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On Fourier series with nonnegative coefficients

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1. Let $f(x)$ be a continuous and 2π periodic function and let

(1) $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series. Denote by $s_n = s_n(x) = s_n(f; x)$ the n-th partial sum of (1).

If $\omega(\delta)$ is a nondecreasing continuous function on the interval $[0, 2\pi]$ having the following properties

$$\omega(0) = 0, \quad \omega(\delta_1 + \delta_2) \leqq \omega(\delta_1) + \omega(\delta_2)$$

for any $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1 + \delta_2 \leqq 2\pi$ then it will be called modulus of continuity. As usually $W^r H^\omega$ and $W^r(H^\omega)^*$ denote the following function classes:

(2) $$W^r H^\omega = \{f \colon \|f^{(r)}(x+h) - f^{(r)}(x)\| = O(\omega(h))\},$$

(3) $$W^r (H^\omega)^* = \{f \colon \|f^{(r)}(x+h) + f^{(r)}(x-h) - 2f^{(r)}(x)\| = O(\omega(h))\},$$

where $f^{(r)}$ denotes the r-th derivative of f, and $\|\cdot\|$ denotes the usual supremum norm. For $r=0$ and $\omega(\delta) = \delta^\alpha$ $H^\omega = H^{\delta^\alpha}$ is called the Lipschitz class of order α.

L. LEINDLER ([3]) defined the so called generalized Lipschitz-classes as follows. For $0 \leqq \alpha \leqq 1$ let $\omega_\alpha(\delta)$ denote a modulus of continuity having the following properties

(i) for any $\alpha' > \alpha$ there exists a natural number $\mu = \mu(\alpha')$ such that

(4) $$2^{\mu\alpha'} \omega_\alpha(2^{-n-\mu}) > 2\omega_\alpha(2^{-n}) \quad \text{holds for all} \quad n \geqq 1;$$

(ii) for every natural ν there exists a natural number $N(\nu)$ such that

(5) $$2^{\nu\alpha} \omega_\alpha(2^{-n-\nu}) \leqq 2\omega_\alpha(2^{-n}) \quad \text{if} \quad n \geqq N(\nu).$$

Using such modulus of continuity, H^{ω_α} defines the generalized Lipschitz class.

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For any positive β and p L. LEINDLER ([2]) defined the following strong means and function classes:

$$h_n(f,\beta,p) = \left\|\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_k(x)-f(x)|^p\right\}^{1/p}\right\|, \tag{6}$$

$$H(\beta,p,r,\omega) = \left\{f\colon h_n(f,\beta,p) = O\left(n^{-r}\omega\left(\frac{1}{n}\right)\right)\right\}, \tag{7}$$

and in [3] and [4] he proved the following relations:

$$\left.\begin{array}{ll} H(\beta,p,r,\omega_\alpha)\equiv W^rH^{\omega_\alpha} & \text{for}\quad 0<\alpha<1; \\ W^rH^{\omega_1}\subset H(\beta,p,r,\omega_1)\equiv W^r(H^{\omega_1})^* & \text{for}\quad \alpha=1\end{array}\right\}\quad \text{if}\quad \beta>(r+\alpha)p. \tag{8}$$

In [8] we gave coefficient-conditions assuring that a function should belong to H^{ω_α} (and so in certain cases to $H(\beta,p,\omega_\alpha)$).

For example the following theorem was proved.

Theorem A (Theorem 1 of [8]). *Let* $\lambda_n\geqq 0$ *and* λ_n *be the Fourier sine or cosine coefficients of* $\varphi(x)$. *Then*

$$\varphi\in H^{\omega_\alpha}\quad (0<\alpha<1)$$

if and only if

$$\sum_{k=n}^{\infty}\lambda_n = O\left(\omega_\alpha\left(\frac{1}{n}\right)\right), \tag{9}$$

or equivalently

$$\sum_{k=1}^{n}k\cdot\lambda_k = O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right). \tag{10}$$

It is clear that in order to obtain coefficient-conditions of type (9) for f to belong to $H(\beta,p,r,\omega_\alpha)$ instead of $H(\beta,p,\omega_\alpha)$ it is sufficient to give conditions assuring that f should belong to $W^rH^{\omega_\alpha}$ or equivalently, under the restriction $\lambda_n\geqq 0$, to $H(\beta,p,r,\omega_\alpha)$. In other words it is sufficient to find coefficient conditions for the derivatives of f to be in H^{ω_α}.

