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Abstract Galois theory and endotheory. II

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5. Abstract fields and endofields; isomorphism and homomorphism theorems

Let $S=(E, R)$ be a structure, and consider the class $\bar{R}$ of relations preserved by each $\sigma\in G(E/S)$ and the class $\bar{\bar{R}}$ of relations stabilized by each $\delta\in D(E/S)$. These classes are closed with respect to the fundamental and direct fundamental operations, respectively, and they are the smallest classes having this property. Really, if $\varrho\supseteqq R$ is a class closed with respect to the fundamental operations and X^0 is a set such that $\operatorname{card} X^0\geqq\operatorname{card} E$ and R is under X^0 then ϱ includes $R_f^{(X^0)}$. By Remark 1 of Section 4, $R_f^{(X^0)}=\bar{R}^{(X^0)}$. As each $r\in\bar{R}$ belongs to some $\bar{R}^{(X^0)}$, we have $\bar{R}\subseteqq\varrho$. The case of $\bar{\bar{R}}$ can be handled similarly. Therefore, for every set R of relations, the *closure* of R with respect to all fundamental or to all direct fundamental operations is well-defined.

Now let ϱ be a class of relations which is closed with respect to fundamental or, respectively, to direct fundamental operations. Let G be the group of permutations of E that preserve each $r\in\varrho$, and let D be the monoid of self-mappings of E that stabilize each $r\in\varrho$. The semi-regular decomposition R_r of each $r\in\varrho$ (cf. Section 2) is included in ϱ. Further, and arbitrary $\sigma\in S(E)$ preserves r iff it preserves every relation in R_r, and an arbitrary $\delta\in D(E)$ stabilizes r iff it stabilizes every relation in R_r. So G and D are completely determined by the semi-regular relations belonging to ϱ. Let r be a semi-regular relation in ϱ, let $P\in t(r)$, and let $\mathfrak{P}$: $\mathfrak{X}\to E$ be a fixed bijective point. Then $(\varepsilon_{P,\mathfrak{P}})\cdot r$ is an $\mathfrak{X}_P$-relation belonging to ϱ, where $\mathfrak{X}_P=\mathfrak{P}^{-1}P\cdot E\subseteqq\mathfrak{X}$. Clearly, the same permutations $\sigma\in S(E)$ preserve and the same $\delta\in D(E)$ stabilize $(\varepsilon_{P,\mathfrak{P}})\cdot r$ as r; and $(\varepsilon_{P,\mathfrak{P}})\cdot r$ is a relation under $\mathfrak{X}$. So G and D are already determined by $\varrho\cap R^{(\mathfrak{X})}$, which is a set of relations under $\mathfrak{X}$. In fact, $G=G(E/\varrho\cap R^{(\mathfrak{X})})$ and $D=D(E/\varrho\cap R^{(\mathfrak{X})})$. Now put $R=\varrho\cap R^{(\mathfrak{X})}$. As $\varrho\subseteqq p-\operatorname{inv} G$

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or $\varrho \subseteqq s-\operatorname{inv} D$, we have $\varrho \subseteqq \bar{R}$ or $\varrho \subseteqq \bar{\bar{R}}$, whence $\varrho = \bar{R}$ or $\varrho = \bar{\bar{R}}$, respectively. So, finally, every closed (with respect to all fundamental or direct fundamental operations) class of relations *is the closure* (with respect to the same operations) *of some set* of relations; this set may even be supposed to be under $\tilde{X}$ where $\tilde{X}$ is an arbitrary set with power card E.

Classes of relations that are closed with respect to all fundamental or all direct fundamental operations will be called *abstract fields* and *abstract endofields* on E (or, in other words, with base set E), respectively. For a structure $S = (E, R)$, $\bar{R} = R_f$ and $\bar{\bar{R}} = R_{df}$ are the smallest abstract field and abstract endofield including R. They will be called the abstract field and abstract endofield generated by S, and will be denoted by $K(S)$ and $K_e(S)$, respectively. If k and K are abstract fields (resp. endofields) and $k \subseteqq K$ then k is said to be a subfield (resp. subendofield) of K or, in other words, K is called an extension or overfield (resp. overendofield) of k. The notation K/k, instead of $k \subseteqq K$, is also used. We have seen that every abstract field or endofield is generated by an appropriate structure S. Two structures, S and S', generate the same abstract field or endofield iff $S \sim S'$ or $S \underset{d}{\sim} S'$, respectively. More generally, $K(S) \subseteqq K(S')$ is equivalent to $S \leqq S'$, while $K_e(S) \subseteqq K_e(S')$ is equivalent to $S \underset{d}{\leqq} S'$. In particular, if K is an abstract field or endofield and $\operatorname{card} X^0 \geqq \operatorname{card} E$ then $K = K(K \cap R(E; X^0))$ or $K = K_e(K \cap R(E; X^0))$, respectively, and $k \subseteqq K$ is clearly equivalent to $k \cap R(E; X^0) \subseteqq \subseteqq K \cap R(E; X^0)$. Given a set F of structures, we say, by abusing the language[1)], that the corresponding abstract fields $K(S)$ or endofields $K_e(S)$, $S \in F$, form a *set*. In this sense, all the abstract fields and all the abstract endofields on E form sets, denoted by $AF(E)$ and $AEF(E)$, respectively. In particular, if some set X^0 with the property $\operatorname{card} X^0 \geqq \operatorname{card} E$ is fixed then any abstract field or endofield K is uniquely determined by its part $K^{(X^0)} = K \cap R(E; X^0)$ under X^0, and the mapping $K \to K \cap \cap R(E; X^0)$ preserves the inclusion. This allows us to say that one set of fields or endofields is included in another, and, also, to speak of mappings, the intersection and the join (alias compositium) of a given set of fields or endofields. That is, for example, $k \subseteqq K$ will mean $k \cap R(E; X^0) \subseteqq K \cap R(E; X^0)$, a mapping $K \cap R(E; X^0) \to \to k \cap R(E; X^0)$ will be considered as a mapping $K \to k$, K will be called the intersection or join of a set F of (endo)fields iff $K^{(X^0)}$ is that of $F^{(X^0)} = \{k^{(X^0)}; k \in F\}$. Note that $\bigvee_{K \in F} K$ is the smallest (endo)field that includes every $K \in F$ and $\bigcap_{K \in F} K$ is the greatest (endo)field included in all $K \in F$. Further, $r \in \bigcap_{K \in F} K$ iff $r \in K$ for all $K \in F$.

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1) In case we want to remain within the frame of Bernays—Gödel axiomatic system. There are other ways to found mathematics where no abuse or not this kind of abuse would occur in the present situation.

For an abstract field K, let $G(E/K)$ denote the group of all permutations of E that preserve each $r\in K$. If K is an abstract endofield, let $D(E/K)$ denote the monoid of self-mappings of E that stabilize each $r\in K$. Clearly, if $K=K(S)$ or $K=K_e(S)$ then $G(E/K)=G(E/S)$ or $D(E/K)=D(E/S)$, respectively. So K is the class of all relations preserved by every $\sigma\in G(E/K)$ or stabilized by every $\delta\in D(E/K)$, respectively. Thus $K\to G(E/K)$ is a bijection of $AF(E)$ onto the set of permutation groups on E, while $K\to D(E/K)$ is a bijection of $AEF(E)$ onto the set of monoids of mappings $E\to E$. These mappings, called *canonical Galois mappings,* are decreasing, i.e., $k\subseteq K$ implies $G(E/k)\supseteq G(E/K)$ or $D(E/k)\supseteq D(E/K)$, respectively.

Now, if K is an abstract endofield such that $D(E/K)$ happens to be a group then K is an abstract field and $G(E/K)=D(E/K)$. Really, if all $\sigma\in G(E/K)$ stabilize a relation then they preserve it (cf. Remark 1 in Section 1). Further, $AF(E)\subseteq AEF(E)$. Therefore a number of results for endofields that will be proved later are automatically valid for abstract fields, too. On the other hand, if K_e is an abstract endofield defined by a structure S, i.e., $K_e=K_e(S)$, then $K=K(S)$ is completely determined by K_e, i.e., K does not depend on the particular choice of S. Really, by Remark 1 of Section 1, $G(E/K)=G(E/S)$ is the greatest permutation group included in $D(E/S)=D(E/K_e)$. When K_e happens to be an abstract field then $K=K_e$. So the mapping $K_e\to K$, from $AEF(E)$ onto $AF(E)$, can be called the *canonical projection.*

Let K and K' be abstract endofields with respective base sets E and E'. (So, the base sets of points, relations, structures, etc. are no longer assumed to be fixed in the rest of this paragraph.) We shall speak of a mapping of K into another endofield K' only if it is describable, in terms of Bernays—Gödel axiomatism, as a class of pairs $(r, r')\in K\times K'$. This is the case if, for an arbitrary $r\in K$, the corresponding r' can be described in terms of set theory. A mapping (assumed to be admissible in the previous sense) $\eta\colon K\to K'$ will be called surjective if for each $r'\in K'$ there is an $r\in K$ such that $r'=\eta\cdot r$, and it is called injective if $r_1\neq r_2\in K$ implies $\eta\cdot r_1\neq\eta\cdot r_2$. This η will be said to be a *homomorphism with respect to a fundamental operation* ω if, with ξ denoting the value of the argument of ω, ω is defined for $\eta\cdot\xi$ if it is defined for ξ and $\eta\cdot\omega(\xi)=\omega(\eta\cdot\xi)$. (Here ξ may be a set of relations in K, then $\eta\cdot\xi$ denotes $\{\eta\cdot r;\ r\in\xi\}$, or a single relation in K.)

Observation 1. If η is a homomorphism with respect to all projections, all contractions and the infinitary union then η is surely a mapping, i.e., η is describable in terms of the Bernays—Gödel system.

To prove this observation, put $D=D(E/K)$ and let $\tilde{P}\colon \tilde{X}\to E$ be a bijective point. By Remark 4 in Section 4, for each $r\in K$ there is a superposition ω of these three kinds of fundamental operations such that $r=\omega(\tilde{D}\cdot\tilde{P})$. But then ω is also defined for $\eta\cdot(\tilde{D}\cdot\tilde{P})$ and $\eta\cdot r=\eta\cdot\omega(\tilde{D}\cdot\tilde{P})=\omega(\eta\cdot(\tilde{D}\cdot\tilde{P}))$, i.e., η is completely determined by $\eta\cdot(\tilde{D}\cdot\tilde{P})$, the image of $\tilde{D}\cdot\tilde{P}$.

In spite of the above argument we should not think that for every $r'\in K'$ there exists a homomorphism with respect to the fundamental operations occurring in Observation 1 that sends $\tilde{D}\cdot\tilde{P}$ to r'. The reason is that $\omega(r')$ is not necessarily defined when $\omega(\tilde{D}\cdot\tilde{P})$ is, or $\omega(\tilde{D}\cdot\tilde{P})=\omega'(\tilde{D}\cdot\tilde{P})$ need not imply $\omega(r')=\omega'(r')$.

Remark 1. If η: $K\to K'$ is a homomorphism with respect to all contractions then the η-image of every X-relation in K is an X-relation again.

Really, a contraction $(\varphi\colon X\to Y)$ is defined only for X-relations and it is defined for all X-relations when it is a floatage (i.e., φ is a bijection), whence the assertion follows easily.

Remark 2. If η: $K\to K'$ is a surjective homomorphism with respect to all projections, all contractions and the infinitary union, as in Observation 1, then there exists a surjective point P': $\tilde{X}\to E'$ belonging to $\eta\cdot(D\cdot\tilde{P})$.

To prove this remark, observe that the operations $\mathrm{pr}_{\bar{X}}^{X}$ and $(\varphi\colon X\to Y)$ are punctual mappings of X-relations. If P: $X\to E$ is an X-point then $(P|\bar{X})\cdot\bar{X}\subseteq P\cdot X$ and $((\varphi)\cdot P)\cdot Y=P\cdot X$. Therefore, if r is an X-relation, $\mathrm{pr}_{\bar{X}}\cdot r$ and $(\varphi\colon X\to Y)\cdot r$ have surjective points only if r has. Similarly, $\bigcup_{r\in R} r$ has some surjective point iff there is an $r\in R$ having one. So any relation obtained from $\eta\cdot(D\cdot\tilde{P})$ by a superposition ω of projections, contractions and infinitary unions has surjective points only if $\eta\cdot(D\cdot\tilde{P})$ has. Since each $r\in K$ is of the form $\omega(D\cdot\tilde{P})$ for such a superposition ω, $\eta\cdot r=\eta\cdot\omega(D\cdot\tilde{P})=\omega(\eta\cdot(D\cdot\tilde{P}))$ and $\eta\cdot r$ has no surjective point when $\eta\cdot(D\cdot\tilde{P})$ does not have. But there are relations in K' having surjectiv points; indeed, the $D(E'/K')$-orbit of any surjective point P' contains P'. This proves Remark 2.

Now let K be an abstract endofield. A non-empty relation $r\in K$ is called *irreducible* in K if $\emptyset\neq r'\subset r$ holds for no relation $r'\in K$. A relation $r\in K$ is said to be *indecomposable* in K if for any set $R\subset K$ $\bigcup\cdot R=r$ implies $r\in R$. Every irreducible relation is clearly indecomposable.

Lemma 1. *A relation $r\in K$ is indecomposable iff it is the D-orbit of some point P: $X\to E$ where $D=D(E/K)$. If the D-orbit of some surjective point P is irreducible in K then D is a permutation group and K is an abstract field. Further, if K is an abstract endofield such that $D=D(E/K)$ is a group then all D-orbits are irreducible.*

Proof. Let $r\in K$ be indecomposable. As $r=\bigcup_{P\in r} D\cdot P$, there exists a point $P\in r$ such that $r=D\cdot P$. It is obvious that $D\cdot P$ is indecomposable. Now let P: $X\to E$ be a surjective point, and suppose D is not a permutation group. If we had $D\delta=D$ for all $\delta\in D$ then each element of the monoid D would have a left inverse and, as it is well-known from the elements of group theory, D would turn out to be a group (of permutations, of course). Hence there is a $\delta \in D$ such that $D\delta$ is a proper subset of D. Then the D-orbit $D\cdot(\delta\cdot P)=D\delta\cdot P$ of $\delta\cdot P$ is a non-empty relation in K and a proper subset of $D\cdot P$. This means that $D\cdot P$ is not irreducible. Hence if $D\cdot P$ is irreducible then D is a subgroup of $S(E)$ and K is an abstract field. Finally, if $D=D(E/K)$ is a group then any two D-orbits are disjoint or coincide, whence every D-orbit is irreducible in K. The proof is complete.

Let K and K' be abstract endofields on E and E', respectively, let $D=D(E/K)$ and $D'=D(E'/K')$ denote the corresponding stability monoids, and let $\eta\colon K\to K'$ be a mapping of K into K'. With these notations fixed, we prove four lemmas.

Lemma 2. *If η is a homomorphism with respect to the infinitary union then it preserves the inclusion $\subseteqq$ between relations and semi-commutes with the infinitary intersection. If, in addition, η is surjective and preserves the argument set of relations then $\eta\cdot\emptyset=\emptyset$, $\eta\cdot I(X,E)=I(X,E')$, and for each point P' on E' the D'-orbit $D'\cdot P'$ is the η-image of $D\cdot P$ for some point P on E.*

Proof. For $r, r'\in K$, $r\subseteqq r'$ we have $\eta\cdot r\cup\eta\cdot r'=\eta\cdot(r\cup r')=\eta\cdot r'$, i.e., $\eta\cdot r\subseteqq \subseteqq\eta\cdot r'$. If R is a set of relations, $r\in R$ and $R\subset K$, then $\bigcap\cdot R\subseteqq r$ for every $r\in R$ and we obtain $\eta\cdot(\bigcap\cdot R)\subseteqq\bigcap_{r\in R}\eta\cdot r=\bigcap\cdot(\eta\cdot R)$. If η is surjective and preserves the argument sets then $\eta\cdot\emptyset=\emptyset$ and $\eta\cdot I(X,E)=I(X,E')$ easily follow from the fact that η preserves the inclusion; the smallest and largest X-relations on E are obviously mapped on the smallest and largest ones on E'. Assume now that $\eta\cdot r= D'\cdot P'$ where $r\in K$ and P' is a point on E'. Then $\eta\cdot r=\eta\cdot\big(\bigcup_{P\in r}D\cdot P\big)=\bigcup_{P\in r}\eta\cdot(D\cdot P)$ and the indecomposability of $\eta\cdot r=D'\cdot P'$ in K' yield the existence of some $P\in r$ such that $D'\cdot P'=\eta\cdot(D\cdot P)$.

Lemma 3. *Suppose η is a homomorphism with respect to the infinitary union and intersection; further let $\eta\cdot\emptyset=\emptyset$ and $\eta\cdot I(X,E)=I(X,E')$ for any X. Then η is also a homomorphism with respect to the negation $\urcorner$, which is a partially defined operation on K. Moreover, if η is surjective and K happens to be an abstract field then the η-images of D-orbits are D'-orbits and K' is also an abstract field.*

Proof. If r is an X-relation and $r, \urcorner\cdot r\in K$ then $\eta\cdot r\cup\eta\cdot(\urcorner\cdot r)=\eta\cdot(r\cup(\urcorner\cdot r))= =\eta\cdot I(X,E)=I(X,E')$ and $\eta\cdot r\cap\eta\cdot(\urcorner\cdot r)=\eta\cdot(r\cap(\urcorner\cdot r))=\eta\cdot\emptyset=\emptyset$, whence $\eta\cdot(\urcorner\cdot r)=\urcorner\cdot(\eta\cdot r)$ follows. Now let η be assumed surjective and let K be an abstract field. For each $r'\in K'$ there is an $r\in K$ with $r'=\eta\cdot r$. As $\urcorner\cdot r$ also belongs to K, $\urcorner\cdot r'=\urcorner\cdot(\eta\cdot r)=\eta\cdot(\urcorner\cdot r)\in K'$, showing that K' is also an abstract field. The surjectivity of η readily yields that η sends indecomposable relations to indecomposable ones. Hence Lemma 1 applies and the proof is complete.

Lemma 4. *Assume that η is a homomorphism with respect to all dilatations and $\eta \cdot I(X, E) = I(X, E')$ for any X. Then the η-image of a multidiagonal $I_C(E) \in K$ is $I_C(E')$, a multidiagonal of the same pattern. If, in addition, $\eta \cdot \emptyset = \emptyset$, η is a homomorphism with respect to the intersection and all points of a relation r in K are injective then so are the points of $\eta \cdot r$.*

Proof. Let C be an equivalence relation on an argument set X, and let ψ denote the canonical surjection $X \to X^* = X/C$. We have $I_C(E) = [\psi] \cdot I(X^*, E)$, whence $\eta \cdot I_C(E) = [\psi] \cdot (\eta \cdot I(X^*, E)) = [\psi] \cdot I(X^*, E') = I_C(E')$. In particular, if $x, y \in X$ then $\mathrm{ext}_X \cdot I(\{x, y\}, E)$ is a simple diagonal and $\eta \cdot (\mathrm{ext}_X \cdot I(\{x, y\}, E)) = \mathrm{ext}_X \cdot I(\{x, y\}, E')$. Observe that, for an X-relation r, all $P \in r$ are injective iff $r \cap \mathrm{ext}_X \cdot I(\{x, y\}, E) = \emptyset$ for any two distinct elements x and y in X; and so this property is preserved by η.

Lemma 5. *If η is a homomorphism with respect to the infinitary union then the following two conditions are equivalent:*

(C) *if $R' \subseteq K'$, $r \in K$ and $r' = \bigcup \cdot R'$ equals $\eta \cdot r$ then there exists a mapping θ: $R' \to K$ such that $r = \bigcup \cdot (\theta \cdot R')$ and, for every $\varrho' \in R'$, $\eta \cdot (\theta \cdot \varrho') \subseteq \varrho'$;*

(D) *the η-image of every D-orbit $D \cdot P$ in K is a D'-orbit (on E') in K'.*

Proof. Assume (C) and let $r = D \cdot P$ be a D-orbit in K. Let $R' \subseteq K'$ be a set of relations such that $\eta \cdot r = \bigcup \cdot R'$. Consider a mapping θ according to (C). Then $r = \bigcup \cdot (\theta \cdot R')$ and the indecomposability of r in K (cf. Lemma 1) yield the existence of a $\varrho' \in R'$ such that $r = \theta \cdot \varrho'$. Therefore $\varrho' \subseteq r' = \eta \cdot r = \eta \cdot (\theta \cdot \varrho') \subseteq \varrho'$, i.e., $r' = \varrho'$. Thus r' is indecomposable in K' and Lemma 1 furnishes (D). Conversely, assume (D) and let $\eta \cdot r$ be equal to $r' = \bigcup \cdot R'$ for some $r \in K$ and $R' \subseteq K'$. As $r = \bigcup_{P \in r} D \cdot P$, we can define a mapping θ: $R' \to K$ by putting $\theta \cdot \varrho' = \bigcup_{\eta \cdot (D \cdot P) \subseteq \varrho'} D \cdot P$ for $\varrho' \in R'$. Then $r = \bigcup \cdot (\theta \cdot R')$ and $\eta \cdot (\theta \cdot \varrho') \subseteq \varrho'$ for every $\varrho' \in R'$. The proof of the lemma is done.

For two abstract endofields K and K', a mapping η: $K \to K'$ will be called an *isomorphism* of K onto K' if it is bijective and is a homomorphism with respect to all fundamental operations. (Note that, by Lemma 3 it is sufficient to require that η be a bijective homomorphism with respect to direct fundamental operations only.) As an isomorphism η is uniquely determined by the η-image of the D-orbit $D \cdot \tilde{P}$ of a bijective point $\tilde{P}$: $\tilde{X} \to E$, there are no logical difficulties in considering these mappings. The image $\eta \cdot (D \cdot \tilde{P})$ is a D'-orbit, as it follows from Lemmas 2 and 3. If K is an abstract field then, by Lemma 3, so is K'. Therefore, if K is an abstract field then D' is a permutation group on E' and $\eta \cdot (D \cdot \tilde{P}) = D' \cdot \tilde{P}'$ for some $\tilde{X}$-point $\tilde{P}'$ of E'. We claim that $\tilde{P}'$ is bijective. Since the points of $D \cdot \tilde{P}$ are injective, the same is true for $\eta \cdot (D \cdot \tilde{P}) = D' \cdot \tilde{P}'$ by Lemma 4. In particular, $\tilde{P}'$ is injective. If $\tilde{P}'$ is not surjective then there are a set $\hat{X} \supset \tilde{X}$ and a point $\hat{P} \in E'^{\hat{X}}$ such that $\hat{P}$ is still injective and $\tilde{P}'=(\hat{P}|\tilde{X})$. By applying the previous argument for $D'\cdot\hat{P}$ and η^{-1} we obtain that $\eta^{-1}\cdot(D'\cdot\hat{P})$ consists of injective points. But then $D\cdot\tilde{P}=$ $=\eta^{-1}\cdot(D'\cdot\tilde{P}')=\eta^{-1}\cdot(\mathrm{pr}^{\hat{X}}_{\tilde{X}}\cdot(D'\cdot\hat{P}))=\mathrm{pr}^{\hat{X}}_{\tilde{X}}\cdot(\eta^{-1}\cdot(D'\cdot\hat{P}))$ would contain no surjective point, which is a contradiction. Therefore $\tilde{P}'$ is surjective, whence it is bijective, indeed.

