On the theory of the mechanical quadrature.

By P. TURÁN in Budapest.

§ 1. In what follows I communicate a few simple remarks on the theory of mechanical quadrature to which also L. FEJÉR[1]) devoted an important paper. These remarks though they are rather naturally connected to the classical theory of mechanical quadrature of GAUSS—JACOBI, seem not to be observed so far. These reveal an interesting property of those polynomials $\pi^*_{n,2l}(x)$ (n, l fixed integers) which minimize the integral

$$I_{2l}(\pi_n) = \int_{-1}^{+1} |\pi_n(x)|^{2l}\, dx \tag{1.1}$$

among the polynomials

$$\pi_n(x) = x^n + a_1 x^{n-1} + \ldots + a_n. \tag{1.2}$$

This polynomial $\pi^*_{n,2l}(x)$ minimizes obviously at the same time also the expression

$$H_{2l}(\pi_n) = \left[\int_{-1}^{+1} |\pi_n(x)|^{2l}\, dx\right]^{\frac{1}{2l}}. \tag{1.3}$$

§ 2. The classical theorem of GAUSS—JACOBI deals with quadrature-formulae of the type

$$\int_{-1}^{+1} f(x)\, dx = \sum_{\nu=1}^{n} f(x_\nu)\, \lambda_\nu, \tag{2.1}$$

where $x_1, \ldots, x_n$ are different, arbitrarily prescribed numbers and the λ's are independent of f. Putting

$$\omega(x) = \prod_{\nu=1}^{n} (x - x_\nu) \quad \text{and} \quad l_\nu(x) = \frac{\omega(x)}{\omega'(x_\nu)(x - x_\nu)} \qquad (\nu = 1, 2, \ldots, n)$$

we have $f(x) = \sum_{\nu=1}^{n} f(x_\nu)\, l_\nu(x)$, for all polynomials $f(x)$ of degree $\leq n-1$,

1) L. FEJÉR, Mechanische Quadraturen mit positiven Cotesschen Zahlen, *Math. Zeitschrift*, **37** (1933), pp. 287–310.

and consequently

$$\int_{-1}^{+1} f(x)\,dx = \sum_{\nu=1}^{n} f(x_\nu) \int_{-1}^{+1} l_\nu(x)\,dx. \tag{2. 2}$$

This is a quadrature-formula of type (2. 1) with the „Cotes-numbers“ $\lambda_\nu = \int_{-1}^{1} l_\nu(x)\,dx$ $(\nu = 1, 2, \ldots, n)$, valid for all polynomials of degree $\leqq n-1$.

Now the above-mentioned theorem of Gauss–Jacobi solves the question how to choose the "fundamental points" $x_1, x_2, \ldots, x_n$ in order that the quadrature-formula (2. 2) be valid for "the greatest possible set" of polynomials. They proved that formula (2. 2) is valid for all polynomials of degree $\leqq 2n-1$ if and only if $x_1, \ldots, x_n$ are the zeros of the nth Legendre-polynomial

$$P_n(x) = [(x^2-1)^n]^{(n)}. \tag{2. 3}$$

§ 3. Now we consider mechanical quadratures of the type

$$\int_{-1}^{+1} f(x)\,dx = \sum_{\nu=1}^{n} f(x_\nu)\,\lambda_\nu^{(0)} + \sum_{\nu=1}^{n} f'(x_\nu)\,\lambda_\nu^{(1)}, \tag{3. 1}$$

where the quantities $\lambda_\nu^{(0)}, \lambda_\nu^{(1)}$ are independent of f. The existence of such a quadrature-formula follows immediately from the formula of FEJÉR[2])

$$f(x) = \sum_{\nu=1}^{n} f(x_\nu)\,l_{\nu,0}(x) + \sum_{\nu=1}^{n} f'(x_\nu)\,l_{\nu,1}(x) \tag{3. 2}$$

valid for all polynomials $f(x)$ of degree $\leqq 2n-1$, where — again with the notation of § 2 —

$$l_{\nu,0}(x) = \left(1 - \frac{\omega''(x_\nu)}{\omega'(x_\nu)}(x-x_\nu)\right) l_\nu^2(x), \quad l_{\nu,1}(x) = (x-x_\nu)\,l_\nu^2(x). \tag{3. 3}$$

