Acta Sci. Math., 50 (1986), 51—57

Bäcklund's theorem and transformation for surfaces V_2 in E_n

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1. Introduction and classical background. Bäcklund's classical transformation gives a way to generate new solutions of the Sine—Gordon equation $\partial^2\psi/\partial u^1\partial u^2 = \sin\psi$ from a given solution. Its geometrical setting uses some basic propositions for surfaces V_2 in E_3, which are the following.

A. If there is a diffeomorphism $V_2 \to V_2^*$, $x \mapsto x^*$, between two distinct surfaces in E_3 such that $\overrightarrow{xx^*} \in T_x V_2 \cap T_{x^*} V_2$, $|\overrightarrow{xx^*}| = r = \text{const}$ and the angle φ between $T_x V_2$ and $T_{x^*} V_2^*$ is a constant, then both V_2 and V_2^* have constant negative Gaussian curvature equal to $-(\sin^2\varphi)/r^2$ (the classical Bäcklund's theorem [1], [2]).

B. This diffeomorphism, called a pseudospherical line congruence, maps asymptotic curves of V_2 to asymptotic curves of V_2^* (i.e. it is a Weingarten congruence or W-congruence).

C. Asymptotic curves of a surface V_2 with Gaussian curvature $K = \text{const} < 0$ (i.e. of an immersion of a piece of the Bolyai—Lobachevsky plane $L_2(K)$ into E_3) form a Chebyshev net: in suitable net parameters u^1 and u^2 the metric of V_2 can be given by $ds^2 = (du^1)^2 + 2\cos\psi \cdot du^1\, du^2 + (du^2)^2$.

D. The net angle ψ of a Chebyshev net of a Riemannian V_2 satisfies the equation $\partial^2\psi/\partial u^1\partial u^2 = -K\sin\psi$, where K is the Gaussian curvature of V_2; in case if V_2 is a piece of $L_2(-1)$ this equation is the Sine—Gordon equation.

Due to C and D, every immersion of a piece of $L_2(-1)$ into E_3 gives a solution ψ of the Sine—Gordon equation and this correspondence is one to one up to rigid motion. Due to A and B, there is a transformation of such a solution to another, the analytical formulation of which gives Bäcklund's classical transformation [2].

The aim of this paper is to give some generalizations of propositions A, B and C to the case of surfaces V_2 in E_n, $n > 3$. Note that D needs no generalization because it does not depend on the immersion.

Received January 2, 1984 and in revised form August 17, 1984.

A generalization of the geometrical Bäcklund theorem and transformation in another direction, for the case of V_m in E_{2m-1}, is given in [4], [5]. If $m=2$ this reduces to the classical one.

2. Main results. The next generalization of Bäcklund's theorem gives some additions to the classical case too.

Theorem 1 [3]. *For two distinct surfaces V_2 and V_2^* in E_n, $n\geqq 3$, let $V_2\to V_2^*$, $x\mapsto x^*$ be a diffeomorphism such that $\overrightarrow{xx^*}\in T_xV_2\cap T_{x^*}V_2^*$ and $|\overrightarrow{xx^*}|=r\neq 0$ for every point $x\in V_2$. Let φ be the angle between T_xV_2 and $T_{x^*}V_2^*$, and let K and K^* be the Gaussian curvatures of V_2 and V_2^* in the corresponding points x and x^*. Then the following four conditions are equivalent:*

(1) $r=const$ *and* $\varphi=const$,

(2) $K=K^*=-(\sin^2\varphi)/r^2=const$,

(3) $K=K^*=-(\sin^2\varphi)/r^2$ *and* $r=const$,

(4) $K=K^*=-(\sin^2\varphi)/r^2$ *and* $\varphi=const$.

Under the assumptions of this theorem the diffeomorphism $V_2\to V_2^*$ is called the line pseudocongruence (if $n=3$ "pseudo" is to be dropped); V_2 and V_2^* are called its focal surfaces. They cannot be arbitrary surfaces, but necessarily must consist of planar points only. Tangent planes T_xV_2 and $T_{x^*}V_2^*$ in corresponding points x and x^* lie in an Euclidean 3-plane $(E_3)_x$. Among the second fundamental tensors of V_2 in normal directions to T_xV_2 we can distinguish the tensor h in the normal direction lying in $(E_3)_x$. A pair of null directions of the tensor h is called a pair of h-asymptotic directions in T_xV_2 and corresponding curves on V_2 are called h-asymptotic curves. The diffeomorphism $V_2\to V_2^*$ is called h-asymptotic if it maps h-asymptotic curves of V_2 to h-asymptotic curves of V_2^*.

