ esse; $g_1\neq 0$

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[22]) Secundum (28) et (26″) fit $r(f_1)=n-1$, quamobrem lineae matricis n^{ti} gradus formae f_1 inter se *lineari modo dependere* debent.

[23]) Haec condicio etiam e regularitate matricis H extemplo concluditur.

tandem (26″) confirmat. Itaque sequitur ratione (29):

(32) $$\eta_1 \equiv c_1 x_1 + \ldots + c_n x_n \equiv \sum_i \frac{\mu_i}{g_1 C} \cdot u_i ;$$

si autem systema lineare

$$a_{i1} x_1 + \ldots + a_{in} x_n \equiv u_i$$
$$(i = 1, 2, \ldots, n)$$

pro litteris x solvatur, quod posito def. $f \neq 0$ unica ratione perfici potest, systema aliud

$$x_i \equiv \alpha_{1i} u_1 + \alpha_{2i} u_2 + \ldots + \alpha_{ni} u_n$$
$$(i = 1, 2, \ldots, n)$$

accipitur, in quo quidem, si per characteres $A_{ik} = A_{ki}$ *cofactores* elementorum $a_{ik} = a_{ki}$ in determinante symmetrico $|f|$ designentur, aequationes

$$\alpha_{ki} = \frac{A_{ki}}{|f|} = \frac{A_{ik}}{|f|} = \alpha_{ik}$$

locum habent. Idcirco

(33) $$\eta_1 \equiv c_1 x_1 + \ldots + c_n x_n \equiv e_1 u_1 + \ldots + e_n u_n ,$$

ubi scriptum est:

$$e_i = \alpha_{i1} c_1 + \alpha_{i2} c_2 + \ldots + \alpha_{in} c_n =$$

(33′) $$= \frac{1}{|f|} \cdot (A_{i1} c_1 + A_{i2} c_2 + \ldots + A_{in} c_n).$$
$$(i = 1, 2, \ldots, n)$$

In transitu inici potest, quod ope formarum linearium

(34) $$\frac{1}{2} \cdot \frac{\partial F}{\partial x_i} \equiv \frac{1}{|f|} (A_{i1} x_1 + \ldots + A_{in} x_n) \equiv U_i(x_1, \ldots, x_n),$$
$$(i = 1, 2, \ldots, n)$$

ad *formam reciprocam*

$$F \equiv \sum_{i,k} \frac{A_{ik}}{|f|} x_i x_k$$

formae f pertinentium, ex (33′) relationes

(33″) $$e_i = U_i(c_1, c_2, \ldots, c_n)$$
$$(i = 1, 2, \ldots, n)$$

nascuntur.

Comperimus deinde secundum (33), respectu (7):

$$\eta_1 \equiv \sum_i c_i x_i \equiv \sum_i e_i u_i(x_1, \ldots, x_n) \equiv \sum_i u_i(e_1, \ldots, e_n) \cdot x_i ,$$

et hinc

$$c_i = u_i(e_1, \ldots, e_n), \tag{35}$$
$$(i = 1, 2, \ldots, n)$$

quod ceterum et ex (33′) aut (33″) manifestum est. Si igitur, pro variabilibus x identice

$$\eta_1 \equiv \sum_i c_i x_i \equiv \sum_i \alpha_i u_i \equiv \sum_i u_i(\alpha_1, \ldots, \alpha_n) \cdot x_i$$

supponatur, sive

$$u_i(\alpha_1, \ldots, \alpha_n) = c_i$$

sumantur, etiam aequationes

$$\alpha_i = e_i$$
$$(i = 1, 2, \ldots, n)$$

locum habere invenientur. Systema enim lineare

$$u_i(x_1, \ldots, x_n) = c_i$$
$$(i = 1, 2, \ldots, n)$$

respectu $|f| \neq 0$, unicam tantum solutionem pro incognitis x permittit, et quidem talem suppeditat complexus $e_1, e_2, \ldots, e_n$ in (33′) determinatus, ratione aequationum (35). Videlicet itaque ope (32) relationes

$$\frac{\mu_i}{g_1 C} = e_i$$
$$(i = 1, 2, \ldots, n)$$

subsistere, unde tandem per (31), adiumento (35) sequitur:

$$\frac{1}{g_1 C} \sum_i \mu_i c_i = \frac{1}{g_1} = \sum_i e_i c_i = \sum_i e_i u_i(e_1, \ldots, e_n) = \tag{36}$$
$$= f(e_1, e_2, \ldots, e_n) \neq 0.$$

Enuntiare denique possumus: elementa $e_1, e_2, \ldots, e_n$, sub (33′) coefficientibus $c_1, c_2, \ldots, c_n$ formae η_1 in summa quadratorum (26^{bis}) determinata, talia esse, ut quaevis iis innisa transformatio Weierstrassiana genetrix elementaris[24]) terminum quadtratum $g_1\eta_1^2$ forma f abscindit.

Aut brevius: ad terminum $g_1\eta_1^2$, per (24^{bis}) ab exordio datum,[25]) transformatio genetrix typi Weierstrassiani pertinet, qua ille generatur.

Et revera — collectis supra repertis — tota condicio sequentibus formulis comprehendi potest:

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[24]) Notum est quidem unumquodque e_i, quod $\neq 0$, ad constructionem cuiusdam passus generatoris adeptum esse.

[25]) Idem vero valet pro quovis termino $g_k \eta_k^2$ summae (24^{bis}).

1⁰) data sunt $g_1 \neq 0$ et forma linearis

$$\eta_1 \equiv c_1 x_1 + \ldots + c_n x_n \not\equiv 0$$ [26];

2⁰) per (33′) — aut si velis per (33″) — numeri certi

$$e_1, e_2, \ldots, e_n \tag{e}$$

determinantur, quibuscum:

3⁰) secundum (35):

$$c_i \equiv u_i(e_1, \ldots, e_n), \quad (i = 1, 2, \ldots, n) \tag{35}$$

et tandem

4⁰) secundum (36):

$$f(e_1, e_2, \ldots, e_n) = \frac{1}{g_1} \neq 0 \tag{36bis}$$

evadent.