Integrating (3. 2) over $[-1, +1]$ we obtain a formula of type (3. 1) with

$$\lambda_\nu^{(0)} = \int_{-1}^{+1} l_{\nu,0}(x)\,dx, \qquad \lambda_\nu^{(1)} = \int_{-1}^{+1} l_{\nu,1}(x)\,dx, \tag{3. 4}$$

valid for all polynomials $f(x)$ of degree $\leqq 2n-1$.

§ 4. The n zeros of the Legendre-polynomial $P_n(x)$ of (2. 3) are real. This fact implies that the Gauss–Jacobi formula is applied in praxis, e. g. in meteorology. Hence it is reasonable to modify a little Gauss—Jacobi's problem asking for a *real* system $(x_1, x_2, \ldots, x_n)$ for which the quadrature-formula (3. 1)—(3. 4) is true for a greater variety of polynomials than those of degree $\leqq 2n-1$. It is easy to show that by *no* choice this formula

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[2]) Implicitly in his paper: Lagrangesche Interpolation und die zugehörigen konjugierten Punkte, *Math. Annalen*, **106** (1932), pp. 1—56. Our notation differs from his one; this change is motivated by the considerations of § 8.

(3. 1)—(3. 4) can be made precise even to all polynomials of degree $\leqq 2n$. The validity of formula (3. 1) for a class A of polynomials means namely that for any $f_1(x)$ and $f_2(x)$ of the class A, for which

$$(4.1) \qquad f_1(x_\nu)=f_2(x_\nu),\ f_1'(x_\nu)=f_2'(x_\nu) \qquad (\nu=1,2,\ldots,n)$$

we have

$$(4.2) \qquad \int_{-1}^{+1} (f_1(x)-f_2(x))\,dx=0.$$

But it follows from (4. 1) that the polynomial $f_1(x)-f_2(x)$ is divisible by $\omega^2(x)$; hence if A is the class of polynomials of degree $\leqq 2n$, we have $f_1(x)-f_2(x)=c\omega^2(x)$, i. e. from (4. 2) we would have for all c

$$c\int_{-1}^{+1}\omega^2(x)\,dx=0.$$

But this is impossible for a proper polynomial $\omega(x)$ with real zeros only.

§ 5. Now we pass a step further. We consider quadrature-formulae of type

$$(5.1) \qquad \int_{-1}^{+1} f(x)\,dx=\sum_{\nu=1}^{n} f(x_\nu)\,\lambda_\nu^{(0)}+\sum_{\nu=1}^{n} f'(x_\nu)\,\lambda_\nu^{(1)}+\sum_{\nu=1}^{n} f''(x_\nu)\,\lambda_\nu^{(2)},$$

where the $\lambda_\nu^{(j)}$'s are independent of f. It is again easy to show the existence of such a quadrature-formula (5. 1), valid for all $f(x)$ polynomials of degree $\leqq 3n-1$. Following namely the reasoning FEJÉR used to determine $l_{\nu 0}(x)$ and $l_{\nu 1}(x)$ in (3. 2), we obtain[8]) the representation

$$(5.2) \qquad f(x)=\sum_{\nu=1}^{n} f(x_\nu)\,l_{\nu,0}(x)+\sum_{\nu=1}^{n} f'(x_\nu)\,l_{\nu,1}(x)+\sum_{\nu=1}^{n} f''(x_\nu)\,l_{\nu,2}(x)$$

valid for all $f(x)$ of degree $\leqq 3n-1$.

