On automorphisms of the subalgebra lattice induced by automorphisms of the algebra

E. FRIED and G. GRÄTZER*

1. Introduction. We are going to prove the following result:

Theorem. *Let G be a group, L an algebraic lattice with more than one element, and let φ be a homomorphism of G into* $\operatorname{Aut} L$. *Then there exists an algebra $\mathfrak{A}$ such that there are isomorphisms $\alpha: G \to \operatorname{Aut} \mathfrak{A}$ and $\beta: L \to \operatorname{Sub} \mathfrak{A}$ satisfying* (see Figure) $\alpha\varphi_{\mathfrak{A}} = \varphi \operatorname{Aut} \beta$, *where* $\operatorname{Aut} \beta$ *is the isomorphism of* $\operatorname{Aut} L$ *and* $\operatorname{Aut} \operatorname{Sub} \mathfrak{A}$ *induced by β.*

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & \operatorname{Aut} L \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\operatorname{Aut}\beta} \\ \operatorname{Aut} \mathfrak{A} & \xrightarrow[\varphi_{\mathfrak{A}}]{} & \operatorname{Aut} \operatorname{Sub} \mathfrak{A} \end{array}$$

To put it simply, $\langle \operatorname{Aut} \mathfrak{A}, \operatorname{Sub} \mathfrak{A}, \varphi_{\mathfrak{A}} \rangle$ is characterized as $\langle G, L, \varphi \rangle$. The exception is that we have to assume that $|L| > 1$. Indeed, if $|L| = 1$, then A is the only subalgebra of $\mathfrak{A}$, that is, every element is an algebraic constant. In this case, $|G| = 1$. Thus $\langle \operatorname{Aut} \mathfrak{A}, \operatorname{Sub} \mathfrak{A}, \varphi_{\mathfrak{A}} \rangle$ is just as independent as $\langle \operatorname{Aut} \mathfrak{A}, \operatorname{Sub} \mathfrak{A} \rangle$ is.

Corollary. (E. T. SCHMIDT [7]) *Given a group G and an algebraic lattice L with more than one element, there exists an algebra $\mathfrak{A}$ satisfying $G \cong \operatorname{Aut} \mathfrak{A}$ and $L \cong \operatorname{Sub} \mathfrak{A}$.*

Proof. Let φ map all of G into the identity element of $\operatorname{Aut} L$. Then the algebra $\mathfrak{A}$ we obtain from the Theorem yields the Corollary.

This Corollary contains earlier results of G. BIRKHOFF [1] characterizing automorphis groups of algebras and of G. BIRKHOFF and O. FRINK [2] characterizing the subalgebra lattices of algebras.

It may be of some interest to note that in Schmidt's construction φ is indeed the constant map. If in our proof φ is the constant map, we obtain a somewhat simplified proof of Schmidt's result.

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2. The construction. Let G, L, and φ be given as in the Theorem. Let C be the set of all compact elements of L. Then C is a join-semilattice with zero, and the ideal lattice, $\mathrm{Id}\, C$, of C is isomorphic to L (see, for instance, [5]). It is also trivial that $\mathrm{Aut}\, C$ and $\mathrm{Aut}\, L$ are isomorphic, hence we can assume that φ is a homomorphism of G into $\mathrm{Aut}\, C$.

Set $A=(G\times(C-\{0\}))\cup\{0\}$. We define some operations on A ($\alpha, \beta\in G$, $a, b\in C-\{0\}$):

k is a constant operation with value 0;

$\vee$ is a binary operation defined by

$$0\vee 0=0,\quad 0\vee\langle\alpha, a\rangle=\langle\alpha, a\rangle\vee 0=\langle\alpha, a\rangle,\quad \langle\alpha, a\rangle\vee\langle\beta, b\rangle=\langle\alpha, a\vee b\rangle;$$

$f_{\alpha,a}$ is a unary operation: $f_{\alpha,a}(0)=0$ and

$$f_{\alpha,a}(\langle\beta, b\rangle)=\begin{cases}\langle\alpha\beta, a(\beta\varphi)\rangle & \text{if } a(\beta\varphi)\leqq b,\\ \langle\alpha\beta, b\rangle & \text{if } b\leqq a(\beta\varphi),\\ 0 & \text{otherwise.}\end{cases}$$

Observe that if $a\neq 0$, then $a(\beta\varphi)$ is the image of a under the automorphism $\beta\varphi$ of C, hence $a(\beta\varphi)\neq 0$. Thus $f_{\alpha,a}$ is an operation on A.

