Acta Sci. Math, 50 (1986), 183—189

Decompositions of completely bounded maps

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1. Introduction. Let $\varphi: A \to B$ be a bounded linear map between C^*-algebras, and let $\varphi \otimes \mathrm{id}_n: A \otimes M_n \to B \otimes M_n$ be the associated maps $(n=1, 2, \ldots)$, where M_n is the full matrix algebra of order n. The map φ is said to be completely positive if each $\varphi \otimes \mathrm{id}_n$ is positive, and completely bounded if $\|\varphi\|_{cb} \equiv \sup_n \|\varphi \otimes \mathrm{id}_n\| < \infty$; in this case $\|\varphi\|_{cb}$ is called the completely bounded norm of φ. It is known that a completely positive map φ is completely bounded and $\|\varphi\|_{cb} = \|\varphi\|$. A linear map $\varphi: A \to B$ between C^*-algebras is said to have a positive (resp. completely positive) decomposition if φ can be written as a linear combination of positive (resp. completely positive) linear maps.

A C^*-algebra A is injective if and only if for any C^*-algebra B such that $B \supseteq A$, there exists a projection of B onto A of norm one [2; Theorem 5.3]. WITTSTOCK [15] proved that every completely bounded map of a unital C^*-algebra into an injective C^*-algebra has a completely positive decomposition (see, also, [8]). In [3] we proved, as a limited converse of Wittstock's theorem, that given a separable C^*-algebra B, every bounded linear map of any C^*-algebra into B has a positive decomposition if and only if B is finite-dimensional, namely, injective. In this paper, we show that given a separable unital C^*-algebra B, every completely bounded map of any unital C^*-algebra into B has a completely positive decomposition if and only if B is finite-dimensional, namely, injective. We also prove that if A and B are separable, infinite-dimensional, unital C^*-algebras and A contains a self-adjoint element such that the set of limit points of its spectrum is infinite, then the span of positive linear maps of A into B is nowhere dense in the Banach space of all bounded linear maps of A into B.

Throughout this paper, it is assumed that all C^*-algebras are unital. If S is a compact Hausdorff space, we denote by $C(S)$ the C^*-algebra of continuous functions on S. We mean by αN the one-point compactification of the set N of positive integers with the point ∞ at infinity.

Received May 20, 1983.

We remark that some of the results of this paper were announced in [12].

The author would like to thank Professor J. Tomiyama for his interest and encouragement.

2. Bounded linear maps between commutative C^*-algebras. For each n in N let $X_n=\{x_{n,1}, \cdots, x_{n,m}, \cdots, x_{n,\infty}\}$ and $Y_n=\{y_{n,1}, \cdots, y_{n,m}, \ldots\}$. Denote by X the one-point compactification of the topological sum of the sequence $\{X_n\}_{n=1}^{\infty}$ of copies αN; denote by Y the one-point compactification of the sequence $\{Y_n\}_{n=1}^{\infty}$ of copies N with the point y_∞ at infinity. The space Y is homeomorphic to αN.

From now on, we use X, Y, X_n, $x_{n,m}$, $x_{n,\infty}$, $y_{n,m}$ and y_∞ in the above situation. We consider linear maps of $C(X)$ into $C(Y)$. Proofs of Lemmas 2 and 3 are based on an idea due to KAPLAN [4] and TSUI [13].

We first recall TSUI's example [13; 1.3.4, Example II].

Lemma 1. *Let* $\Phi: C(X)\to C(Y)$ *be the self-adjoint linear map defined by* $\Phi(f)(y_{n,m})=f(x_{n,m})-f(x_{n,m+1})$ *and* $\Phi(f)(y_\infty)=0$. *Then* Φ *has no positive decomposition.*

Lemma 2. *For any positive integer* k, *there exists a self-adjoint linear map* $\Psi_k: C(X)\to C(Y)$ *with* $\|\Psi_k\|=1$ *satisfying the following properties.*

(1) Ψ_k *has a positive decomposition.*

(2) *If* Ψ_k *is decomposed as the difference of two positive linear maps* Ψ^+, Ψ^-, *then* $\|\Psi^+\|\geqq k/2$ *and* $\|\Psi^-\|\geqq k/2$.