An obvious example of isomorphism of abstract endofields is the *transportation of structures*. For definition, let K be an abstract endofield on E and let $s\colon E\to E'$ be a bijection. This bijection induces a mapping $(s)\colon r\to s\cdot r$ of K, and the class $s\cdot K=\{s\cdot r;\ r\in K\}$ is visibly closed under all direct fundamental operations, whence it is an abstract endofield. Further, (s) is a bijection of K onto $s\cdot K$, which, by Proposition 1 (1) of Section 3, commutes with all fundamental operations. Therefore (s) is an isomorphism of K onto $s\cdot K$, called the *transportation of structure* induced by s. If K is an abstract field then, clearly, so is $s\cdot K$.

Lemma 6. *If $s\colon E\to E'$ is a bijection and K is an abstract endofield on E then $D(E'/s\cdot K)=sD(E/K)s^{-1}$.*

Proof. Let $r\in K$ and let δ be a self-mapping of E. Then $\delta\cdot r\subseteq r$ iff $s\delta 1_E\cdot r\subseteq s\cdot r$ iff $s\delta(s^{-1}s)\cdot r\subseteq s\cdot r$ iff $s\delta s^{-1}\cdot(s\cdot r)\subseteq s\cdot r$, which proves the lemma.

Consequence. *When K happens to be an abstract field then $G(E'/s\cdot K)=$ $=sG(E/K)s^{-1}$.*

Theorem (the isomorphism theorem of abstract Galois theory). *Every isomorphism of an abstract field is a transportation of structure.*

Proof. We have seen that each $\tilde{X}$-point of E is of the form $\delta\cdot\tilde{P}$ for a suitable $\delta\colon E\to E$. Considering $\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}=(\tilde{P}^{-1}|\delta\cdot E)(\delta\cdot\tilde{P})$ we have $(\varepsilon_{\delta\cdot\tilde{P},\tilde{P}})\cdot(\delta\cdot\tilde{P})=(\tilde{P}|\tilde{X}_{\delta\cdot\tilde{P}})$, where $\tilde{X}_{\delta\cdot\tilde{P}}=\tilde{P}^{-1}\cdot(\delta\cdot E)$, and $\delta\cdot\tilde{P}=[\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\cdot(\tilde{P}|\tilde{X}_{\delta\cdot\tilde{P}})$. Now, if σ is a permutation of E, so $\sigma\cdot E=E$ and $\tilde{X}_{\sigma\cdot\tilde{P}}=\tilde{X}$, these formulas turn into $\sigma\cdot\tilde{P}=[\varepsilon_{\sigma\cdot\tilde{P},\tilde{P}}]\cdot\tilde{P}$; i.e. $\sigma\circ\tilde{P}=P\circ\varepsilon_{\sigma\cdot\tilde{P},\tilde{P}}$ and $\varepsilon_{\sigma\cdot\tilde{P},\tilde{P}}$ is a permutation of $\tilde{X}$. So for any permutation σ of E there exists one (and only one) permutation $\varepsilon(\sigma)=\varepsilon_{\sigma\cdot\tilde{P},\tilde{P}}$ of $\tilde{X}$ such that $\sigma\circ\tilde{P}=$ $\tilde{P}\circ\varepsilon(\sigma)$, $\varepsilon(\sigma)$ being clearly dependent on the choice of $\tilde{P}$; further, for every permutation ε of $\tilde{X}$ there exists one and only one permutation $\sigma(\varepsilon)$ of E such that $\sigma(\varepsilon)\circ\tilde{P}=$ $=\tilde{P}\circ\varepsilon$. We obviously have $\varepsilon(\sigma)=\tilde{P}^{-1}\circ\sigma\circ\tilde{P}$ and $\sigma(\varepsilon)=\tilde{P}\circ\varepsilon\circ\tilde{P}^{-1}$. Let G stand for $G(E/K)$ and put $r=G\cdot\tilde{P}$. For $\varepsilon\in S(X)$ we have $[\varepsilon]\cdot r=[\varepsilon]\cdot(G\cdot\tilde{P})=G\cdot([\varepsilon]\cdot\tilde{P})=$ $=G\cdot(\sigma(\varepsilon)\cdot\tilde{P})=G\sigma(\varepsilon)\cdot\tilde{P}$ and, as $\tilde{P}$ is surjective, $[\varepsilon]\cdot r\cap r=G\sigma(\varepsilon)\cdot\tilde{P}\cap G\cdot\tilde{P}=$ $=(G\sigma(\varepsilon)\cap G)\cdot\tilde{P}$, which is either $r=G\cdot\tilde{P}$ or $\emptyset$ depending on $\sigma(\varepsilon)\in G$ or $\sigma(\varepsilon)\notin G$. But $\sigma(\varepsilon)\in G$ iff $\varepsilon=\varepsilon(\sigma(\varepsilon))=\tilde{P}^{-1}\circ\sigma(\varepsilon)\circ\tilde{P}$ belongs to $\tilde{P}^{-1}\circ G\circ\tilde{P}$. So $[\varepsilon]\cdot r\cap r=r$ if $\varepsilon\in\tilde{P}^{-1}G\tilde{P}$ and $[\varepsilon]\cdot r\cap r=\emptyset$ if $\varepsilon\notin\tilde{P}^{-1}G\tilde{P}$.

Now if $\eta\colon K\to K'$ is an isomorphism of an abstract field K on E onto an abstract endofield K' on E' then K' is also an abstract field and $\eta\cdot(G\cdot\tilde{P})=G'\cdot\tilde{P}'$ where $G'=G(E'/K')$ and $\tilde{P}'\colon\tilde{X}\to E'$ is a bijective point. For a permutation ε of E an analogous reasoning shows that $[\varepsilon]\cdot r'\cap r'$ is r' or $\emptyset$ according to $\varepsilon\in\tilde{P}'^{-1}G\tilde{P}'$ or $\varepsilon\notin\tilde{P}'^{-1}G\tilde{P}'$, where $r'=G'\cdot P'$.

Put $s=\tilde{P}'\tilde{P}^{-1}\colon E\to E'$, which is a bijection of E onto E'. We have $(s)\cdot(G\cdot\tilde{P})=$ $=s\cdot(G\cdot\tilde{P})=sG\cdot\tilde{P}=sGs^{-1}\cdot(s\cdot\tilde{P})=sGs^{-1}\cdot(\tilde{P}'\tilde{P}^{-1}\cdot\tilde{P})=sGs^{-1}\cdot\tilde{P}'\tilde{P}^{-1}\tilde{P}=sGs^{-1}\cdot\tilde{P}'$. Since η is a $K\to K'$ isomorphism, we have $[\varepsilon]\cdot r\cap r=r\Rightarrow[\varepsilon]\cdot(\eta\cdot r)\cap(\eta\cdot r)=\eta\cdot r$ and $[\varepsilon]\cdot r\cap r=\emptyset\Rightarrow[\varepsilon]\cdot(\eta\cdot r)\cap(\eta\cdot r)=\eta\cdot\emptyset=\emptyset$. So $\varepsilon\in\tilde{P}^{-1}G\tilde{P}$ iff $\varepsilon\in\tilde{P}'^{-1}G'\tilde{P}'$. Therefore we have $\tilde{P}^{-1}G\tilde{P}=\tilde{P}'^{-1}G'\tilde{P}'$ and $G'=(\tilde{P}'\tilde{P}^{-1})G(\tilde{P}'\tilde{P}^{-1})^{-1}=sGs^{-1}$. Thus $\eta\cdot(G\cdot\tilde{P})=$ $=G'\cdot\tilde{P}'=sGs^{-1}\cdot\tilde{P}'=sG\cdot(s^{-1}\cdot\tilde{P}')=sG\cdot(\tilde{P}\tilde{P}'^{-1}\cdot\tilde{P}')=sG\cdot\tilde{P}=s\cdot(G\cdot\tilde{P})=(s)\cdot(G\cdot\tilde{P})$. As $\eta\cdot(G\cdot\tilde{P})$ determines the isomorphism η, we have $\eta=(s)$, which completes the proof.

Starting from this theorem, it is easy to develop a formalism for abstract field extensions that I have already done in [1], i.e., an analogous counterpart of the classical Galois theory. Indeed, let K/k be an extension of abstract fields. An isomorphism $\eta\colon K\to K$ is called an *isomorphism of* K/k or an *isomorphism with respect to* k if its restriction to k is the identical mapping 1_k. If η is an isomorphism of K/k then it is induced by a bijection $\sigma\colon E\to E$ which preserves all $r\in k$, i.e., by a $\sigma\in G(E/k)$. Two isomorphisms of K/k, say (σ) and (τ) induced by $\sigma,\tau\in G(E/k)$, coincide if and only if for every $r\in K$ we have $\sigma\cdot r=(\sigma)\cdot r=(\tau)\cdot r=\tau\cdot r$, i.e. $\sigma^{-1}\tau\cdot r=r$, which is equivalent to $\sigma^{-1}\tau\in G(E/K)$ and also to $\tau\in\sigma G(E/K)$. Therefore if $G(K/k)$ denotes the set of isomorphisms of K/k then $\eta\to\{\sigma\in G(E/k);\ (\sigma)=\eta\}$ is a bijection of $G(K/k)$ onto $G(E/k)/G(E/K)$, the set of left residue classes of $G(E/k)$ modulo $G(E/K)$. The cardinal number $[K{:}k]=\operatorname{card}G(K/k)$ is called the *Galois degree* of K/k. Note that $[K{:}k]$ is equal to the index $(G(E/k){:}G(E/K))$ of $G(E/K)$ in $G(E/k)$. In case L, K and k are abstract fields and $L\supseteq K\supseteq k$ then L/k is called an (abstract) overextension of K/k while K/k is an (abstract) subextension of L/k. Every $\eta\in G(K/k)$ is induced by some $\eta'\in G(L/K)$; really, η is a transposition of structures induced by some $\sigma\in G(E/k)$, which induces an appropriate isomorphism η' of L/k. Clearly, $[L{:}k]=(G(E/k){:}G(E/L))=$ $=(G(E/k){:}G(E/K))(G(E/K){:}G(E/L))=[L{:}K][K{:}k]$. An abstract field extension K/k is called *normal* if $\eta\cdot K=K$ holds for every $\eta\in G(K/k)$, i.e., if every isomorphism of K/k is an automorphism. In case K/k is an abstract field extension then K/k is normal iff $\sigma\cdot K=K$ for all $\sigma\in G(E/k)$ (here we put (σ) instead of η), i.e., iff $\sigma G(E/K)\sigma^{-1}=G(E/\sigma\cdot K)=G(E/K)$. So K/k is normal iff $G(E/K)$ is invariant in $G(E/k)$. Let K/k be a normal extension; the second isomorphism theorem of group theory readily yields that the mapping $L\to G(K/L)$ is a decreasing bijection from the set $\{L;\ K\supseteq L\supseteq k\}$ of all intermediate abstract fields onto the set of all subgroups of $G(K/k)$, and L/k is normal iff $G(K/L)$ is invariant in $G(K/k)$. In case L/k is normal then each $\eta\in G(K/k)$ induces an automorphism $\bar{\eta}=(\eta|L)$ of L/k, and the mapping $\eta\to\bar{\eta}$ is a homomorphism of $G(K/k)$ onto $G(L/k)$ with the kernel $G(K/L)$. So $G(L/k)$ is canonically isomorphic to $G(K/k)/G(K/L)$.

Definition. Let K and K' be abstract endofields with respective base sets E and E'. A mapping η: $K \to K'$ is called a *homomorphism* of K *onto* K' if it is surjective, it is a homomorphism with respect to the infinitary union, all projections, all extensions, all contractions and all dilatations, and, further, it satisfies the following condition

(C) If $r' = \eta \cdot r = \bigcup \cdot R'$ for an arbitrary $r \in K$ and a set $R' \subseteqq K'$ then there exists a mapping θ: $R' \to K$ such that all $\theta \cdot \varrho'$ ($\varrho' \in R'$) have the same argument set, $r = \bigcup \cdot (\theta \cdot R')$ and, for every $\varrho' \in R'$, $\eta \cdot (\theta \cdot \varrho') \subseteqq \varrho'$.

Before formulating and proving a "homomorphism theorem" of abstract Galois endotheory, some special kinds of homomorphisms will be introduced.

1. Representative homomorphisms. Let D be a subsemigroup of $D(E)$, i.e., a semigroup of self-mappings of E. A surjection f: $E \to E'$ will be called a *representation* of D if $f \cdot x = f \cdot y$ implies $f \cdot (\delta \cdot x) = f \cdot (\delta \cdot y)$, for every $x, y \in E$ and $\delta \in D$.

When f is a representation of D and $e' \in E'$ then there is an $e \in E$ such that $e' = f \cdot e$ and $f \cdot (\delta \cdot e)$ does not depend on the particular choice of e. So δ^f: $e' = f \cdot e \to f \cdot (\delta \cdot e)$ is a self-mapping of E' such that $f\delta = \delta^f f$. Clearly, a surjection f: $E \to E'$ is a representation of D if and only if for each $\delta \in D$ there exists a δ^f such that the diagram

$$\text{(D)} \qquad \begin{array}{ccc} E & \xrightarrow{\delta} & E \\ \downarrow f & & \downarrow f \\ E' & \xrightarrow{\delta^f} & E' \end{array}$$

commutes. We will write $D^f = \{\delta^f;\ \delta \in D\}$.

Proposition 1. *Let K be an abstract endofield with base set E, put $D = D(E/K)$, and let f: $E \to E'$ be a representation of D. Then the mapping (f): $r \to f \cdot r$ is a homomorphism of K onto an endofield K' where K' is the endofield determined by the property $D^f = D(E'/K')$.*

These kinds of homomorphisms will be called *representative*.

The proof requires the axiom of choice. By Proposition 1 of Section 3, f commutes with all operations required by the definition of homomorphisms between endofields. For any point P of E we have $(f) \cdot (D \cdot P) = f \cdot (D \cdot P) = fD \cdot P = \{f\delta \cdot P;\ \delta \in D\} = \{\delta^f f \cdot P;\ \delta \in D\} = D^f f \cdot P = D^f \cdot (f \cdot P)$. So the (f)-image of the D-orbit of P is the D^f-orbit of $f \cdot P$. Hence (f) satisfies (C) by Lemma 5. We have seen that (f) is a mapping of K into the abstract endofield K' defined by $D(E'/K') = D^f$. Now it has remained to show that this mapping is surjective, i.e., there is a point P: $X \to E$ such that $f \cdot P$ is a bijective point of E'. Take a bijective point $\tilde{P}'$: $\tilde{X}' \to E'$. As $f \cdot E = E'$, the axiom of choice yields the existence of a mapping $h\colon E' \to E$ such that $f \circ h = 1_{E'}$. So by putting $P = h \cdot \tilde{P}' = h \circ P'$ we have $f \cdot P = f \circ P = f \circ (h \circ \tilde{P}') = (f \circ h) \circ \tilde{P}' = 1_{E'} \circ \tilde{P}' = \tilde{P}'$, completing the proof.

2. *Norms and pseudo-norms.* Let K/k be an extension of abstract endofields with a base set E. For $r \in K$ the set of relations $\varrho \in k$ that includes r (as a subset) is not empty. The intersection of all these ϱ also belongs to k and it is the smallest relation in k that includes r. This relation will be called the *norm* of r in K/k (or, in other words, with respect to k), and will be denoted by $N_{K/k}(r)$. Yet, we need to consider a more general situation, too. Let $\bar{E}$ be a subset of E and let K and k be abstract endofields with respective base sets $\bar{E}$ and E. Put $\bar{D} = D(\bar{E}/K)$, $\Delta = D(E/k)$ and $\Delta_{\bar{E}} = \{\delta \in \Delta;\ \delta \cdot \bar{E} \subseteq \bar{E}\}$. If $\bar{D}$ is a submonoid of $(\Delta_{\bar{E}}|\bar{E}) = \{(\delta|\bar{E});\ \delta \in \Delta_{\bar{E}}\}$ then K will be said to be a *pseudo-extension* of k, and K/k will be called a *pseudo-extension* of abstract endofields. As $\bar{E} \subseteq E$, the relations on $\bar{E}$ are relations on E as well. So, for each $r \in K$ there is a smallest relation in k that includes r, and it will still be denoted by $H_{K/k}(r)$ and called the *pseudo-norm* of r in K/k (or with respect to k). Clearly, $N_{K/k}(r) = \Delta \cdot r$. In particular, if $r = \bar{D} \cdot \bar{P}$ is the $\bar{D}$-orbit of some point $\bar{P}$ of $\bar{E}$ then $N_{K/k}(\bar{D} \cdot \bar{P}) = \Delta \cdot \bar{P}$. It is obvious that the mapping $N_{K/k}\colon r \to N_{K/k}(r)$ is a homomorphism with respect to the infinitary union, all projections, all contractions and all dilatations. But, generally, the pseudo-norm is not a homomorphism with respect to extensions and it is not a surjection of K onto k. (Note that the norm is always a surjection of K onto k since it is the identity mapping when restricted to k.) We shall study necessary and sufficient conditions for $N_{K/k}$ commuting with extensions or being surjective. As the pseudo-norm of a $\bar{D}$-orbit is a Δ-orbit, condition (C) is satisfied by $N_{K/k}$ in virtue of Lemma 5.

Lemma 7. *A pseudo-norm $N_{K/k}$ is a homomorphism with respect to all extensions if and only if $(\Delta|\bar{E}) = \Delta^{(E \to \bar{E})}\bar{D}$ where $\Delta^{(E \to \bar{E})} = \{\delta \in \Delta;\ \delta \cdot \bar{E} = E\}$. In particular, for a norm $N_{K/k}$, iff $\Delta = \Delta^{(s)} \cdot D$ where $D = D(E/K)$ and $\Delta^{(s)}$ is the monoid of all self-surjections of E.*

Proof. (The necessity part requires the axiom of choice.) As pseudo-norms commute with the infinitary unions, it suffices to prove the lemma only for $\bar{D}$-orbits. Let $\bar{P}\colon X \to \bar{E}$, $r = \bar{D} \cdot \bar{P}$, and let X' be a disjoint union $X' = X \dot{\cup} Y$. Then we have $N_{K/k}(r) = \Delta \cdot \bar{P}$, $\operatorname{ext}_{X'} \cdot r = \bar{D} \cdot \bar{P} \times \bar{E}^Y$, $\operatorname{ext}_{X'} \cdot N_{K/k}(r) = \Delta \cdot \bar{P} \times E^Y$ and $N_{K/k}(\operatorname{ext}_{X'} \cdot r) = \Delta \cdot (\bar{D} \cdot \bar{P} \times \bar{E}^Y)$. Denoting by ϱ this last relation, let us calculate it. According to the usual conventions, an X'-point P' will be written as an ordered pair (P, P^*) where $P = (P'|X)$ is an X-point and $P^* = (P'|Y)$ is a Y-point. Then

$$\varrho = \bigcup_{\delta \in \Delta} \delta \cdot (\bar{D} \cdot \bar{P} \times \bar{E}^Y) = \{(\delta \cdot (\bar{\delta} \cdot \bar{P}), \delta \cdot P^*);\ (\delta, \bar{\delta}, P^*) \in \Delta \times \bar{D} \times \bar{E}^Y\} =$$

$$= \bigcup_{\delta \in \Delta} \bigcup_{\bar{\delta} \in \bar{D}} (\{\delta\bar{\delta} \cdot \bar{P}\} \times (\delta \cdot \bar{E})^Y) = \bigcup_{P \in \Delta \cdot \bar{P}} (\{P\} \times \bigcup_{\delta \in \theta(P)} (\delta \cdot \bar{E})^Y)$$

where $\theta(P)=\{\delta\in\Delta;\ (\exists\bar\delta\in\bar D)(\delta\bar\delta\cdot\bar P=P)\}$ and, as $\bar P\cdot X\subseteq\bar E$, $\delta\bar\delta\cdot\bar P=(\delta\bar\delta|\bar E)\cdot\bar P$. If $P\in\Delta\cdot\bar P=(\Delta|\bar E)\cdot\bar P$ then there exists a $\hat\delta$: $\bar E\to E$, $\hat\delta\in(\Delta|\bar E)$, such that $P=\hat\delta\cdot\bar P$. Further, if $\bar P$: $X\to\bar E$ is surjective then this $\hat\delta$ is unique and $\hat\delta\cdot\bar P=\delta\bar\delta\cdot\bar P$ implies $\hat\delta=\delta\bar\delta$. Hence, in this case, $\theta(P)=\theta(\hat\delta)=\{\delta\in\Delta;\ (\exists\bar\delta\in\bar D)(\delta\bar\delta=\hat\delta)\}$. In the general case we have $\theta(\hat\delta)\subseteq\theta(P)$. Since $\mathrm{ext}_{X'}\cdot N_{K/k}(r)=\Delta\cdot\bar P\times E^Y=\bigcup_{P\in\Delta\cdot P}(\{P\}\times E^Y)$, the equality $N_{K/k}(\mathrm{ext}_{X'}\cdot r)=\mathrm{ext}_{X'}\cdot N_{K/k}(r)$ holds for every $X'\supseteq X$ and for every D-orbit $r=\bar D\cdot\bar P$ with argument set X if and only if for every $P\in\Delta\cdot\bar P$ and for every set Y we have $E^Y=\bigcup_{\delta\in\theta(\delta)}(\delta\cdot\bar E)^Y$. When $\bar P$: $X\to\bar E$ is surjective, this condition turns into the following one: for every $\hat\delta\in(\Delta|\bar E)$ and for every set Y we have $E^Y=\bigcup_{\delta\in\theta(\hat\delta)}(\delta\cdot\bar E)^Y$. Note that this later condition implies the former one for each $P\in\Delta\cdot\bar P$. So this is a condition we were looking for, i.e., a necessary and sufficient condition for $N_{K/k}$ commuting with all extensions. Further, this commutativity holds for all $r\in K$ if it holds for the $\bar D$-orbit of only one surjective point $\bar P$. This condition is certainly satisfied if, for each $\hat\delta\in(\Delta|\bar E)$, there exists a $\delta\in\theta(\hat\delta)$ such that $\delta\cdot\bar E=E$, i.e., if $\hat\delta=\delta\bar\delta\in\Delta^{(E\to E)}\bar D$, i.e., if $(\Delta|\bar E)=\Delta^{(E\to E)}\bar D$. But, by Cantor's diagonal method and using the axiom of choice, we will prove that if $\delta\cdot\bar E\neq E$ holds for some fixed $\hat\delta\in(\Delta|\bar E)$ with all $\delta\in\theta(\hat\delta)$, i.e., if $(\Delta|\bar E)\neq\Delta^{(E\to E)}\bar D$, then there exists a Y such that

$$E^Y\neq\bigcup_{\delta\in\theta(\delta)}(\delta\cdot\bar E)^Y.$$

Indeed, if for all $\delta\in\theta(\hat\delta)$ we have $\delta\cdot\bar E\neq E$, take a Y with card $Y\geqq\operatorname{card}\theta(\hat\delta)$. Then there exists an injection ψ: $\theta(\hat\delta)\to Y$. The set $(\delta\cdot\bar E)^Y$ consists of all Y-points Q: $Y\to E$ satisfying $Q\cdot y\in\delta\cdot\bar E$ for any $y\in Y$. But if all $\delta\cdot\bar E$ differ from E then, by the axiom of choice, there is a Y-point Q of E such that $Q\cdot(\psi\cdot\delta)\in\delta\cdot\bar E$ for no $\delta\in\theta(\hat\delta)$. So Q cannot belong to any $(\delta\cdot\bar E)^Y$ and, consequently, does not belong to the union $\bigcup_{\delta\in\theta(\delta)}(\delta\cdot\bar E)^Y$. Hence this union cannot be E^Y.