Perspicitur enim ratione promulgationum CAPITIS PRIMI formulas nunc conscriptas condicionem constituere, per quam transformatio quaedam genetrix Weierstrassiana elementis $e_1, e_2, \ldots, e_n$ superstructa atque formae f adhibita terminum $g_1\eta_1^2$ generabit.

Demonstratio vero generalis Theorematis I. per hactenus evoluta nondum est perfecta; perscrutari proinde debemus, quomodo deinde termini ceteri quadrati $g_2\eta_2^2, \ldots$, itemque semper methodo Weierstrassiana, alter post alterum abscindi possent?

Construamus hanc ob rem „anulum“ quendam transformationis Weierstrassianae ad elementa (e) pertinentem. Tam (35) quam (36bis) certe confirmat unum minimum ex numeris (e) non evanescere; puta iam $e_1 \neq 0$,[27] tunc substitutio ad substitutionem

$$\begin{aligned}
\eta_1 &= c_1 x_1 + \ldots + c_n x_n, \\
y_2 &= -\frac{e_2}{e_1} x_1 + x_2 \\
&\ldots\ldots\ldots\ldots \\
y_n &= -\frac{e_n}{e_1} x_1 \quad + x_n,
\end{aligned}$$

aut curte

$$[\eta_1, y_2, \ldots, y_n] = X[x], \tag{37}$$

[26]) Haec inaequalitas etiam *a priori* nota est, quia, sicut hunc nexum iam antea (adnot. [23])) indicavimus, matrices H et ideo H^{-1} regulares supponuntur.

[27]) Permutatione simplici indicum capi potest.

secundum praescripta formulae (12) CAPITIS PRIMI constructam, *reciproca,*[28]) formam $f(x_1, \ldots, x_n)$ in formam

(37') $$\bar{f}(\eta_1, y_2, \ldots, y_n) \equiv g_1\eta_1^2 + \varphi(y_2, \ldots, y_n)$$

transmutabit. Itaque respectu (26^{bis}):

(38) $$\varphi(y_2, \ldots, y_n) \equiv g_2\eta_2^2 + \ldots + g_n\eta_n^2,$$

identitatem quidem *pro variabilibus* x intellegendo, postquam valores y secundum (37), valores η autem secundum (η) per variabiles x expressi sunt. Si igitur in aequatione (38) variabiles x secundum (24^{bis}) transformantur, pars dextera eius intacta manebit,[29]) sinistra contra per substitutionem

(39) $$[\eta_1, y_2, \ldots, y_n] = XH[\eta],$$

ex (37) et (24^{bis}) compositam, in formam quandam

$$\bar{\varphi}(\eta_1, \eta_2, \ldots, \eta_n)$$

transmutabit; quapropter, pro variabilibus η identice

$$\bar{\varphi}(\eta_1, \eta_2, \ldots, \eta_n) \equiv g_2\eta_2^2 + \ldots + g_n\eta_n^2$$

evadet.

Inde sequitur, — ratione independentiae variabilium $\eta_1, \ldots, \eta_n$, — omnes coefficientes formae $\bar{\varphi}$ ad terminos typi $\eta_1\eta_\alpha$ ($\alpha = 1, \ldots, n$) pertinentes evanescere; quocirca autem scribi potest:

(40) $$\bar{\varphi}(0, \eta_2, \ldots, \eta_n) \equiv g_2\eta_2^2 + \ldots + g_n\eta_n^2.$$

Manifestum est praeterea eandem formam $\bar{\varphi}(0, \eta_2, \ldots, \eta_n)$ nancisci via substitutionis $\eta_1 = 0$ *a limine,* id est expressiones variabilium $y_2, \ldots, y_n$ ex (39) sumptas — respectu $\eta_1 = 0$ — in formam $\varphi(y_2, \ldots, y_n)$ introducendo. Hic processus autem evidenter transformationem formae $\varphi(y_2, \ldots, y_n)$ per substitutionem $n-1$ variabilium

(41) $$[y_2, \ldots, y_n] = B[\eta_2, \ldots, \eta_n]$$

significat, in qua charactere B matricem $(n-1)^{mi}$ gradus, ex XH ommissione primae lineae atque primae columnae remanentem nuncupavimus. Superest persuasio de regularitate matricis B. Et

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[28]) Secundum (13) resp. (36): det. $X = \dfrac{1}{g_1 e_1} \neq 0.$

[29]) Patet enim

$$[\eta] = H^{-1}[x] = H^{-1}H[\eta] = [\eta]$$

esse.

revera protinus perspicitur matricem XH aspectum

$$XH=\left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline \alpha_2 & & & \\ \vdots & & B & \\ \alpha_n & & & \end{array}\right)$$

induere, quia enim substitutio (39) variabilem η_1 certe immutatam conservat; habemus itaque propter regularitatem matricis XH:

$$\text{det. } XH = \text{det. } B \neq 0;$$

q. e. d.

Ex his omnibus concludimus: substitutio Weierstrassiana

$$[x] = W[\eta_1, y_2, \dots, y_n],$$

ubi quidem $W = X^{-1}$, formam f in formam (37') transfert, in qua forma residua

$$\varphi(y_2, \dots, y_n),$$

secundum (17') ordinis $n-1$, talis est, ut per substitutionem

(41) $$[y_2, \dots, y_n] = B[\eta_2, \dots, \eta_n]$$

in summam quadratorum

$$\bar{\varphi} \equiv \varphi B \equiv g_2\eta_2^2 + \dots + g_n\eta_n^2$$

transformatur. Itaque substitutio (41) forma φ regulari ex toto eundem nexum praebet, quem supra inter (24$^{\text{bis}}$) et formam f repperimus. Denuo habebitur ergo — antea stabilitis quidem utendo — passus Weierstrassianus, quo terminus $g_2\eta_2^2$ forma φ abscinditur; etc., etc. — Compositione horum passuum transformatio tandem Weierstrassiana

(42) $$[x] = W[\eta]$$

accipitur, per quam

(43) $$fW \equiv \sum_{i=1}^{n} g_i\eta_i^2$$

exhibetur.