Here we have with the notation of § 2

$$(5.3) \qquad \begin{aligned} l_{\nu,0}(x)&=\left\{1-3l_\nu'(x_\nu)(x-x_\nu)+\frac{3}{2}\left[5(l_\nu'(x_\nu))^2-l_\nu''(x_\nu)\right](x-x_\nu)^2\right\} l_\nu^3(x),\\ l_{\nu,1}(x)&=(x-x_\nu)\left[1-3l_\nu'(x_\nu)(x-x_\nu)\right] l_\nu^3(x),\quad l_{\nu,2}(x)=\frac{1}{2}(x-x_\nu)^2\, l_\nu^3(x), \end{aligned}$$

i. e. we obtain (5. 1) with

$$(5.4) \qquad \lambda_\nu^{(0)}=\int_{-1}^{+1} l_{\nu,0}(x)\,dx,\quad \lambda_\nu^{(1)}=\int_{-1}^{+1} l_{\nu,1}(x)\,dx,\quad \lambda_\nu^{(2)}=\int_{-1}^{+1} l_{\nu,2}(x)\,dx.$$

§ 6. Now we raise again the question to determine those systems $(x_1,\ldots,x_n)$ of n different points for which the quadrature-formula (5. 1)—(5. 4) is valid for all polynomials of degree $\leqq 4n-1$. If B denotes this class of polynomials

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8) The same formula was established also by Mr. I. RAISZ in an unpublished paper.

and $f_1(x)$, $f_2(x)$ denote any two members of the class B with

(6.1) $$f_1(x_\nu)=f_2(x_\nu),\quad f_1'(x_\nu)=f_2'(x_\nu),\quad f_1''(x_\nu)=f_2''(x_\nu) \qquad (\nu=1,2,\dots,n),$$

then we must have

(6.2) $$\int_{-1}^{+1}(f_1(x)-f_2(x))\,dx=0.$$

Fixing $f_1(x)$ in B and choosing

(6.3) $$f_2(x)=f_1(x)+\omega^3(x)\,h(x)$$

where $h(x)$ is an arbitrary polynomial of degree $\leqq n-1$, $f_2(x)$ belongs obviously to the class B and satisfies (6.1); hence by (6.2) we must have

(6.4) $$\int_{-1}^{+1}\omega^3(x)\,h(x)\,dx=0$$

for all polynomials $h(x)$ of degree $\leqq n-1$. Since any two polynomials with the property (6.1) fulfill the relation (6.3), the condition (6.4) is necessary and sufficient to the validity of the quadrature-formula (5.1)—(5.4).

§ 7. Now we have to determine whether or not there is an $\omega(x)$ with the "higher orthogonality-property" (6.4). We suppose that such an $\omega(x)$ exists. We show first that all the zeros of $\omega(x)$ lie in the interior of the interval $[-1,+1]$ and are simple. To prove this by a classical argument we remark that from (6.4) obviously

(7.1) $$\int_{-1}^{+1}\omega^3(x)\,x^\nu\,dx=0 \qquad (\nu=0,1,\dots,n-1).$$

If $\omega(x)$ would have in the interior of $[-1,+1]$ only $k<n$ sign-changing places, say $-1<\zeta_1<\zeta_2<\dots<\zeta_k<+1$, then we would have

$$\omega^3(x)\sum_{\nu=0}^{k}c_\nu x^\nu=\omega^3(x)\prod_{\nu=1}^{k}(x-\zeta_\nu)\geqq 0$$

in $[-1,+1]$ and hence

$$0<\int_{-1}^{+1}\omega^3(x)\prod_{\nu=1}^{k}(x-\zeta_\nu)\,dx=\sum_{\nu=0}^{k}c_\nu\int_{-1}^{1}\omega^3(x)\,x^\nu\,dx=0$$

owing to (7.1); an obvious contradiction. Hence $k=n$ and the assertion concerning the zeros of $\omega(x)$ is proved. This implies of course that the coefficients of $\omega(x)$ are all real too. Since any polynomial $\pi_n(x)=x^n+\dots+a_n$ may be written in the form $\pi_n(x)=\omega(x)+h(x)$ with an $h(x)$ of degree $\leqq n-1$ we have

$$\Delta=\int_{-1}^{+1}|\pi_n(x)|^4dx-\int_{-1}^{+1}|\omega(x)|^4dx=\int_{-1}^{+1}[\omega(x)+h(x)]^2\,[\omega(x)+\bar h(x)]^2\,dx-\int_{-1}^{+1}|\omega(x)|^4dx$$

where $\bar h(x)$ denotes that polynomial whose coefficients are conjugate-complex to those of $h(x)$. Hence

$$\Delta=2\int_{-1}^{+1}\omega^3(x)\,(h(x)+\bar{h}(x))\,dx+\int_{-1}^{1}[|h(x)|^2+\omega(x)\,(h(x)+\bar{h}(x))]^2\,dx+ \tag{7.2}$$
$$+2\int_{-1}^{+1}|h(x)|^2\,\omega^2(x)\,dx.$$