Proof. We define the map $\Psi_k: C(X)\to C(Y)$ by

$$\Psi_k(f)(y_{n,m}) = (1/2)\big(f(x_{n,m})-f(x_{n,m+1})\big) \quad \text{if} \quad n \leqq k,$$

$$\Psi_k(f)(y_{n,m}) = 0 \quad \text{if} \quad n > k,$$

$$\Psi_k(f)(y_\infty) = 0.$$

It is easy to check that $\Psi_k(f)$ is continuous on Y and $\|\Psi_k\|=1$.

(1) We define k positive linear maps $\psi_i: C(X)\to C(Y)$ by

$$\psi_i(f)(y_{n,m}) = 0 \quad \text{if} \quad n+m \leqq i,$$

$$\psi_i(f)(y_{n,m}) = (1/2)f(x_{i,n+m-i}) \quad \text{if} \quad n+m > i,$$

$$\psi_i(f)(y_\infty) = (1/2)f(x_{i,\infty}).$$

Since $\psi_i(f)(y_{i,m})=(1/2)f(x_{i,m})$, we have $\psi_1+\ldots+\psi_k\geqq\Psi_k$, and hence Ψ_k has a positive decomposition.

(2) For a linear map $\psi: C(X)\to C(Y)$, let $\psi_{(n,m)}$ denote the linear functional on $C(X)$ defined by $\psi_{(n,m)}(f)=\psi(f)(y_{n,m})$. If $n\leqq k$, then $(1/2)\delta(x_{n,m})\leqq\Psi^+_{(n,m)}$ and

$(1/2)\delta(x_{n,m+1}) \leqq \Psi^{-}_{(n,m)}$, where $\delta(x)$ denotes the point measure at x. Let e_n be the characteristic function of the subset X_n of X. Then

$$1/2 = (1/2)\delta(x_{n,m})(e_n) \leqq \Psi^{+}_{(n,m)}(e_n) = \Psi^{+}(e_n)(y_{n,m}),$$

$$1/2 = (1/2)\delta(x_{n,m+1})(e_n) \leqq \Psi^{-}_{(n,m)}(e_n) = \Psi^{-}(e_n)(y_{n,m}).$$

If $j=+,-$, we have

$$1/2 \leqq \lim_m \Psi^j(e_n)(y_{n,m}) = \Psi^j(e_n)(y_\infty).$$

Therefore,

$$\Psi^j(1)(y_\infty) \geqq \sum_{n=1}^{k} \Psi^j(e_n)(y_\infty) \geqq k/2,$$

so that $\|\Psi^+\| \geqq k/2$ and $\|\Psi^-\| \geqq k/2$.

Lemma 3. *With Φ as in Lemma* 1, *if φ is a bounded linear map of $C(X)$ into $C(Y)$ satisfying $\|\varphi - \Phi\| < 1/2$, then φ has no positive decomposition.*

Proof. Suppose that φ has a positive decomposition. Then the self-adjoint part τ of φ has a positive decomposition $\tau = \tau^+ - \tau^-$. For a linear map $\psi\colon C(X) \to C(Y)$, as in Lemma 2, we define the linear functional $\psi_{(n,m)}$ on $C(X)$ by $\psi_{(n,m)}(f) = \psi(f)(y_{n,m})$. Since X is countable, $\tau^j_{(n,m)}$ can be written as

$$\tau^j_{(n,m)} = \sum_{x\in X} \beta^j(x)\delta(x), \quad 0 \leqq \beta^j(x) \in \mathbf{R}, \quad j = +, -,$$

where $\delta(x)$ denotes the point measure at x. We then have

$$1/2 > \|\varphi - \Phi\| \geqq \|\tau - \Phi\| \geqq \|\tau_{(n,m)} - \Phi_{(n,m)}\| =$$

$$= \Big\|\sum_{x\in X} \beta^+(x)\delta(x) - \sum_{x\in X} \beta^-(x)\delta(x) - \delta(x_{n,m}) + \delta(x_{n,m+1})\Big\| \geqq$$

$$\geqq |\beta^+(x_{n,m}) - \beta^-(x_{n,m}) - 1|.$$

Hence $(1/2)\delta(x_{n,m}) \leqq \tau^+_{(n,m)}$. Let e_n be the characteristic function of the subset X_n of X. Then

$$1/2 = (1/2)\delta(x_{n,m})(e_n) \leqq \tau^+_{(n,m)}(e_n) = \tau^+(e_n)(y_{n,m}),$$

so that $1/2 \leqq \lim_m \tau^+(e_n)(y_{n,m}) = \tau^+(e_n)(y_\infty)$. Therefore

$$\tau^+(1)(y_\infty) \geqq \tau^+\Big(\sum_{n=1}^{k} e_n\Big)(y_\infty) \geqq k/2$$

for any positive integer k. This implies the unboundedness of τ^+.

