On the differentiability of semi-group operators.

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1. Let $\mathfrak{X}$ be a complex (B)-space, A a closed linear operator on $\mathfrak{X}$ to itself whose domain $\mathfrak{D}[A]$ is dense in $\mathfrak{X}$. Suppose that the operator $\lambda I - A$ has a bounded inverse $R(\lambda; A)$ for each fixed real positive value of λ and that

$$\lambda \| R(\lambda; A) \| \leqq 1, \quad \lambda > 0. \tag{1.1}$$

Under these assumptions it is known (see E. HILLE [1], p. 238 and K. YOSIDA [2], p. 15) that A is the infinitesimal generator of a semi-group $\mathfrak{S} = \{T(\xi)\}$, $\xi > 0$, of linear bounded operators $T(\xi)$ with the properties

(i) $T(\xi_1)\, T(\xi_2) = T(\xi_1 + \xi_2)$, $\xi_1 > 0$, $\xi_2 > 0$,

(ii) $\| T(\xi) \| \leqq 1$,

(iii) $\lim\limits_{\xi \to 0} T(\xi) x = x$, $x \in \mathfrak{X}$.

Here (iii) implies the further property $\lim\limits_{\xi \to \xi_0} T(\xi) x = T(\xi_0) x$ for $\xi_0 > 0$. Conversely, if a semi-group $\mathfrak{S}$ with the properties (i), (ii) and (iii) is given, then it has an infinitesimal generator A which is a linear closed operator whose domain $\mathfrak{D}[A]$ is dense in $\mathfrak{X}$. Further the resolvent $R(\lambda; A)$ exists for $\mathfrak{R}(\lambda) > 0$ and

$$\sigma \| R(\sigma + i\tau; A) \| \leqq 1, \quad \lambda = \sigma + i\tau, \quad \sigma > 0. \tag{1.2}$$

For $x \in \mathfrak{D}[A]$ we have

$$\lim_{\delta \to 0} \frac{1}{\delta} [T(\xi + \delta) x - T(\xi) x] = T'(\xi) x = A\, T(\xi) x = T(\xi) A x, \tag{1.3}$$

in the sense of strong convergence.

The assumptions on A stated above imply that $T(\xi)$ is strongly continuous for $\xi \geqq 0$ and no further continuity properties may be asserted in general. Similarly, for $\xi > 0$ the operator $T'(\xi) = A\, T(\xi)$ is ordinarily an unbounded operator whose domain of definition contains $\mathfrak{D}[A]$ and may coincide with $\mathfrak{D}[A]$. For the higher derivatives we have a similar situation; the domain of $T^{(n)}(\xi) = A^n\, T(\xi)$ contains $\mathfrak{D}[A^n]$ which is dense in $\mathfrak{X}$ for every n and $\bigcap_n \mathfrak{D}[A^n]$ is also dense in $\mathfrak{X}$.

Thus if we want to get semi-group operators with stronger continuity and differentiability properties, we must impose stronger restrictions on A. Two sets of such conditions were given in § 12.2 of [1]. The first set give continuity of $T(\xi)$ in the uniform operator topology for $\xi > 0$, but not for $\xi = 0$, nor the existence of bounded derivatives, while the second set implies the existence of derivatives of all orders for $\xi > 0$ but not analyticity. In view of this situation we shall investigate the existence of derivatives of semi-group operators in the present note.

2. We start with

Theorem 1. *If for a positive ξ_0 the semi-group operator $T(\xi_0)$ maps $\mathfrak{X}$ upon a subset of $\mathfrak{D}[A]$, then $T'(\xi) = AT(\xi)$ exists as a bounded operator for $\xi \geqq \xi_0$. Moreover, $T^{(n)}(\xi) = A^n T(\xi)$ exists as a bounded operator for $\xi \geqq n\xi_0$, $n = 1, 2, 3, \ldots$.*

Proof. $AT(\xi_0)$ is a linear closed operator which is defined everywhere in $\mathfrak{X}$, hence it is bounded. Since $AT(\xi) = AT(\xi_0)\,T(\xi - \xi_0)$ for $\xi > \xi_0$, it follows that $AT(\xi)$ is also bounded. Further, for $\xi \geqq n\xi_0$ we have

$$T^{(n)}(\xi) = A^n T(\xi) = \left[AT\left(\frac{\xi}{n}\right)\right]^n, \quad n = 1, 2, 3, \ldots \tag{2. 1}$$

so that the higher derivatives exist as asserted.