But the first integral in (7. 2) is 0 owing to (6. 4) and hence we have $\Delta \geqq 0$; equality only in the case $h(x)\equiv 0$. *Hence if a polynomial $\omega(x)$ with property* (6. 4) *exists, then it minimizes the integral $I_4(\pi_n)$ of* (1. 1). But the existence and uniqueness of a solution of this extremal-problem was proved by JACKSON[4]). Hence we proved the following

Theorem I. *Among the quadrature-formulae* (5. 1) *valid for all polynomials $f(x)$ of degree $\leqq 3n-1$ there is exactly one choice of $(x_1,\dots,x_n)$ for which the formula is valid for all polynomials of degree $\leqq 4n-1$. This $(x_1,\dots,x_n)$-system consists of the n real distinct zeros in the interior of $[-1,1]$ of that polynomial $\pi^*_{n,4}(x)=x^n+\dots$ which minimizes the integral $I_4(\pi_n)$ of* (1. 1) *in the class* (1. 2).

§ 8. The generalisation of the quadrature-formula (5. 1) is immediate. Given any system of n distinct points $(x_1,\dots,x_n)$ and n integers $m_1, m_2,\dots, m_n$, HERMITE[5]) proved the existence and uniqueness of polynomials

$$l_{\nu 0}(x),\ l_{\nu 1}(x),\dots,l_{\nu, m_\nu-1}(x)\qquad (\nu=1,2,\dots,n),$$

each of degree $\leqq (m_1+m_2+\dots+m_n-1)$, such that

$l^{(h)}_{\nu k}(x_\mu)=0$ for all ν,μ,k,h $(1\leqq\nu\leqq n,\ 1\leqq\mu\leqq n,\ 0\leqq k\leqq m_\nu-1,\ 0\leqq h\leqq m_\mu-1)$,

except for $\nu=\mu$ and $k=h$; in the latter case

$$l^{(k)}_{\nu k}(x_\nu)=1.$$

Then we have the representation

$$f(x)=\sum_{\nu=1}^{n}[f(x_\nu)\,l_{\nu 0}(x)+f'(x_\nu)\,l_{\nu 1}(x)+\dots+f^{(m_\nu-1)}(x_\nu)\,l_{\nu,m_\nu-1}(x)], \tag{8.1}$$

valid for all polynomials of degree $\leqq (m_1+\dots+m_n-1)$, for the difference of the expressions on both sides of (8. 1) is divisible by $(x-x_1)^{m_1}(x-x_2)^{m_2}\dots(x-x_n)^{m_n}$, i. e. identical zero. Hence integrating (8. 1) over $[-1,+1]$ we obtain the quadrature-formula of L. TSCHAKALOFF[6])

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[4]) D. JACKSON, On functions of closest approximation, *Transactions American Math. Society*, **22** (1921), pp. 117–128.

[5]) CH. HERMITE, Sur la formule d'interpolation de Lagrange, *Journal für die reine und angewandte Math.*, **84** (1878), pp. 70—79.

[6]) L. TSCHAKALOFF, Über eine allgemeine Quadraturformel, *Comptes Rendus de l'Acad. Bulgare des Sciences*, **1** (1948), pp. 9–12. The point of his paper is a method for computation of the $\lambda^{(i)}_j$'s.

$$\int_{-1}^{+1} f(x)\,dx = \sum_{\nu=1}^{n} [f(x_\nu)\,\lambda_\nu^{(0)} + f'(x_\nu)\,\lambda_\nu^{(1)} + \ldots + f^{(m_\nu-1)}(x_\nu)\,\lambda_\nu^{(m_\nu-1)}]$$

valid for all polynomials of degree $\leqq (m_1 + \ldots + m_n - 1)$, which was the starting point of these investigations.