When the condition of this lemma is satisfied, the corresponding pseudo-norm or norm $N_{K/k}$ is said to be *regular*.

Lemma 8. *The pseudo-norm $N_{K/k}$: $K\to k$ is surjective if and only if there exist a subset $\bar E^*$ of $\bar E$ and $\delta,\delta'\in\Delta$ such that $(\delta|\bar E^*)$: $\bar E^*\to E$ is bijective and $\delta'(\delta|\bar E^*)=1_{E^*}$.* (Note that in case $N_{K/k}$ is a norm, this condition is always satisfied by $\bar E^*=\bar E=E$ and $\delta=\delta'=1_E$.)

Proof. Let $\tilde P$: $\tilde X\to E$ be a bijective point of E, and assume that $N_{K/k}$ is surjective. Then there is an $r\in K$ such that $N_{K/k}(r)=\Delta\cdot\tilde P$. Further, by Lemma 2, this r can be chosen to be a $\bar D$-orbit $\bar D\cdot\bar P$. But then $N_{K/k}(\bar D\cdot\bar P)=\Delta\cdot\bar P$ implies $\Delta\cdot\bar P=$ $=\Delta\cdot\tilde P$. As $\bar P\in\Delta\cdot\bar P$ and $\tilde P\in\Delta\cdot\tilde P$, this equality yields the existence of some δ

and δ' in Δ such that $\tilde{P}=\delta\cdot\bar{P}$ and $\bar{P}=\delta'\cdot\tilde{P}$. Since $\Delta\delta$ and $\Delta\delta'$ are subsets of Δ, the existence of these δ and δ' is, in fact, equivalent to the equation $\Delta\cdot\bar{P}=\Delta\cdot\tilde{P}$. The point $\bar{P}$ is a mapping of $\tilde{X}$ into $\bar{E}$, whence $\bar{E}^*=\bar{P}\cdot\tilde{X}\subset\bar{E}$. So $\delta\cdot\bar{E}^*=\delta\cdot(\bar{P}\cdot\tilde{X})=$ $=(\delta\cdot\bar{P})\cdot\tilde{X}=\tilde{P}\cdot\tilde{X}=E$, and from the injectivity of $\tilde{P}=\delta\cdot\bar{P}=(\delta|\bar{P}\cdot\tilde{X})\cdot\bar{P}=(\delta|\bar{E}^*)\cdot\bar{P}$ we obtain that both $\bar{P}$ and $(\delta|\bar{E}^*)$ must be injective. That is, $(\delta|\bar{E}^*)$ is a bijection of $\bar{E}^*$ onto E. We have $\delta'(\delta|\bar{E}^*)\cdot\bar{P}=\delta'\cdot((\delta|\bar{E}^*)\cdot\bar{P})=\delta'\cdot(\delta\cdot\bar{P})=\delta'\cdot\tilde{P}=\bar{P}$. As $\bar{P}$ is injective and $\bar{E}^*=\bar{P}\cdot\tilde{X}$, we have $\delta'(\delta|\bar{E}^*)=1_{\bar{E}^*}$. Conversely, let $\bar{E}^*$, δ and δ' satisfy the conditions of the lemma. Then, as $\delta\in\Delta^{(\bar{E}^*\to E)}$, we have $(\delta|\bar{E}^*)\cdot\bar{E}^*=E$, and $\delta'(\delta|\bar{E}^*)=1_{\bar{E}^*}$ implies $\delta'\cdot E=\bar{E}^*\subseteqq\bar{E}$. So $\bar{P}=\delta'\cdot\tilde{P}$ is a point of $\bar{E}$, because $\bar{P}\cdot\tilde{X}=(\delta'\cdot\tilde{P})\cdot\tilde{X}=\delta'\cdot(\tilde{P}\cdot\tilde{X})=\delta'\cdot E=\bar{E}^*$. Now $\delta'(\delta|\bar{E}^*)=1_{\bar{E}^*}$ yields that both δ' and $(\delta|\bar{E}^*)$ are bijective and they are inverses of each other. So $(\delta|\bar{E}^*)\delta'=1_E$ and $\delta\cdot\bar{P}=$ $=(\delta|\bar{E}^*)\cdot(\delta'\cdot\tilde{P})=(\delta|\bar{E}^*)\delta'\cdot\tilde{P}=1_E\cdot\tilde{P}=\tilde{P}$ and $N_{K/k}(\bar{D}\cdot\bar{P})=\Delta\cdot\tilde{P}$. Since $\Delta\cdot\tilde{P}$ generates k (cf. Remark 4 in Section 4), $N_{K/k}$ is surjective, indeed.

A pseudo-norm satisfying the conditions of Lemma 8 will be called a *quasi-norm* while the corresponding pseudo-extension will be called a *quasi-extension*. In particular, norms are always quasi-norms. We have seen that a pseudo-norm is a homomorphism of endofields iff it is a regular quasi-norm.

Remark 3. Let K/k be a pseudo-extension, and let E and $\bar{E}\subseteqq E$ be the base sets of k and K, respectively. Assume further that K/k is either a regular or a quasi-extension. Then card $\bar{E}=$card E.

To check this remark it is sufficient to show that card $\bar{E}\geqq$card E. If K/k is regular and $\Delta=D(E/k)$ then $\Delta^{(\bar{E}\to E)}$ is not empty. Hence, by the axiom of choice, the assertion follows. In the other case, when K/k is a quasi-extension, there are a set $\bar{E}^*\subseteqq\bar{E}$ and a $\delta\in\Delta$ such that $(\delta|\bar{E}^*)$ is a bijection of $\bar{E}^*$ onto E and card $E=$card $\bar{E}^*\leqq$card $\bar{E}$.

Theorem. (Homomorphism theorem of abstract Galois endotheory.) *Let K and K' be abstract endofields with base sets E and E' and endomorphism monoids $D=D(E/K)$ and $D'=D(E'/K')$, respectively. Let η: $K\to K'$ be a homomorphism of K onto K'. Then there is a representation f: $E\to\bar{E}'\subseteqq E'$ of D such that $(f\cdot K)/K'$ is a regular quasi-extension and $\eta=N_{(f\cdot K)/K'}\circ(f)$.*

Proof. Let $\tilde{P}$: $\tilde{X}\to E$ be a bijective point. Then, by Lemma 5, η maps the D-orbit $D\cdot\tilde{P}$ onto the D'-orbit $D'\cdot P'$ of some $\tilde{X}$-point P': $\tilde{X}\to E'$. Let $f=P'\tilde{P}^{-1}$: $E\to E'$ and $\bar{E}'=P'\cdot\tilde{X}=f\cdot E$. Clearly, f is a surjection of E onto $\bar{E}'\subseteqq E'$ and $P'=f\cdot P$.

Let an arbitrary δ belong to D. Then $\delta\cdot\tilde{P}=[\varepsilon_{\delta\cdot P,\bar{P}}]\cdot(\tilde{P}|\bar{X})$ where $\bar{X}=\tilde{P}^{-1}\cdot(\delta\cdot E)$. But the D-orbit of $\delta\cdot\tilde{P}$ is $D\cdot(\delta\cdot\tilde{P})=D\delta\cdot\tilde{P}\subseteqq D\cdot\tilde{P}$ as $D\delta\subseteqq D$. Since all mappings commute with projections and dilatations, we have $D\cdot(\delta\cdot\tilde{P})=D\cdot([\varepsilon_{\delta\cdot P,\bar{P}}]\,\mathrm{pr}_{\bar{X}}\cdot\tilde{P})=$ $=[\varepsilon_{\delta\cdot P,\bar{P}}]\,\mathrm{pr}_{\bar{X}}\cdot(D\cdot\tilde{P})$. But η is a homomorphism with respect to the same operations. So

$$\eta\cdot(D\cdot(\delta\cdot\tilde{P}))=[\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\,\mathrm{pr}_{\bar{X}}\cdot(\eta\cdot(D\cdot\tilde{P}))=$$
$$=[\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\,\mathrm{pr}_{\bar{X}}\cdot(D'\cdot P')=D'\cdot([\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\cdot(P'|\bar{X})).$$

As the mapping f also commutes with projections and dilatations, we have

$$[\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\cdot(P'|\bar{X})=[\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\cdot(f\cdot\tilde{P}|\bar{X})=$$
$$=f\cdot([\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\cdot(\tilde{P}|\bar{X}))=f\cdot(\delta\cdot\tilde{P})=f\delta\cdot\tilde{P},$$

whence $f\delta\cdot\tilde{P}\in\eta\cdot(D\cdot(\delta\cdot\tilde{P}))$. But, as $D\cdot(\delta\cdot\tilde{P})\subseteqq D\cdot\tilde{P}$ and η is a homomorphism with respect to $\cup$, Lemma 2 yields $\eta\cdot(D\cdot(\delta\cdot\tilde{P}))\subseteqq\eta\cdot(D\cdot\tilde{P})=D'\cdot P'=D'f\cdot\tilde{P}$. So there exists a $\delta'\in D'$ such that $f\delta\cdot\tilde{P}=\delta' f\cdot\tilde{P}=(\delta'|\bar{E}')f\cdot\tilde{P}$. The bijectivity of $\tilde{P}$ implies $f\delta=(\delta'|\bar{E}')f$. On the other hand, as $f\cdot E=\bar{E}'$ and $f\delta\cdot E=f\cdot(\delta\cdot E)\subseteqq\subseteqq f\cdot E=\bar{E}'$, we have $(\delta'|\bar{E}')\cdot\bar{E}'=(\delta'|\bar{E}')f\cdot E=f\delta\cdot E\subseteqq\bar{E}'$. That is, $(\delta'|\bar{E}')$ is a self-mapping of $\bar{E}'$. By this we have seen that, for each $\delta\in D$, there is a self-mapping $\bar{\delta}'$ of $\bar{E}'$ making the diagram (D) commutative, i.e., satisfying $f\delta=\bar{\delta}'f$. So f is a representation of D, and the preceeding $(\delta'|\bar{E}')$ is just δ^f. Further, as $\delta'\in D'$ and $\delta'\cdot\bar{E}'\subseteqq\bar{E}'$, we have that $(\delta'|\bar{E}')\in(D'_{\bar{E}'}|\bar{E}')$ and $D^f\subseteqq(D'_{\bar{E}'}|\bar{E}')$. Therefore $(f\cdot K)/K'$ is a pseudo-extension.

Consider a D-orbit $D\cdot P$. We have

$$D\cdot P=D\cdot([\varepsilon_{P,\tilde{P}}]\cdot(\tilde{P}|\tilde{X}_P))=[\varepsilon_{P,\tilde{P}}]\,\mathrm{pr}_{\tilde{X}_P}\cdot(D\cdot\tilde{P}).$$

So

$$\eta\cdot(D\cdot P)=[\varepsilon_{P,\tilde{P}}]\,\mathrm{pr}_{\tilde{X}_P}\cdot(D'\cdot P')=D'\cdot([\varepsilon_{P,\tilde{P}}]\,\mathrm{pr}_{\tilde{X}_P}\cdot(f\cdot\tilde{P}))=$$
$$=D'\cdot(f\cdot([\varepsilon_{P,\tilde{P}}]\,\mathrm{pr}_{\tilde{X}_P}\cdot\tilde{P}))=D'f\cdot P\supseteqq D^f f\cdot P=(f)\cdot(D\cdot P).$$

So, for any D-orbit $r=D\cdot P$, $\eta\cdot r$ is a D'-orbit containing the D^f-orbit $D^f\cdot(f\cdot P)=$ $=(f)\cdot r$, whence it is the least relation in K' containing $(f)\cdot r$. Therefore $\eta\cdot r=$ $=N_{(f\cdot K)/K'}((f)\cdot r)$; the same is true for every $r\in K$ since r is a union of D-orbits and both f and η commute with this union. We have seen that $\eta=N_{(f\cdot K)/K'}\circ(f)$.

It is easy to see that $N_{(f\cdot K)/K'}$ is a surjective mapping of $f\cdot K$ into K'; really, if $r'\in K'$ then there is an $r\in K$ such that $r'=\eta\cdot r=N_{(f\cdot K)/K'}((f)\cdot r)$ and $(f)\cdot r=$ $=f\cdot r\in f\cdot K$. So $(f\cdot K)/K'$ is a quasiextension and $N_{(f\cdot K)/K'}$ is a quasi-norm. In order to show that it is also regular, let Y be arbitrary. Then

$$\eta\cdot(D\cdot\tilde{P}\times E^Y)=\eta\cdot(D\cdot\tilde{P})\times E'^Y=N_{(f\cdot K)/K'}(f\cdot(D\cdot\tilde{P}))\times E'^Y$$

and, as (f) commutes with dilatations,

$$\eta\cdot(D\cdot\tilde{P}\times E^Y)=N_{(f\cdot K)/K'}((f)\cdot(D\cdot\tilde{P}\times E^Y))=N_{(f\cdot K)/K'}(f\cdot(D\cdot\tilde{P})\times\bar{E}'^Y).$$

So $N_{(f\cdot K)/K'}(f\cdot(D\cdot\tilde{P})\times\bar{E}'^Y)=N_{(f\cdot K)/K'}(f\cdot(D\cdot\tilde{P}))\times\bar{E}'^Y$. But $f\cdot(D\cdot\tilde{P})=fD\cdot\tilde{P}=$ $=D^f f\cdot\tilde{P}=D^f\cdot(f\cdot\tilde{P})$, and $f\cdot\tilde{P}\colon\tilde{X}\to\bar{E}'$ is a surjective point of $\bar{E}'$. So $N_{(f\cdot K)/K'}$ commutes with all extensions of the D^f-orbit of some surjective point, and we have seen in the proof of Lemma 7 that then the same is true for every relation $r\in f\cdot K$. Hence $N_{(f\cdot K)/K'}$ is regular. The proof of the homomorphism theorem of Galois endotheory is complete.

It would be interesting to see what the possible decompositions of a given homomorphism $\eta\colon K\to K'$, as products of a quasi-norm and a representative homomorphism, are. The formulate a result of this kind, let $\eta=N_{(f\cdot K)/K'}\circ(f)$ be one of these decompositions, and put $\bar{E}'=f\cdot E$. Then we have

Proposition 2. *Let*

$$\Delta'=\{\delta'\in D'=D(E'/K');\ (\exists\delta''\in D')((\delta''\delta'|\bar{E}')=(\delta''|\delta'\cdot\bar{E}')(\delta'|\bar{E}')=1_{E'})\}.$$

Then the set of all desired decompositions of η *is* $\{N_{(\delta'f\cdot K)/K'}\circ(\delta'f);\ \delta'\in\Delta'\}$.

Proof. Let P'' be some generating point of $\eta\cdot(D\cdot\tilde{P})=D'\cdot P'$, i.e., let P'' be a point with the property $D'\cdot P''=D'\cdot P'$. By the preceeding proof, if $f'=P''\tilde{P}^{-1}$ then $\eta=N_{(f'\cdot K)/K'}\circ(f')$. Now let $\eta=N_{(f'\cdot K)/K'}\circ(f')$ where f' is a representation of $D=D(E/K)$. Then $(f')\cdot(D\cdot\tilde{P})=D^{f'}\cdot(f'\cdot\tilde{P})$ and $D'\cdot P'=\eta\cdot(D\cdot\tilde{P})=$ $=N_{(f'\cdot K)/K'}(D^{f'}\cdot(f'\cdot\tilde{P}))=D'\cdot(f'\cdot\tilde{P})$. So $P''=f'\cdot\tilde{P}$ is a generating point of $\eta\cdot(D\cdot\tilde{P})$ and $f'=P''\tilde{P}^{-1}$. Therefore the considered decompositions correspond to different generating points of $\eta\cdot(D\cdot\tilde{P})$.

But if $P''\in\eta\cdot(D\cdot\tilde{P})=D'\cdot P'$ then there exists a $\delta'\in D'$ such that $P''=\delta'\cdot P'=$ $=\delta'f\cdot\tilde{P}$. Further, P'' is a generating point of $D'\cdot P'$ iff there exists a $\delta''\in D'$ such that $\delta''\cdot P''=P'$. As $P'\cdot\tilde{X}=\bar{E}'$, this means that $(\delta''\delta'|\bar{E}')=1_{E'}$. In this case $f'=P''\tilde{P}^{-1}=(\delta'\cdot P')\tilde{P}^{-1}=\delta'\circ(P'\tilde{P}^{-1})=\delta'\circ f=\delta'f$. Thus the proposition is proved.

Case of finite base sets. In the finite base set case $\operatorname{card}\bar{E}'=\operatorname{card}E'$ and $\operatorname{card}\bar{E}'\leq\operatorname{card}\tilde{X}=\operatorname{card}\bar{E}$ imply that $\bar{E}'=E'$. So every quasi-norm is a norm. If $\eta\colon K\to K'$ is a homomorphism then, consequently, *we have* $\eta=N_{(f\cdot K)/K'}\circ(f)$ *where* $N_{(f\cdot K)/K'}$ *is a regular norm and* $f\colon E\to E'$ *is a representation of* D.

Proposition 3 (P. Lecomte). *If* $\eta=N_{(f\cdot K)/K'}\circ(f)$ *is a* $K\to K'$ *homomorphism such that* $f\colon E\to E'$ *is a representation of* $D=D(E/K)$ *(i.e.,* $N_{(f\cdot K)/K'}$ *is a regular norm) and if* η *is bijective then* f *is bijective, too, and* $f\cdot K=K'$. *That is, in this case* η *is a transportation of structures and, in particular, it is an isomorphism of* K *onto* K'.

Proof. If η is bijective then so is (f), too. But if $f\colon E\to E'$ is not bijective then there are $e_1, e_2\in E$, $e_1\neq e_2$, such that $f\cdot e_1=f\cdot e_2$. Let $\tilde{P}\colon\tilde{X}\to E$ be a bijective point and $x_1, x_2\in\tilde{X}$ such that $\tilde{P}\cdot x_1=e_1$ and $\tilde{P}\cdot x_2=e_2$. Consider the point $P\colon\tilde{X}\to E$ defined by $(P|\tilde{X}\setminus\{x_1,x_2\})=(\tilde{P}|\tilde{X}\setminus\{x_1,x_2\})$ and $P\cdot x_1=P\cdot x_2=e_1$. The D-orbits of P and $\tilde{P}$ are different, because $\tilde{P}\in D\cdot\tilde{P}$ is injective but no point in $D\cdot P$ can be injective. But if $e'=f\cdot e_1=f\cdot e_2$ then $(f)\cdot(D\cdot\tilde{P})=D^f\cdot(f\cdot\tilde{P})$ and $(f)\cdot(D\cdot P)=$ $=D^f\cdot(f\cdot P)$ coincide since for any $x\in\tilde{X}\setminus\{x_1,x_2\}$ we have $(f\cdot\tilde{P})\cdot x=f\cdot(\tilde{P}\cdot x)=$

$=f\cdot(P\cdot x)=(f\cdot P)\cdot x$ while $(f\cdot\tilde{P})\cdot x_i=f\cdot(\tilde{P}\cdot x_i)=f\cdot e_i=e'$ and $(f\cdot P)\cdot x_i= =f\cdot(P\cdot x_i)=f\cdot e_1=e'$, $i=1, 2$. That is, when f is not injective then (f) is not either.

We have seen that (f): $K\to f\cdot K$ is a bijection, whence it is a surjection of K onto $f\cdot K$. But then $N_{(f\cdot K)/K'}$: $f\cdot K\to K'$ must be a bijection of $f\cdot K$ onto K'. If $r\in f\cdot K$ then $N_{(f\cdot K)/K'}\cdot r=N_{(f\cdot K)/K'}(N_{(f\cdot K)/K'}\cdot r)$ and the injectivity of $N_{(f\cdot K)/K'}$ yields $r=N_{(f\cdot K)/K'}\cdot r$. Thus $N_{(f\cdot K)/K'}$ is the identity mapping of K and $K'= =N_{(f\cdot K)/K'}\cdot K=K$. So $\eta=(f)$ is a transportation of structures. (Hence η is an isomorphism and so is η^{-1}.)

Consequence. *If K is an abstract endofield with finite base set then every bijective homomorphism of K is a transportation of structure (whence it is an isomorphism).*

Indeed, every quasi-norm is a norm in this case.

To close this paragraph we mention some open problems. Given an arbitrary bijective homomorphism η: $K\to K'$ of abstract endofields, is it always true that

(α) it is a transportation of structure?

(β) it is an isomorphism?

(γ) η^{-1} is a homomorphism of K' onto K?

and

(δ) the condition (γ) implies (α) and (β)? (In other words, can a regular quasi-norm be injective without being a representative homomorphism?)

6. Abstract Galois set theory

Let k be an abstract endofield on E. For $e\in E$ the relation $(x; e)=\{\{x\to e\}\}$ is independent of the particular choice of x up to restricted floating equivalence; and it will be identified with e if considered modulo this equivalence. So the endo-extension of k generated by $(x; A)=\{(x; a); a\in A\}$ does not depend on the choice of x; it will be denoted by $k(A)$ and called the *set-extension of k generated by A* (or *by the adjunction of A*). Extensions of the form $k(A)/k$, where $A\subseteqq E$, are called set extensions; their study is called *abstract Galois set theory*. One of the main problems in this theory is to describe the set $\bar{A}_k=\{\bar{a}\in E;\ (x; \bar{a})\in k(A)\}$ in terms of k and A. This set $\bar{A}_k$ will be called the *rationality domain* of $k(A)$. Clearly, $\bar{A}_k$ is the set of all $e\in E$ preserved by every $\delta\in D(E/k(A))$ that fixes the points of A. Another problem, which has been studied only in some particular cases and will not be considered here, is to characterize the monoids of the form $D(E/k(A))$ or groups of the form $G(E/k(A))$ where $A\subseteqq E$.

Theorem 1. *Let A be a subset of E. The set-extension $k(A)$ is the class of all relations that are (infinitary) unions of relations of the form*

$$(\varrho)\qquad \mathrm{pr}_X\cdot\Big(r\cap\Big(\bigcap_{x\in X_0}\mathrm{ext}_X\cdot(x;\ \theta\cdot x)\Big)\Big)$$

where $\bar{X}$ is an argument set, $\bar{X}$ and $\bar{X}_0$ are subsets of X, $X=\bar{X}\cup X_0$, r is an X-relation in k, and θ: $X_0\to A$ is a mapping of X_0 into A.

Proof. Clearly, every relation of the considered form belongs to $k(A)$ as we have only used direct fundamental operations to obtain it from $r\in k\subseteq k(A)$ and from certain $(x; a)$, $a\in A$. Further, every $r\in k$ is of this form (take $\bar{X}_0=\emptyset$ and $\bar{X}=X$), and so is every $(x; a)$, $a\in A$ (take $X=\{x\}=\bar{X}=X_0$ and θ: $x\to a$). So, to prove the theorem, it suffices to show that the considered class of relations is closed with respect to all direct fundamental operations. But before doing so let us make some remarks.