3. The main results. We recall that X and Y are the one-point compactifications of the topological sums of sequences of copies αN and N, respectively. In order to extend maps obtained in Section 2 to non-commutative C^*-algebras, we construct completely positive maps with range algebras $C(X)$, $C(Y)$.

Lemma 4. *If a separable C^*-algebra A contains a self-adjoint element a such that the set of limit points of the spectrum of a is infinite, then there exist unital completely positive maps $\pi_A\colon A\to C(X)$ and $\nu_A\colon C(X)\to A$ such that $\pi_A\circ\nu_A$ is the identity map on $C(X)$.*

Proof. Let S denote the spectrum of a. Since S is a compact subset of real numbers, choose a point s_∞ and a sequence $\{s_n\}_{n=1}^\infty$ of limit points of S such that $3|s_\infty-s_{n+1}|<|s_\infty-s_n|$ for all n. For each n take a sequence $\{s_{n,i}\}_{i=1}^\infty$ of distinct points of S such that $3|s_n-s_{n,i}|<|s_\infty-s_n|$ for all i and $\lim_i s_{n,i}=s_n$. Put $S_n=\{s_{n,1},\dots,s_{n,m},\dots,s_n\}$ and $\tilde{S}=\{s_\infty\}\cup(\bigcup_{n=1}^\infty S_n)$. If $s\in\tilde{S}$, we choose a state g_s on A such that $g_s(f)=f(s)$ for all f in $C(S)$ because $C(S)$ is the C^*-subalgebra generated by a and 1. We then define the positive linear map π of A into the C^*-algebra of all bounded functions on $\tilde{S}$ by $\pi(b)(s)=g_s(b)$ for s in $\tilde{S}$ and b in A. Since A is separable, so is the C^*-subalgebra $C^*(\pi(A))$ generated by $\pi(A)$. There exists a compact metric space T with metric d such that $C(T)=C^*(\pi(A))$. Then $\tilde{S}$ is canonically regarded as a subset of T. For each n let t_n be a limit point of the subset S_n of T and choose a subsequence $\{\tilde{s}_{n,i}\}_{i=1}^\infty$ of $\{s_{n,i}\}_{i=1}^\infty$ such that $\lim_i \tilde{s}_{n,i}=t_n$. If $f\in C(S)$, then

$$\pi(f)(t_n)=\lim_i \pi(f)(\tilde{s}_{n,i})=\lim_i f(\tilde{s}_{n,i})=f(s_n).$$

Hence $t_n\neq t_m$ if $n\neq m$.

We again choose a point t_∞ and a subsequence $\{t_{h(n)}\}_{n=1}^\infty$ of $\{t_n\}_{n=1}^\infty$ such that $3d(t_\infty, t_{h(n+1)})<d(t_\infty, t_{h(n)})$. For each n take a subsequence $\{t_{h(n),i}\}_{i=1}^\infty$ of $\{\tilde{s}_{h(n),i}\}_{i=1}^\infty$ such that $3d(t_{h(n)}, t_{h(n),i})<d(t_\infty, t_{h(n)})$. Put $T_n=\{t_{h(n),1},\dots,t_{h(n),m},\dots,t_{h(n)}\}$ and $\tilde{X}=\{t_\infty\}\cup(\bigcup_{n=1}^\infty T_n)$. By its construction, $\tilde{X}$ is canonically regarded as the space X.

Let $\varphi\colon X=\tilde{X}\to\tilde{S}\subseteqq S$ be defined by

$$\varphi(t_{h(n),i})=t_{h(n),i},\quad \varphi(t_{h(n)})=s_{h(n)},\quad \varphi(t_\infty)=s_\infty.$$

For f in $C(S)$, $\pi(f)(t_{h(n)})=f(s_{h(n)})$ and $\pi(f)(t_\infty)=f(s_\infty)$. The map φ is one-to-one and continuous. Then there exists, by [1; Theorem 3.11], a unital positive linear map $\nu_A\colon C(X)\to C(S)\subseteqq A$ such that $\nu_A(f)\circ\varphi=f$ for all f in $C(X)$.