§ 9. Specializing $m_1 = m_2 = \ldots = m_n = k$ we obtain the quadrature-formula

$$\int_{-1}^{+1} f(x)\,dx = \sum_{\nu=1}^{n} [f(x_\nu)\,\lambda_\nu^{(0)} + f'(x_\nu)\,\lambda_\nu^{(1)} + \ldots + f^{(k-1)}(x_\nu)\,\lambda_\nu^{(k-1)}], \tag{9. 1}$$

valid for all polynomials $f(x)$ of degree $\leqq kn-1$. In this case the functions $l_{\nu j}(x)$ of § 8 can explicitly be represented following FEJÉR's procedure[2]) and so the quantities $\lambda_\nu^{(j)}$. Asking by which choice of the x_ν's the formula (9. 1) will be exact for all polynomials $f(x)$ of degree $\leqq (k+1)\,n-1$ we obtain similarly that there is no real system $(x_1, \ldots, x_n)$ if k is even, and for odd k if and only if $x_1, \ldots, x_n$ are the zeros of the minimizing polynomial $\pi^*_{n,k+1}(x)$ of § 1.

§ 10. Is this result compatible with the theorem of Gauss—Jacobi, explained in § 2 which corresponds to the special case $k-1$? It is a well-known property of the nth Legendre-polynomial (2. 3) that, when properly normalized, it minimizes the integral $I_2(\pi_n)$ of (1. 1). Hence our results constitute a generalization of GAUSS—JACOBI's theorem.

§ 11. All these considerations can be applied to the theory of mechanical quadrature of MEHLER—CHRISTOFFEL—CHEBYSHEV—STIELTJES, where a weight-function is permitted, i. e. of quadrature-formulae of the type

$$\int_{-1}^{+1} f(x)\,p(x)\,dx = \sum_{\nu=1}^{n} f(x_\nu)\,\lambda_\nu^{(0)} \tag{11. 1}$$

where the $\lambda_\nu^{(0)}$'s are independent of $f(x)$ but dependent in general on $p(x)$. The theorems so obtained will not be formulated explicitly except in the case

$$p(x) = \frac{1}{\sqrt{1-x^2}}. \tag{11. 2}$$

In this case — as S. BERNSTEIN discovered[7]) — the Chebyshev-polynomial $T_n(x)$ with

$$T_n(\cos\vartheta) = \frac{1}{2^{n-1}} \cos n\vartheta \tag{11. 3}$$

minimizes *all* functionals

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[7]) S. BERNSTEIN, Sur les polynomes orthogonaux relatifs à un segment fini, *Journal de Math.*, (9) **9** (1930), pp. 127—177 et (9) **10** (1931), pp. 219—286.

$$J_k(\pi_n)=\int_{-1}^{+1}\frac{|\pi_n(x)|^k}{\sqrt{1-x^2}}\,dx \qquad (k\geqq 1,\ \text{fixed}) \tag{11.4}$$

in the class (1.2). Hence we obtained the following new property of the zeros of $T_n(x)$:

Theorem II. *Given an arbitrary odd integer k we can determine the numbers $\lambda_\nu^{(j)}$ so that quadrature-formula*

$$\int_{-1}^{+1} f(x)\,dx=\sum_{\nu=1}^{n}\Bigg[f\left(\cos\frac{2\nu-1}{2n}\pi\right)\lambda_\nu^{(0)}+f'\left(\cos\frac{2\nu-1}{2n}\pi\right)\lambda_\nu^{(1)}+ \\ +\ldots+f_\nu^{(k-1)}\left(\cos\frac{2\nu-1}{2n}\pi\right)\lambda_\nu^{(k-1)}\Bigg] \tag{11.5}$$

is valid for all polynomials $f(x)$ of degree $\leqq(k+1)n-1$.