Let F consist of k, $\vee$, and all the $f_{\alpha,a}$, $\alpha\in G$, $a\in C-\{0\}$ and set $\mathfrak{A}=\langle A; F\rangle$.

3. Verification. Now we prove that $\mathfrak{A}$ satisfies the conditions of the Theorem.

Claim 1. Let $B\subseteqq A$. B is closed under all the operations in F iff $B=(G\times(I-\{0\}))\cup\{0\}$, where $I\in\mathrm{Id}\, C$.

Proof. Checking the definition of the operations, it is clear that, for $I\in\mathrm{Id}\, C$,

$$(G\times(I-\{0\}))\cup\{0\}$$

is closed under all the operations in F.

Now let $B\subseteqq A$ and let B be closed under all the operations in F. Since $k\in F$, we obtain $0\in B$. Define

$$I=\{a\mid a\in C \text{ and } \langle\alpha, a\rangle\in B \text{ for some } \alpha\in G\}\cup\{0\}.$$

If $B=\{0\}$, then $I=\{0\}$ is an ideal. Now let $B\neq\{0\}$. Obviously, if $a, b\in I$, then $a\vee b\in I$. Let $b\in I$ and $c\leqq b$; we wish to prove that $c\in I$. If $c=0$, then $0\in I$ by definition. If $c\neq 0$, then $b\neq 0$, hence we can choose a $\beta\in G$ such that $\langle\beta, b\rangle\in B$ by the definition of I. Thus, for any $\alpha\in G$,

$$f_{\alpha\beta^{-1}, c(\beta\varphi)^{-1}}(\langle\beta, b\rangle)=\langle\alpha, c\rangle,$$

since $c(\beta\varphi)^{-1}(\beta\varphi)=c\leqq b$. We conclude that $\langle\alpha, c\rangle\in B$, since $c\in I$. Therefore, $I\in\mathrm{Id}\, C$. Since we have $\langle\alpha, c\rangle\in B$ for all $\alpha\in G$, we also conclude that $B=(G\times(I-\{0\}))\cup\{0\}$, verifying the claim.

Claim 2. Sub $\mathfrak{A} \cong L$.

Proof. It is clear from Claim 1 that $I \to (G \times (I - \{0\})) \cup \{0\}$ is an isomorphism between $\operatorname{Id} C$ and $\operatorname{Sub} \mathfrak{A}$. Since $\operatorname{Id} C \cong L$, the claim follows.

Claim 3. For every $\gamma \in G$, the map T_γ: $\langle \beta, b \rangle \to \langle \beta\gamma, b(\gamma\varphi) \rangle$, $0 \to 0$ is an automorphism of $\mathfrak{A}$.

Proof. It is trivial that $0T_\gamma = 0$, $(x \vee y)T_\gamma = xT_\gamma \vee yT_\gamma$, for $x, y \in A$. Since right-multiplication of G and $\gamma\varphi$ on C are permutations, so is T_γ. It remains to prove that $f_{\alpha,a}(xT_\gamma) = f_{\alpha,a}(x)T_\gamma$. This is obvious for $x = 0$. Now let $x = \langle \beta, b \rangle$. If $a(\beta\varphi)$ and b are not comparable, then $(a(\beta\varphi))(\gamma\varphi)$ and $b(\gamma\varphi)$ are not comparable, that is, $a((\beta\gamma)\varphi)$ and $b(\gamma\varphi)$ are not comparable, hence

$$f_{\alpha,a}(\langle \beta, b \rangle)T_\gamma = 0T_\gamma = 0 = f_{\alpha,a}(\langle \beta\gamma, b(\gamma\varphi) \rangle) = f_{\alpha,a}(\langle \beta, b \rangle T_\gamma).$$

The other two cases $(a(\beta\varphi) \leqq b$ and $b \leqq a(\beta\varphi))$ are similar.

Claim 4. Every automorphism of $\mathfrak{A}$ is of the form T_γ for a unique $\gamma \in G$.

