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Normalcy is a superfluous condition in the definition of G-finiteness*

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Dedicated to Professor Károly Tandori on his 60th birthday

Let M be a W^*-algebra and let G be a group of $*$-automrophisms of M. In [2] we have proved that if there exists a faithful G-invariant *normal* state φ on M, then for every $t \in M$, the w^*-closure of the convex hull of the orbit of t under G contains a unique G-invariant element t^G and the mapping $t \to t^G$ is normal. (In fact, we have proved this result under the more general assumption that the family of G-invariant normal states on M is faithful, i.e., M is G-finite [2]. If M is σ-finite, for example, if M is an operator algebra in a separable Hilbert space, then this assumption obviously implies the existence of a faithful G-invariant normal state on M.) In the present paper we shall prove that the assumption of normalcy of φ is superfluous in this theorem (cf. Theorem). Under additional hypotheses, we shall also prove that φ itself is a normal state (cf. Corollary 1). Furthermore, we shall prove some converse results (cf. Corollaries 2 and 3).

For the general theory of W^*-algebras, we refer the reader to [1] and [3].

At the end of the paper we shall make two comments on our paper [4].

Theorem. *Let M be a W^*-algebra and let G be a group of $*$-automorphisms of M. If there exists a faithful G-invariant state φ on M, then there exists a faithful G-invariant normal state ψ on M, i.e., M is G-finite.*

Proof**. Let $\varphi = \varphi_n + \varphi_S$ be the canonical decomposition of φ into normal part φ_n and singular part φ_s [3]. Consider an element $g \in G$. Then $\varphi_n(g\cdot)$ is normal due to

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the continuity properties of g. On the other hand, $\varphi_s(g\cdot)$ is singular, since a positive linear form μ on M is singular if and only if every nonzero projection $p\in M$ majorizes a nonzero projection $q\in M$ such that $\mu(q)=0$ [3]. Since φ is g-invariant and the decomposition into normal and singular parts is unique, we obtain that φ_n is g-invariant (for all $g\in G$). Furthermore, φ_n is faithful. For let p be a nonzero projection in M. Since φ_s is singular, there exists a nonzero subprojection q of p in M, such that $\varphi_s(q)=0$. Then $\varphi_n(p)\geqq\varphi_n(q)=\varphi(q)-\varphi_s(q)=\varphi(q)>0$ because φ was assumed to be faithful. Summing up, we can choose $\psi=\varphi_n$.

Corollary 1. *Let M be a W^*-algebra and G a group of $*$-automorphisms of M. Suppose that for every $t\in M$, the norm-closed convex hull of the orbit Gt of t under G contains at least one G-invariant element. If φ is a G-invariant faithful state on M and the restriction of φ to the fixed-point algebra M^G is normal, then φ is normal.*

Proof. According to Theorem, M is G-finite [2]. Consequently, the G-invariant element, say t^G, in the norm-closed convex hull of Gt is unique [2]. Moreover, the mapping $t\to t^G\colon M\to M^G$ is normal [2]. Since φ is G-invariant and norm-continuous, $\varphi(t)=\varphi(t^G)$ $(t\in M)$. Therefore, the mapping $t\to\varphi(t)\colon M\to\mathbf{C}$ is the composite mapping of $t\to t^G\colon M\to M^G$ and $t\to\varphi(t)\colon M^G\to\mathbf{C}$. Since both of these mappings are normal, φ is normal.

Corollary 2. *Let M be a W^*algebra and G a group of $*$-automorphisms of M. Suppose that for every $t\in M$, the w^*-closed (norm-closed) convex hull of the orbit Gt of t under G contains exactly one G-invariant element, say t^G. If $t^G\neq 0$ for $t\geqq 0$, $t\neq 0$, then M is G-finite.*

Proof. The mapping $t\to t^G\colon M\to M^G$ is linear. In the case of the norm-closed convex hull, this can be proved as follows. The homogeneity of the mapping $t\to t^G$ is obvious. To prove its linearity, let t, $s\in M$ and let $\varepsilon>0$ be a given number. There exists a v_0 in the convex hull conv G of G, such that $\|v_0(t)-t^G\|<\varepsilon/2$. Similarly, there exists $v_1\in\operatorname{conv} G$, such that $\|v_1v_0(s)-s^G\|<\varepsilon/2$. Since every element of G has norm 1, we have $\|v_1v_0(t)-t^G\|<\varepsilon/2$. Consequently, $\|v_1v_0(t+s)-(t^G+s^G)\|\leqq\leqq\|v_1v_0(t)-t^G\|+\|v_1v_0(s)-s^G\|<\varepsilon$. Since $\varepsilon>0$ was arbitrary, this proves that $(t+s)^G=t^G+s^G$.