Corollary. *If $T(\xi)[\mathfrak{X}] \subset \mathfrak{D}[A]$ for each $\xi > 0$, then $T^{(n)}(\xi)$ exists as a bounded operator on $\mathfrak{X}$ to $\mathfrak{X}$ for each $\xi > 0$ and $n = 1, 2, 3, \ldots$*

Conversely, if $T'(\xi)$ exists as a bounded operator for $\xi = \xi_0$, then the limit in the first member of (1. 3) exists for all x when $\xi = \xi_0$, that is, $T(\xi_0)[\mathfrak{X}] \subset \mathfrak{D}[A]$ so that the condition of Theorem 1 is necessary as well as sufficient.

Theorem 2. *If $T(\xi)$ satisfies* (ii) *and if $T'(\xi)$ exists as a bounded operator for $\xi > \xi_0$, then $\|T'(\xi)\|$ is a monotone decreasing function of ξ in (ξ_0, ∞).*

For if $\delta > 0$ then $\|T'(\xi+\delta)\| = \|T'(\xi)\,T(\delta)\| \leqq \|T'(\xi)\|$. The same conclusion obviously holds for $\|T^{(n)}(\xi)\|$ in $(n\xi_0, \infty)$.

In particular, it follows that $\|T'(\xi)\|$ tends to a finite or infinite limit when ξ decreases to ξ_0. If $\xi_0 > 0$, it may very well happen that $\lim\limits_{\xi \to \xi_0} \|T'(\xi)\|$ is finite. In order to see this, we shall introduce a class of operators which will be used repeatedly in the following (cf. [1], Theorems 18. 2. 1 and 18. 4. 1). Let $\mathfrak{X} = L_2(-\infty, \infty)$, let $F(u)$ be the Fourier transform of $f(t) \in \mathfrak{X}$ and set

$$T(\xi)[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\xi\varrho(u) + itu} F(u)\, du, \tag{2. 2}$$

where $\varrho(u)$ is a continuous function whose real part is never negative and the integral exists in the sense of mean convergence. This defines a semi-group of linear operators in $\mathfrak{X}$ and $\|T(\xi)\|$ is the essential supremum of

$|e^{-\xi\varrho(u)}|$ and hence $\leqq 1$. Here we choose

$$\varrho(u) = |u| + ie^{\alpha|u|},\ \alpha > 0.$$

A simple calculation shows that $T'(\xi)$ exists as a bounded operator for $\xi \geqq \alpha$ but not for $\xi < \alpha$. The situation is different for $\xi_0 = 0$ since $T'(0)$ is bounded if and only if A is bounded in which case $T(\xi) = \exp(\xi A)$, that is, $T(\xi)$ is an entire function of ξ of exponential type.

We shall now consider the case in which $\xi_0 = 0$, that is, $T'(\xi)$ exists as a bounded operator for $\xi > 0$. The rate of growth of $\|T'(\xi)\| \equiv g(\xi)$ as $\xi \to 0$ is of fundamental importance for the following. We have observed above that $g(\xi)$ can stay bounded if and only if A is a bounded operator. Actually a sharper result holds.

Theorem 3. *A is bounded and $T(\xi) = \exp(\xi A)$ if*

$$\limsup_{\xi \to 0} \xi \|T'(\xi)\| < \frac{1}{e}. \tag{2.3}$$

Proof. Since $T(\xi)$ has derivatives of all orders, we may use Taylor's theorem obtaining

$$T(\xi) = \sum_{k=0}^{n-1} \frac{(\xi-\alpha)^k}{k!} A^k T(\alpha) + \frac{1}{(n-1)!} \int_\alpha^\xi (\xi-\eta)^{n-1} A^n T(\eta)\, d\eta. \tag{2.4}$$

Using (2.1) and (2.3) we see that the remainder tends to zero when $n \to \infty$ provided $0 < \alpha \leqq \xi < \alpha\left(1 + \frac{1}{\varrho}\right)$ where ϱ equals e times the left member of (2.3). It follows that the Taylor expansion converges for these values of ξ and represents $T(\xi)$. But the power series converges for $|\xi - \alpha| < \frac{\alpha}{\varrho}$ and in this circle it defines a semi-group operator which is the analytic continuation of $T(\xi)$. It follows that $T(\xi)$ is analytic in some neighborhood of the origin and this requires that $T(\xi)$ is an entire function so that $T(\xi) = \exp(\xi A)$ with a bounded operator A.