Remark 1. Let $X\supseteq X'\supseteq X_0$, let r be an X-relation on E, and let r_0 be an X_0-relation on E. Then $\mathrm{pr}_{X'}\cdot(r\cap \mathrm{ext}_X\cdot r_0)=\mathrm{pr}_{X'}\cdot r\cap \mathrm{ext}_{X'}\cdot r_0$.

Indeed, let P': $X'\to E$ be an X'-point. We have $P'\in \mathrm{pr}_{X'}\cdot(r\cap \mathrm{ext}_X\cdot r_0)$ iff there exists an X-point $P\in r$ such that $P'=(P|X')$ and $(P|X_0)\in r_0$. But this means that $P'\in \mathrm{pr}_{X'}\cdot r$ and $(P|X_0)=((P|X')|X_0)=(P'|X_0)$. So the additional condition $(P|X_0)\in r_0$ is equivalent to $(P'|X_0)\in r_0$, proving the remark.

Remark 2. A relation of the form (ϱ) but hurting the condition $X=\bar{X}\cup X_0$ can be represented as a relation fully being of the form (ϱ).

Indeed, as $\mathrm{ext}_X\cdot(x; \theta\cdot x)=\mathrm{ext}_X^{X_0}\,\mathrm{ext}_{X_0}\cdot(x; \theta\cdot x)$ and

$$\bigcap_{x\in X_0} \mathrm{ext}_X\cdot(x;\ \theta\cdot x) = \mathrm{ext}_X^{X_0}\cdot \bigcap_{x\in X_0} \mathrm{ext}_{X_0}\cdot(x;\ \theta\cdot x),$$

the relation (ϱ) can also be written as $\mathrm{pr}_{\bar{X}}\cdot(r\cap \mathrm{ext}_X\cdot r_0)$ with $r_0=\bigcap_{x\in X_0}\cdot(x;\theta\cdot x)$. Suppose $X\neq\bar{X}\cup X_0$ and put $X'=\bar{X}\cup X_0\supseteq X_0$. Then, by Remark 1,

$$\mathrm{pr}_{\bar{X}}\cdot(r\cap \mathrm{ext}_X\cdot r_0) = \mathrm{pr}_{\bar{X}}^{X'}\,\mathrm{pr}_{X'}\cdot(r\cap \mathrm{ext}_X\cdot r_0) =$$

$$= \mathrm{pr}_{\bar{X}}^{X'}\cdot(\mathrm{pr}_{X'}\cdot r\cap \mathrm{ext}_{X'}\cdot r_0) = \mathrm{pr}_{\bar{X}}\cdot(\mathrm{pr}_{X'}\cdot r\cap(\bigcap_{x\in X_0} \mathrm{ext}_{X'}\cdot(x;\ \theta\cdot x))).$$

Since $\mathrm{pr}_{X'}\cdot r\in k$, this last expression is also of the form (ϱ) with $X'=\bar{X}\cup X_0$ instead of X.

Remark 3. Let r be an X-relation, and let $\bar{X}\subseteq X$, $X\cap Y=\emptyset$. Then $\mathrm{ext}_{\bar{X}\cup Y}\,\mathrm{pr}_{\bar{X}}\cdot r= =\mathrm{pr}_{\bar{X}\cup Y}\,\mathrm{ext}_{X\cup Y}\cdot r$.

Indeed, as $\bar{X}\cap Y=\emptyset$, an $(\bar{X}\cup Y)$-point $\bar{P}^*$ can be represented by a uniquely determined pair $(\bar{P}, Q)$ where $\bar{P}$ is an $\bar{X}$-point, Q is a Y-point, and $\bar{P}^*=(\bar{P}, Q)\in \mathrm{ext}_{\bar{X}\cup Y}\,\mathrm{pr}_{\bar{X}}\cdot r$ is equivalent to $\bar{P}\in \mathrm{pr}_{\bar{X}}\cdot r$. But then $(\bar{P}, Q)=((P, Q)|\bar{X}\cup Y)$, and $P\in r$ is equivalent to $P^*=(P, Q)\in \mathrm{ext}_{X\cup Y}\cdot r$. So $\bar{P}^*\in \mathrm{ext}_{\bar{X}\cup Y}\,\mathrm{pr}_{\bar{X}}\cdot r$ is equivalent to the existence of an $(X\cup Y)$-point P^* such that $\bar{P}^*=(P^*|\bar{X}\cup Y)$ and $P^*\in \mathrm{ext}_{X\cup Y}\cdot r$, i.e., to $\bar{P}^*\in \mathrm{pr}_{\bar{X}\cup Y}\,\mathrm{ext}_{X\cup Y}\cdot r$. This proves the remark.

Now we can start to prove the theorem. The closedness with respect to the infinitary union needs no argument. As infinitary intersections distribute over infinitary unions and the rest of the direct fundamental operations commute with the infinitary union, it will be sufficient to prove that these direct fundamental operations applied to relations of the form (ϱ) yield relations of the same form.

(α) The case of the infinitary intersection. Let us have a set R of relations with a common argument set $\bar{X}$ and assume that each $\varrho \in R$ is equal to

$$\mathrm{pr}_{\bar{X}} \cdot (r(\varrho) \cap (\bigcap_{x \in X_0(\varrho)} \mathrm{ext}_{X(\varrho)} \cdot (x; \theta_\varrho \cdot x)))$$

where $r(\varrho) \in k$ is an $X(\varrho)$-relation, $X_0(\varrho) \subseteq X(\varrho) = \bar{X} \cup X_0(\varrho)$ and θ_ϱ is a mapping of $X_0(\varrho)$ into A. We will write $\varrho^* = r(\varrho) \cap (\bigcap_{x \in X_0(\varrho)} \cdot (\mathrm{ext}_{X(\varrho)} \cdot (x; \theta_\varrho \cdot x)))$. By Lemma 1 of Section 2 we have

$$\cap \cdot R = \bigcap_{\varrho \in R} \mathrm{pr}_{\bar{X}} \cdot \varrho^* = \mathrm{pr}_{\bar{X}} \cdot (\cap_f^{(\bar{X})} \cdot R^*)$$

where $\cap_f^{(\bar{X})}$ denotes the semi-free intersection of anchor $\bar{X}$ and $R^* = \{\varrho^*; \varrho \in R\}$. Let us study this semi-free intersection. Without changing ϱ, let us float the arguments in $X(\varrho) \setminus \bar{X}$ so that the sets $Y(\varrho) = X(\varrho) \setminus \bar{X}$ become pairwise disjoint; we can assume that this has already been done. Then $\cap_f^{(\bar{X})}$ turns, up to canonical identification, into the ordinary intersection. On the other hand, a floatage (φ) of ϱ^* does not affect the form of this relation; really, we have

$$(\varphi) \cdot \varrho^* = (\varphi) \cdot r(\varrho) \cap (\bigcap_{x \in X_0(\varrho)} \mathrm{ext}_{\varphi \cdot X(\varrho)} \cdot (\varphi \cdot x;\ \theta_\varrho \cdot x)) =$$

$$= (\varphi) \cdot r(\varrho) \cap (\bigcap_{y \in \varphi \cdot X_0(\varrho)} \mathrm{ext}_{\varphi \cdot X(\varrho)} \cdot (y, \theta_\varrho \varphi^{-1} \cdot y)).$$

As $x \in X(\varrho) \setminus \bar{X}$ are the only floating arguments, the previous floatage preserves $\bar{X}$ and $\theta_\varrho \varphi^{-1} \cdot x = \theta_\varrho \cdot x$ holds for every $x \in \bar{X}$. Suppose that this preliminary floatage has already been done and let us return to the previous notations. Let $Y(\varrho) = X(\varrho) \setminus \bar{X}$, $\bar{X}_0(\varrho) = \bar{X}_0(\varrho) \cap \bar{X}$, $Y = \bigcup_{\varrho \in R} Y(\varrho)$, $\bar{X}_0 = \bigcup_{\varrho \in R} \bar{X}_0(\varrho)$, $X_0 = \bar{X}_0 \dot{\cup} Y$ and $X = \bigcup_{\varrho \in R} X(\varrho) = \bar{X} \dot{\cup} Y$ where $\dot{\cup}$ stands for the disjoint union. We have

$$\cap_f^{(\bar{X})} \cdot R^* = \bigcap_{\varrho \in R} \mathrm{ext}_X \cdot \varrho^* = \bigcap_{\varrho \in R} (\mathrm{ext}_X \cdot r(\varrho) \cap (\bigcap_{x \in X_0(\varrho)} \mathrm{ext}_X \cdot (x;\ \theta_\varrho \cdot x))) =$$

$$= (\bigcap_{\varrho \in R} \mathrm{ext}_X \cdot r(\varrho)) \cap (\bigcap_{\varrho \in R} \bigcap_{x \in X_0(\varrho)} \mathrm{ext}_X \cdot (x;\ \theta_\varrho \cdot x)).$$

For $x \in \bar{X}_0$, let $R(x) = \{\varrho \in R;\ x \in \bar{X}_0(\varrho)\}$. Then the preceding expression turns into

$$(\bigcap_{\varrho \in R} \mathrm{ext}_X \cdot r(\varrho)) \cap (\bigcap_{x \in \bar{X}_0} \bigcap_{\varrho \in R(x)} \mathrm{ext}_X \cdot (x;\ \theta_\varrho \cdot x)) \cap (\bigcap_{\varrho \in R} \bigcap_{x \in Y(\varrho)} \mathrm{ext}_X \cdot (x;\ \theta_\varrho \cdot x)).$$

If this relation is empty then it belongs to k, and so does its $\bar{X}$-projection $\cap \cdot R$. If $\cap \cdot R \neq \emptyset$ then, for every $x \in \bar{X}_0$, $\bigcap_{\varrho \in R(x)} \mathrm{ext}_X^{\{x\}} \cdot (x; \theta_\varrho \cdot x) = \mathrm{ext}_X \cdot \bigcap_{\varrho \in R(x)} (x; \theta_\varrho \cdot x) \neq \emptyset$ and $\bigcap_{\varrho \in R(x)} (x; \theta_\varrho \cdot x) \neq \emptyset$. This means that $\theta_\varrho \cdot x$ does not depend on $\varrho \in R(x)$, so it will be denoted by $\theta \cdot x$. As Y is the disjoint union of $Y(\varrho)$, $\varrho \in R$, each $y \in Y$ belongs to exactly one $Y(\varrho)$; let $\theta \cdot y$ stand for the corresponding $\theta_\varrho \cdot y$. So $\theta\colon x \to \theta \cdot x$ is a mapping of $X_0 = \bar{X}_0 \cup Y$ into A, and $\bigcap_{\varrho \in R(x)} (x; \theta_\varrho \cdot x) = (x; \theta \cdot x)$ for $x \in \bar{X}_0$ while $(x; \theta_\varrho \cdot x) = (x; \theta \cdot x)$ for $x \in Y$. Therefore, in case $\cap \cdot R \neq \emptyset$, we have

$$\cap_f^{(\bar{X})} \cdot R^* = \Big(\bigcap_{\varrho \in R} \mathrm{ext}_X \cdot r(\varrho)\Big) \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x; \theta \cdot x)\Big).$$

Hence, putting $r = \bigcap_{\varrho \in R} \mathrm{ext}_X \cdot r(\varrho) \in k$, we have $\cap \cdot R = \mathrm{pr}_{\bar{X}} \cdot \big(r \cap \big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x; \theta \cdot x)\big)\big)$. Consequently, $\cap \cdot R$ is of the form (ϱ). Besides, we have $X_0 \cup \bar{X} \supseteq Y \cup \bar{X} = X$, implying $X = X_0 \cup \bar{X}$.

(β) Projections. For $\hat{X} \subseteq \bar{X}$,

$$\mathrm{pr}_{\hat{X}} \cdot \Big(\mathrm{pr}_{\bar{X}} \cdot \Big(r \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x;\ \theta \cdot x)\Big)\Big)\Big) = \mathrm{pr}_{\hat{X}} \cdot \Big(r \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x;\ \theta \cdot x)\Big)\Big),$$

which is of the form (ϱ) again except that $\hat{X} \cup X_0$ may differ from X. But this is not essential by Remark 2 of this section.

(γ) Extensions. If $\bar{X}' \supseteq \bar{X}$ then, by Remark 3 of this section,

$$\mathrm{ext}_{\bar{X}'}^{\bar{X}} \mathrm{pr}_{\bar{X}} \cdot \Big(r \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x; \theta \cdot x)\Big)\Big) = \mathrm{pr}_{\bar{X}'} \cdot \Big(\mathrm{ext}_{X \cup \bar{X}'} \cdot \Big(r \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x; \theta \cdot x)\Big)\Big)\Big) =$$
$$= \mathrm{pr}_{\bar{X}'} \cdot \Big(\mathrm{ext}_{X \cup \bar{X}'} \cdot r \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_{X \cup \bar{X}'} \cdot (x;\ \theta \cdot x)\Big)\Big).$$

(ϱ) Contractions. Let $\varrho = \mathrm{pr}_{\bar{X}} \cdot \varrho^*$, where $\varrho^* = r \cap \big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x; \theta \cdot x)\big)$, and let the contraction $(\bar{\varphi}) = (\bar{\varphi}\colon \bar{X} \to \bar{Y})$ be applicable to ϱ. Let $T(\bar{\varphi})$ be the type of $\bar{\varphi}$, and let $\varphi\colon X \to Y = \bar{Y} \cup (X \setminus \bar{X})$ be a surjection such that $(\varphi | \bar{X}) = \bar{\varphi}$ and $(\varphi | (X \setminus \bar{X}))$ is the identity. ($\bar{Y}$ and $X \setminus \bar{X}$ are assumed to be disjoint as otherwise we may perform a floatage of the arguments in $X \setminus \bar{X}$.) Now $T(\bar{\varphi})$ and $T(\varphi)$ coincide on $\bar{X}$, and $T(\varphi)$ induces the discrete (i.e. the smallest) equivalence on $X \setminus \bar{X}$. As $\mathrm{pr}_{\bar{X}} \cdot \varrho^*$ is compatible with $\bar{\varphi}$, ϱ^* is compatible with $T(\varphi)$. So

$$\varrho^* = (\varrho^* \cap I_{T(\varphi)}(E)) = (r \cap I_{T(\varphi)}(E)) \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x, \theta \cdot x)\Big),$$

and $r' = r \cap I_{T(\varphi)}(E)$ is also an X-relation of k. If $\varrho^* \neq \emptyset$ then we have $\theta \cdot x_1 = \theta \cdot x_2$ for $x_1, x_2 \in X_0 \setminus \bar{X}$, since otherwise $I_{T(\varphi)}(E) \cap \mathrm{ext}_X \cdot (x_1; \theta \cdot x_1) \cap \mathrm{ext}_X \cdot (x_2; \theta \cdot x_2)$ would be empty. As (φ) commutes with $\cap$ and $(\varphi) \cdot I_{T(\varphi)}(E) = E^Y$, we have

$$(\bar{\varphi}) \cdot \varrho = \mathrm{pr}_{\bar{\varphi}, \bar{X}} \cdot ((\varphi) \cdot \varrho^*) = \mathrm{pr}_{\bar{Y}} \cdot \Big((\varphi) \cdot r' \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_Y \cdot (\varphi \cdot x;\ \theta \cdot x)\Big)\Big).$$

This means that $(\bar\varphi)\cdot\varrho$ is of the form (ϱ), where $X, \bar X$ and X_0 are replaced by $Y=\varphi\cdot X=\bar\varphi\cdot\bar X\cup(X\setminus\bar X)$, $\bar Y=\bar\varphi\cdot\bar X$ and $Y_0=\varphi\cdot X_0$, r is replaced by $(\varphi)\cdot r'= =(\varphi)\cdot(r\cap I_{T(\varphi)}(E))$, and the X_0-point $\theta\colon X_0\to A$ is replaced by $(\varphi|X_0)\cdot\theta$, which is well-defined as θ is compatible with $(\varphi|X_0)$.

(ε) Dilatations. We preserve the meanings of X, $\bar X$, X_0, ϱ, ϱ^*, r and θ, but consider a surjection $\bar\psi\colon \bar Y\to\bar X$ instead of $\bar\varphi\colon \bar X\to\bar Y$. We suppose that, by some preliminary semi-free floatage of anchor $\bar X$, X is already transformed so that $(X\setminus\bar X)\cap\bar Y=\emptyset$. Let $Y=\bar Y\,\dot\cup\,(X\setminus\bar X)$, and let $\psi\colon Y\to X$ be the surjection for which $(\psi|\bar X)=\bar\psi$ and $(\psi|(X\setminus\bar X))$ is the identity. Obviously, we have $[\bar\psi]\cdot\varrho=\mathrm{pr}_Y\cdot([\psi]\cdot\varrho^*)$. But

$$[\psi]\cdot\varrho^*=[\psi]\cdot r\cap(\bigcap_{x\in X_0}[\psi]\cdot\mathrm{ext}_X\cdot(x;\ \theta\cdot x)),$$

and $[\psi]\cdot r\in k$. As it is easy to see,

$$\begin{aligned}[\psi]\cdot\mathrm{ext}_X\cdot(x;\ \theta\cdot x)&=[\psi]\cdot\mathrm{ext}_X\cdot\{\{x\to\theta\cdot x\}\}=\\ =I_{T(\psi)}(E)&\cap(\bigcap_{y\in\psi^{-1}\cdot x}\mathrm{ext}_Y\cdot\{\{y\to\theta\cdot x=\theta\psi\cdot y\}\}=\\ &=I_{T(\psi)}(E)\cap(\bigcap_{y\in\psi^{-1}\cdot x}\mathrm{ext}_Y\cdot(y;\ \theta\psi\cdot y)).\end{aligned}$$

So, if $Y_0=\psi^{-1}\cdot X_0$ and $\hat r=[\psi]\cdot r\cap I_{T(\psi)}(E)$, we have

$$\begin{aligned}[\psi]\cdot\varrho^*=[\psi]\cdot r\cap I_{T(\psi)}(E)\cap(\bigcap_{x\in X_0}\bigcap_{y\in\psi^{-1}\cdot x}\mathrm{ext}_Y\cdot(y;\ \theta\psi\cdot y))=\\ =\hat r\cap(\bigcap_{y\in Y_0}\mathrm{ext}_Y(y;\ \theta\psi\cdot y)).\end{aligned}$$

This means that $[\bar\psi]\cdot\varrho$ is still of the form (ϱ), where X, X_0, $\bar X$, r and θ are replaced by $Y=\psi^{-1}\cdot X$, $Y_0=\psi^{-1}\cdot X_0$, $\bar Y=\psi^{-1}\cdot\bar X$, $\hat r=[\psi]\cdot r\cap I_{T(\psi)}(E)$ and $\theta\psi\colon Y\to A$. The theorem is proved.

Remark 4. The relation $\varrho^*=r\cap(\bigcap_{x\in X_0}\mathrm{ext}_X\cdot(x;\theta\cdot x))$ is the set of all points $P\in r$ which extend the X_0-point $\theta\colon X_0\to A$ on X.

Remark 5. If k is an abstract field then so is $k(A)$. Therefore, in this case, the class of infinitary unions of relations of the form (ϱ) is also closed with respect to the negation $\neg$.

Really, every $\delta\in D(E/k)=G(E/k)$ is a permutation on E. But $\delta\cdot(x;a)\subseteq(x;a)$ holds iff $\delta\cdot a=a$, which implies $\delta\cdot(x;a)=(x;a)$. Therefore every $\delta\in D(E/k(A))$ not only stabilizes but preserves every $(x;a)$, $a\in A$. Thus $D(E/k(A))=G(E/k(A))$ and $k(A)$ is an abstract field. The rest of the remark can be proved directly; note that even in case r is of the form (ϱ), $\neg r$ is an infinitary union of relations of the form (ϱ) in general.

Operations generated by relations. Let D be a subset of E^X. A mapping $\omega\colon D\to E$ will be called an *X-operation* (or *partial X-composition*) on E. The set D will be called the *(definition) domain* of ω. When $D=E^X$, ω is said to be a *complete* X-operation on E; the terms *X-function* of E and *X-polymapping* of E are also used.

Let $X'=X\dot{\cup}\{y\}$ and let $r\subseteq E^{X'}$ be an X'-relation on E. The relation r and the argument y will define an X-operation $\omega_r^{(y)}\colon D_r^{(y)}\to E$ in the following way. An X'-point P' will often be denote by $(P, \{y\to e\})$ where $P=(P'|X)$ and $\{y\to e\}= =(P'|\{y\})$. If $P'\in r$ then $e=(P'|\{y\})\cdot y$ is called a *prolongation* of $P=(P'|X)$ in r. Clearly, $P\in E^X$ has prolongations in r iff $P\in \mathrm{pr}_X\cdot r$. Let $D_r^{(y)}$ be the set of all $P\in E^X$ that have exactly one prolongation in r. For $P\in D_r^{(y)}$ let $\omega_r^{(y)}\cdot P$ be the unique prolongation of P in r.

It is not hard to express $D_r^{(y)}$ from r by means of fundamental operations. Let y' be an argument, not in X', and let $\varphi\colon X'\to X\cup\{y'\}$ be the floatage for which $(\varphi|X)$ is the identity and $\varphi\cdot y=y'$. Put $X''=X\cup\{y, y'\}=X'\cup\{y'\}$, and let us consider the set of all points $P''\in \mathrm{ext}_{X''}\cdot r\cap \mathrm{ext}_{X''}\cdot((\varphi)\cdot r)$ such that $(P''|X)=P$. This is clearly the set of all points of the form $(P, \{y\to e\}, \{y'\to e'\})$, where e and e' are arbitrary prolongations of P in r. An X-point $P\in \mathrm{pr}_X\cdot r$ has several distinct prolongations in r iff

$$\mathrm{ext}_{X''}\cdot r\cap \mathrm{ext}_{X''}\cdot((\varphi)\cdot r) \quad\text{and}\quad \{P\}\times(\neg D_{y,y'}(E)) = \mathrm{ext}_{X''}\cdot \{P\}\cap(\neg \mathrm{ext}_{X''}\cdot D_{y,y'}(E))$$

are not disjoint. So the set of all these points is $\mathrm{pr}_X\cdot(\mathrm{ext}_{X''}\cdot r\cap \mathrm{ext}_{X''}\cdot((\varphi)\cdot r)\cap$ $\cap(\neg \mathrm{ext}_{X''}\cdot D_{y,y'}(E)))$. Therefore we have

$$D_r^{(y)} = (\mathrm{pr}_X\cdot r)\cap(\neg \mathrm{pr}_X\cdot(\mathrm{ext}_{X''}\cdot r\cap \mathrm{ext}_{X''}\cdot((\varphi)\cdot r)\cap(\neg \mathrm{ext}_{X''}\cdot D_{y,y'}(E)))).$$

In case X is empty, i.e. $X'=\{y\}$, $D_r^{(y)}$ is non-empty iff the unique "empty" point $P_\emptyset$ has a unique prolongation e in r, i.e., $r=(y; e)$. Then $\omega_r^{(y)}\cdot P_\emptyset=e$, and $\omega_r^{(y)}$ is, in fact the adjunction of the element $e\in E$.

We say that $A\subseteq E$ is closed with respect to an X-operation $\omega\colon D\to E$ if $\omega\cdot P\in A$ holds whenever $P\colon X\to A$ and $P\in D$.