Theorem 2. *Under the same assumptions as in Theorem* 1 *the next three conditions are equivalent to each other and also to each of the conditions* (1)—(4):

(5) $K=-(\sin^2\varphi)/r^2=const$ *and* $V_2\to V_2^*$ *is h-asymptotic,*

(6) $K=-(\sin^2\varphi)/r^2$, $r=const$ *and* $V_2\to V_2^*$ *is h-asymptotic,*

(7) $K=-(\sin^2\varphi)/r^2$, $\varphi=const$ *and* $V_2\to V_2^*$ *is h-asymptotic.*

Here the Gaussian curvature K of V_2 can be replaced of course by the Gaussian curvature K^* of V_2^*.

Theorem 3. *Under the same assumptions as in Theorem* 1 *let one of the conditions* (1)—(7) *be satisfied (and hence each of them). Let the field of distinguished normal directions (i.e. belonging in each $x\in V_2$ to $(E_3)_x$) be parallel along the curves tangent to directions of $\overrightarrow{xx^*}$ with respect to normal connection of V_2. Then the net of h-asymptotic curves on V_2 is a Chebyshev net.*

Theorem 3, due to proposition D, gives a possibility to find a solution of the Sine—Gordon equation by the special immersion of a piece of the Bolyai—Lobachevsky plane $L_2(-1)$ into E_n. Theorems 1 and 2 show how this solution can be then transformed.

The well-known Hilbert's theorem [6] states, that there is no solution $\psi: R^2 \to R$ of the Sine—Gordon equation, which is different from 0 and π in every point $(u^1, u^2) \in R^2$. It follows, that the class of surfaces V_2, satisfying the assumptions of Theorem 3, does not include the Bolyai—Lobachevsky plane $L_2(-1)$, globally immersed into E_n with regular h-asymptotic net. That gives a contribution to the theorems about classes of surfaces V_2 in E_n, which does not contain a V_2 isometric with $L_2(-1)$ (see [7]).

Here it is important that a surface V_2, satisfying the assumptions of Theorem 3, can be defined by following conditions, without turning to $V_2^* \subset E_n$: 1) V_2 consists of planar points only, 2) the field of normal curvature directions, corresponding to the lines of conjugated net family of V_2, is parallel along the lines of the same family with respect to normal connection, 3) invariants r and φ (which can be expressed in terms of V_2 only) are constants and $r^2 = \sin^2 \varphi$. In this paper we cannot give the complete explanation of the question about impossibility to realize $L_2(-1)$ by such a V_2. It needs a new publication.

3. Frame restriction. A local field of orthonormal frames will be choosen so that the origin is $x \in V_2$ and $e_1, e_2 \in T_x V_2$. In formulae $dx = e_I \theta^I$, $de_I = e_K \theta_I^K$; $I, K, \ldots = 1, \ldots, n$; $d\theta^I = \theta^K \wedge \theta_K^I$, $d\theta_K^I = \theta_K^L \wedge \theta_L^I$, $\theta_I^K + \theta_K^I = 0$ for the field of orthonormal frames in E_n we have then $\theta^3 = \ldots = \theta^n = 0$, and hence $\theta^1 \wedge \theta_1^\alpha + \theta^2 \wedge \theta_2^\alpha = 0$; $\alpha, \beta, \ldots = 3, \ldots, n$. By Cartan's lemma we may write $\theta_i^\alpha = b_{ij}^\alpha \theta^j$, $b_{ij}^\alpha = b_{ji}^\alpha$, $i, j, \ldots = 1, 2$. From the assumptions of Theorem 1 it follows that the tangent planes $T_x V_2$ and $T_{x^*} V_2^*$ lie in an Euclidean 3-plane $(E_3)_x$ because $T_x V_2 \cap T_{x^*} V_2^* \ni \overrightarrow{xx^*} \neq 0$. The frame can be choosen so that $e_3 \in (E_3)_x$ in each point $x \in V_2$ and $e_1 = (1/r)\overrightarrow{xx^*}$. Then the point $x^* \in V_2^*$, corresponding to $x \in V_2$, has the radius vector $x^* = x + re_1$ and from

$$dx^* = (\theta^1 + dr)e_1 + (\theta^2 + r\theta_1^2)e_2 + r(\theta_1^3 e_3 + \theta_1^\varrho e_\varrho), \tag{3.1}$$

$$\varrho, \sigma, \ldots = 4, \ldots, n,$$

it follows that by such a choice of the frame we have $\theta_1^\varrho = 0$. Thus $b_{11}^\varrho = b_{12}^\varrho = 0$.