Define the unital positive linear map $\pi_A\colon A\to C(X)$ by $\pi_A(b)=\pi(b)|_X$, the restriction to $X=\tilde{X}$ of $\pi(b)$. Then $\pi_A(f)=f\circ\varphi$ for all f in $C(S)$. Hence $\pi_A\circ\nu_A$

is the identity map on $C(X)$. Both π_A and ν_A are completely positive [10; Chapter IV, Corollary 3.5].

Lemma 5. *If B is a separable, infinite-dimensional C^*-algebra, then there exist unital completely positive maps $\pi_B\colon B\to C(Y)$ and $\nu_B\colon C(Y)\to B$ such that $\pi_B\circ\nu_B$ is the identity map on $C(Y)$.*

Proof. There exists a self-adjoint element a in B with infinite spectrum S [7]. Denote by $C^*(a, 1)$ the C^*-subalgebra generated by a and 1. Then $C(S)=C^*(a, 1)$. Since S is a compact metrizable space, we choose a point s_∞ and a sequence $\{s_n\}_{n=1}^\infty$ of distinct points in S with $\lim_n s_n=s_\infty$ and $\{a_n\}_{n=1}^\infty$ of $C(S)$ such that $a_n(s_n)=1$, $0\leqq a_n\leqq 1$ and $a_p a_q=0$ for $p\neq q$.

Put $\tilde{S}=\{s_1, \dots, s_n, \dots, s_\infty\}$. If $s\in\tilde{S}$, we take a state g_s on B such that $g_s(f)=f(s)$ for all f in $C(S)$. We define the unital positive linear map π of B into the C^*-algebra of all bounded functions on $\tilde{S}$ by $\pi(b)(s)=g_s(b)$. Since B is separable, so is the C^*-subalgebra $C^*(\pi(B))$ generated by $\pi(B)$. There then exists a compact metrizable space T such that $C(T)=C^*(\pi(B))$. Then $\tilde{S}$ is canonically regarded as a subset of T.

We choose a point $s_{h(\infty)}$ in T and a subsequence $\{s_{h(n)}\}_{n=1}^\infty$ of $\{s_n\}_{n=1}^\infty$ such that $\lim_n s_{h(n)}=s_{h(\infty)}$. Put $\tilde{Y}=\{s_{h(1)}, \dots, s_{h(n)}, \dots, s_{h(\infty)}\}$. Then $\tilde{Y}$ is canonically regarded as the space Y because $\tilde{Y}$ is homeomorphic to αN.

We define the unital positive linear map $\pi_B\colon B\to C(Y)$ by $\pi_B(b)=\pi(b)|_Y$, the restriction to $Y=\tilde{Y}$ of $\pi(b)$. We also define the unital positive linear map $\nu_B\colon C(Y)\to C(S)\subseteq B$ by

$$\nu_B(b) = \sum_{n=1}^{\infty} [b(s_{h(n)})-b(s_{h(\infty)})]a_{h(n)}+b(s_{h(\infty)})1.$$

Then $\pi_B\circ\nu_B$ is the identity map on $C(Y)$ and both π_B and ν_B are completely positive [10; Chapter IV, Corollary 3.5].

Theorem 6. *Let A and B be separable, infinite-dimensional C^*-algebras. Assume that A contains a self-adjoint element a such that the set of limit points of the spectrum of a is infinite. Then*

(1) *There exists a completely bounded map $\tilde{\Phi}\colon A\to B$ such that each bounded linear map $\varphi\colon A\to B$ with $\|\varphi-\tilde{\Phi}\|<1/2$ has no positive decomposition.*

(2) *There exists a self-adjoint linear map $\tilde{\Psi}\colon A\to B$ having a completely positive decomposition such that for any positive decomposition $\tilde{\Psi}=\tilde{\Psi}^+-\tilde{\Psi}^-$ we have $\|\tilde{\Psi}^+\|>\|\tilde{\Psi}\|_{cb}$ and $\|\tilde{\Psi}^-\|>\|\tilde{\Psi}\|_{cb}$.*

Proof. We use maps Φ, Ψ_4, π_A, ν_A, π_B and ν_B constructed in Lemmas 1, 2, 4 and 5.