Theorem 2. *The rationality domain $\bar{A}_k$ of $k(A)$ is the closure of A with respect to all operations $\omega_r^{(y)}$ such that $r\in k$ and y belongs to the argument set of r.*

Proof. Let X_r be the argument set of a relation $r\in k$, and let $y\in X_r$. Put $X=X_r\setminus\{y\}$, and let $P\colon X\to \bar{A}_k$ be a point belonging to $D_r^{(y)}$. If $e=e(P)=\omega_r^{(y)}\cdot P$ then

$$\mathrm{pr}_{\{y\}}\cdot((\{P\}\times E^{\{y\}})\cap r) = \{\{y\to e\}\} = (y;\, e).$$

As $\{P\}\times E^{\{y\}} = \bigcap_{x\in E} \mathrm{ext}_{X_r}\cdot(x; P\cdot x)$ and $P\cdot x\in \bar{A}_k$, we have $(x; P\cdot x)\in k(A)$. Since $(y; e)$ is obtained from $(x; P\cdot x)\in k(A)$, $x\in X$, and from $r\in k(A)$ by direct fundamental operations, $(y; e)$ belongs to $k(A)$. Therefore $\omega_r^{(y)}\cdot P=e(P)\in \bar{A}_k$, and $\bar{A}_k$ is closed with respect to all the mentioned operations $\omega_r^{(y)}$.

Now suppose that $e \in \bar{A}_k$, i.e. $(y; e) \in k(A)$. By the previous theorem, $(y; e)$ is the infinitary union of an appropriate set of $\{y\}$-relations of the form

$$\varrho = \mathrm{pr}_{\{y\}} \cdot (r \cap (\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x; \theta \cdot x)))$$

where $X = \{y\} \cup X_0$, r is an X-relation in k and $\theta\colon X_0 \to A$ is a mapping of X_0 into $A \subseteqq \bar{A}_k$. As $(y; e)$ is irreducible, it must be equal to some of these relations. So we assume that $(y; e)$ is the above-mentioned ϱ. If $y \in X_0$ then $e = \varrho \cdot y = \theta \cdot y \in A$. If $y \notin X_0$ then $\varrho^* = r \cap (\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x; \theta \cdot x))$ is the set of X-points $P = (P_0 = (P|X_0), (P|\{y\}) = \{\{y \to e'\}\})$ such that $P_0 = \theta$ and e' is a prolongation of θ in r. Therefore ϱ is the set of all $\{y\}$-points $\{\{y \to e'\}\}$ such that e' is a prolongation of θ in r. But, by the assumption, $\varrho = (y; e)$, whence e is the only prolongation of θ in r, $\theta \in D_r^{(y)}$ and $e = \omega_r^{(y)} \cdot \theta$. As θ is a point of A, e belongs to the closure of A with respect to $\omega_r^{(y)}$. Therefore $\bar{A}_k$ is included in the closure of A with respect to all $\omega_r^{(y)}$, which proves the theorem.

Remark 6. If $\omega_r^{(y)}$ is an $\emptyset$-operation with $D_r^{(y)} \neq \emptyset$ and $r \in k$ then $D_r^{(y)} = \{P_\emptyset\}$ and $r = (y; \omega_r^{(y)} \cdot P_\emptyset)$. In this case $e = \omega_r^{(y)} \cdot P_\emptyset$ belongs to the rationality domain $\bar{\emptyset}_k$ of k, and this operation is the mere adjunction of $e \in E$ belonging to this domain, i.e. preserved by all $\delta \in D(E/k)$. Therefore $\bar{A}_k$ can also be characterized as the closure of $A \cup \bar{\emptyset}_k$ with respect to all X-operations $\omega_r^{(y)}$ such that $X \neq \emptyset$, $r \in k$ and $y \in X_r$.

Remark 7. If an X-relation $\varrho \in k$ is the infinitary union $\bigcup \cdot R$ of a set R of relations in k and $y \in X$ then for each $\bar{P} \in D_\varrho^{(y)}$ there exists a relation $r \in R$ such that $\bar{P} \in D_r^{(y)}$ and $\omega_r^{(y)} \cdot \bar{P} = \omega_\varrho^{(y)} \cdot \bar{P}$. Further, this r can be chosen so that r is semi-regular, $r \subseteqq \varrho$ and $P = (\bar{P}, \{y \to \omega_\varrho^{(y)} \cdot \bar{P}\})$ belongs to $t(r)$, the head of r. Moreover, the $D(E/k)$-orbit $D(E/k) \cdot P$ of P is such a semi-regular relation r.

Indeed, if $e = \omega_\varrho^{(y)} \cdot \bar{P}$ then $P = (\bar{P}, \{y \to e\}) \in \varrho$ and e is the only prolongation of $\bar{P}$ in ϱ. Since $\varrho = \bigcup \cdot R$, there exists an $r \in R$ such that $P \in r$ and every prolongation of $\bar{P}$ in r is a prolongation of $\bar{P}$ in ϱ. So e is the unique prolongation of $\bar{P}$ in r, which implies $\bar{P} \in D_r^{(y)}$ and $e = \omega_r^{(y)} \cdot \bar{P}$. Since, by Lemma 2 of Section 2, each relation $\varrho \in k$ can be decomposed into a set R of semi-regular relations belonging to k such that any $P \in \varrho$ belongs to $t(r)$ for some $r \in R$, the rest of Remark 7 follows.

Remark 8. Let $r \in K$ be a semi-regular X-relation, let $y \in X$, and assume that $P = (\bar{P}, \{y \to \omega_r^{(y)} \cdot \bar{P}\}) \in t(r)$ for some $\bar{P} \in D_r^{(y)}$. Then there exists a semi-regular relation $r' \in k$ such that $T(r')$, the type of r', is the discrete equivalence on X', the argument set of r', and there are $y' \in X'$ and a point $\bar{P}' \in D_{r'}^{(y')}$ with $\bar{P}' \cdot (X' \setminus \{y'\}) \subseteqq \subseteqq \bar{P} \cdot (X \setminus \{y\})$, $\omega_r^{(y)} \cdot \bar{P} = \omega_{r'}^{(y')} \cdot \bar{P}'$ and $P' = (\bar{P}', \{y' \to \omega_{r'}^{(y')} \cdot \bar{P}'\}) \in t(r')$.

Indeed, let X be the argument set of r, and put $e = \omega_r^{(y)} \cdot \bar{P}$. Let φ be the canonical mapping of X onto $X' = X/T(r)$. Then $r' = (\varphi) \cdot r$ is well-defined and $T(r')$

is the discrete equivalence on X'. As $T(P)=T(r)$, we have $T((\varphi)\cdot P)=T(r')$. Hence $(\varphi)\cdot P$ is an injective point and belongs to $t(r')$. If $y'=\varphi\cdot y$ then the value $P'\cdot y'$ of the point $P'=(\varphi)\cdot P$ at y' is equal to $P\cdot y=e$. Put $\bar{X}=X\setminus\{y\}$, $\bar{X}'=X'\setminus\{y'\}$ and $\bar{P}'=(P'|\bar{X}')$. It is clear that $\bar{P}'\cdot\bar{X}'\subseteq\bar{P}\cdot\bar{X}$ (in particular, if $\bar{P}$ is a point of $\bar{A}_k$ then so is $\bar{P}'$), and e is the only prolongation of $\bar{P}'$ in r'. Therefore $\bar{P}'\in D_{r'}^{(y')}$ and $\omega_{r'}^{(y')}\cdot\bar{P}'=e=\omega_r^{(y)}\cdot\bar{P}$.

Remark 9. It follows from the preceding remarks that a subset of E is closed with respect to all $\omega_r^{(y)}$, $r\in k$, iff it is closed with respect to the operations $\omega_r^{(y)}$ such that r has an injective point. Moreover, if X^0 is an argument set with $\operatorname{card} X^0\geqq\operatorname{card} E$ then a subset of E is closed with respect to all $\omega_r^{(y)}$ iff it is closed with respect to those that are determined by relations of k under X^0. In particular, $\bar{A}_k$ is the closure of $A\subseteq E$ with respect to this last variety of operations.

Indeed, if an X_r-relation r has an injective point then $\operatorname{card} X_r\leqq\operatorname{card} E$. On the other hand, if (φ) is a floatage then $D_{(\varphi)\cdot r}^{(\varphi\cdot y)}=(\varphi|(X_r\setminus\{y\}))\cdot D_r^{(y)}$ and $\omega_{(\varphi)\cdot r}^{(\varphi\cdot y)}\cdot((\varphi|(X_r\setminus\{y\})\cdot\bar{P}))=\omega_r^{(y)}\cdot\bar{P}$ where $\bar{P}\in D_r^{(y)}$.

Let $S=(E, R)$ be a structure, and let $k=K_e(S)$ be the corresponding abstract endofield. We have seen that $\bar{A}_k$, the closure of $A\subseteq E$, is the closure of A with respect to the operations $\omega_r^{(y)}$ such that the r are relations in k under a fixed X^0 with $\operatorname{card} X^0\geqq\operatorname{card} E$. That is, these r belong to $\bar{R}^{(X^0)}=R_{\mathrm{df}}^{(X^0)}$ and to k. Now the question is whether a sufficiently wide class of structures can be defined such that the "huge set" $R_{\mathrm{df}}^{(X^0)}$ can be replaced by the (much smaller) set R in case of these structures. The answer is positive; an appropriate class, the class of the so-called *eliminative structures*, can be defined. I will not speak about these structures in the present paper — it will be done in some other publication, which will contain the necessary proofs. However, the structure (E, R) of classical Galois theory is eliminative, and from this fact, accepted here without proof, we are going to deduce the fundamental theorem of classical Galois theory. In other words, let E be a normal algebraic or an algebraically closed field extension of some basic field k, let $R=\{(f=0); f\in k[x_1, x_2, \ldots, x_n, \ldots]\}$, and let A be a subset of E; we take it for granted that the rationality domain of the abstract set extension $(k)(A)$, obtained by adjoining A to the abstract endofield (or field) (k) defined by the structure (E, R), is the closure of A with respect to all operations defined by the relations $(f=0)$, $f\in k[x_1, x_2, \ldots, x_n, \ldots]$, of this structure. In order to deduce the fundamental theorem of classical Galois theory, first we introduce some constructions that yield operations from operations.

(1) Let X be an argument set and let U be a set of X-operations $\omega\colon D_\omega\to E$. An X-operation $\omega^*\colon D_{\omega^*}\to E$ is called a *mosaic* of the operations $\omega\in U$ if there

exists a partition $\{d_\omega;\ \omega\in U\}$ of D_{ω^*}, i.e. $\omega\neq\omega'\Rightarrow d_\omega\cap d_{\omega'}=\emptyset$ and $\bigcup_{\omega\in U} d_\omega=D_{\omega^*}$, such that $d_\omega\subseteq D_\omega$ and $(\omega^*|d_\omega)=(\omega|d_\omega)$ hold for every $\omega\in U$.

Lemma 1. *If $A\subseteq E$ is closed with respect to all $\omega\in U$ then it is closed with respect to every mosaic ω^* of $\omega\in U$.*

Proof. Let $P\colon X\to A$ be a point in D_{ω^*}. Then there exists one and only one $\omega\in U$ such that $P\in d_\omega\subseteq D_\omega$ and $\omega^*\cdot P=\omega\cdot P$. Now the assertion follows from $\omega\cdot P\in A$.

(2) Let $\Omega\colon D\to E$ be an X-operation, let Y be an argument set, and for each $x\in X$ let $\omega^x\colon d_x\to E$ be a Y-operation. For $Q\in\bigcap_{x\in X} d_x$, let $\omega\cdot Q$ be the X-point $\{x\to\omega^x\cdot Q;\ x\in X\}$. Let d be the set of all $Q\in\bigcap_{x\in X} d_x$ such that $\omega\cdot Q\in D$. Then a Y-operation, denoted by $\Omega(\{x\to\omega^x\})$, can be defined in the following way. The domain of $\Omega(\{x\to\omega^x\})$ is d, and for every $Q\in d$ we put $\Omega(\{x\to\omega^x\})\cdot Q=\Omega\cdot(\omega\cdot Q)$. This Y-operation will be called the *superposition* of Ω and the "operation point" $\omega\colon x\to\omega^x$, $x\in X$.

Lemma 2. *If $A\subseteq E$ is closed with respect to Ω and to all ω^x, $x\in X$, then it is also closed with respect to $\Omega(\{x\to\omega^x\})$.*

Proof. Let $P\colon Y\to A$ belong to d. Then, for every $x\in X$, $Q\in d_x$ and $\omega^x\cdot Q\in A$. So $\omega\cdot Q\colon x\to\omega^x\cdot Q$ is an X-point of A. On the other hand, $\omega\cdot Q\in D$. Therefore $\Omega(\{x\to\omega^x\})\cdot Q=\Omega\cdot(\omega\cdot Q)\in A$, proving the lemma.

For a set U of operations and a subset B of U, B will be called a *basis* of U if each $\omega\in U$ can be obtained from the operations of B by a combination of mosaics and superpositions represented by a tree of finite height. Clearly, a subset A of E is closed with respect to all $\omega\in B$ iff it is closed with respect to all $\omega\in U$.

To conclude the paper, we determine a simple basis of operations $\omega^{(y)}_{f=0}$ defined by the relations $(f=0)$, $f\in k[x_1, x_2, \dots, x_n, \dots]$, of classical Galois theory. Clearly, floatages do not change, up to floatages of arguments, the operations defined by a relation. So we can consider only the polynomials $f(x_1, x_2, \dots, x_s, y)\in k[x_1, \dots, x_s, y]$ (where s can be arbitrary) and the corresponding operations $\omega^{(y)}_{f=0}$. Let n be the degree of such an f for y, i.e.,

$$f(x_1, \dots, x_s, y)=\sum_{0\leq i\leq n} f_i(x_1, \dots, x_s)y^i.$$

Let $P\colon X_s=\{x_1, \dots, x_s\}\to E$ be an X_s point of E, which will be represented by the system $(\xi_1, \dots, \xi_s)$ of its values $\xi_i=P\cdot x_i\in E$, $i=1, \dots, s$. The point P has exactly one prolongation in $r=(f=0)$ iff the polynomial $f(\xi_1, \dots, \xi_s, y)\in E[y]$ has exactly one root, i.e., for some j, $1\leqq j\leqq n, f(\xi_1, \ldots, \xi_s, y)=a(y-\eta)^j$, $a, \eta\in E$ and $a\neq 0$, i.e.

$$(\mathrm{C}_j)\qquad \begin{cases} f_i(\xi_1, \ldots, \xi_s)=0 & \text{if } j<i \\ f_j(\xi_1, \ldots, \xi_s)=a\neq 0 & \\ f_i(\xi_1, \ldots, \xi_s)=(-1)^{j-1}\binom{j}{j-i}a\eta^{j-i} & \text{if } i<j. \end{cases}$$

If the characteristic of k is 0 then $f_{j-1}(\xi_1, \ldots, \xi_s)=-ja\eta$; i.e., (C_j) implies $jf_j(\xi_1, \ldots, \xi_s)\eta+f_{j-1}(\xi_1, \ldots, \xi_s)=0$ and $f_j(\xi_1, \ldots, \xi_s)\neq 0$. Therefore, if we consider the polynomial

$$g_j(x_1, \ldots, x_s, y)=f_{j-1}(x_1, \ldots, x_s)+jf_j(x_1, \ldots, x_s)y$$

then the operation $\omega^{(y)}_{g_j=0}$ is defined on $D_j=D^{(y)}_{g_j=0}=\neg(f_j=0)$. Clearly, D_j includes the set d_j of all points $P=(\xi_1, \ldots, \xi_s)\in E^{X_s}$ that satisfy (C_j). In other words, d_j, the set of all P for which $f(\xi_1, \ldots, \xi_s, y)$ is of the form $a(y-\eta)^j$ for some $a, \eta\in E$, $a\neq 0$, is included in D_j. For $P\in d_j$ we clearly have

$$(\alpha)\qquad \omega^{(y)}_{f=0}\cdot P=\eta=\omega^{(y)}_{g_j=0}\cdot P=-f_{j-1}(\xi_1, \ldots, \xi_s)(jf_j(\xi_1, \ldots, \xi_s))^{-1}.$$

Now consider the case when the characteristic p of k is different from 0. Let $j=hp^{s(j)}$ where h is not divisible by the prime number p. We have, in k, $\binom{j}{i}=0$ if $0<i<p^{s(j)}$ and $\binom{j}{i}=h$ if $i=p^{s(j)}$. In particular, we have $f_{j-p^{s(j)}}(\xi_1, \ldots, \xi_s)=$ $=-ha\eta^{p^{s(j)}}$. Let

$$g_{j,p}(x_1, \ldots, x_s, y)=hf_j(x_1, \ldots, x_s)y^{p^{s(j)}}+f_{j-p^{s(j)}}(x_1, \ldots, x_s).$$

Then (C_j) implies that $hf_j(\xi_1, \ldots, \xi_s)\neq 0$ and η is the unique root of $g_{j,p}(\xi_1, \ldots, \xi_s, y)$. Therefore $\omega^{(y)}_{g_{j,p}=0}$ is defined on the same $D_j=\neg(f_j=0)$, $d_j\subseteqq D_j$ and, for every $P\in d_j$, we have

$$(\beta)\quad \omega^{(y)}_{f=0}\cdot P=\eta=\omega^{(y)}_{g_{j,p}}=0\cdot P=(-f_{j-p^{s(j)}}(\xi_1, \ldots, \xi_s)(hf_j(\xi_1, \ldots, \xi_s))^{-1})^{p^{-s(j)}}.$$

In both cases, $(d_1, d_2, \ldots, d_n)$ is a partition of $D^{(y)}_{f=0}$. Therefore, $\omega^{(y)}_{f=0}$ is a mosaic of the operations $\omega_1, \omega_2, \ldots, \omega_n$ where these ω_j are defined on the respective sets $D_j=\neg(f_j=0)$ in the following way: for $P=(\xi_1, \ldots, \xi_s)\in D_j$ we put

$$\omega_j\cdot P=-f_{j-1}(\xi_1, \ldots, \xi_s)(jf_j(\xi_1, \ldots, \xi_s))^{-1}$$

when k is of zero characteristic, and we put

$$\omega_j\cdot P=(-f_{j-p^{s(j)}}(\xi_1, \ldots, \xi_s)(hf_j(\xi_1, \ldots, \xi_s))^{-1})^{p^{-s(j)}}$$

when the characteristic of k is $p\neq 0$ (cf. (α) and (β)). It is clear that the ω_j are superpositions of the operations $(x_1, x_2)\to x_1+x_2$, $(x_1, x_2)\to x_1x_2$, $x_1\to x_1^{-1}$ (defined on $E\setminus\{0\}$), $x_1\to\sqrt[p]{x_1}$ if $p\neq 0$ (defined on $\{x^p; x\in E\}$), and the adjunctions $P_\emptyset\to\alpha$, $\alpha\in k$.

Therefore these operations form a basis of $\{\omega^{(\nu)}_{f=0};\ f\in k[x_1, \ldots, x_s, \ldots]\}$. If the afore-mentioned theorem about eliminative structures is proved and it is shown that the considered structure is eliminative then it follows that the rationality domain of $(k)(A)$, i.e. the set of all $e\in E$ that are preserved by any automorphism of E/k preserving every $a\in A$, is the closure of $A\cup k = A\cup\overline{\emptyset}$ with respect to addition, multiplication, inversion and, if $p\neq 0$, forming p-th roots. This is one of the classical formulations of the first Galois theorem.

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Lattice ordered binary systems

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By a *groupoid* we mean an *algebra* $(G, \cdot)=:\mathfrak{G}$ of *type* (2). By a *binary system* we mean a groupoid weaker than a *group*. Special binary systems are *semigroups, quasigroups,* and *loops*. The notion binary system was introduced by R. H. BRUCK [13].

A binary system is called *partially (lattice-) ordered* if G is partially (lattice-) ordered by an order relation $\leqq$ satisfying:

(0) $$a \leqq b \rightarrow xa \leqq xb \;\&\; ax \leqq bx.$$

If $(G, \cdot, \leqq)$ is lattice-ordered we call $(G, \cdot, \leqq)$ briefly a *lattice groupoid.* By a *lattice semigroup* we mean a lattice groupoid satisfying $(ab)c=a(bc)$. Analogously we speak of a *lattice quasigroup* if all equations $ax=b$ and $ya=b$ have unique solutions $a\diagdown b$ in the first and $b\diagup a$ in the latter case. Accordingly by a *lattice loop* we mean a lattice quasigroup with *unit* 1. A loop is said to have the *inverse property* if for each x there exists an x^{-1} such that for any a the identities $x^{-1}(xa)=a$ and dually $a=(ax)x^{-1}$ are valid. If $(G, \cdot)$ is an *inverse loop* we have in addition the equations $(x^{-1})^{-1}=x$ and $(xy)^{-1}=y^{-1}x^{-1}$, as is easily checked by the reader.

There is no lack of lattice quasigroups. To see this consider $(R^n, \leqq)$ with respect to $a\circ b:=a+2b$. Furthermore there is an abundance of lattice loops, since starting from a lattice quasigroup $(Q, \circ, \wedge, \vee)$ we get a lattice loop by putting $a\cdot b:=(a\diagup x)\circ(y\diagdown b)$, where x, y are fixed elements. And, above all, it should be emphasized that any free loop admits not only a lattice but even a total order [13, 22].

Lattice-ordered binary systems are *congruence distributive* in any case and *congruence permutable* in many cases. Thus the theory of lattice-ordered binary systems is rich from the purely algebraic point of view. On the other hand, however, there are not too many *lattice groupoid* results arising from order theoretic or combined apects although G. BIRKHOFF [5] and L. FUCHS [19] as well state problems of such type. Nevertheless, at least a fruitful lattice loop theory should be possible as indicated

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already in [5], and even suggested by results of EVANS and HARTMAN [17] who had a first breakthrough after several contributions of different authors like ZELINSKY [41], [42], Kaplansky, Ingraham and Birkhoff (cf. [5]), and ACZÉL [1].

An element a of a partially ordered binary system is called *positive* iff it satisfies $ax \geqq x \leqq xa$ $(\forall x \in G)$. The subset C^+ of all positive elements is called the *(positive) cone*. Dually *negative* elements and the *negative cone* are defined. Both the positive and the negative cone are closed under *multiplication, join* and *meet*, and if in addition a unit element is present the positive cone C^+ coincides with $\{x \mid x \geqq 1\}$, and the negative cone C^- is equal to the subset $\{x \mid x \leqq 1\}$.

The central structure of this paper is that of a *divisibility semiloop*, i.e. *a cancellation groupoid with unit* 1 whose *carrier* is *semilattice-ordered* such that $ax \leqq b \to \exists u$: $au = b$ and $ya \leqq b \to \exists v$: $va = b$. Hence a divisibility semiloop is a common abstraction of the lattice loop and the lattice loop cone.