§ 12. These results make desirable to find an explicit expression of the extremal-polynomials $\pi^*_{n,k}(x)$ (for all $k\geqq 1$) which minimize (1.3) with k instead of $2l$ in the class (1.2) or develop a similar asymptotical theory of them which exists[8]) in the case $k=2$. As to the explicit representation the following cases are only known to me.

$k=1$. The minimizing polynomial is the polynomial $U_n(x)$ with

$$U_n(\cos\vartheta)=\frac{1}{2^n}\,\frac{\sin(n+1)\vartheta}{\sin\vartheta}.$$

(Result of KORKINE and ZOLOTAREFF.)

$k=2$. The minimizing polynomial is the nth Legendre-polynomial.

$k=+\infty$. The minimizing polynomial is the polynomial $T_n(x)$ of (11.3). (Classical result of CHEBYSHEV.)

As to the asymptotical theory of these polynomials little is known. Among the four main questions of the theory, namely

a) asymptotic behaviour on the segment $[-1,+1]$,

b) asymptotic behaviour outside the segment $[-1,+1]$,

c) asymptotic determination of the individual zeros,

d) uniform distribution of the zeros,

only the last one is in a somewhat satisfactory shape. As we have shown[9]), the zeros of $\pi^*_{n,k}(x)$ are "uniformly-distributed on the unit-circle" in the sense that writing them in the form (k and n fixed)

$$x_\nu=x_{\nu,n}=\cos\vartheta_{\nu,n}=\cos\vartheta_\nu \qquad (\nu=1,2,\ldots,n)$$

we have for all $0\leqq\alpha<\beta\leqq\pi$

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[8]) See e. g. G. SZEGŐ, *Orthogonal polynomials* (New York, 1939).

[9]) P. ERDŐS and P. TURÁN, On the uniformly dense distribution of certain sequences of points, *Annals of Math.*, 41 (1940), pp. 162–173.

$$\left|\sum_{\alpha\leqq\vartheta_\nu\leqq\beta} 1 - \frac{\beta-\alpha}{\pi} n\right| < c(k)\sqrt{n \log n}.$$

As to the question *b)* we obtained certain results, but — as Prof. G. SZEGŐ mentioned in a conversation — he has found a sharper asymptotical formula for $\pi^*_{n,k}(z)$ outside the interval $[-1, +1]$. Essentially the same is quite recently announced by GERONIMUS [10]).

§ 13. Finally we return to the quadrature-formula (5. 1). FEJÉR[1]) has shown the importance of the fact that the Cotes-numbers λ_ν in (2. 1) are non-negative in some cases. The same advantages can be derived for the quadrature-formula (5. 1) if the numbers $\lambda_\nu^{(0)}, \lambda_\nu^{(1)}, \lambda_\nu^{(2)}$ are non-negative. In what follows I shall show that if the x_ν's are the zeros of $\pi^*_{n,4}(x)$, then

(13. 1) $$\lambda_\nu^{(2)} > 0 \qquad (\nu = 1, 2, \ldots, n).$$

The corresponding questions for $\lambda_\nu^{(0)}$ and $\lambda_\nu^{(1)}$ remain open.

To prove the assertion (13. 1) we remark that from (5. 3) and (5. 4)

$$\lambda_\nu^{(2)} - \frac{1}{2}\int_{-1}^{+1} (x-x_\nu)^2\, l_\nu^4(x)\, dx = \frac{1}{2[\omega'(x_\nu)]^3}\int_{-1}^{+1} \omega^3(x) \frac{1-l_\nu(x)}{x-x_\nu}\, dx.$$

But $[1-l_\nu(x)]/(x-x_\nu)$ is obviously a polynomial of degree $n-2$, i. e., from the orthogonality-property (6. 4), the last integral is 0. Hence

$$\lambda_\nu^{(2)} = \frac{1}{2}\int_{-1}^{+1} (x-x_\nu)^2\, l_\nu^4(x)\, dx > 0. \qquad \text{Q. e. d.}$$

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[10]) JA. L. GERONIMUS, On asymptotic properties of polynomials deviating least from zero in the space L^p_σ, *Doklady Akad. Nauk SSSR*, **62** (1948), pp. 9—12. I know this paper only from its review in the *Math. Reviews.*