Hac via Theorema I. omni ex parte demonstratum est.

*

Hoc theorema lucis plenioris gratia etiam ita concipi potest:

T h e o r e m a I$^{\text{bis}}$. *Transformatio Weierstrassiana instrumentum quoddam generalissimum reductionis formae $f(x_1, \dots, x_n)$ non singularis, variabiles numero n involventis, in summam quadratorum constituit; id est: per transformationes Weierstrassianas omnes* *repraesentationes formae f regularis per summam terminorum solum quadratorum capi possunt.*

Corollarium. Posita forma $f(x_1, \ldots, x_n)$ singulari, id est

$$(0 <) \, r(f) = r < n,$$

substitutione quadam regulari

$$[x] = H[\eta]$$

habeatur

(a) $$fH \equiv g_1\eta_1^2 + g_2\eta_2^2 + \ldots g_r\eta_r^2,$$

ubique

$$g_1 g_2 \ldots g_r \neq 0;$$

tunc substitutiones Weierstrassianae

$$[x] = W[y]$$

et

$$[y] = \overline{W}[\eta]$$

reperiri possunt, quarum composita:

$$[x] = W\overline{W}[\eta]$$

formam f in summam

$$fW\overline{W} \equiv fH \equiv g_1\eta_1^2 + \ldots + g_r\eta_r^2$$

traducet.[30])

Ante omnia hoc duplex Lemma est cognoscendum:

Lemma: I.) *Si $0 < r < n$, et respectu matricibus diagonalibus*

$$A_r = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_r \end{pmatrix}, \quad B_r = \begin{pmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \ddots & \\ & & & \beta_r \end{pmatrix}$$

r^{ti} gradus matrix M n^{ti} gradus ita se habet, ut

(A) $$B = \begin{array}{r} \\ \\ \\ n-r \\ \text{lineae} \end{array} \left. \begin{pmatrix} \beta_1 & & & & \\ & \ddots & & & \\ & & \beta_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \right.^{31)} = M'AM,$$

$$\underbrace{\qquad}_{n-r \text{ columnae}}$$

ubi quidem

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[30]) Ergo: $H = \Theta W\overline{W}$, ubi Θ matricem transformationis cuiusdam *automorphae* formae f significat.

[31]) Locis vacuis ubique nota 0 est fingenda.

$$A = \begin{array}{c} \\ \\ \\ \left.\begin{array}{r} n-r \\ \text{lineae} \end{array}\right\{ \end{array} \left(\begin{array}{cc|cc} \alpha_1 & & & \\ & \ddots & & \\ & & \alpha_r & \\ \hline & & & 0 \\ & & & \ddots \\ & & & & 0 \end{array}\right)^{31)}, \quad \underbrace{\qquad}_{n-r \text{ columnae}}$$

et M′ transpositam matricis M designant, tunc etiam

$$(\alpha) \qquad B_r = M_r' A_r M_r$$

concluditur, si enim M_r *matricem* r^{ti} *gradus angulum superiorem sinistrum matricis M occupantem,* M_r' *quidem illius transpositam significant.*

II.) *Si autem locum habeat* (α), *simul concludetur:*[32]

$$(\beta) \qquad B = M_1' A M_1,$$

cum matrice n^{ti} *gradus*

$$M_1 = \left(\begin{array}{c|ccc} M_r & & & \\ \hline & 1 & & \\ & & \ddots & \\ & & & 1 \end{array}\right) \begin{array}{l} \\ \\ \left.\right\} \begin{array}{l} n-r \\ \text{lineae} \end{array} \end{array} \quad \underbrace{\qquad}_{n-r \text{ columnae}}$$

eiusque transposita[33]

$$M_1' = \left(\begin{array}{c|ccc} M_r' & & & \\ \hline & 1 & & \\ & & \ddots & \\ & & & 1 \end{array}\right).$$

D e m o n s t r a t i o a d I. Compositione directa comperimus matricem AM n^{ti} gradus in angulo suo superiore sinistro matricem $A_r M_r$ r^{ti} gradus continere, et simul ultimis $n-r$ lineis elementa tantum 0 ostentare; idcirco in matrice $M'(AM)$ matrix r^{ti} gradus in angulo superiore sinistro emergens prorsus $M_r' A_r M_r$ erit, unde quidem (α) secundum (A) iam innotescit.

D e m o n s t r a t i o a d II. Et nunc ut supra $A_r M_r$ est matrix r^{ti} gradus, quae angulum superiorem sinistrum matricis compositae AM_1 occupat, et haec in $n-r$ lineis ultimis elementa solum 0 continet; manifestum itaque, quod $M_r' A_r M_r$, matrix r^{ti} gradus, an-

[32]) Obvium est theorema II.) Lemmatis quodam sensu inversum theorematis I.) effingere.

[33]) In campos vacuos notae 0 sunt scribendae.

gulum superiorem sinistrum matricis $M_1'(AM_1)$ explebit ac posterior in suis lineis, numero $n-r$ ultimis — respectu structurae matricis M_1' — elementa omnino 0 habebit. Quoniam tandem *symmetrica* est matrix $M_1'AM_1$,[34]) ideo vero secundum (α):

$$M_1'AM_1 = \begin{pmatrix} B_r & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix} = B;$$

q. e. d.[35])

Stabilito hoc Lemmate duplici, demonstremus Corollarium.