The linear span of normal curvature vectors $b_{ij}^\alpha X^i X^j e_\alpha$ with arbitrary unit vector $X^i e_i \in T_x V_2$ is called the first normal space $N_x^1 V_2$. Now it has dimension two because it is spanned on $b_{11} = b_{11}^3 e_3$, $b_{12} = b_{12}^3 e_3$ and $b_{22} = b_{22}^\alpha e_\alpha$, the first two of which are collinear. We can finally restrict our choice of the frame by the condition that $e_4 \in N_x^1 V_2$ in each point $x \in V_2$. Then

$$b_{22}^5 = \ldots = b_{22}^n = 0;$$

and so we have

$$\theta_i^3 = h_{ij}\theta^j, \quad h_{21} = h_{12}, \quad \theta_1^4 = 0, \quad \theta_2^4 = k_{22}\theta^2, \quad \theta_i^5 = \ldots = \theta_i^n = 0,$$

where the notations $h_{ij}=b_{ij}^3$ and $k_{22}=b_{22}^4$ are used.

4. Gaussian curvatures. The above restriction can be done for the surface V_2^* choosing the frame vectors e_I^* at the point $x^*\in V_2^*$ in a similar way. Then

$$e_1^* = e_1,$$
$$e_2^* = e_2\cos\varphi + e_3\sin\varphi,$$
$$e_3^* = -e_2\sin\varphi + e_3\cos\varphi,$$
$$e_4^* = e_4, \ldots, e_n^* = e_n,$$

and

$$dx^* = e_1^*\theta^{*1} + e_2^*\theta^{*2} = e_1\theta^{*1} + (e_2\cos\varphi + e_3\sin\varphi)\theta^{*2}.$$

Comparing with (3.1) we have

$$\theta^{*1} = \theta^1 + dr, \quad \theta^{*2}\cos\varphi = \theta^2 + r\theta_1^2, \quad \theta^{*2}\sin\varphi = r\theta_1^3.$$

Here $\sin\varphi$ cannot be 0 because this would lead to $\theta_1^3=0$ and V_2 would be a torse with line generators xx^* and we had $V_2^*=V_2$ what is excluded by the assumptions of Theorem 1. Therefore

$$(1/r)\theta^2 + \theta_1^2 = \cot\varphi\cdot\theta_1^3.$$

From this, by exterior differentiation and using well-known formulae,

$$d\theta_1^2 = -K\theta^1\wedge\theta^2, \quad \theta_1^3\wedge\theta_2^3 = K\theta^1\wedge\theta^2 \tag{4.1}$$

we have (see [3])

$$K = -((\sin^2\varphi)/r^2)(1+r_1) + (h_{12}\varphi_1 - h_{11}\varphi_2), \tag{4.2}$$

where $dr = r_1\theta^1 + r_2\theta^2$, $d\varphi = \varphi_1\theta^1 + \varphi_2\theta^2$.

For the surface V_2^*,

$$\theta_1^{*3} = de_1^*\cdot e_3^* = ((\sin\varphi)/r)\theta^2, \quad \theta_2^{*3} = de_2^*\cdot e_3^* = \theta_2^3 + d\varphi,$$

and now the second formula in (4.1) for V_2^* gives ([3])

$$K^* = -\frac{\sin^2\varphi}{r^2}\,\frac{h_{12}+\varphi_1}{h_{12}(1+r_1) - h_{11}r_2}. \tag{4.3}$$

These formulae (4.2) and (4.3) for the Gaussian curvatures K and K^* will be used in the proof of Theorems 1 and 2, but they also have their own significance.

5. Proof of Theorem 1. If $r=\text{const}$ and $\varphi=\text{const}$, then from (4.2) and (4.3) we obtain (2), (3) and (4) immediately. Conversely, let

$$K = K^* = -(\sin^2\varphi)/r^2.$$

Then the same formulae (4.2) and (4.3) give correspondingly

$$\begin{aligned} h_{12}\varphi_1 - h_{11}\varphi_2 &= ((\sin^2\varphi)/r^2)r_1, \\ h_{12}r_1 - h_{11}r_2 &= \varphi_1. \end{aligned} \tag{5.1}$$

In case of (2) we have $K=\text{const}$ and from $Kr^2+\sin^2\varphi=0$ it follows that $dr=r\cot\varphi\cdot d\varphi$ and the last two equations give $\sin^2\varphi\cdot\varphi_1=0$. Therefore $\varphi_1=r_1=0$ and $h_{12}\varphi_2=h_{11}r_2=0$. Here $h_{11}=0$ would lead to $\varphi=0$ what is excluded, and we have (1).

In case of (2) or (3), when $r=\text{const}$ or $\varphi=\text{const}$, the same equations give (1). Theorem 1 is proved.