Proof. Let T be an automorphism of $\mathfrak{A}$. Define the functions f and g on $C - \{0\}$ by

$$\langle 1, c \rangle T = \langle f(c), g(c) \rangle,$$

where 1 is the identity of G. Then, for $c, d \in C - \{0\}$,

$$\langle f(c \vee d), g(c \vee d) \rangle = \langle 1, c \vee d \rangle T = (\langle 1, c \rangle \vee \langle 1, d \rangle) T =$$
$$= \langle 1, c \rangle T \vee \langle 1, d \rangle T = \langle f(c), g(c) \rangle \vee \langle f(d), g(d) \rangle = \langle f(c), g(c) \vee g(d) \rangle.$$

Thus, for any $c, d \in C - \{0\}$,

$$f(c) = f(c \vee d) = f(d),$$

that is, $f(c)$ is a constant function, $f(c) = f \in C - \{0\}$. Thus $\langle 1, c \rangle T = \langle f, g(c) \rangle$ and $g(c \vee d) = g(c) \vee g(d)$, implying that g is an automorphism of $C - \{0\}$. Set $c = a \vee g^{-1}(a(f\varphi))$. Since $a \leqq c$ the first clause of the definition of $f_{\alpha,a}$ applies so we have

$$\langle \alpha, a \rangle T = f_{\alpha,a}(\langle 1, c \rangle) T = f_{\alpha,a}(\langle 1, c \rangle T) = f_{\alpha,a}(\langle f, g(c) \rangle) = \langle \alpha f, a(f\varphi) \rangle,$$

where, in the last step, the first clause of the definition of $f_{\alpha,a}$ again applies since $a(f\varphi) \leqq g(c)$.

This proves that $T = T_f$ since they agree on $A - \{0\}$, and obviously agree at 0. The uniqueness of f is obvious.

Claim 5. $G \cong \operatorname{Aut} \mathfrak{A}$.

Proof. $f \to T_f$ is the required isomorphism by Claims 3 and 4.

We have verified all but the last statement of the Theorem. Let $\alpha: G \to \operatorname{Aut} \mathfrak{A}$ and $\beta: L \to \operatorname{Sub} \mathfrak{A}$ be defined as in Claim 5 and Claim 2. Let $\gamma \in G$. Then $\gamma\varphi$ is an automorphism of C. An ideal I of C is carried to $(G\times(I-\{0\}))\cup\{0\}$ by Aut β and thus $(\gamma\varphi)$Aut β is an automorphism of Sub $\mathfrak{A}$ mapping $(G\times(I-\{0\}))\cup\{0\}$ to $(G\times(I(\gamma\varphi)-\{0\}))\cup\{0\}$. Now $\gamma\alpha$ is an automorphism of $\mathfrak{A}$, namely, T_γ. Thus $(\gamma\alpha)\varphi_{\mathfrak{A}}$ is an automorphism of Sub $\mathfrak{A}$ carrying a subalgebra B to BT_γ, that is, $(G\times(I-\{0\}))\cup\{0\}$ to $((G\times(I-\{0\}))\cup\{0\})T_\gamma=(G\times(I(\gamma\varphi)-\{0\}))\cup\{0\}$ (this equality follows from the definition of T_γ). This completes the proof of the Theorem.

4. Concluding remarks. Let $\mathfrak{m}$ be an infinite regular cardinal. The finitary concepts ($\mathfrak{m}=\aleph_0$) of the Theorem generalize naturally (see G. GRÄTZER [3] and [4]) to the concepts: $\mathfrak{m}$-algebraic lattice and algebra of characteristic $\mathfrak{m}$. Subalgebra lattices of algebras of characteristic $\mathfrak{m}$ can be characterized, up to isomorphism, as $\mathfrak{m}$-algebraic lattices. The Theorem of this note generalizes to $\mathfrak{m}$-algebraic lattices and algebras of characteristic $\mathfrak{m}$. In the proof, it is only necessary to replace the binary operation $\vee$ by infinitary joins of less than $\mathfrak{m}$ elements.

It is a curious fact that the algebra $\mathfrak{A}$ constructed has no endomorphisms other than the automorphisms.

Similarly to the definition of $\varphi_{\mathfrak{A}}$, we can define $\psi_{\mathfrak{A}}\colon \mathrm{Aut}\,\mathfrak{A}\to\mathrm{Aut}\,\mathrm{Con}\,\mathfrak{A}$, where Con $\mathfrak{A}$ is the congruence lattice of $\mathfrak{A}$ and we can ask for a characterization of $\langle \mathrm{Aut}\,\mathfrak{A}, \mathrm{Con}\,\mathfrak{A}, \psi_{\mathfrak{A}}\rangle$. (For the most recent accounting of the characterization problems connected with Con $\mathfrak{A}$, see G. GRÄTZER and W. A. LAMPE [6].) Even harder is the characterization problem of

$$\langle \mathrm{Aut}\,\mathfrak{A}, \mathrm{Sub}\,\mathfrak{A}, \mathrm{Con}\,\mathfrak{A}, \varphi_{\mathfrak{A}}, \psi_{\mathfrak{A}}\rangle.$$

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UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
CANADA R3T 2N2