Imprimis videtur — ratione disquisitionum CAPITIS PRIMI — transformationem Weierstrassianam regularem

$$[x] = W[y]$$

exsistere, qua forma f r^{ti} ordinis in formam

(b) $$fW \equiv \omega_1 y_1^2 + \ldots + \omega_r y_r^2$$

redigitur, ubi quidem

$$\omega_1 \omega_2 \ldots \omega_r \neq 0.$$

Et si ad analogiam praecedentium ponantur:

$$G_r = \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_r \end{pmatrix} \quad \text{et} \quad G = \begin{pmatrix} G_r & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix},$$

et porro:

$$\Omega_r = \begin{pmatrix} \omega_1 & & & \\ & \omega_2 & & \\ & & \ddots & \\ & & & \omega_r \end{pmatrix} \quad \text{et} \quad \Omega = \begin{pmatrix} \Omega_r & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix};$$

ubi quidem G et Ω n^{ti} gradus sunt; aequationes per matrices expressae atque identitatibus (a) resp. (b) aequivalentes manifeste hae erunt:

(γ) $$G = H'(f)H, \text{ resp. } \Omega = W'(f)W;$$

et ideo ordinatim:

(c) $$\begin{aligned} G &= H'(f)H = H'W'^{-1}(W'(f)W)W^{-1}H = \\ &= (W^{-1}H)'\Omega(W^{-1}H) = M'\Omega M, \end{aligned}$$

---

[34]) Matrix A enim evidenter symmetrica est, quapropter

$$(M_1'AM_1)' = M_1'AM_1.$$

[35]) Admonere convenit regulas compositionis matricum hic adhibitas ope theoriae compositionis matricum *dissolubilium* ex unico fonte hauriri posse.

si quidem scriptum est

$$W^{-1}H=M.$$

Haec autem relatio (c) omni respectu eandem structuram revelat, quam in aequatione (A) partis I.) Lemmatis cognovimus, et ideo habemus ex (c), secundum (α):

$$\text{(d)} \qquad G_r = M_r' \Omega_r M_r,$$

et praeterea matricem M_r *non-singularem* fore, quia enim ex (d)

$$0 \neq \det. G_r = g_1 g_2 \ldots g_r = \det. \Omega_r (\det. M_r)^2$$

concluditur. Relatio (d) igitur indicat substitutionem regularem

$$\text{(e)} \qquad [y_1, \ldots, y_r] = M_r [\eta_1, \ldots, \eta_r]$$

variabiles r involventem exsistere, qua forma non-singularis

$$\omega_1 y_1^2 + \ldots + \omega_r y_r^2$$

in formam non-singularem

$$g_1 \eta_1^2 + \ldots + g_r \eta_r^2$$

transformatur. Hinc autem sequitur — in Theoremate I.) pro n numerum r ponendo — relationi (e) et matrice W_r characteris Weierstrassiani satisfieri posse, qua quidem relatio

$$\text{(d')} \qquad G_r = W_r' \Omega_r W_r$$

aequationi (d) respondens locum habebit. Utendo denique hac relatione (d'), sicut *suppositione praemissa* partis II.) Lemmatis antecedentis, typum quidem (α) habenti, invenimus secundum (β)

$$\text{(g)} \qquad G = \overline{W}' \Omega \overline{W},$$

ubi enim $\overline{W}$ matricem n^{ti} gradus

$$\overline{W} = \begin{pmatrix} W_r & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

significat.[86]) Oritur tandem ex (g) ope (γ):

$$G = H'(f) H = \overline{W}' W'(f) W \overline{W} = (W\overline{W})'(f,(W\overline{W}),$$

unde revera:

$$fH \equiv fW\overline{W};$$

q. e. d.

*

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[86]) Matrice W_r nucleum matricis $\overline{W}$ efficiente transformationem matricis $\overline{W}$ item Weierstrassianam aspectare licebit.

Hoc cum theoremate est sine dubio coniunctum sequens aliud, cuius quidem demonstrationem — ope promulgationum antecedentium — quisque facillime perficiet.

T h e o r e m a II. *Si formas ingularis* f r^{ti} *ordinis substitutione regulari*

$$[x] = H[\eta]$$

in quadratorum summam (a) *transferatur, etiam transformatio generalis „d i c h o t o m a"*[37])

$$[x] = \Gamma[y],$$

et insuper alia transformatio Weierstrassiana

$$[y] = W[\eta]$$

inveniri poterunt, quarum compositione, scilicet

$$[x] = \Gamma W[\eta],$$

identitas

$$f\Gamma W \equiv fH$$

evadat.[38])

CAPUT TERTIUM.

Investigationes circa problema parametrorum.

Haud superfluum credimus considerationes sequentes — nexum vero rerum ad tempus ferme tantum explanantes — praemittere.[39]) Ante omnia videlicet haec duo problemata:

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[37]) Supposito enim in forma f r^{ti} ordinis determinante

$$\begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \cdots & \cdots & \cdots \\ a_{r1} & \cdots & a_{rr} \end{vmatrix} \neq 0,$$

transformationem regularem

$$[x] = \Gamma[y],$$

cuius ope

$$f\Gamma \equiv f(y_1, \ldots, y_r, 0, \ldots, 0) \equiv \sum_{i,k}^{r} a_{ik} y_i y_k$$

reperitur, *dichotomam*, [ex διχοτομέω = dissecare, diffindere], — ad f pertinentem — vocamus; haec quidem denominatio nobis haud absona videbatur. Pertractationem transformationum huius generis fusiorem vide ex. gr. apud T. J. I'A. Bromwich, *Quadratic forms and their classification by means of invariant-factors* (Cambridge, 1906), p. 9 sqq.

[38]) Proinde scribi potest $\Gamma W = \Theta H$, ubi Θ matricem transformationis cuiusdam *automorphae* ad formam f pertinentis significat.

[39]) Cf. ex. gr. J. A. Serret, *Cours d'Algèbre supérieure*, tome I, 6. éd. (Paris, 1910), p. 548, art. 247, et R. F. Scott — G. B. Mathews, *The Theory of Determinants*, 2. ed. (Cambridge 1904), p. 189, art. 11.

