An assertion which is equivalent to the generalized continuum hypothesis.

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Let E be a set of power $2^{\aleph_\alpha}$. Denote by B the set of all subsets of E consisting of two elements only.

We shall prove that *the generalized continuum hypothesis H is equivalent to the following assertion*:

A. There exists a (many-to-one) mapping T of B into E such that

1) *for any $r=\{x,y\}\in B$ we have either $Tr=x$ or $Tr=y$,*

2) *for any subset E_1 with power $>\aleph_\alpha$, the union of the sets $r\in B$ for which $Tr\in E_1$, is equal to E:*

$$\cup T^{-1}E_1=E.^{1)}$$

Proof. Let the elements of E be arranged in a transfinite sequence of ordinal type φ

$$x_0, x_1, x_2, \ldots, x_\omega, x_{\omega+1}, \ldots, x_\zeta, \ldots \qquad (\zeta<\varphi) \tag{1}$$

where φ is the initial ordinal of power $2^{\aleph_\alpha}$.

First we shall show that A is a consequence of H. Suppose that H is true. The desired mapping T^* which fulfils the assertion A is defined in the following way. Let, for every $r\in B$, T^*r be equal to the element of r which has a greater subscript in the sequence (1). Consider an arbitrary subset D of power $>\aleph_\alpha$ of E. The sum of the elements $r\in B$ for which $T^{*-1}x=r$ for arbitrary element $x=x_\gamma$ of D is equal to the set $\{x_\varkappa\}_{\varkappa\leqq\gamma}$: $\cup T^{*-1}x_\gamma=\{x_\gamma\}_{\varkappa\leqq\gamma}$. By the continuum hypothesis H, D is a subsequence of type φ of the sequence (1). It follows that the sum of the elements r of B for which $T^*r\in D$ is equal to E.

Next we prove that H is a consequence of A. Suppose T is a mapping which fulfil the conditions of A. Let x_η be a given element of E. We prove that the power of set of those $\{x_\eta, x_\gamma\}\in B$ for which $\gamma>\eta$ and $T\{x_\eta, x_\gamma\}=x_\eta$, is

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1) For any $F\subset E$, we denote by $\cup T^{-1}F$ the union of all those sets $r(\in B)$ for which $Tr\in F$.

$\leqq \aleph_\alpha$. In fact, admitting the contrary the power of the set V of the elements x_γ for which $\gamma > \eta$ and $T\{x_\eta, x_\gamma\} = x_\eta$ is greater than $\aleph_\alpha$. It follows then from the property 2) of the mapping T that

$$\cup T^{-1} V = E,$$

i. e., in particular, $\cup T^{-1} V \ni x_\eta$. Thus there exists at least one $r \in B$ such that $Tr \in V$, $Tr = x_\varkappa$ say ($\varkappa > \eta$), and $r \ni x_\eta$. By the property 1) of T we have necessarily $r = \{x_\eta, x_\varkappa\}$. But $x_\varkappa$ being an element of V, it follows by the definition of V that $Tr = x_\eta$, and this contradicts to $Tr = x_\varkappa$.

Consider now the section $C = \{x_\gamma\}_{\gamma < \omega_{\alpha+1}}$ of the sequence (1). It is evident that for any $\gamma < \omega_{\alpha+1}$ the power of the set of sets $\{x_\gamma, x_\beta\}$ ($\beta < \gamma$) for which $T\{x_\gamma, x_\beta\} = x_\gamma$ is $\leqq \aleph_\alpha$. According to that has been seen above the power of the set of sets $\{x_\gamma, x_\beta\}$. ($\beta > \gamma$) for which $T\{x_\gamma, x_\beta\} = x_\gamma$ is $\leqq \aleph$ too. As $\bar{C} = \aleph_{\alpha+1}$ it follows that

$$\aleph_{\alpha+1} \leqq \cup T^{-1} C \leqq \aleph_{\alpha+1} \cdot \aleph_\alpha.$$

As $\aleph_{\alpha+1} \cdot \aleph_\alpha = \aleph_{\alpha+1}$ we have therefore $\cup T^{-1} C = \aleph_{\alpha+1}$. But, by the condition 2) of T, we have

$$\cup T^{-1} C = E.$$

Consequently

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

(Received October 1, 1952.)