Corollary. *If $T(\xi)$ is a proper semi-group operator which has a bounded derivative for each positive ξ then*

$$\limsup_{\xi \to 0} \xi \|T'(\xi)\| \geqq \frac{1}{e}. \tag{2.5}$$

This is actually the best possible result of its kind, for if we take $\varrho(u) = |u|$ in (2.2), that is, if we form the Poisson-Abel transform of the Fourier integral, then $\xi\|T'(\xi)\| \equiv \frac{1}{e}$ for $\xi > 0$. On the other hand, there is no upper limit for the rate of growth of $\|T'(\xi)\|$ when $\xi \to 0$. In order to see this, we have merely to choose $\varrho(u) = |u| + ie^{\varphi(u)}$ where $\varphi(u) = o(u)$. The slower $u^{-1}\varphi(u) \to 0$ when $|u| \to \infty$, the faster grows the norm of $T'(\xi)$ when

$\xi \to 0$ and by a suitable choice of $\varphi(u)$ we can achieve that the norm grows faster than a preassigned function of ξ.

3. Conditions (i), (ii) and (iii) imply that $R(\lambda; A)$ exists for $\sigma > 0$ and satisfies (1. 2). Ordinarily no more may be asserted, but if $T(\xi)$ has derivatives, the resolvent set becomes more extensive.

Theorem 4. *Suppose that $T(\xi)$ is differentiable for $\xi > 0$ and set*

(3. 1) $$\| T'(\xi) \| = g(\xi) = \xi G(\xi).$$

Let $\delta(\tau)$ be the distance of the point $1 + \tau i$ from the spectrum of A. For large values of $|\tau|$ we have

(3. 2) $$\delta(\tau) > \frac{1}{3\eta(\tau)},$$

where $\eta(\tau)$ is the unique root of the equation

(3. 3) $$G(\eta) = |\tau|.$$

Proof. Since

$$R(\lambda; A) = \sum_{n=0}^{\infty} [R(\lambda_0; A)]^{n+1} (\lambda_0 - \lambda)^n$$

converges for $|\lambda - \lambda_0| \, \|R(\lambda_0; A)\| < 1$, it follows that

$$\delta(\tau_0) \, \|R(\lambda_0; A)\| \geqq 1, \quad \lambda_0 = 1 + i\tau_0.$$

Thus (3. 2) is implied by

(3. 4) $$\|R(1 + \tau i; A)\| < 3\eta(\tau).$$

But

$$R(\lambda; A) = \int_0^{\infty} e^{-\lambda\xi} T(\xi)\, d\xi = \int_0^{\eta} + \int_{\eta}^{\infty} = J_1 + J_2,$$

where $\eta = \eta(\tau)$ is chosen as indicated above. We note that $G(\xi)$ is strictly decreasing from $+\infty$ to 0 when ξ goes from 0 to $+\infty$, so the equation (3. 3) has a unique root and if $|\tau|$ is sufficiently large, $\eta(\tau)$ is less than one. Without restricting the generality, we may assume that (2. 5) holds so that $\eta(\tau)$ is at least $O(|\tau|^{-\frac{1}{2}})$ when $|\tau| \to \infty$. For J_1 we have the trivial estimate $\|J_1\| \leqq \eta$. An integration by parts gives

$$\|J_2\| \leqq \frac{1}{|\lambda|} [1 + g(\eta)].$$

In view of the choice of $\eta(\tau)$ we see that (3. 4) holds for large values of $|\tau|$ and consequently also the desired relation (3. 2).

The resulting estimate is not particularly accurate, mainly on account of the crude estimate used for J_1. The latter integral is of the same order of magnitude as

$$\int_0^{\eta} \left[T\left(\xi + \frac{\pi}{|\tau|}\right) - T(\xi) \right] d\xi.$$

This suggests that a study of the modulus of continuity of $T(\xi)$ in $L(0, \eta)$ might lead to further improvements of the estimate. We shall not pursue this possibility here, however. The same method as used above leads to

Theorem 5. *If* $T(\xi)$ *is differentiable for* $\xi > 0$ *and if* $\log g(\xi) \in L(0,1)$, *then*

$$\left\{ \int_{-\infty}^{-1} + \int_{1}^{\infty} \right\} \| R(1+\tau i; A) \| \frac{d\tau}{|\tau|} < \infty. \tag{3.5}$$

Proof. If $\log g(\xi) \in L(0, 1)$ so does $\log G(\xi)$. Without restricting the generality we may suppose that $G(\xi)$ is absolutely continuous, since otherwise we may replace $G(\xi)$ by an absolutely continuous dominant having the same integrability properties. In view of (3. 4) the integral in (3. 5) is dominated by a constant multiple of

$$\int_1^\infty \eta(\tau) \frac{d\tau}{\tau} = - \int_0^{\eta_0} \eta \frac{G'(\eta)}{G(\eta)} d\eta = \int_0^{\eta_0} \log G(\eta)\, d\eta.$$

Here we have used the fact that $\eta \log G(\eta) \to 0$ with η and that $G(\eta_0) = 1$. This completes the proof.