I) exhibeatur substitutio non-singularis

$$[x] = Z[\zeta],$$

per quam forma regularis n variabilium

$$f \equiv \sum_{i,k}^{n} a_{ik} x_i x_k$$

in summam quadratorum

(44) $$fZ \equiv \zeta_1^2 + \zeta_2^2 + \ldots + \zeta_n^2$$

redigatur, aut

II) determinentur elementa *ignota* α — numero n^2 — matricis regularis n^{ti} gradus

$$Z^{-1} = (\alpha_{ik}),$$

relationi

$$\sum_{i,k}^{n} a_{ik} x_i x_k \equiv \sum_{s=1}^{n} (\alpha_{s1} x_1 + \alpha_{s2} x_2 + \ldots + \alpha_{sn} x_n)^2$$

pro variabilibus x identicae satisfacientia, — unum idemque desiderium exprimere.

Haec autem identitas ultima $\frac{n(n+1)}{2}$ relationibus aequabit, quae systema sequens constituunt:

(45) $$\sum_{s=1}^{n} \alpha_{si} \alpha_{sk} = a_{ik}. \quad (1 \leq i \leq k \leq n)$$

Quoniam autem numerus aequationum huius systematis *quadratici* $\frac{n(n+1)}{2}$ est, quantitatum ignotarum quidem n^2, ideo „generaliter" expectandum, recte *sperandum* esset, valores α numero $\frac{n(n+1)}{2}$ per relictos α numero

(N) $$n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$$

exprimi posse. Secundum hanc ratiocinationem quantitates α nunc ipsum commemoratae numero $\frac{n(n-1)}{2}$ in matrice Z^{-1} sicut parametri independentes fungentur, ideoque elementa matricis Z identitatem (44) proferentis nihilo minus quam illae α functiones parametrorum numero $\frac{n(n-1)}{2}$ libere eligendarum erunt.

His observationibus praemissis superest deinde, ut transformationem Weierstrassianam et ex parte quaestionis parametrorum paulo accuratius perscrutemur.

Utamur primo quadam simplificatione, per quam enim una transformationis Weierstrassianae effigies certa uniformis orietur, quippe quum tali modo forma f iam ab ovo in sua forma *normali*

$$\zeta_1^2+\zeta_2^2+\ldots+\zeta_r^2, \qquad (r=r(f))$$

prodiret. — Ceterum supponatur nunc formam f non-singularem, id est

$$r(f)=n$$

fore; respectu autem designationum symboli in CAPITE PRIMO introducti conserventur. Si deinde

$$f(e_1, e_2, \ldots, e_n)=\gamma \neq 0 \tag{2}$$

una cum $e_1 \neq 0$ accipiantur, ponamus

$$e_i=\sqrt{\gamma}\,.\,\varepsilon_i, \tag{46}$$
$$(i=1, 2, \ldots, n)$$

ubi quidem $\sqrt{\gamma}$ duorum valorum radicis *utrum determinatum* significat. Habetur ideo ex (2) secundum (46):

$$f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)=1, \tag{47}$$

insuper vero

$$\varepsilon_1=\frac{e_1}{\sqrt{\gamma}}\neq 0.$$

Scribendo praeterea

$$\varkappa_i=u_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)=a_{i1}\varepsilon_1+a_{i2}\varepsilon_2+\ldots+a_{in}\varepsilon_n,$$
$$(i=1, 2, \ldots, n)$$

concluditur:

$$c_i=u_i(e_1, e_2, \ldots, e_n)=a_{i1}e_1+a_{i2}e_2+\ldots+a_{in}e_n=$$
$$=\sqrt{\gamma}\,.\,(a_{i1}\varepsilon_1+a_{i2}\varepsilon_2+\ldots+a_{in}\varepsilon_n)=\sqrt{\gamma}\,.\,\varkappa_i,$$

porroque

$$y_1=c_1x_1+c_2x_2+\ldots+c_nx_n=\sqrt{\gamma}\,.\,(\varkappa_1x_1+\varkappa_2x_2+\ldots+\varkappa_nx_n).$$

His omnibus respectis, quia secundum (9) $g\gamma=1$ et

$$\frac{e_i}{e_1}=\frac{\varepsilon_i}{\varepsilon_1}, \quad (i=2, 3, \ldots, n)$$

habetur, identitas (W) CAPITIS PRIMI in relationem

$$f(x_1, x_2, \ldots, x_n) \equiv (\varkappa_1 x_1 + \varkappa_2 x_2 + \ldots + \varkappa_n x_n)^2 +$$
$$+ f\left(0, x_2 - \frac{\varepsilon_2}{\varepsilon_1} x_1, \ldots, x_n - \frac{\varepsilon_n}{\varepsilon_1} x_1\right) - \left(\sum_{i=2}^{n} \varkappa_i \left(x_i - \frac{\varepsilon_i}{\varepsilon_1} x_1\right)\right)^2$$

transiet, quocirca autem, si hic inversa substitutionis

$$(48) \quad \begin{cases} \varkappa_1 x_1 + \varkappa_2 x_2 + \ldots + \varkappa_n x_n = x_1', \\ -\frac{\varepsilon_2}{\varepsilon_1} x_1 + x_2 \qquad\qquad\qquad = x_2', \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -\frac{\varepsilon_n}{\varepsilon_1} x_1 \qquad\qquad + x_n = x_n', \end{cases}$$

curte

$$(49) \qquad [x] = W_1 [x']$$

adhibeatur, ubi quidem[40])

$$(50) \qquad \text{det. } W_1 = \frac{1}{\text{det. (48)}} = \frac{\varepsilon_1}{\varkappa_1 \varepsilon_1 + \ldots + \varkappa_n \varepsilon_n} = \varepsilon_1 \neq 0,$$ [41])

transformata sub (15) scriptae respondens

$$(51) \qquad \bar{f}(x_1', x_2', \ldots, x_n') \equiv x_1'^2 + f(0, x_2', \ldots, x_n') - \left(\sum_{i=2}^{n} \varkappa_i x_i'\right)^2 \equiv$$
$$\equiv x_1'^2 + \varphi(x_2', \ldots, x_n')$$

nanciscetur, in qua igitur aequatio identica

$$(52) \qquad \varphi(x_2', \ldots, x_n') \equiv f(0, x_2', \ldots, x_n') - \left(\sum_{i=1}^{n} \varkappa_i x_i'\right)^2 \equiv$$
$$\equiv \sum_{i=2}^{n} \sum_{k=2}^{n} \alpha_{ik} x_i' x_k',$$

cum coefficientibus

$$(53) \qquad \alpha_{ik} = a_{ik} - \varkappa_i \varkappa_k$$
$$(i, k = 2, 3, \ldots, n)$$

locum habebit.[42]) Ratione evolutionum respondentium CAPITIS PRIMI notum est

$$(53') \qquad r(\varphi) = r(f) - 1 = n - 1$$

fore.