It is a folklore today that any *lattice group* is a *quotient extension* of its cone such that the structure of the whole is completely determined by the structure of the cone. This is quite different in the lattice loop case where not even a *total and complete* order yields any connection between the positive and the negative cone. To verify this the reader may consider the real line with respect to $a \circ b := a + b$ if one of the components is not negative and $a \circ b := a - ab + b$ otherwise, [22]. Hence the situation seems to be hopeless. Nevertheless it is possible to prove a result shedding some light as far as isolated cones are considered, namely: The lattice loop cones are exactly the *positive* divisibility semiloops $(G = C^+)$, and every lattice loop cone is the cone even of an inverse lattice loop. This extends a theorem and answers a question of J. v. NEUMANN (cf. [4]).

Thus a chance might be given to settle general lattice loop problems via inverse lattice loops.

Given a lattice ordered binary system the first order problem to arise is the question what the *descending chain condition* (for *closed intervals*) is equivalent to from the purely algebraic point of view. Hence this question has been treated for different algebraic systems several times, especially for semigroups by ARNOLD [2], CLIFFORD [14], [15], LORENZEN [28] and others (cf. [20]), and for lattice groups by BIRKHOFF [4] and WARD [40]. But the problem remained open for lattice loops until EVANS [16] showed that lattice loops, satisfying the D.C.C. are abelian lattice groups with the *prime factorization property* (P.F.P.). This yields as a corollary that every lattice quasigroup with D.C.C. is the isotope of a free abelian group. See also TESTOV [38]. Therefore a similar investigation of divisibility semiloops is motivated, and it is by no means surprising that an analogue of Evans' theorem remains valid. However it is not the result by which Section 3 is legitimatized in the author's opinion, but the method of proof that justifies this part.

There are two natural generalizations of the D.C.C. and the P.F.P. respectively namely *completeness* (for closed intervals) on the one hand and *representability* on the other hand, i.e. the property to admit a *subdirect decomposition into totally ordered factors.*

As far as completeness is considered we shall prove that *power-associative* divisibility semiloops are *associative* and *commutative* thus carrying over IWASAWA's theorem [27] to our structure. Furthermore it is shown in Section 4 that completeness combined only with *mon*associativity is a too weak requirement with respect to the associativity or commutativity property.

As another topic in the context of completeness we take up the problem of characterizing divisibility semiloops admitting a *complete extension.* This has been done for lattice group like systems several times and it seems to the author that ARNOLD [2] and VAN DER WAERDEN [39] were the first to settle a problem of this type in general, followed by others like LORENZEN [29], CLIFFORD [14], [15], and EVERETT and ULAM [18], the first to treat a noncommutative case. But no nonassociative analysis was given before 1972 when P. A. HARTMAN [22], [23] settled the problem for partially ordered quasigroups and loops. Of course, there are further results, consult for instance [5] and [19], above all the initial contribution of RICHARD DEDEKIND (cf. [5]). Hence characterizing divisibility semiloops with complete extensions is a most natural additional step according to a long lasting development (Section 5).

Finally we turn to representable divisibility semiloops.

There are various results concerning lattice-ordered structures of such type, the historical one being Stone's celebrated decomposition theorem for boolean algebras, afterwards extended to distributive lattices (cf. [5]), for instance: LORENZEN [28], CLIFFORD [15], RIBENBOIM [32] (abelian lattice-ordered groups); LORENZEN [29], ŠIK [34], BANASCHEWSKI [3] (arbitrary lattice-ordered groups); SWAMY [37] (abelian residuated lattice-ordered semigroups); BOSBACH [8], [10] (complementary semigroups); TH. MERLIER [30] (abelian lattice-ordered monoids); FUCHS [20] (general lattice-ordered algebras); FUCHS [21] (positive abelian lattice-ordered monoids); BIRKHOFF and PIERCE [6] (lattice-ordered rings); EVANS and HARTMAN [17] (lattice-ordered loops).

But a general solution is still outstanding and also special problems have remained unsolved up to now although they were stated several times, like the lattice semigroup problem [19], [21] or the lattice groupoid and the lattice quasigroup problem [17]. Therefore Section 6 will be devoted not only to divisibility semiloops with a representation, but also to general lattice-ordered binary systems of this type, the principal result being a decomposition theorem that solves the problems mentioned above in a one cast manner.

The notation of this paper is standard in general, but sometimes $\hat{:}$ will stand for *"such that"* and $a \cdot bc$ for $a(bc)$. Consequently $a \cdot \cdot b \cdot cd$ f.i. will mean $a(b(cd))$.

The basic concepts of algebra and order theory are to be found in [5]. The later paragraphs are based only on Section 1.

Finally we give a most important hint. There will appear *dualities* of various kinds, for instance right/left dualities or $\geqq/\leqq$ dualities. Hence there will be propositions holding necessarily together with their dual. So the reader should realize this situation whenever it comes up. Nevertheless he will be requested from time to time to take that fact into account.

1. Divisibility semiloops

1.1. Definition. By a *divisibility semiloop* we mean an algebra $\mathfrak{G} := (G, \cdot, \wedge, 1)$ of type $(2, 2, 0)$ satisfying

(DSL 1) $(G, \cdot)$ is a cancellation groupoid,
(DSL 2) 1 is unit of $(G, \cdot)$,
(DSL 3) $(G, \wedge)$ is a semilattice,
(DSL 4) $x(a \wedge b) \cdot y = xa \cdot y \wedge xb \cdot y$

(observe that (DSL 4) requires right- and left-distributivity because of axiom (DSL 2)),

(DSL 5) $ax \leqq b \to \exists u\colon au = b,\ ya \leqq b \to \exists v\colon va = b$

(observe furthermore that the negative cone of any divisibility semiloop is itself a positive divisibility semiloop with respect to $\vee$).
Classical examples of a divisibility semiloop are the lattice loop and the lattice loop cone. Therefore the divisibility semiloop is a common abstraction of these two structures.

For the sake of convenience we start from an arbitrary but fixed divisibility semiloop.

1.2. Lemma. $\forall a, b, x, y\colon a \leqq b \to ax \leqq bx \,\&\, ya \leqq yb$ *and*

$$ax \leqq bx \vee ya \leqq yb \to a \leqq b.$$

Proof. Obviously we may confine ourselves to the left-sided cases. But these follow by $a \leqq b \to ya \wedge yb = y(a \wedge b) = ya$ for the left-right direction and from $ya = ya \wedge \wedge yb \to ya = y(a \wedge b) \to a = a \wedge b$ otherwise.

1.3. Lemma. $b \geqq 1 \,\&\, a''(a \wedge c) = a \to a \wedge bc = (a'' \wedge b)(a \wedge c)$.

Proof. $b \geqq 1 \to a \wedge bc = a''(a \wedge c) \wedge ba \wedge bc = (a'' \wedge b)(a \wedge c)$.

As an immediate consequence of 1.3 we get.

1.3′. Lemma. $x \leqq bc \,\&\, b \geqq 1 \rightarrow x = x_b x_c \,\hat{:}\, x_b \leqq b \,\&\, x_c \leqq c$.

1.4. Proposition. $(a \wedge b)a' = a \rightarrow ba' = \sup(a, b) =: a \vee b$.

Proof. Suppose $(a \wedge b)a' = a \geqq (a \wedge b)1$. Then we can infer $a' \geqq 1$ and thereby: $ba' \geqq a$ & $ba' \geqq b$. On the other hand any c with $c \geqq a, b$ satisfies for some x the implication: $c = bx$ & $a = (a \wedge b)(a' \wedge x) \rightarrow a' = x \wedge a' \rightarrow a' \leqq x \rightarrow ba' \leqq bx = c$ which had to be proved. (Similarly one shows that $(a \vee b)a' = a$ & $(a \vee b)b' = b$ implies $ab' = a \wedge b$. This is possible by means of (DSL 5).)

1.5. Lemma. $x(a \vee b) \cdot y = xa \cdot y \vee xb \cdot y$.

Proof. Suppose $xa \vee xb = (xa)c$. Then by (DSL 5) there is an element u such that $xu = xa \vee xb$ from which follows $u \geqq a \vee b$ and thereby $x(a \vee b) = xa \vee xb$. The rest follows by duality.

1.6. Lemma. $(a \wedge b)a' = a \,\&\, (a \wedge b)b' = b \rightarrow (a \wedge b)a' \cdot b' = (a \wedge b)(a' \vee b')$.

Proof. $(a \wedge b)a' \cdot b' = ab' = a \vee b = (a \wedge b)(a' \vee b')$.

1.7. Corollary. $b \wedge c = 1 \vee b \vee c = 1 \rightarrow ab \cdot c = ac \cdot b = a \cdot bc$.

Proof. Indeed, $b \wedge c = 1 \rightarrow ab \wedge ac = a$ and $b \vee c = 1 \rightarrow ab \vee ac = a$.

1.8. Corollary. $a \wedge b = 1 \rightarrow ab = a \vee b = ba$.

1.9. Lemma. $ab = cd \rightarrow ab = (a \wedge c)(b \vee d) = (a \vee c)(b \wedge d)$.

Proof. $ab = cd \rightarrow ab \geqq (a \wedge c)b \vee (a \wedge c)d = (a \wedge c)(b \vee d)$

$$\& \, ab \leqq a(b \vee d) \wedge c(b \vee d) = (a \wedge c)(b \vee d).$$

1.10. Corollary. $a = (1 \wedge a)(1 \vee a) = (1 \vee a)(1 \wedge a)$.

1.11. Definition. By *the positive part of* a we mean the element $1 \vee a =: a^+$, by *the negative part of* a we mean the element $1 \wedge a =: a^-$. By a^* we denote the uniquely determined element x satisfying $a^- x = 1$, and we define dually $a^{\cdot}$, satisfying $a^{\cdot} a^- = 1$.

There is a series of crucial lemmata interlinking these notions.

1.12. Lemma. $ab = ab^+ \cdot b^- = ab^- \cdot b^+$.

Proof. Write $ab = a1 \cdot b = ab \cdot 1$ and apply Lemma 1.9.

1.13. Lemma. $a^+ \wedge a^* = 1$.

Proof. $a^+ = aa^* \,\&\, aa^* \wedge a^* = (a \wedge 1)a^* = 1$.

1.14. Lemma. $c \leqq 1 \,\&\, b \wedge c^* = 1 \rightarrow a \cdot bc = ab \cdot c = ac \cdot b$.

Proof. $b \wedge c^* = 1 \rightarrow 1 \wedge cb = (1 \wedge c)(c^* \wedge b) = 1 \wedge c$ by the dual of Lemma 1.3. Thus, if moreover c is negative, we may infer $c = (cb)^-$ and $b = (cb)^+$ from which we get $a \cdot bc = ab \cdot c = ac \cdot b$ by Lemma 1.12.

1.15. Lemma. $u\wedge a^*=1\to a^-u\wedge 1=a^-\leftrightarrow 1\vee a^-u=u\to u\wedge a^*=1$.

This implies nearly immediately

1.13′. Lemma. $y\leqq 1\leqq x\,\&\,x\wedge y^*=1\,\&\,xy=a\to x=a^+\,\&\,y=a^-$.

Moreover 1.15 is essential for part (i) of the subsequent statement.

1.16. Lemma. (i) $(ab)^+=(1\vee a^+b^-)(1\vee a^-b^+)$, $(ab)^-=(1\wedge a^-b^+)(1\wedge a^+b^-)$,
(ii) $(a\wedge b)^+=a^+\wedge b^+\,\&\,(a\wedge b)^-=a^-\wedge b^-$,
(iii) $(a\vee b)^+=a^+\vee b^+\,\&\,(a\vee b)^-=a^-\vee b^-$.

Proof. Ad (i). By 1.14 we have

$$ab = a^+a^-\cdot b^+b^- = (a^+\cdot a^-b^+)b^- = \big(a^+(1\vee a^-b^+)\big)\big((1\wedge a^-b^+)\,b^-\big) =$$
$$= \big((1\wedge a^-b^+)\cdot a^+b^-\big)(1\vee a^-b^+),$$

from which (i) follows by repeating the method on the grounds of

$$u\wedge a^* = 1 = u\wedge b^* \to (1\vee a^-u)(1\vee b^-u) = uu = u\vee(a^-b^-\cdot u)\,u.$$

(We shall come back to this implication in Chapter 4.)

Ad (ii) & (iii). $x\leqq a\leftrightarrow 1\wedge x\leqq 1\wedge a\,\&\,1\vee x\leqq 1\vee a$ and $(a^-\wedge b^-)^*=a^*\vee b^*$ and $(a^-\vee b^-)^*=a^*\wedge b^*$ by 1.9.

1.17. Lemma. $a\wedge b^{:}=1\leftrightarrow a\wedge b^*=1$.

Proof. $a\wedge b^{:}=1\to ab^-\wedge 1=b^-\to a(b^-b^*)\wedge b^*=1\to a\wedge b^*=1$.

We now introduce two further operations.

1.18. Definition. x is called *the right complement* $a*b$ *of* a *in* b if $(a\wedge b)x=b$. Dually we define *the left complement* $b:a$ *of* a *in* b.

Because of $(a\wedge b)(a*b\wedge b*a)=a\wedge b$ we get immediately $a*b\wedge b*a=1$. Next we have

1.19. Lemma. $a\wedge b=a/(b*a)=(b:a)\backslash b$ *and* $a\vee b=a(a*b)=(a:b)\,b$.

1.20. Lemma. $a\leqq b\to x*a\leqq x*b\,\&\,a*x\geqq b*x$.

Furthermore we obtain

1.21. Lemma. $a*(b\vee c)=a*b\vee a*c$.

Proof. $a(a*b\vee a*c)=a(a*b)\vee a(a*c)=a\vee b\vee a\vee c=a\vee(b\vee c)$.

1.22. Lemma. $(a\wedge b)*c=a*c\vee b*c$.

Proof. $(a\wedge b)*c\geqq a*c\vee b*c\,\&\,(a\wedge b)(a*c\vee b*c)\geqq(a\wedge b)\vee c$.

1.23. Lemma. $a*(b\wedge c)=a*b\wedge a*c$.

Proof. We have $a*(b\wedge c)\leqq a*b\wedge a*c$ and

$$(a\wedge b\wedge c)(a*b\wedge a*c) \leqq (a\wedge b)(a*b)\wedge(a\wedge c)(a*c) = b\wedge c$$

whereby $a*b\wedge a*c\leqq a*(b\wedge c)$.

1.24. Lemma. $(a\vee b)*c=a*c\wedge b*c$.

Proof. We have $(a\vee b)*c\leqq a*c\wedge b*c$ and

$$(a\vee b)(a*c\wedge b*c) \leqq a(a*c)\vee b(b*c) = a\vee b\vee c$$

whereby $a*c\wedge b*c\leqq(a\vee b)*c$.

The reader should check that 1.21 through 1.24 remain valid if we replace $*$ by $\backslash$ and $:$ by $/$, provided the "results" under consideration do exist. Now, applying Lemma 1.23 we are able to prove

1.25. Proposition. $(G, \wedge, \vee)$ *is distributive.*

Proof. $a\vee(b\wedge c)=a\big(a*(b\wedge c)\big)=a(a*b)\wedge a(a*c)=(a\vee b)\wedge(a\vee c)$ (and, alternatively, by applying 1.24, $a\wedge(b\vee c) = a/\big((b\vee c)*a\big) = a/(b*a)\vee a/(c*a) = (a\wedge b)\vee(a\wedge c)$).

In the remainder of this section special situations are considered with respect to later paragraphs.

1.26. Definition. We say that *a covers b* if a satisfies $a>b$ and no element of G lies strictly between a and b. By an *atom* we mean any p which covers 1.

1.27. Lemma. *Every atom is prime, i.e. every atom satisfies the implication* $p\leqq a^+b^+\rightarrow p\leqq a^+\vee p\leqq b^+$.

Proof. $p\leqq a^+b^+\,\&\,p\not\leqq b^+\rightarrow p=(p\wedge a^+)(p\wedge b^+)=p\wedge a^+$ by (1.3).

Recall that the standard meaning of p^n is $\big(\cdots((pp)p)p\cdots\big)$.

1.28. Lemma. *Every atom p satisfies* $ap\cdot p^n=a\cdot pp^n$.

Proof. $ap\cdot p^n=a\cdot qp^n$ & $p\neq q$ implies $ap\cdot p^n=aq\cdot p^n$ because of Lemma 1.7, since $ap\cdot p^n$ covers ap^n, whence q is an atom.

1.29. Corollary. *The natural powers of any atom p form a subsemigroup.*

Proof. This is easily shown by induction on the grounds of 1.28.

1.30. Lemma. *Every atom satisfies* $px=1\leftrightarrow xp=1$.

Proof. We prove the left-right direction: 1 covers x and moreover we have $x \leqq 1 \leqq p$ & $x \leqq xp \leqq p$ whence we can infer

$$1 \wedge xp = 1 \rightarrow xp = 1 \quad \text{because of} \quad xp < p$$

$$\vee\ 1 \wedge xp = x \rightarrow 1 \vee xp = p \rightarrow xp = px = 1.$$

We are now turning to rules relevant for Section 4.

1.31. Lemma. *Let the right inverses a^r and b^r exist. Then $a \wedge b$ and $a \vee b$ are right invertible, too, and they satisfy the formulas*

$$(a \wedge b)^r = a^r \vee b^r \quad \textit{and} \quad (a \vee b)^r = a^r \wedge b^r.$$

Proof. $aa^r = 1 = bb^r \rightarrow (a \wedge b)(a^r \vee b^r) = 1 = (a \vee b)(a^r \wedge b^r)$.

Furthermore we shall need some implications for *orthogonal pairs* a, b, i.e. pairs with $a \wedge b = 1 \leftrightarrow: a \perp b$. Here we obtain:

1.32. Lemma. *If $\mathfrak{G}$ is positive, i.e. $G = G^+$, then*

$$a \perp b \rightarrow a * bc = b(a * c) \ \& \ cb : a = (c : a)b.$$

Proof. Making use of 1.3 and 1.7 we get

$$a \perp b \rightarrow (a \wedge bc)\big(b(a * c)\big) = (a \wedge c)\big(b(a * c)\big) = b \cdot (a \wedge c)(a * c) = bc$$

and the rest follows by duality.

1.33. Lemma. *If $\mathfrak{G}$ is positive, then*

$$a \perp c \rightarrow ab * c = b * c = ba * c \,\&\, c : ab = c : b = c : ba.$$

Proof. $a \perp c \rightarrow (ab \wedge c)x = c \rightarrow (b \wedge c)x = c$ by Lemma 1.3, and the rest follows by duality.

1.34. Lemma. *If $\mathfrak{G}$ is positive, then*

$$a \perp b \rightarrow xa * xb = b \,\&\, bx : ax = b.$$

Proof. $a \perp b \rightarrow (xa \wedge xb)y = xb \rightarrow x(a \wedge b) \cdot y = xb \rightarrow y = b$, the rest following by duality.

1.35. Lemma. *If $\mathfrak{G}$ is positive and associative then $\mathfrak{G}$ satisfies*

(i) $ab * c = b * (a * c)$,

(ii) $a * (b : c) = (a * b) : c$,

(iii) $a * bc = (a * b)\big((b * a) * c\big)$.

Proof. These formulas were developed already in earlier papers of the author but for the sake of selfcontainedness we give short proofs in spite of this.

Ad (i): $abx \geqq c \leftrightarrow bx \geqq a*c \leftrightarrow x \geqq b*(a*c)$,
Ad (ii): $ax \geqq b:c \leftrightarrow axc \geqq b \leftrightarrow cx \geqq a:b$,
Ad (iii): $ax \geqq bc \leftrightarrow x = (a*b)y \leftrightarrow a(a*b)y = b(b*a)y \geqq bc$.

Henceforth we consider (conditionally) complete divisibility semiloops. Here we obtain analogously to the finite case:

1.36. Lemma. *If* $\mathfrak{G}$ *is complete then* $\mathfrak{G}$ *satisfies the equation:*
(i) $x(\bigvee a_i)\cdot y = \bigvee(xa_i\cdot y) \,\&\, x(\bigwedge a_i)\cdot y = \bigwedge(xa_i\cdot y)$, *implying*
(ii) $x\diagdown(\bigvee a_i) = \bigvee(x\diagdown a_i) \,\&\, x\diagdown(\bigwedge a_i) = \bigwedge(x\diagdown a_i)$ *and*
(iii) $(\bigvee a_i)\diagdown x = \bigwedge(a_i\diagdown x) \,\&\, (\bigwedge a_i)\diagdown x = \bigvee(a_i\diagdown x)$, *implying*
(iv) $a\wedge\bigvee b_i = \bigvee(a\wedge b_i) \,\&\, a\vee\bigwedge b_i = \bigwedge(a\vee b_i)$.

Proof. The proof is left to the reader since it is analogous to the corresponding proofs of the finite cases. (Of course, (ii) and (iii) are valid as far as the objects under consideration do exist.)

Finally we remark

1.37. Lemma. $\mathfrak{G}$ *is already complete if its (positive) cone is complete. More precisely:* $s \leqq a_i \rightarrow \bigwedge(1\vee a_i)\cdot\bigwedge(1\wedge a_i) = \bigwedge a_i$.

Proof. This is an immediate consequence of $x \leqq a_i$ if and only if $1\vee x \leqq 1\vee a_i$ & $(1\wedge x)^* \geqq (1\wedge a_i)^*$ which implies for lower bounded sets a_i $(i\in I)$ the formula stated above.

2. Lattice loop cones

The structure of a lattice group is completely determined (up to isomorphism) by the structure of its cone. The question arises whether the same is true in the lattice loop case. Obviously the situation is pleasant as far as the underlying lattice is considered (1.31). But it was already shown in the introduction, that non-isomorphic lattice loops may have isomorphic cones. Hence the question is reduced to the problem whether it is possible to characterize those divisibility semiloops which admit some lattice loop extension. To this end we start from a positive divisibility semiloop $\mathfrak{C}$.

2.1. Definition. By L we denote the set of all *orthogonal pairs* $(a|b)$ $(a\perp b,$ $a, b\in C)$. Furthermore $\mathfrak{L}$ will symbolize the structure $(L, \circ, \wedge)$ the operations of which are defined by

$$(a|b)\circ(c|d) := ((a:d)(b*c)|(d:a)(c*b))$$

and

$$(a|b)\wedge(c|d) := (a\wedge c|b\vee d).$$

Obviously $\circ$ is defined in a right left dual manner. This means: a proposition and its proof remain true if $(x|y)$ is replaced by $(y|x)$ and $a*b$ by $b:a$, $c:d$ by $d*c$. Furthermore by Lemma 1.3 $\circ$ is an operation.

2.2. Lemma. $(L, \wedge)$ *is a semilattice.*

Proof. We have to show $a \perp b \ \& \ c \perp d \rightarrow a\wedge c \perp b\vee d$, which follows from $(a\wedge c)\wedge(b\vee d)=(a\wedge c\wedge b)\vee(a\wedge b\wedge d)$

2.3. Lemma. $\mathfrak{L}$ *satisfies*

$$(a|b) \leqq (c|d) \rightarrow (a|b)\circ(x|y) \leqq (c|d)\circ(x|y) \,\&\, (x|y)\circ(a|b) \leqq (x|y)\circ(a|b).$$

Proof. This is an immediate consequence of Lemma 1.20.

2.4. Lemma. $(a|b)\circ(c|d)=((a|b)\circ(c|1))\circ(1|d)=(a|1)\circ((1|b)\circ(c|d))$.

Proof. By 1.32 and 1.33

$$((a:d)(b*c)|(d:a)(c*b)) = (a(b*c):d|(d:a)(c*b))=$$
$$= (a(b*c):d|(d:a(b*c))(c*b)) = (a(b*c)|c*b)\circ(1|d) =$$
$$= ((a|b)\circ(c|1))\circ(1|d),$$

the rest following by duality.