The condition $\log g(\xi) \in L(0, 1)$ is probably far too restrictive for the desired conclusion. In fact there are transformations of type (2. 2) for which (3. 5) holds and merely $\log\log g(\xi) \in L(0, 1)$. For this class of transformations the condition $\log\log g(\xi) \in L(0, 1)$ is the best possible of its kind which will ensure convergence of (3. 5). In order to prove an improved version of Theorem 5 with $\log g(\xi)$ replaced by $\log\log g(\xi)$ it would be sufficient to prove that

$$\|J_1\| \leqq \frac{\eta}{\log|\tau|}. \tag{3.6}$$

4. Theorem 4 suggests that if $\delta(\tau)$ grows sufficiently fast with $|\tau|$, then the semi-group operator $T(\xi)$ generated by A might be differentiable. We shall prove

Theorem 6. *If the operator* A *satisfies the conditions of section* 1, *if* $\|R(\lambda; A)\| \to 0$ *when* $\lambda \to \infty$ *in such a manner that the distance of* λ *from the spectrum of* A *becomes infinite, and if, for every fixed positive* K, *the inequality* $\delta(\tau) > K \log|\tau|$ *holds except in a set of intervals over which the total variation of* τ^2 *is finite, then the operator* $T(\xi)$ *generated by* A *has derivatives of all orders for* $\xi > 0$.

Proof. We know in advance that $T(\xi)$ exists and has the properties (i), (ii), and (iii). By Theorem 1 it is sufficient to prove the existence of $T'(\xi)$ for $\xi > 0$. For this purpose we consider the integral $\int e^{\lambda\xi} R(\lambda; A)\, d\lambda$ taken along a closed contour $PQRSP$ where PQ, QR, and RS are straight line segments and SP is an arc of the curve Γ: $\lambda = 1 - \frac{1}{2}\delta(\tau) + \tau i$, $Q = 1 - \omega i$

$R=1+\omega i$ while the imaginary part of P equals that of Q and the imaginary part of S equals that of R. We let $\omega\to\infty$; using the fact that $\|R(\lambda;A)\|$ tends uniformly to zero on and to the right of Γ, one sees that the integrals along the horizontal line segments PQ and RS tend to zero. That the integral along the arc PS of the curve Γ tends to a limit when $\omega\to\infty$ follows from the absolute convergence of the resulting integral which in its turn follows from the absolute convergence of the integral (4. 2) discussed below. It follows that the integral from Q to R along the vertical line tends to a limit in the uniform operator topology when $\omega\to\infty$. But for $x\in\mathfrak{D}[A]$ we have (see Theorem 11. 7. 1 of [1])

$$T(\xi)x=\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{1-\omega i}^{1+\omega i} e^{\lambda\xi}R(\lambda;A)x\,d\lambda,$$

whence it follows that

$$T(\xi)=\frac{1}{2\pi i}\int_{\Gamma} e^{\lambda\xi}R(\lambda;A)\,d\lambda. \tag{4. 1}$$

Formally the derivative of $T(\xi)$ is given by

$$T'(\xi)=\frac{1}{2\pi i}\int_{\Gamma} e^{\lambda\xi}\lambda R(\lambda;A)\,d\lambda, \tag{4. 2}$$

and all we have to do in order to prove the theorem is to show the absolute convergence of this integral for $\xi>0$. The norm of this integral is dominated by a constant multiple of

$$e^{\xi}\int_{-\infty}^{\infty}|\tau|\,e^{-\frac{1}{2}\xi\delta(\tau)}\,d\tau. \tag{4. 3}$$

The range of integration may now be split into two subsets E_1 and E_2. In E_1 we shall have $\delta(\tau)>(6/\xi)\log|\tau|$ and $|\tau|>1$ so that the integral over E_1 converges as $\int_1^\infty \tau^{-2}\,d\tau$, while the integral over E_2 is dominated by the total variation of τ^2 over E_2 which is finite by assumption. This completes the proof.

References.

[1] E. Hille, Functional Analysis and Semi-groups. *American Math. Society Colloquium Publications,* Vol. XXXI, (New York, 1948).

[2] K. Yosida, On the differentiability and the representation of one-parameter semi-group of linear operators, *Journal Math. Society of Japan,* 1 (1948), pp. 15—21.

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