In the special cases $\omega(\delta)=\delta^\alpha$ coefficient-conditions for $f\in H^{\delta^\alpha}$ and $f\in W^rH^{\delta^\alpha}$ were given by G. G. LORENTZ ([7]), R. P. BOAS ([1]) and LING-YAU CHAN ([6]).

2. Theorems. Throughout the rest of the paper we shall assume that the Fourier coefficients a_n, b_n are nonnegative and

$$g(x) = \sum_{k=1}^{\infty}b_k\sin kx,\quad f(x) = \sum_{k=1}^{\infty}a_k\cos kx,$$

furthermore f and g are continuous functions on $[0,\pi]$.

Theorem 1. *If* $0<\alpha<1$ *then for any* $r\geqq 1$

$$g\in W^rH^{\omega_\alpha}$$

if and only if

$$\sum_{k=1}^{n} k^{r+1} b_k = O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right),$$

or equivalently

$$\sum_{k=n}^{\infty} k^r b_k = O\left(\omega_\alpha\left(\frac{1}{n}\right)\right).$$

Theorem 2. *If* $0<\alpha<1$ *then for any* $r \geqq 1$

$$f \in W^r H^{\omega_\alpha}$$

if and only if

$$\sum_{k=1}^{n} k^{r+1} a_k = O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right),$$

or equivalently

$$\sum_{k=n}^{\infty} k^r a_k = O\left(\omega_\alpha\left(\frac{1}{n}\right)\right).$$

Theorem 3. *If* $\alpha=1$ *and* r *is odd, then*

$$g \in W^r (H^{\omega_1})^*$$

if and only if

$$\sum_{k=1}^{n} k^{r+2} b_k = O\left(n^2 \omega_1\left(\frac{1}{n}\right)\right).$$

Theorem 4. *If* $\alpha=1$ *and* r *is even* $(r \geqq 2)$, *then*

$$g \in W^r H^{\omega_1}$$

if and only if

$$\sum_{k=1}^{n} k^{r+1} b_k = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

Theorem 5. *If* $\alpha=1$ *and* r *is even* $(r \geqq 0)$, *then*

$$f \in W^r (H^{\omega_1})^*$$

if and only if

$$\sum_{k=1}^{n} k^{r+2} a_k = O\left(n^2 \omega_1\left(\frac{1}{n}\right)\right).$$

Theorem 6. *If* $\alpha=1$ *and* r *is odd, then*

$$f \in W^r H^{\omega_1}$$

if and only if

$$\sum_{k=1}^{n} k^{r+1} a_k = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

Theorem 7. *If* $\alpha=1$ *and* r *is odd, then*

$$g\in W^r H^{\omega_1}$$

if and only if

$$\sum_{k=1}^{n} k^{r+2} b_k = O\left(n^2\omega_1\left(\frac{1}{n}\right)\right) \quad \text{and} \quad \left\|\sum_{k=1}^{n} k^{r+1} b_k \sin kx\right\| = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

Theorem 8. *If* $\alpha=1$ *and* r *is even* $(r\geqq 0)$, *then*

$$f\in W^r H^{\omega_1}$$

if and only if

$$\sum_{k=1}^{n} k^{r+2} a_k = O\left(n^2\omega_1\left(\frac{1}{n}\right)\right) \quad \text{and} \quad \left\|\sum_{k=1}^{n} k^{r+1} a_k \sin kx\right\| = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

3. Lemmas.

Lemma 1 (Lemma 2 of [8]). *If* $\mu_k\geqq 0$ *and* $\delta>\beta>0$, *then*

$$\sum_{k=1}^{n} k\cdot\mu_k = O\left(n^\delta \omega_{\delta-\beta}\left(\frac{1}{n}\right)\right)$$

if and only if

$$\sum_{k=n}^{\infty} \mu_k = O\left(\omega_{\delta-\beta}\left(\frac{1}{n}\right)\right).$$

Lemma 2. (Lemma 2 of [6]). *For each integer* $j\geqq 0$ *the quantity*

$$G(j, u) = \sin u - u + \frac{u^3}{3!} - \dots + \frac{(-1)^{j+1}}{(2j+1)!} u^{2j+1}$$

is of constant sign for all $u>0$. *Furthermore if* $0<u\leqq 1$, *then*

$$|G(j, u)| \geqq \frac{u^{2j+3}}{(2j+3)!\,2}.$$

Lemma 3 (Lemma 3 of [6]). *For each integer* $j\geqq 0$ *the quantity*

$$F(j, u) = \cos u - 1 + \frac{u^2}{2!} - \dots + (-1)^{j+1}\frac{u^{2j}}{(2j)!}$$

is of constant sign for all $u>0$. *Furthermore, if* $0\leqq u\leqq 1$, *then*

$$|F(j. u)| \geqq \frac{u^{2j+2}}{(2j+2)!\,2}.$$

Lemma 4 (Theorem 2 of [8]).