2.5. Lemma. $((a|b)\circ(1|x))\circ(x|1)=(a|b) = (1|x)\circ((x|1)\circ(a|b))$.

Proof. We have

$$((a:x)|(x:a)\,b)\circ(x|1) = ((a:x)((x:a)\,b*x)|x*(x:a)\,b) =$$
$$= ((a:x)((x:a)\,b*(x:a)(a\wedge x))|(x:a)(a\wedge x)*(x:a)\,b) = ((a:x)(x\wedge a)|b) = (a|b)$$

by 1.34, the rest following by duality.

2.6. Lemma. $((a|b)\circ(x|1))\circ(1|x)=(a|b)=(x|1)\circ((1|x)\circ(a|b))$.

Proof. We have

$$(a(b*x)|(x*b)\circ(1|x)) = (a(b*x):x|(x:a(b*x))(x*b)) =$$
$$= (a(b*x):(b\wedge x)(x*b)|((x:(b*x)):a)(x*b)) = (a|(x\wedge b)(x*b)) = (a|b)$$

by 1.34, the rest following by duality.

2.7. Lemma. $((a|b)\circ(x|y))\circ(y|x)=(a|b)=(x|y)\circ((y|x)\circ(a|b))$.

Proof. We have

$$((a|b)\circ(x|y))\circ(y|x) = ((((a|b)\circ(x|1))\circ(1|y))\circ(y|1))\circ(1|x) =$$
$$= ((a|b)\circ(x|1))\circ(1|x) = (a|b),$$

the rest following by duality.

2.8. Lemma. *$(a|b)\circ(x|y)\doteq(c|d)$ and $(u|v)\circ(a|b)\doteq(c|d)$ have uniquely determined solutions.*

Proof. Apply Lemma 2.7. It follows that $(x|y)=((b|a)\circ(c|d))$ in the first case and $(u|v)=((c|d)\circ(b|a))$ in the second case are the only solutions.

2.9. Lemma. $(a|b)\circ(1|1)=(a|b)=(1|1)\circ(a|b)$.

Proof. $(a:1|1*b)=(a|b)=(1*a|b:1)$.

2.10. Lemma. *$(a|1)\circ(b|1)=(ab|1)$ and $(a|1)\wedge(b|1)=(a\wedge b|1)$.*

Proof. Obvious.

Hence summarizing the lemmata proven so far we get

2.11. Proposition. *A partially ordered groupoid is the cone of some lattice loop if and only if it is a positive divisibility semiloop.*

2.12. Definition. By an *inverse* loop we mean a loop having the inverse property, i.e. satisfying $\forall a\,\exists a^{-1}$: $a^{-1}(ab)=b=(ba)a^{-1}$.

Obviously inverse loops satisfy $xx^{-1}=1=x^{-1}x$ and furthermore one can infer $(xy)^{-1}=y^{-1}x^{-1}$, since $(xy)y^{-1}=x\rightarrow y^{-1}=(xy)^{-1}x\rightarrow y^{-1}x^{-1}=(xy)^{-1}$. In general a lattice loop is far from being inverse. However we can prove

2.13. Proposition. *Any lattice loop cone is the cone of an inverse lattice loop.*

Proof. We define $(x|y)^{-1}:=(y|x)$. Then the assertion is proven by Lemma 2.7.

Let us consider now the extension $\mathfrak{L}$ of the cone $\mathfrak{C}$. We shall show that $\mathfrak{L}$ is uniquely determined up to isomorphism provided inverse lattice loops are considered. Furthermore we shall prove some other extension properties concerning congruence relations and order.

2.14. Proposition. *$\mathfrak{L}$ is uniquely determined provided inverse extensions are considered.*

Proof. Let $\mathfrak{J}$ denote an inverse lattice loop. Then by Lemma 1.16 we can infer $ab^{-1}\cdot cd^{-1}=a(1\vee b^{-1}c)\cdot(1\wedge b^{-1}c)d$ and by the rules of lattice loop arithmetic we get $1\vee a^{-1}b=a*b$ since $a(1\vee a^{-1}b)=a\vee b$, and $1\vee ba^{-1}=b:a$ by duality. Thus $1\wedge a^{-1}b=(1\vee b^{-1}a)^{-1}=(b*a)^{-1}$ and $1\wedge ba^{-1}=(a:b)^{-1}$ by duality, whence

$$ab^{-1}=(1\vee ab^{-1})(1\wedge ab^{-1})=(a:b)(b:a)^{-1}.$$

But applying these formulas and 1.16 we obtain

$$ab^{-1}\cdot cd^{-1} = a(1\vee b^{-1}c)\cdot(1\wedge b^{-1}c)\,d^{-1} = a(b*c)\cdot(c*b)^{-1}d^{-1} =$$
$$= (c*b)^{-1}\cdot\big(a(b*c)\cdot d^{-1}\big) = (c*b)^{-1}\cdot\big(ad^{-1}\cdot(b*c)\big) =$$
$$= (c*b)^{-1}\cdot\big((a:d)(d:a)^{-1}\cdot(b*c)\big) = (a:d)(b*c)\cdot(c*b)^{-1}(d:a)^{-1} =$$
$$= (a:d)(b*c)\cdot\big((d:a)(c*b)\big)^{-1}.$$

Hence the function $(a|b)\to ab^{-1}$ is an isomorphism of $\mathfrak{L}$ and $\mathfrak{J}$ if the cone $\mathfrak{C}$ is isomorphic to the cone of $\mathfrak{J}$.

We now turn to elementary algebraic properties like associativity, commutativity, etc., the first result of this type being nearly obvious:

2.15. Lemma. *If $\mathfrak{C}$ is commutative then $\mathfrak{L}$ is commutative, too.*

Proof. If $\mathfrak{C}$ is commutative then $x:y$ is equal to $y*x$ which yields

$$(a|b)\circ(c|d) = \big((a:d)(b*c)|(d:a)(c*b)\big) =$$
$$= \big((b*c)(d*a)|(c*b)(a*d)\big) = (c|d)\circ(a|b).$$

A loop $\mathfrak{L}$ is called *monassociative* if every $a\in L$ generates a subsemigroup of $(L,\,\cdot)$. A loop is called *power-associative* if every $a\in L$ generates a subgroup of $(L,\,\cdot,\diagdown,\diagup)$.

2.16. Lemma. *If $\mathfrak{C}$ is monassociative then $\mathfrak{L}$ is power-associative.*

Proof. By Lemma 1.3 we get $(a|b)^n=(a^n|b^n)$ $(n\in\mathbb{N})$ and by the inverse property we have $(a|b)^{-n}=((a|b)^{-1})^n$.

2.17. Lemma. *If $\mathfrak{C}$ is associative then $\mathfrak{L}$ is associative, too.*

Proof. We show

$$\big((a|1)\circ(c|d)\big)\circ(1|v) = (a|1)\circ\big((c|d)\circ(1|v)\big),$$
$$\big((1|b)\circ(c|d)\big)\circ(1|v) = (1|b)\circ\big((c|d)\circ(1|v)\big),$$
$$\big((1|b)\circ(c|d)\big)\circ(u|1) = (1|b)\circ\big((c|d)\circ(u|1)\big).$$

(Observe that line 3 can be considered as a dual of line 1, since putting $a\cdot b:=ba$ we get a dual divisibility semiloop with $(a|b)\bullet(c|d)=(c|d)\circ(a|b)$. Hence line 3 results from line 1 for the dual structure.)

Equivalently

$$\big((a:d)\,c:v|(v:(a:d)\,c)(d:a)\big) = \big((a:(v:c)\,d)(c:v)|(v:c)\,d:a\big),$$
$$\big((b*c):v|(v:(b*c))\,d(c*b)\big) = \big(b*(c:v)|(v:c)\,d((c:v)*b)\big),$$

and

$$\big((b*c)(d(c*b)*u)|u*d(c*b)\big) = \big(b*c(d*u)|(u*d)(c(d*u)*b)\big).$$

But lines 1 and 3 follow from Lemma 1.35 and its duals, and the left components of the second equation are equal because of 1.35, too. So it remains to show

$$(v:(b*c))\,d(c*b)*(v:c)\,d((c:v)*b) = 1,$$
$$(v:(b*c))\,d(c*b):(v:c)\,d((c:v)*b) = 1.$$

Now, the second equation is the right-left dual of the first one. Therefore it suffices to settle the first case. Here we obtain:

$$\begin{aligned}(v:(b*c))\,d(c*b)*(v:c)\,d((c:v)*b) &=\\ = d(c*b)*(((v:c)*(v:(b*c)))*d((c:v)*b)) &=\\ = d(c*b)*(((c:v)*(c:(b*c)))*d((c:v)*b)) &=\\ = d(c*b)*d(((c:v)*(c:(b*c)))*((c:v)*b)) &=\\ = (c*b)*(((c\wedge b)*(c:v))*((c\wedge b)*b)) &=\\ = (c*b)*(((c\wedge b)*(c:v))*(c*b)) &= 1.\end{aligned}$$

The second, third, and fourth equalities follow from 1.35, 1.32 and 1.19, 1.35, respectively. Hence the proof is completed by

$$\begin{aligned}((a|b)\circ(c|d))\circ(u|v) &= (((a|1)\circ((1|b)\circ(c|d)))\circ(u|1))\circ(1|v) =\\ &= ((a|1)\circ(((1|b)\circ(c|d))\circ(u|1)))\circ(1|v) =\\ &= (a|1)\circ((((1|b)\circ(c|d))\circ(u|1))\circ(1|v)) =\\ &= (a|1)\circ(((1|b)\circ((c|d)\circ(u|1)))\circ(1|v)) =\\ &= (a|1)\circ((1|b)\circ(((c|d)\circ(u|1))\circ(1|v))) = (a|b)\circ((c|d)\circ(u|v)).\end{aligned}$$

We continue our investigation by two further results concerning the order relation.

2.18. Lemma. *If $\mathfrak{C}$ is totally ordered then $\mathfrak{L}$ is totally ordered, too.*

Proof. $a\leqq b\to(a|b)=(1|b)$ and $a\geqq b\to(a|b)=(a|1)$. Furthermore we get $(a|1)\geqq$ $\geqq(1|b)$ for all $a, b\in C$.

2.19. Lemma. *If $\mathfrak{C}$ is completely ordered then $\mathfrak{L}$ is completely ordered, too,*

Proof. Apply Lemma 1.37.

Finally we consider congruences. Here we can show

2.20. Proposition. *The congruences of $(C, \cdot, *, :)$ are uniquely extended to $\mathfrak{L}$.*

Proof. Let $\equiv$ be a congruence of $(C, \cdot, *, :)$. We define $(a|b)\equiv(c|d)$ iff $a\equiv c$ & $b\equiv d$. This provides a congruence on $\mathfrak{L}$ as is easily checked by the reader. On the other hand for any extension ϱ of $\equiv$ from $(C, \cdot, *, :)$ to $\mathfrak{L}$ we get $(a|b)\,\varrho\,(c|d)\leftrightarrow ad\equiv bc$ which implies $a\equiv c$ & $b\equiv d$ because of Lemma 1.3.

3. The chain condition

Obviously a divisibility semiloop satisfies the descending chain condition for any $[a, b)$ iff it satisfies the ascending chain condition for any $(a, b]$. Hence we may speak of models with chain condition (C.C.). Suppose in this section that $\mathfrak{G}$ has the C.C.-property. Then every positive element a is a product of atoms since otherwise there would be a minimal one to fail, a contradiction. Furthermore for every $a>1$ and arbitrary atom p there exists a maximal number $p(a)$ such that $p^{p(a)} \leqq a$. Finally for any pair of different atoms p, q we get $p^m \perp q^n$ $(m, n \in \mathbf{N})$ because of 1.3, and thereby $p^m \cdot q^n = p^m \vee q^n$. This provides a uniquely determined prime factorization for any positive $a \in G$ (see f.i. [16]).

The purpose of this paragraph is to show that C.C. implies commutativity and associativity. This is nearly obvious for C^+ and by duality also for C^- (consult 1.29 and the remark above). But the general case requires some additional calculation.

3.1. Lemma. *Let $\bar{q}$ be the right inverse of q and let p, q be two atoms. Then every p^m commutes with every $\bar{q}^n$.*

Proof. It suffices to prove $p\bar{p}=1 \rightarrow p^m \cdot \bar{p}^m = 1$, because of 1.14, 1.30. But this is shown by induction since 1.28 implies $p^m p \cdot \bar{p}\bar{p}^m = p^m(p\bar{p} \cdot \bar{p}^m)$.

3.2. Lemma. *If $\mathfrak{G}$ satisfies* C.C. *then $\mathfrak{G}$ is associative and commutative.*

Proof. By 3.1 and the distributivity laws we get $a^+ \cdot b^- = b^- \cdot a^+$ whence $a^+ \cdot b = a^+ b^+ \cdot b^- = b^- \cdot b^+ a^+ = b \cdot a^+$ and dually $a \cdot b^- = b^- \cdot a$. Hence we obtain $a \cdot b = a^- \cdot a^+ b = ba^+ \cdot a^- = ba$. Furthermore we have $ab^- \cdot c^- = a \cdot b^- c^-$. Thus we get $ab \cdot c = (a^+ b^+ \cdot a^- b^- \cdot \cdot c^-)c^+ = c^+(a^+ b^+ \cdot a^- b^- c^-) = c^+ a^+ b^+ \cdot a^- b^- c^- = a \cdot bc$.

Summarizing the preceding remarks and results we get

3.3. Theorem. *A divisibility semiloop satisfies the chain condition for closed intervals* $[a, b]$ *if and only if it is a direct sum of copies of* $(\mathbf{Z}, +, \min)$ *and* $(\mathbf{N}^0, +, \min)$ *respectively.*

4. Complete divisibility semiloops

In this section we shall prove that power-associative complete divisibility semiloops are even associative and commutative. This was done for loops with the real line as underlying lattice by ACZÉL [1], and for totally ordered loops in general by HARTMAN [22].

4.1. Definition. $\mathfrak{G}$ is called *power-associative* if any element a generates a subsemigroup and any pair a^-, a^* generates a subgroup of $(G, \cdot)$.

4.2. Definition. Extending the relation $\perp$, henceforth by $u \perp x$ we shall mean $u^+u^* \wedge x^+x^* = 1$. Furthermore $U^\perp$ will denote the set of all x satisfying $u \perp x$, where u is running through U.

It is easily checked by Lemma 1.16 and Lemma 1.31 that $U^\perp$ is a multiplicatively closed sublattice of $\mathfrak{C}$.

4.3. Lemma. *Let $\mathfrak{C}_1 \times \mathfrak{C}_2$ be a direct decomposition of $(C^+, \cdot, \wedge, \vee)$. Then $\mathfrak{C}_1^\perp \times \mathfrak{C}_2^\perp$ is a direct decomposition of $\mathfrak{G}$.*

Proof. We denote $C_1^\perp$ by G_2 and $C_2^\perp$ by G_1. Then every element a is a product of type $a_1 a_2$ where the indices indicate the components G_1, G_2. To see this we consider a^-. There is a decomposition $a^* = a_1^* a_2^*$ and we have $a^- a_1^* \leqq 1$ and $a^- a_2^* \leqq 1$ whence there are elements a_1^{*l} and a_2^{*l} with $(a_1^{*l} \cdot a_2^{*l}) \cdot (a_1^* \cdot a_2^*) = 1$. Hence $a_1^{*l} a_2^{*l}$ is equal to a^- and by definition a_1^{*l} and a_2^{*l} are contained in G_1 and G_2 respectively. But this yields

$$a^+a^- = a_1^+a_2^+ \cdot a_1^{*l}a_2^{*l} = a_1^+(a_2^+ \cdot a_1^{*l}a_2^{*l}) = a_1^+(a_1^{*l} \cdot a_2^+a_2^{*l}) = a_1^+a_1^{*l} \cdot a_2^+a_2^{*l}$$

by means of 1.14, 1.17, 1.3, and, applying 1.14, 1.3, we obtain furthermore

$$a_1a_2 = b_1b_2 \to a_1^+a_2^+ \cdot a_1^-a_2^- = b_1^+b_2^+ \cdot b_1^-b_2^- \to$$
$$\to a_1^+a_2^+ = b_1^+b_2^+ \,\&\, a_1^-a_2^- = b_1^-b_2^- \to a_1^+ = b_1^+ \ldots a_2^- = b_2^-,$$

since $a_1^- a_2^- \cdot a_1^* a_2^* = 1$, which implies $a_1^+ a_2^+ \perp (a_1^- a_2^-)^*$.

Hence G may be considered as the cartesian product of G_1 and G_2. We now show that the operations $\cdot$ and $\wedge$ may be carried out pointwise. First of all we recall $a_1 a_2 = a_1^+ a_2^+ \cdot a_1^- a_2^-$ which was stated above on the grounds of Lemma 1.14. This implies with respect to multiplication

$$a \cdot b_1b_2 = (a \cdot b_1^+b_2^+) \cdot b_1^-b_2^- = (a \cdot b_2^+b_1^+) \cdot b_2^-b_1^- =$$
$$= (ab_1^+ \cdot b_2^+)b_1^- \cdot b_2^- = (ab_1^+ \cdot b_1^-)b_2^+ \cdot b_2^- = ab_1 \cdot b_2 = ab_2 \cdot b_1$$

(in the third step 1.7 was applied), from which it follows that

$$a_1a_2 \cdot b_1b_2 = (a_1a_2 \cdot b_1)b_2 = (a_1b_1 \cdot a_2)b_2 = a_1b_1 \cdot a_2b_2.$$

Recall now $a_1^+ a_2^+ = a_1^+ \vee a_2^+$ and $a_1^- a_2^- = a_1^- \wedge a_2^-$ (1.8). One can infer:

$$a_1a_2 \wedge b_1b_2 = (a_1^+a_2^+ \wedge b_1^+b_2^+) \cdot (a_1^-a_2^- \wedge b_1^-b_2^-) =$$
$$= (a_1^+ \wedge b_1^+)(a_2^+ \wedge b_2^+) \cdot (a_1^- \wedge b_1^-)(a_2^- \wedge b_2^-) =$$
$$= (a_1^+ \wedge b_1^+)(a_1^- \wedge b_1^-) \cdot (a_2^+ \wedge b_2^+)(a_2^- \wedge b_2^-) = (a_1 \wedge b_1) \cdot (a_2 \wedge b_2).$$

Thus out proof is complete.

4.4. Lemma. *Let $\mathfrak{G}$ be complete and $a \not\leqq b$ & $b \not\leqq a$. Then there is a direct decomposition $\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2$ with $\bar{a}_1 \leqq \bar{b}_1$ & $\bar{a}_2 \leqq \bar{b}_2$.*

Proof. By Lemma 4.2 it suffices to verify the assertion for positive divisibility semiloops. In this case we define $C_1:=(a*b)^{\perp}$ and $C_2:=C_1^{\perp}$. Then C_1 and C_2 are 1-disjoint and every c has a decomposition c_1c_2 with $c_1=\mathrm{Sup}\,(x|x\leqq c\ \&\ x\in C_1)$. (This idea seems to go back to RIESZ [33]. See also BIRKHOFF [4].) Observe: $y\in C_1\to\to c_1\cdot(c_2\wedge y)=c_1\cdot 1$. Furthermore this decomposition is unique and the operations may be carried out pointwise since $a\wedge b=1\to a\cdot b=a\vee b$.

Now we are ready to prove:

4.5. Theorem. *A power-associative and complete divisibility semiloop $\mathfrak{L}$ is associative and commutative. But if a complete divisibility semiloop is only monassociative it need neither be associative nor commutative even though $\mathfrak{G}$ should be a complete totally ordered loop.*

Proof. We shall verify our assertion by constructing a series of models and specializing the situation until $ab\cdot c\neq a\cdot bc$ leads to a contradiction.

By Lemma 4.3 we may start from a model $\mathfrak{G}_1$ with $ab\cdot c<a\cdot bc$ for some triple a, b, c. Furthermore, by the same lemma, we may suppose that a, b and c are strictly positive or negative, and that $\{a, b, c\}$ is totally ordered. We consider $1<t\leqq d:=ab\cdot c*a\cdot bc$ and some $x>1$. There exists a natural number n such that $t^n\leqq x\ \&\ t^{n+1}\nleqq x$, since otherwise $\mathrm{Sup}\,(t^n|n\in\mathbf{N})=:\Omega$ would exist and satisfy $\Omega t=\Omega$, a contradiction. Hence in any case there exists a model $\overline{\mathfrak{G}}_1$, with $\bar{t}^n\leqq\bar{x}<\bar{t}^{n+1}$ satisfying $\bar{a}\bar{b}\cdot\bar{c}<\bar{a}\cdot\bar{b}\bar{c}$ because of $\bar{1}<\bar{x}*\bar{t}^{n+1}\leqq\bar{a}\bar{b}\cdot\bar{c}*\bar{a}\cdot\bar{b}\bar{c}$.

Consequently we may suppose a model $\mathfrak{G}_2$ containing a triple u, v, w with $uv\cdot w<u\cdot vw$ and $1<s\leqq uv\cdot w*u\cdot vw$ such that $\{s^n|n\in\mathbf{Z}\}\cap\{u, v, w\}$ is totally ordered: Apply the method above successively to $a\vee a^*$, $b\vee b^*$, $c\vee c^*$. None of these elements is equal to 1 and if for instance a is (strictly) negative, then according to (DSL 5) $\bar{r}:=\bar{t}\wedge\bar{a}^*$ is invertible whence we can continue the procedure with $\bar{r}$ satisfying $\bar{1}<\bar{r}\leqq\bar{d}$. So in $\mathfrak{G}_2$ we have $1<s\leqq uv\cdot w*u\cdot vw<s^3$. But this implies that the proof is complete if we deduce $1<g^4\leqq xy\cdot z*x\cdot yz$ for some triple x, y, z in some model $\mathfrak{H}$.

To this end we start w.l.o.g. from $s^n<u\vee u^*=:\bar{u}<s^{n+1}$. This leads to $1<u*s^{n+1}=:f<s$ and further to $f(f*s)=s$ whence we get one of the three relations $1<f^2\leqq s$ or $1<(f\wedge(f*s))^2\leqq s$ or $f^2\nleqq s\ \&\ f\perp f*s$. Obviously in the first two cases there is some f_1 in G_2 satisfying the inequality $1<f_1^2\leqq s$ in $\mathfrak{G}_2$. We now show that also the third case provides some model of this type. Indeed, $f\perp f*s$ implies $s\nleqq f^2$ since $f(f*s)\leqq ff$ would yield $1<f*s<f$. Hence we get $f^2\nleqq s\nleqq f^2$, and thereby a direct decomposition $\mathfrak{G}_2=\overline{\mathfrak{G}}_2\times\overline{\overline{\mathfrak{G}}}_2$ with $\bar{f}^2\geqq\bar{s}$ in $\overline{\mathfrak{G}}_2$ and $\bar{\bar{f}}^2\leqq\bar{\bar{s}}$ in $\overline{\overline{\mathfrak{G}}}_2$. Suppose now that $\bar{f}$ is equal to $\bar{\bar{1}}$. Then $\bar{f}$ is different from $\bar{1}$ and hence $\overline{\mathfrak{G}}_2$ is a model satisfying $\bar{f}*\bar{s}=\bar{f}*\bar{s}\wedge\bar{f}^2=\bar{1}$ whence we get $\bar{f}=\bar{s}$ and thereby $\bar{\bar{u}}=\bar{s}^n$. Hence continuing the procedure with $\bar{v}$ or $\bar{w}$ in the role of u (above), in any case we arrive

at a direct factor $\mathfrak{G}'$ of $\mathfrak{G}$ with $1'<f'^2 \leqq s' \leqq u'v' \cdot w' * u' \cdot v'w'$. Therefore starting from this new situation with f' in the role of s we finally do obtain a model $\mathfrak{H}$ with a triple x, y, z satisfying the inequality $1 < g^4 \leqq xy \cdot z * x \cdot yz$, a contradiction.