[40]) Cf. (13) supra.

[41]) Est enim propter (47):

$$(47') \qquad \sum_{i=1}^{n} \varkappa_i \varepsilon_i = \sum_{i=1}^{n} \varepsilon_i u_i(\varepsilon_1, \ldots, \varepsilon_n) = f(\varepsilon_1, \ldots, \varepsilon_n) = 1.$$

[42]) Itaque coefficientes α_{ik} polynomia respectu elementorum ε secundi gradus offerunt. Quod ad valorem det. φ spectat, vide infra ADNOTATIONEM.

Et si nunc numeri $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ ubicunque velut *parametri variabiles* conspiciantur, et has quantitates ε relationibus

(54) $$f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = 1 \quad \text{et} \quad \varepsilon_1 \neq 0$$

solum satisfacere debere memoremus, transformationem (49) tamquam $n-1$ elementa e complexu

(ε) $$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$$

idonee sumpta atque parametros libere eligendas repraesentantia involventem aspicere licebit. Hic ideo status est post passum primum transformationis Weierstrassianae; perlustremus adhuc breviter tabulatum consequens reductionis. Sumantur ad exemplum praecedentium numeri

$$\varepsilon_2', \varepsilon_3', \ldots, \varepsilon_n',$$

quibus

(55) $$\varphi(\varepsilon_2', \varepsilon_3', \ldots, \varepsilon_n') = 1$$

et eodem tempore

(55') $$\varepsilon_2' \neq 0$$

inveniantur. Hoc autem casu supponere convenit valores — antecedentibus satisfacientes — seriei (ε) ad tempus *fixos* esse, et ideo secundum (53') $\varphi \not\equiv 0$ fore atque relationi (55) numeris non omnibus evanescentibus satisfieri posse.[43]) Si deinde scribatur

$$\varkappa_i' = \alpha_{i2}\varepsilon_2' + \alpha_{i3}\varepsilon_3' + \ldots + \alpha_{in}\varepsilon_n', \quad (i = 2, 3, \ldots, n)$$

ita ut fiat

$$\varkappa_2'\varepsilon_2' + \varkappa_3'\varepsilon_3' + \ldots + \varkappa_n'\varepsilon_n' = \varphi(\varepsilon_2', \ldots, \varepsilon_n') = 1,$$

per substitutionem inversam substitutionis

$$\begin{array}{llll} x_1' & & & = x_1'', \\ & \varkappa_2' x_2' + \varkappa_3' x_3' + \ldots + \varkappa_n' x_n' & & = x_2'', \\ & -\dfrac{\varepsilon_3'}{\varepsilon_2'} x_2' + x_3' & & = x_3'', \\ & \ldots\ldots\ldots\ldots\ldots & & \\ & -\dfrac{\varepsilon_n'}{\varepsilon_2'} x_2' & + x_n' & = x_n'' \end{array}$$

variabilium totalitatem implicantis, puta

(56) $$[x'] = W_2[x''],$$

---

[43]) Status $\varepsilon_2 \neq 0$ transignatione apta capi potest.

forma

$$\bar{f}(x_1', x_2', \ldots, x_n')$$

in formam

(57) $$\bar{\bar{f}}(x_1'', x_2'', \ldots, x_n'') \equiv x_1''^2 + x_2''^2 + \psi(x_3'', \ldots, x_n'')$$

transmutabitur, ubi quidem

$$r(\psi) = n - 2$$

et

$$\det. W_2 = \varepsilon_2' \neq 0,$$

coefficientes autem formae

$$\psi(x_3'', \ldots, x_n'') \equiv \sum_{i=3}^{n} \sum_{k=3}^{n} \beta_{ik} x_i'' x_k''$$

per analogiam (53):

$$\beta_{ik} = \alpha_{ik} - \varkappa_i' \varkappa_k' = a_{ik} - \varkappa_i \varkappa_k - \varkappa_i' \varkappa_k'$$
$$(i, k = 3, 4, \ldots, n)$$

erunt. Expressiones β igitur polynomia secundi gradus respectu quantitatum ε', quarti gradus contra respectu elementorum ε — ob valores α in formis $\varkappa'$ occurrentes — constituunt.[44])

Innitentes ergo supra circumscriptis $n-2$ valores adeptos ex serie

(ε') $$\varepsilon_2', \varepsilon_3', \ldots, \varepsilon_n'$$

electos, sicut parametros *independenter variabiles* aspectare et ideo enuntiare possumus: *in substitutione lineari*

$$[x] = W_1 W_2 [x''],$$

ex substitutionibus (49) *et* (56) *composita elementa matricis* $W_1 W_2$ *a parametris numero* $(n-1)+(n-2) = 2n-3$ *liberis pendent.*

Hac via autem in constructione transformationis Weierstrassianae porro progredientes, statim perspicimus unumquemque passum generatorem, quo terminus quadratus — alter post alterum — a forma principali abscinditur, seriem ulteriorem parametrorum novarum introducere, atque has series recentiores — multitudinem ordinatim unitate imminutam parametrorum continentes — praecedentibus cum seriebus — in colligatione ex supra traditis elucenti — se coniungere. Momentum quidem implicationem superpositarum serierum parametrorum succedentium efficiens in eo conspicari potest, quod coefficientes formae cuiusvis residuae poly-