(1) We put $\tilde{\Phi}=\nu_B\circ\Phi\circ\pi_A$. By [6; Lemma 1], $\tilde{\Phi}$ is completely bounded. Then

$$\|\pi_B\circ\varphi\circ\nu_A-\Phi\| = \|\pi_B\circ\varphi\circ\nu_A-\pi_B\circ\tilde{\Phi}\circ\nu_A\| \leqq \|\varphi-\tilde{\Phi}\| < 1/2.$$

By Lemma 3, $\pi_B\circ\varphi\circ\nu_A$ has no positive decomposition. If φ has a positive decomposition, so does $\pi_B\circ\varphi\circ\nu_A$. This is a contradiction.

(2) We put $\tilde{\Psi}=\nu_B\circ\Psi_4\circ\pi_A$. By Lemma 2 and [10; Chapter IV, Corollary 3.5], $\tilde{\Psi}$ is a self-adjoint linear map of A into B having a completely positive decomposition and

$$1 = \|\Psi_4\| = \|\pi_B\circ\tilde{\Psi}\circ\nu_A\| \leqq \|\tilde{\Psi}\|_{cb} = \|\nu_B\circ\Psi_4\circ\pi_A\|_{cb} \leqq \|\Psi_4\|_{cb}=\|\Psi_4\|,$$

where the last equality follows from [6; Lemma 1]. Hence $\|\tilde{\Psi}\|_{cb}=1$. If $\tilde{\Psi}$ has a positive decomposition $\tilde{\Psi}=\tilde{\Psi}^+-\tilde{\Psi}^-$, then Ψ_4 has a positive decomposition $\Psi_4= =\pi_B\circ\tilde{\Psi}^+\circ\nu_A-\pi_B\circ\tilde{\Psi}^-\circ\nu_A$. By Lemma 2 we have

$$\|\tilde{\Psi}^+\| \geqq \|\pi_B\circ\tilde{\Psi}^+\circ\nu_A\| \geqq 4/2 > \|\tilde{\Psi}\|_{cb},$$

and similarly,

$$\|\tilde{\Psi}^-\| \geqq 2 > \|\tilde{\Psi}\|_{cb}.$$

Remark 7. Let A_1 and B_1 be C^*-algebras. Suppose that there exist unital completely positive maps $\pi_1: A_1\to C(X)$, $\nu_1: C(X)\to A_1$, $\pi_2: B_1\to C(Y)$, $\nu_2: C(Y)\to B_1$ such that $\pi_1\circ\nu_1$ and $\pi_2\circ\nu_2$ are the identity maps on $C(X)$ and $C(Y)$, respectively. If we replace A and B by A_1 and B_1, Theorem 6 remains true from the same argument in its proof (cf. [9; Theorem 2.6]).

We recall that the set of self-adjoint elements of an injective C^*-algebra is conditionally complete [11; Theorem 7.1]. Hence a separable C^*-algebra A is injective if and only if A is finite-dimensional.

Corollary 8. *Let B be a separable C^*-algebra. The following statements are equivalent;*

(1) *B is injective;*

(2) *Every completely bounded map of any C^*-algebra into B has a completely positive decomposition;*

(3) *Every linear map φ having a completely positive decomposition of any C^*-algebra into B has a completely positive decomposition such that $\varphi=\varphi_1-\varphi_2+i(\varphi_3-\varphi_4)$ with $\|\varphi_i\|\leqq\|\varphi\|_{cb}$ $(i=1, \cdots, 4)$.*

Proof. By [15; Satz 4.5] we have (1)⇒(2) and (1)⇒(3). Combining the above remark about injective, separable C^*-algebras with Theorem 6, we see that (2)⇒(1) and (3)⇒(1).

In the category of partially ordered Banach spaces, Wickstead [14, Theorem 3.15] obtained a result similar to Corollary 8.

Addition. After this paper was written, the author discovered an example of a non-injective, non-separable C^*-algebra B such that every completely bounded map of any C^*-algebra into B has a completely positive decomposition [16].

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