6. Proof of Theorem 2. The h-asymptotic curves of V_2 are the null curves of the second fundamental form in the direction e_3. This form is

$$\text{II}^3 = \theta^1\theta^3_1+\theta^2\theta^3_2 = h_{ij}\theta^i\theta^j.$$

For V_2^* it is

$$\text{II}^{*3} = \theta^{*1}\theta^{*3}_1+\theta^{*2}\theta^{*3}_2 = (\theta^1+dr)\frac{\sin\varphi}{r}\theta^2+\frac{r}{\sin\varphi}\theta^3_1(\theta^3_2+d\varphi).$$

Using here that $h_{11}h_{22}-h_{12}^2=K$ and (4.2), we have

$$\text{II}^{*3} = \frac{rh_{12}}{\sin\varphi}\text{II}^3+\Phi,$$

where

$$\Phi = \frac{r\varphi_1}{\sin\varphi}[h_{11}(\theta^1)^2+2h_{12}\theta^1\theta^2]+\left(\frac{\sin\varphi}{r}r_2+\frac{r}{\sin\varphi}h_{12}\varphi_2\right)(\theta^2)^2.$$

If $r=\text{const}$ and $\varphi=\text{const}$, then $\Phi=0$, and we have (5). Conversely, let $V_2\to V_2^*$ be h-asymptotic. Then Φ must be proportional to II^3, and therefore

$$h_{22}\varphi_1 = \frac{\sin^2\varphi}{r^2}r_2+h_{12}\varphi_2. \tag{6.1}$$

In case of (5) we have $r_i=r\cot\varphi\cdot\varphi_i$. Now (5.1) and (6.1) give

$$\left(h_{12}-\frac{\sin 2\varphi}{2r}\right)\varphi_1-h_{11}\varphi_2 = 0,$$

$$h_{22}\varphi_1-\left(h_{12}+\frac{\sin 2\varphi}{2r}\right)\varphi_2 = 0.$$

Here the determinant is :

$$K+\frac{\sin^2 2\varphi}{4r^2} = -\frac{\sin^4\varphi}{r^2} \neq 0$$

and hence (1) holds.

In case of (6), from (5.1) and (6.1) it follows that

$$h_{12}\varphi_1 - h_{11}\varphi_2 = 0,$$
$$h_{22}\varphi_1 - h_{12}\varphi_2 = 0,$$

where the determinant is $-h_{12}^2 + h_{11}h_{22} = K \neq 0$. In case of (7) it follows analogously that (1) holds. Theorem 2 is proved.

7. Proof of Theorem 3. If the field of directions e_3 is parallel along the integral curves of the equation $\theta^2=0$ with respect to normal connection, then

$$\theta_3^4 = \lambda\theta^2. \tag{7.1}$$

Taking the unit vectors

$$\hat{e}_1 = e_1\cos\alpha + e_2\sin\alpha, \quad \hat{e}_2 = -e_1\sin\alpha + e_2\cos\alpha$$

in h-principal directions, bisecting h-asymptotic directions we have $\hat{h}_{12}=0$ and besides this

$$\begin{aligned} \hat{\theta}_1^4 &= \theta_2^4\sin\alpha = k_{22}\sin\alpha\cdot\theta^2, \\ \hat{\theta}_2^4 &= \theta_2^4\cos\alpha = k_{22}\cos\alpha\cdot\theta^2, \\ \hat{\theta}_i^\varrho &= 0. \end{aligned} \tag{7.2}$$

The local parameters v^1 and v^2 on V_2 can be choosen so that

$$\hat{\theta}^1 = a_1dv^1, \quad \hat{\theta}^2 = a_2dv^2, \quad \hat{\theta}_1^3 = b_1a_1dv^1, \quad \hat{\theta}_2^3 = b_2a_2dv^2,$$

where $b_1=\hat{h}_{11}$, $b_2=\hat{h}_{22}$. Then from the formulae

$$d\hat{\theta}_1^3 = \hat{\theta}_1^2\wedge\hat{\theta}_2^3, \quad d\hat{\theta}_2^3 = -\hat{\theta}_1^2\wedge\hat{\theta}_1^3,$$

which hold due to (7.1) and (7.2), using the well-known expression

$$\hat{\theta}_1^2 = \frac{1}{a_1}\frac{\partial a_2}{\partial v^1}dv^2 - \frac{1}{a_2}\frac{\partial a_1}{\partial v^2}dv^1,$$

we have

$$\frac{1}{b_i-b_j}\frac{\partial b_i}{\partial v^j} = -\frac{\partial(\ln a_i)}{\partial v^j}, \quad i \neq j.$$

The same computation as in [2] leads us to parameters w^1, w^2, in which for V_2

$$ds^2 = \cos^2\chi(dw^1)^2 + \sin^2\chi(dw^2)^2,$$
$$\mathrm{II}^3 = \sin\chi\cos\chi[(dw^1)^2-(dw^2)^2]$$

and now by $u^1 = w^1 + w^2$, $u^2 = w^1 - w^2$ we get

$$ds^2 = (du^1)^2 + 2\cos\psi\, du^1\, du^2 + (du^2)^2,$$

$$\mathrm{II}^3 = 2\sin\psi\, du^1\, du^2.$$

The h-asymptotic net is a Chebyshev net. Theorem 3 is proved.

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