[44]) Valorem det. ψ vide infra in ADNOTATIONE.

nomia *omnium parametrorum praecedentium* repraesentant. Patet idcirco spatia varietatis pro seriebus singulis parametrorum inter se non independentia fore. Itaque haud dubium est determinationem concinnam spatiorum variandi valorum singularum parametrorum independentium disquisitionem specialem et seorsum perficiendam requirere.[45])

Adducimur attamen ipsis discussionibus eatenus traditis ad sequentem affirmationem — magis vero superficiarii generis — nunc quidem unice appetitam: transformatio Weierstrassiana secundum praescripta antecedenter evoluta constructa, puta

$$[x] = W[\zeta],$$

qua enim forma $f(x_1, \ldots, x_n)$ n^{ti} ordinis in suam formam normalem

$$fW \equiv \zeta_1^2 + \zeta_2^2 + \ldots + \zeta_n^2$$

redigatur, omnino

$$(n-1) + (n-2) + \ldots + 3 + 2 + 1 = \frac{n(n-1)}{2}$$

parametros „independentes" in elementis matricis W implicatas continebit.

ADNOTATIO.

Ex identitate (51) protinus derivatur:

$$\text{det.}\,\bar{f} = \text{det.}\,\varphi = (\text{det.}\,W_1)^2\,.\,\text{det.}\,f = \varepsilon_1^2\,.\,\text{det.}\,f,$$

aut quod idem est:

$$\underset{(i,\,k\,=\,2,\,3,\ldots,\,n)}{|a_{ik} - \varkappa_i \varkappa_k|} = \varepsilon_1^2\,.\,\underset{(j,\,l\,=\,1,\,2,\ldots,\,n)}{|a_{jl}|};$$

haec autem relatio ope aequationis

$$\varkappa_1 \varepsilon_1 + \varkappa_2 \varepsilon_2 + \ldots + \varkappa_n \varepsilon_n = 1$$

etiam per transformationes simplices determinantium facillime verificari posset. Eodemque modo accipitur ex (57):

$$\text{det.}\,\bar{\bar{f}} = \text{det.}\,\psi = \underset{(i,\,k\,=\,3,\,4,\ldots,\,n)}{|a_{ik} - \varkappa_i \varkappa_k - \varkappa_i' \varkappa_k'|} = \varepsilon_1^2 \varepsilon_2'^2\,.\,\text{det.}\,f;$$

etc., etc.

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[45]) Observare convenit transformationem Darbouxianam *a limine* vero aliquid obvolutiorem esse, quippe quod minus aperiat accessum ad iudicium non solum de numero parametrorum *independentium* in transformatione inclusarum, sed etiam de modo praesentiae harum parametrorum.

CAPUT QUARTUM.

Transformatio Jacobiana sicut casus specialis.[46])

Fiat

$$\text{det.}\, f = |a_{ik}| = A$$
$$(i, k = 1, 2, \ldots, n)$$

et ponatur:

$$A^{(i)} = \begin{vmatrix} a_{n-i+1,\, n-i+1} & a_{n-i+1,\, n-i+2} & \ldots & a_{n-i+1,\, n} \\ a_{n-i+2,\, n-i+1} & a_{n-i+2,\, n-i+2} & \ldots & a_{n-i+2,\, n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n,\, n-i+1} & a_{n,\, n-i+2} & \ldots & a_{nn} \end{vmatrix} = \tag{58}$$

$$= \frac{\partial^{n-i} A}{\partial a_{11}\, \partial a_{22} \ldots \partial a_{n-i,\, n-i}}, \quad (i = 1, 2, \ldots, n-1),$$

denique unitatis gratia: $A^{(n)} = A$. Habebitur itaque ordinatim

$$A^{(1)} = a_{nn},\ A^{(2)} = \begin{vmatrix} a_{n-1,\, n-1} & a_{n-1,\, n} \\ a_{n,\, n-1} & a_{nn} \end{vmatrix}, \ldots, A^{(n-2)} = \frac{\partial^2 A}{\partial a_{11}\, \partial a_{22}},$$

$$A^{(n-1)} = \frac{\partial A}{\partial a_{11}} = A_{11},\ A^{(n)} = A;$$

si quidem specialiter A_{11} cofactorem in A ad a_{11} pertinentem significet. Minores capitales sic definitos „minores subcardinales“ resp.

$$1., 2., \ldots, i., \ldots, n.$$

gradus determinantis A appellare volumus. In subsequentibus omnes hos minores capitales „cifra“ differentes, id est

$$A^{(1)} A^{(2)} \ldots A^{(n-1)} A^{(n)} \neq 0 \tag{59}$$

supponemus.

Construamus deinde ad scopum nostrum reductam quandam specialem formae f summam quadratorum ostentantem, cuius vero identitatem repraesentatione Jacobiana statim agnoscere possimus. Ponamus enim hanc ob rem:[47])

$$e_1 = A_{11} = A^{(n-1)},\ e_2 = A_{12}, \ldots, e_n = A_{1n}, \tag{60}$$

ubi quidem A_{ik} cofactorem in A ad a_{ik} pertinentem designat; reperietur ideo:

$$c_1 = a_{11} e_1 + a_{12} e_2 + \ldots + a_{1n} e_n = a_{11} A_{11} + a_{12} A_{12} + \ldots + a_{1n} A_{1n} = A,$$

et porro

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[46]) Cf. WEIERSTRASS: [1]) p. 18—20.