In the case of the w^*-closed convex hull, the linearity of $t\to t^G$ can be proved as follows. The homogeneity of $t\to t^G$ is obvious. Let us verify its additvity. Let s, $t\in M$. Then there exists a net v_n in conv G, such that $\lim_n v_n(s)=s^G$. Since the unit ball of M is w^*-compact, there exists a subnet v_k of v_n, such that $t_k=\lim_k v_k(t)$ exists. Then $\lim_k v_k(s+t)=\lim_k v_k(s)+\lim_k v_k(t)=\lim_n v_n(s)+t_k=s^G+t_k$ belongs to the w^*-closed convex hull of $G(s+t)$. By the definition of $(t_k)^G$, there is a net w_n in conv G, such that $\lim_n w_n(t_k)=(t_k)^G$. Then $\lim_n w_n(s^G+t_k)=\lim_n[w_n(s^G)+w_n(t_k)]=\lim_n[s^G+$

$+w_n(t_k)]=s^G+\lim_n w_n(t_k)=s^G+(t_k)^G$. Consequently, $(s^G+t_k)^G=s^G+(t_k)^G$. Since s^G+t_k belongs to the w^*-closed convex hull of $G(s+t)$, we have $(s+t)^G=(s^G+t_k)^G$. Therefore, $(s+t)^G=s^G+(t_k)^G$. Similarly, since t_k belongs to the w^*-closed convex hull of $G(t)$, we have $t^G=(t_k)^G$. Summing up, we have obtained that $(s+t)^G= =s^G+(t_k)^G=s^G+t^G$, which was to be proved.

So far we have proved that $t\to t^G$: $M\to M^G$ is linear. On the other hand, it is evident that $[g(t)]^G=t^G$ for every $g\in G$, $t\in M$ and $t^G=t$ for $t\in M^G$, the G-fixed-point algebra in M.

Now let φ_0 be a normal state on M^G. Let $\varphi(t)=\varphi_0(t^G)$ for $t\in M$. Then φ is a G-invariant state on M. Let p be the support of φ_0. Then $p\in M^G$ and $(ptp)^G=pt^Gp$. Consequently, φ is faithful on pMp, by the hypotheses of the corollary and by the faithfulness of φ_0 on pM^Gp. Since φ is invariant under the restriction of G to pMp, Theorem can be applied. We obtain that pMp is finite with respect to the restriction of G to pMp. This implies [2] that φ is a G-invariant normal state on M with support p. Since $\sup p=1$ if φ_0 runs over all normal states of M^G, we obtain that M is G-finite [2].

Corollary 3. *Let M be a W^*-algebra and G a group of $*$-automorphisms of M. If τ: $M\to M^G$ is a G-invariant faithful positive linear mapping which leaves M^G elementwise fixed, then M is G-finite.*

Proof. It is similar to the end of the proof of Corollary 2.

Remarks. 1. The proof of one half of Corollary 2 does not require Theorem:

Let M be a W^*-algebra and G a group of $*$-automorphisms of M. Suppose that for every $t\in M$, the norm-closed convex hull of the orbit Gt of t under G contains exactly one G-invariant element, say t^G. If $t^G\neq 0$ for $t\geqq 0$, $t\neq 0$, then M is G-finite (and $t\to t^G$ is a normal positive linear mapping of M onto M^G).

Proof. As in the proof of Corollary 2, we first prove that $t\to t^G$ is a linear mapping. This done, let φ_0 be a normal positive linear form on M^G and let p denote the support of φ_0. Then $(ptp)^G=pt^Gp$ and $t\to\varphi_0(t^G)$ is a faithful positive linear form φ on pMp, invariant under the restriction of G to pMp. Let e be a nonzero projection in pMp, such that $\varphi(e\cdot e)$ is normal [1]. Then $\varphi(\cdot e)\in M^G$. Let $v_n\in\operatorname{conv} G$ be such that $\|v_n(e)-e^G\|\to 0$ as $n\to\infty$. We have $\varphi(\cdot v_n(e))\in M^G$ by the G-invariance of φ and by the fact that $\varphi(\cdot e)M_*$. Then the norm limit of $\varphi(\cdot v_n(e))$ in M_* is $\varphi(\cdot e^G)$, since $\varphi\in M^*$. Therefore, $\varphi(\cdot e^G)\in M_*$. Consequently, $t\to\varphi(e^Gte^G)=\varphi_0((e^Gte^G)^G)=\varphi_0(e^Gt^Ge^G)$ is a normal positive linear form on M. Since $e^G\leqq p$, $e^G\in M^G$, we obtain that $t\to e^Gt^Ge^G$ is normal on M. If φ_0 runs over all normal forms on M^G, we obtain that every nonzero projection $p\in M^G$ majorizes a nonzero projection $e\in M$ (it is $e^G\in M^G$) such that $t\to te^Ge$ is normal. This implies that $t\to t^G$ is normal on M and thus M is G-finite [2].

2. The assumption of Theorem that φ is faithful is essential. Indeed, let G be an abstract infinite Abelian group. Then G acts naturally on $M=l^\infty(G)$ as a group of $*$-automorphisms. A G-invariant state on M is noting else but an invariant mean on G. We know that there are ifinitely many invariant means on G, none of wich are normal (actually, they are singular).

Finally, the author would like to make two comments on his paper [4]. The first comment is that in Proposition 2 and in its corollary the assumption that M is σ-finite should be replaced by the assumption that the predual of M is separable.

The second comment is that all the results of the above mentioned paper remain valid if G is only assumed to be an amenable group (instead of an Abelian one). Indeed, if $U_n\subset G$ is a summing sequence [5], then it is easy to prove that under the hypotheses of Lemma 1, the sequence $\frac{1}{|U_n|}\sum_{g\in U_n} g(t)$ w^*-converges to t^G for every $t\in B^*$. The remaining results of the paper can be extended to amenable groups G without altering the proofs.

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