$$g\in H^{\omega_1}$$

if and only if

$$\sum_{k=1}^{n} kb_k = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

Lemma 5 (Theorem 3 of [8]).

$$f\in(H^{\omega_1})^*$$

if and only if

$$\sum_{k=n}^{\infty} a_k = O\left(\omega_1\left(\frac{1}{n}\right)\right).$$

Lemma 6 (Theorem 4 of [8]).

$$f\in H^{\omega_1}$$

if and only if

$$\sum_{k=n}^{\infty} a_k = O\left(\omega_1\left(\frac{1}{n}\right)\right) \quad \textit{and} \quad \left\|\sum_{k=1}^{n} ka_k \sin kx\right\| = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

4. Proofs. Since the proofs of all theorems above mentioned can be done in the same way as LING-YAU CHAN did in [6] (by using Theorem A and Lemma 1—Lemma 6 instead of those used in [6]) we here show only the proof of Theorem 1 for $r=1$.

Let us suppose that $0<\alpha<1$ and

$$\sum_{k=1}^{n} k^2 b_k = O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right). \tag{11}$$

By Lemma 1 we get that (11) is equivalent to

$$\sum_{k=n}^{\infty} kb_k = O\left(\omega_\alpha\left(\frac{1}{n}\right)\right). \tag{12}$$

So $\sum_{k=1}^{\infty} kb_k$ is convergent series, that is, the series

$$\sum_{k=1}^{\infty} kb_k \cos kx \tag{13}$$

is convergent uniformly which allows us to differentiate the series

$$\sum_{k=1}^{\infty} b_k \sin kx \tag{14}$$

term by term, which gives that

$$g'(x) = \sum_{k=1}^{\infty} kb_k \cos kx. \tag{15}$$

Using Theorem A and (11) we have that

$$g'\in H^{\omega_\alpha},$$

that is

$$g \in W^1 H^{\omega_\alpha},$$

which proves our Theorem 1 in the case $r=1$ in one direction.

For the other direction we assume that

$$g' \in H^{\omega_\alpha}, \tag{16}$$

that is,

$$g \in W^1 H^{\omega_\alpha}.$$

From (16) it is obtained that

$$|g'(t)-g'(0)| = O(\omega_\alpha(t)). \tag{17}$$

Integrating both sides over $(0, x]$ we have

$$|g(x)-xg'(0)| = O(x\omega_\alpha(x)). \tag{18}$$

Using (18) we have that

$$g(x) = O(x). \tag{19}$$

But (19), by using Lemma 4 (for $\omega_1(\delta)=\delta$) and the fact that

$$|g(x)-g(0)| = O(x)$$

implies $g(x) \in H^\delta$, what gives that

$$\sum_{k=1}^{n} kb_k = O(1) \tag{20}$$

that gives that the series

$$\sum_{k=1}^{\infty} kb_k \cos kx \tag{21}$$

is convergent uniformly, so the series

$$g(x) = \sum_{k=1}^{\infty} b_k \sin kx \tag{22}$$

can be differentiated term by term, that is,

$$g'(x) = \sum_{k=1}^{\infty} kb_k \cos kx \quad \text{and} \quad g'(0) = \sum_{k=1}^{\infty} kb_k. \tag{23}$$

Combining (18) and (23) we have

$$\sum_{k=1}^{\infty} b_k(\sin kx - kx) = O(x\omega_\alpha(x)). \tag{24}$$

Using Lemma 2 (for $u=kx$) we get from (24)

$$\sum_{k=1}^{[1/x]} b_k(\sin kx - kx) = O(x\omega_\alpha(x)). \tag{25}$$

Using again Lemma 2 (for $u=kx$) we have

$$\sum_{k=1}^{[1/x]} k^3 b_k x^3 = O(x\cdot\omega_\alpha(x)). \tag{26}$$

Putting $\left[\frac{1}{x}\right]=n$ we have that

$$\sum_{k=1}^{n} k^3 b_k = O\left(n^2\omega_\alpha\left(\frac{1}{n}\right)\right). \tag{27}$$

Using Lemma 1 from (27) we obtain the desired

$$\sum_{k=1}^{n} k^2 b_k = O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right).$$

The proof of Theorem is completed for $r=1$.

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