[47]) Cf. undique designationes et evolutiones CAPITIS PRIMI.

$$c_i = a_{i1} e_1 + \dots + a_{in} e_n = a_{i1} A_{11} + \dots + a_{in} A_{1n} = 0;$$
$$(i = 2, 3, \dots, n)$$

habebitur itaque

$$f(e_1, \dots, e_n) = \gamma = c_1 e_1 + \dots + c_n e_n = A A_{11} = A A^{(n-1)} \neq 0,$$

quapropter

$$g = \frac{1}{\gamma} = \frac{1}{A A^{(n-1)}};$$

denique autem

$$y_1 = c_1 x_1 + \dots + c_n x_n = c_1 x_1 = A x_1.$$

His cum valoribus proinde identitas (W) CAPITIS PRIMI formam

$$f(x_1, \dots, x_n) \equiv \frac{A}{A^{(n-1)}} \cdot x_1^2 + f(0, y_2, \dots, y_n) \tag{61}$$

induet, in qua

$$y_i = x_i - \frac{A_{1i}}{A_{11}} x_1 \tag{61'}$$
$$(i = 2, 3, \dots, n)$$

intellegendum est.

In aequalitate (61) matrix formae $n-1$ variabilium

$$f(0, y_2, \dots, y_n) \equiv \varphi(y_2, \dots, y_n)$$

evidenter

$$(A^{(n-1)}) = (a_{ik})$$
$$(i, k = 2, 3, \dots, n)$$

erit, eiusque determinans

$$\text{det. } \varphi = A^{(n-1)} \neq 0.$$

Ponamus deinde cum cofactoribus $A_{2k}^{(n-1)}$ ad elementa primae lineae determinantis $A^{(n-1)}$ pertinentibus:

$$e_2' = A_{22}^{(n-1)} = A^{(n-2)} \neq 0, \quad e_3' = A_{23}^{(n-1)}, \dots, e_n' = A_{2n}^{(n-1)},$$

quo autem fit

$$c_2' = a_{22} e_2' + \dots + a_{2n} e_n' = \text{det. } \varphi = A^{(n-1)}$$

et

$$c_i' = a_{i2} e_2' + \dots + a_{in} e_n' = 0;$$
$$(i = 3, 4, \dots, n)$$

praeterea fluent inde

$$\varphi(e_2', \ldots, e_n') = \gamma' = c_2' e_2' + \ldots + c_n' e_n' = A^{(n-1)} \,.\, A^{(n-2)} \neq 0,$$

atque

$$g' = \frac{1}{\gamma'}; \quad y_2' = c_2' y_2 + \ldots + c_n' y_n = A^{(n-1)} \,.\, y_2.$$

Hanc per viam oritur transformata formae φ, expressioni (61) similiter constructa:

$$\varphi(y_2, \ldots, y_n) \equiv \frac{A^{(n-1)}}{A^{(n-2)}} \cdot y_2^2 + \varphi(0, y_3', \ldots, y_n'), \tag{62}$$

ubi quidem variabiles y_i et y_i' aequationibus

$$y_i' = y_i - \frac{A_{2i}^{(n-1)}}{A^{(n-2)}} \cdot y_2 \tag{62'}$$

$$(i = 3, 4, \ldots, n)$$

coniunctae sunt.

Per has evolutiones certe satis persuademur methodum reductionis nuperrime perlustratam — respectu enim suppositionis (59) — etiam porro adhiberi et tali modo tandem ad repraesentationem

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) \equiv \frac{A^{(n)}}{A^{(n-1)}} \cdot x_1^2 + \frac{A^{(n-1)}}{A^{(n-2)}} \cdot y_2^2 + \frac{A^{(n-2)}}{A^{(n-3)}} \cdot y_3'^2 + \\ + \ldots + \frac{A^{(2)}}{A^{(1)}} \cdot (y_{n-1}^{(n-3)})^2 + A^{(1)} . (y_n^{(n-2)})^2 \end{aligned} \tag{63}$$

formae f perveniri posse; hic autem habetur

$$A^{(n)} = A, \; A^{(1)} = a_{nn}.$$

Restituantur deinde ubique in summa (63) quadratorum variabiles x principales; cum notatione

$$\alpha_{21} = -\frac{A_{12}}{A_{11}}$$

habemus ex (61')

$$y_2 = x_2 + \alpha_{21} x_1;$$

secundum (62') autem, ratione (61'):

$$y_3' = y_3 + \beta y_2 = x_3 + \alpha_{32} x_2 + \alpha_{31} x_1,$$

ubi constantes α una cum β variabilibus non pendebunt. Ita continuando simili ratione perspicietur valores $y_k^{(k-2)}$ per variabiles x

expressos typum

$$y_k^{(k-2)} = x_k + \alpha_{k,\,k-1} x_{k-1} + \ldots + \alpha_{k1} x_1$$
$$(k = 3, 4, \ldots, n)$$

ostentaturos esse. Identitas (63) ergo in sequenti forma adscribi poterit:

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) \equiv a_{nn} (x_n + \alpha_{n,\,n-1} x_{n-1} + \ldots + \alpha_{n2} x_2 + \alpha_{n1} x_1)^2 + \\ + \frac{A^{(2)}}{a_{nn}} (x_{n-1} + \ldots + \alpha_{n-1,\,2} x_2 + \alpha_{n-1,\,1} x_2)^2 + \\ + \ldots\ldots\ldots + \frac{A^{(n-1)}}{A^{(n-2)}} (x_2 + \alpha_{21} x_1)^2 + \\ + \frac{A}{A^{(n-1)}} \cdot x_1^2; \end{aligned}$$

unde autem revera lucramur hanc formam transformatam ab ea repraesentatione Jacobiana formae f non diversam fore, quae ad successionem

$$x_n, x_{n-1}, \ldots, x_2, x_1$$

variabilium x pertinebit.[48])

Manifestum enim est inversione successionis *et* linearum *et* columnarum cuiusdam determinantis valorem eius non mutari ideoque minores *sub*-cardinales determinantis principalis cum minoribus graduum respondentium *supra*-cardinalibus determinantis inversi ordinatim congruere.

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[48]) Cf. JACOBI, Ueber eine elementare Transformation eines in Bezug auf jedes von zwei Variablen-Systemen linearen und homogenen Ausdrucks, *Gesammelte Werke*, III (Berlin, 1884), pp. 585—590.