Hence $\mathfrak{G}$ is associative and in the same manner one verifies that $\mathfrak{G}$ is also commutative.

It remains to show that there are complete totally ordered loops which are neither associative nor commutative. To this end we consider the real line with respect to some derived operations:

(i) We define $a \circ b := a+b$ except for the case $a \leqq 0 \leqq b$, where we put $a \circ b := a + b/2$ if $a + b/2 \leqq 0$ & $a \circ b := 2a + b$ otherwise. This provides a monassociative but non-associative and non-commutative complete and totally ordered loop. Observe:

$$((-1)\circ 2)\circ(-1) = -1 \neq -1/2 = (-1)\circ(2\circ(-1)).$$

(ii) We define $a \circ b := a+b$, except for the case $a, b \leqq 0$, where we put $a \circ b := a - ab + b$ [22]. This provides a commutative monassociative but non-associative complete and totally ordered loop. Observe:

$$(1\circ(-1))\circ(-1) = -1 \neq -2 = 1\circ((-1)\circ(-1)).$$

5. Completion

The goal of this section is a characterization of divisibility semiloops admitting a complete extension. Nearly obviously such models have to satisfy for lower bounded subsets A the implications

(i) $x, y \mid_l A \;\&\; x \backslash A \downarrow y \backslash A \rightarrow x = y$,

(ii) $x, y \mid_r A \;\&\; A / x \downarrow A / y \rightarrow x = y$,

(iii) $A \mid_l x, y \;\&\; A \backslash x \uparrow A \backslash y \rightarrow x = y$,

(iv) $A \mid_r x, y \;\&\; x / A \uparrow y / A \rightarrow x = y$,

where $\mid_l$ and $\mid_r$ stand for *left-divisor* and *right-divisor* respectively, and $\downarrow$ and $\uparrow$ stand for *coinitial* and *cofinal* respectively. For instance (i) follows from $x \backslash A \downarrow y \backslash A \rightarrow x \backslash \bigwedge A = \bigwedge x \backslash A = \bigwedge y \backslash A = y \backslash \bigwedge A$.

Thus a characterization of models with complete extensions is given provided that (i) through (iv) guarantee such an extension. In order to verify this we start by giving some symbols and notions. Henceforth (A) will denote the set of all upper bounds of A and dually $[A]$ will stand for the set of all lower bounds of A. Furthermore by p we shall mean a multiplication polynomial in one variable, i.e. a polynomial of type $\ldots a_4((a_2(xa_1))a_3)\ldots$. (Recall that $\mathfrak{G}$ has a unit.) Consequently $p(A)$ will denote the set of all $p(a)$ $(a \in A)$.

As an immediate consequence of (DSL 5) we notice that $p^{-1}(v)$ exists if there is an a such that $v \geqq p(a)$.

5.1. Definition. A subset A of G is called a *t-ideal* if A contains all elements c with $v \geqq p(A) \to v \geqq p(c)$.

It is easily checked that t-ideals are lattice ideals. Furthermore the reader straightforwardly verifies that G is a t-ideal and that the intersection of all t-ideals containing $A \neq \emptyset$ is a t-ideal, too. This yields that there is a smallest t-ideal $\bar{A}$ containing $A \neq \emptyset$ and moreover the definition $\bar{A} \cdot \bar{B} = \overline{AB}$ provides a unique multiplication since $\bar{A} = \bar{C}$ & $\bar{B} = \bar{D}$ implies $v \geqq p(AB) \leftrightarrow v \geqq p(CD)$. Henceforth we shall denote $\bar{A}$ also by $\mathbf{A}$.

Let us suppose now that the set X of elements x with $\mathbf{A}x \subseteqq \mathbf{B}$ is not empty. Then $X =: \mathbf{A} * \mathbf{B}$ is a t-ideal which follows from the following implication:

$$v \geqq p(X) \to v \geqq p(c) \quad \text{implies} \quad w \geqq q(\mathbf{B}) \to w \geqq q(AX) \to w \geqq q(Ac),$$

which implies $Ac \subseteqq B$.

5.2. Lemma. $\mathfrak{G}$ *satisfies* $\mathbf{A} = [(A)]$.

Proof. Obviously $\mathbf{A}$ is contained in $[(A)]$. Furthermore any $c \in [(A)]$ satisfies the implication $v \geqq p(A) \to p^{-1}(v) \geqq A \to p^{-1}(v) \geqq c \to v \geqq p(c)$ whence each c of $[(A)]$ is contained in $\mathbf{A}$.

5.3. Lemma. $\mathbf{a} := \bar{a}$ *is equal to the set of all x below a. Hence $\mathfrak{G}$ is embedded in the structure formed by the t-ideals with respect to $\cdot$ and inclusion.*

Proof. Left to the reader.

5.4. Lemma. $\mathfrak{G}$ *satisfies* $\mathbf{A} \cdot \mathbf{X} \subseteqq \mathbf{b} \to \mathbf{A} \cdot (\mathbf{A} * \mathbf{b}) = \mathbf{b}$.

Proof. By assumption $\mathbf{A} * \mathbf{b}$ exists. We suppose $\mathbf{A} \cdot (\mathbf{A} * \mathbf{b}) \leqq c \leqq b$. Then there exists an element v with $A \cdot v \leqq c \leqq b$, whence there is also an element u with $\mathbf{A} \leqq u$ & $us = b$. But for any such u we get:

$$us = b \to \mathbf{A}s \leqq b \to \mathbf{A}s \leqq c \to \mathbf{A} \leqq c/s = u_c|_l b.$$

Hence for any u with $A \leqq u$ we find an u_c with $A \leqq u_c$ such that $us = b$ implies $u_c s = c$. But this means that the set U of all u with $\mathbf{A} \leqq u$ & $u|_l b$ satisfies $U \searrow b \uparrow U \searrow c$ which yields $c = b$.

5.5. Lemma. $\mathfrak{G}$ *satisfies* $\mathbf{A} \subseteqq \mathbf{B} \to \mathbf{A} \cdot (\mathbf{A} * \mathbf{B}) = \mathbf{B}$.

Proof. Consider an arbitrary element $b \in \mathbf{B}$. Then the t-ideal $\mathbf{A}_b$ generated by all $a \wedge b$ $(a \in A)$ satisfies $\mathbf{A}_b \cdot \mathbf{X}_b = \mathbf{b}$ for $\mathbf{X}_b = \mathbf{A}_b * \mathbf{b}$. We consider the t-ideal $\mathbf{X}$

generated by all $\mathbf{X}_b$. Then $\mathbf{A}\cdot\mathbf{X}\supseteqq\mathbf{B}$ is obvious and moreover for any pair a, x ($a\in A$, $x\in X_b$) we can infer

$$(a\wedge b)x \leqq b \to x \leqq a*b \to ax \leqq a(a*b) = a\vee b\in\mathbf{B},$$

whence $\mathbf{A}\cdot\mathbf{X}$ is also contained in $\mathbf{B}$.

5.6. Lemma. (i), ..., (iv)$\Rightarrow\mathbf{a}\cdot\mathbf{X}\subseteqq\mathbf{B}\to\exists\mathbf{Z}$: $\mathbf{a}\cdot\mathbf{Z}=\mathbf{B}$.

Proof. By 5.5 there is a t-ideal $\mathbf{Y}$ with $(\mathbf{a}\cdot\mathbf{X})\cdot\mathbf{Y}=\mathbf{B}$, and for every pair x, y ($x\in\mathbf{X}$, $y\in\mathbf{Y}$) there exists an element z with $(ax)y\leqq az=b\in\mathbf{B}$ since $ax\in\mathbf{B}$ & $(ax)y\in\mathbf{B}$ implies $(ax)(1\vee y)\in\mathbf{B}$. Hence the t-ideal $\mathbf{Z}$ generated by these elements z satisfies $\mathbf{a}\cdot\mathbf{Z}=\mathbf{B}$.

5.7. Lemma. (i), ..., (iv)$\Rightarrow s\geqq A$ & $\mathbf{A}\cdot\mathbf{X}=\mathbf{A}\cdot\mathbf{Y}\to\mathbf{X}=\mathbf{Y}$.

Proof. Suppose $v\geqq\mathbf{X}$. It follows $\mathbf{A}\cdot v\geqq\mathbf{A}\cdot y$ for all $y\in\mathbf{Y}$, and thereby $\mathbf{A}\cdot\overline{(v\vee y)}=\mathbf{A}\cdot\mathbf{v}=:\mathbf{B}$ (5.2). But this yields $\mathbf{B}/v=\mathbf{B}/(y\vee v)$ whence we get $v=y\vee v$. It follows $v\geqq\mathbf{Y}$ and thereby $\mathbf{X}\supseteqq\mathbf{Y}$. Thus the proof is complete by duality.

5.8. Lemma. (i), ..., (iv)$\Rightarrow\mathbf{a}\cdot\bigwedge\mathbf{X}_i=\bigwedge(\mathbf{a}\cdot\mathbf{X}_i)$.

Proof. By 5.6 there is a t-ideal $\mathbf{Z}$ with $\mathbf{a}\cdot\mathbf{Z}=\bigwedge(a\cdot\mathbf{X}_i)$ ($i\in I$). Furthermore by 5.2 the t-ideal generated by all $a\vee b$ ($a\in\mathbf{A}$, $b\in\mathbf{B}$) satisfies $\overline{\{a\vee b\mid a\in\mathbf{A}, b\in\mathbf{B}\}}=\overline{\{\mathbf{A},\mathbf{B}\}}$. Consequently for upper bounded t-ideals $\mathbf{A}$ the following implication holds: $\mathbf{A}\cdot\mathbf{X}\subseteqq\mathbf{A}\cdot\mathbf{Y}\to\mathbf{X}\subseteqq\mathbf{Y}$. Thus $\mathbf{Z}$ is contained in every $\mathbf{X}_i$, which implies the assertion.

Once more we emphasize that we consider a proposition to be proven once its dual has been verified.

Up to now we have been concerned with t-ideals. But obviously there is a dual notion, called v-ideal, which is defined by writing (in 5.1) the symbol $\leqq$ instead of the symbol $\geqq$. We shall denote v-ideals by $\underline{A}$ or A. The proofs, however, given here so far do not carry over in any case since the structure under consideration is not $\geqq/\leqq$-dual. Nevertheless the reader will easily verify that the part up to 5.2 (excluded) can straightforwardly be dualized. Thus there is a product $\mathsf{A}\circ\mathsf{B}=\overline{\underline{AB}}$ and a right-quotient $\mathsf{A}*\mathsf{B}:=\{x|\mathsf{A}x\subseteqq\mathsf{B}\}$ (a left-quotient $\mathsf{B}:\mathsf{A}:=\{x|x\mathsf{A}\subseteqq\mathsf{B}\}$).

We now return to the t-ideal-extension of $\mathfrak{G}$. We wish to show that (DSL 5) is valid. To this end we denote the principal t-ideal $\mathbf{t}$ also by t, the t-ideals in general by lower case greek letters. Furthermore we shall write (α) for $\{v|v\in G$ & $v\geqq\alpha\}$ and define $[\alpha]$ dually. Thus we consider an upper-continuous cut extension Σ of $\mathfrak{G}$ satisfying:

$$x\alpha\leqq\beta\to\beta=x\varkappa\ \&\ \alpha x\leqq\beta\to\beta=\lambda x \quad\text{and}\quad a\bigwedge\beta_i=\bigwedge(a\beta_i).$$

5.9. Lemma. *There are no other (lower bounded) v-ideals of $\mathfrak{G}$ than the subsets (α) of Σ, which means in particular that* $\mathsf{A}=([A])$.

Proof. Consider a lower bounded A with $\bigwedge A=\alpha$. Then $\mathbf{A}\subseteq(\alpha)$ is valid since Σ is a cut extension, and $(\alpha)\subseteq\mathbf{A}$ follows from $t\leqq p(A)\rightarrow t\leqq p(\alpha)$ (5.8)$\rightarrow t\leqq\leqq p(c)$ $(c\geqq\alpha)$.

5.10. Lemma. *If* $\mathbf{B}$ *is contained in* $\mathbf{A}$ *then* $\mathbf{A}$ *is a left (right) divisor of* $\mathbf{B}$.

Proof. Consider a fixed $b\in B$. Then, for $C=A\wedge b$, $\mathbf{A}$ is equal to $\mathbf{C}$. Let X_b be the set of all x satisfying $Ax\geqq b$ and suppose $b\leqq c\leqq AX_b$. We abbreviate Inf(A) by α. It follows $Ax\geqq b\rightarrow\alpha x\geqq b\rightarrow\alpha x\geqq c$. But according to our previous remark there are elements β, γ such that $\alpha\beta=b$ & $\alpha\gamma=c$, whence $x\geqq\beta\rightarrow\alpha x\geqq b\rightarrow\rightarrow\alpha x\geqq c\rightarrow x\geqq\gamma$. This yields $\beta=\gamma$ from which results $b=c$. Therefore any d with $d\leqq AX_b$ satisfies $d\vee b=b$. Hence the ideal $\mathbf{X}$ generated by all X_b satisfies $\mathbf{A}\circ\mathbf{X}=\mathbf{B}$.

So far we have shown that the v-ideals form a lower continuous extension of $\mathfrak{G}$ with respect to $\leqq:=\supseteq$. We shall now show that Σ and the v-ideal extension are isomorphic. Doing this we shall implicitly verify, too, that there is a complete extension satisfying also axiom (DSL 5) which results from $\mathbf{A}\cdot\mathbf{B}\subseteq\mathbf{C}\rightarrow\mathbf{A}\cdot\mathbf{B}\subseteq\mathbf{c}$ $(c\geqq\mathbf{C})$ (cf. 5.4) by lower continuity.

5.11. Lemma. *Σ satisfies* $\bigwedge(\alpha)\circ\bigwedge(\beta)=\bigwedge(\alpha\beta)$.

Proof. Define $\alpha\circ\beta=\gamma$ if $(\alpha)\circ(\beta)=(\gamma)$. Then $\alpha\circ d$ and αd are equal because of Lemma 5.8. Suppose now $\alpha\beta\leqq c$ and $s\leqq ab$ for all $a,b\in(\alpha)\times(\beta)$ and $c=\alpha\circ\gamma$. Then $\alpha\circ c_i=\alpha c_i\geqq\alpha\beta\rightarrow c_i\geqq\beta$ for all $c_i\geqq\gamma$, whence we get by assumption $s\leqq\alpha\circ c_i$ and hereby furthermore $s\leqq\alpha\circ\gamma=c$.

5.12. Proposition. *A divisibility semiloop satisfying* (i), ..., (iv) *has a cut extension isomorphic to the lower bounded v-ideal extension if* $\leqq:=\supseteq$, *as well as to the upper bounded t-ideal extension if* $\leqq:=\subseteq$.

Proof. By 5.11 $[(A)]\rightarrow(A)$ is a homomorphism, and by definition this mapping is bijective.

Thus summarizing we can state:

5.13. Theorem. *A divisibility semiloop admits a complete (cut-) extension if and only if it satisfies the conditions* (i) *through* (iv).

Let now $\mathfrak{G}$ be a divisibility semiloop satisfying (i) through (iv), and let Σ be its cut extension in the sense of above. Then we can show in addition:

5.14. Corollary. *If* $\mathfrak{G}$ *is power-associative, then* Σ *is power-associative, too.*

Proof. If c is equal to a product built by factors a_i $(1\leqq i\leqq n)$ satisfying $a_i\leqq\alpha$ we can infer $c\leqq(a_1\vee\ldots\vee a_n)^n\leqq\alpha^n$.

5.15. Corollary. *If $\mathfrak{G}$ is a lattice loop then Σ is a lattice loop, too. If in addition $\mathfrak{G}$ is inverse then Σ is inverse, too.*

Proof. $\alpha \leqq a$ & $b \leqq \beta \to \alpha(a \diagdown b) \leqq \beta$, and starting from $\alpha = \bigvee a_i$ $(i \in I,\ a_i \in G)$ we get:

$$(ba_i)a_i^{-1} = b \to \bigvee ba_i \bigwedge a_i^{-1} = (b\alpha)\alpha^{-1} = b,$$

from which the general inverse property follows by upper continuity.

5.16. Corollary. *A lattice group admits a complete extension if and only if it is archimedean.*

Proof. Obviously the condition is necessary. On the other hand, if $\mathfrak{G}$ is a lattice group, (i) through (iv) are satisfied if $Ax{\downarrow}A \to x=1$ and its left dual are valid. But this is a consequence of the archimedean property, since

$$Ax{\downarrow}A \geqq s \to Ax^{-n}{\downarrow}A{\downarrow}Ax^n \to x^{-n} \leqq s^{-1}a \geqq x^n \quad (a \in A,\ n \in \mathrm{N}) \to$$

$$\to (x^*)^n \leqq s^{-1}a \,\&\, (x^+)^n \leqq s^{-1}a,$$

by application of Lemma 1.3. Hence Σ is a complete lattice group since associativity follows from $\mathbf{A} \circ \mathbf{B} = \underline{AB}$.

6. Congruences

In this section we are interested in cancellative congruences of an underlying divisibility semiloop $\mathfrak{G}$. The reader will easily remember that there was given a first result already in Section 4, namely the direct decomposition extension result of Lemma 4.2. The main purpose of this section is to analyze under what conditions $\mathfrak{G}$ is representable, that is, is a subdirect product of totally ordered factors.

Observe that cancellative congruences are also $*$, : congruences.

6.1. Lemma. *If U is the positive part of the class $1\equiv$ of some cancellative congruence then U is a multiplicatively closed convex subset satisfying*

(i) $aU = Ua$, (ii) $ab \cdot U = a \cdot bU$, (iii) $U \cdot ab = Ua \cdot b$.

Proof. $u \in U$ implies $a \equiv au = va \to v \equiv 1$, and

$$ab \equiv ab \cdot u = a \cdot bv \to bv \equiv b1 \to v \equiv 1, \quad ab \equiv a \cdot bu = ab \cdot v \to \ldots \to v \equiv 1,$$

whence (i) through (iii) are satisfied, the rest being obvious.

Every multiplicatively closed convex positive subset of G containing 1 and satisfying (i) through (iii) will be called a *kernel*.

6.2. Lemma. *If U is a kernel then $x \equiv y\ (U)$ iff $x \leqq yu$ & $y \leqq xv$ for some $u, v \in U$ defines a cancellative congruence such that the positive part of $1 \equiv$ coincides with U.*

Proof. Straightforward by definition.

Thus we getas a first result.

6.3. Proposition. In every divisibility semiloop $\mathfrak{G}$ the cancellative congruence relations $\equiv$ are uniquely represented by the kernels U via the following definition: $a \equiv b\ (U)$ iff $a \leqq bu$ & $b \leqq au$.

Hint. $a \equiv b \rightarrow a \leqq b(a*b \vee b*a)$ & $b \leqq a(a*b \vee b*a)$ $(a*b,\ b*a \equiv 1)$.

Evans and Hartman [17] gave a characterization of lattice loops admitting a subdirect decomposition into totally ordered ones. This result can be extended to divisibility semiloops. To this end we consider two orthogonal elements a, b. By (DSL 5) they obviously satisfy the equivalence

$$a \wedge (bx \cdot y)/xy = 1 \leftrightarrow a \cdot xy \wedge bx \cdot y = xy \leftrightarrow ((a \cdot xy)/y)/x \wedge b = 1.$$

Hence requiring the first equality means requiring: $u \perp v$ implies u and $(vx \cdot y)/xy$ are orthogonal, too. And the validity of the third equality means: if u, v are orthogonal then u and $((v \cdot xy)/y)/x$ are orthogonal, too. So, if $u \wedge v = 1$ and $U = (u^{\perp})^{+}$, we can deduce from the validity of each of these equalities

$$(Ux \cdot y)/xy \subseteqq U, \quad \text{whence} \quad Ux \cdot y \subseteqq U \cdot xy,$$

and

$$(((U \cdot xy)/y)/x) \subseteqq U, \quad \text{whence} \quad U \cdot xy \subseteqq Ux \cdot y.$$

Similarly we get $Ux = xU$ from $u \wedge v = 1 \rightarrow u \wedge (xv)/x = 1$.

6.4. Theorem. *A divisibility semiloop $\mathfrak{G}$ is representable if and only if it satisfies the conditions*

(i) $(a * b) \cdot xy \wedge (b * a) x \cdot y = xy$,

(ii) $xy \cdot (a * b) \wedge x \cdot y(b * a) = xy$, and

(iii) $x \cdot (a * b) \wedge (b * a) \cdot x = x$.

Proof. Obviously a and b are orthogonal iff $a*b = b$ & $b*a = a$. Hence the conditions above require that the positive part of any $u^{\perp}$ forms a kernel. Suppose now that U is maximal in the set of kernels $M \not\ni c$. Then $\mathfrak{G}/U =: \mathfrak{H}$ is totally ordered since otherwise $\mathfrak{H}$ would contain a pair p, q with $p*q \neq 1 \neq q*p$. But then $U_1 := ((p*q)^{\perp})^{+}$ and $U_2 := (U_1^{\perp})^{+}$ would be two kernels satisfying $U_1 \cap U_2 = \{1\}$, although U_1 and U_2 differ from $\{1\}$ by construction. Therefore the conditions under consideration are sufficient.

On the other hand our conditions are necessary as is easily checked by the reader.

By 6.4. the subdirect products of totally ordered divisibility semiloops are characterized in a classical manner. But it is obvious that this method relies strongly on (DSL 5) and $a*b \perp b*a$. Hence, in order to find a method working also in more general cases, we have to leave orthogonality conditions and to look for $\cdot/\leqq$-conditions. This will be done in the remainder of this section.

Nearly immediately we get:

6.5. Theorem. *A divisibility semiloop* $\mathfrak{G}$ *is representable if and only if it satisfies the condition*

(0) $$p(a)\wedge q(b) \leqq p(b)\vee q(a)$$

for any pair p, q of multiplication polynomials.

Proof. Obviously condition (0) is necessary. So let condition (0) be satisfied. Then putting $(c^+x\cdot y)/xy := c^+\theta$ we infer for orthogonal elements a, b,

$$a\wedge b\theta \leqq b\vee a\theta \rightarrow a\wedge b\theta = (a\wedge b\theta)\wedge(b\vee a\theta) =$$
$$= (a\wedge b\theta\wedge b)\vee(a\wedge b\theta\wedge a\theta) = (1\wedge b\theta)\vee(a\wedge 1) = 1,$$

whence (i) is valid. And in an analogous manner one can deduce (ii) and (iii).

We now show that the condition (0) provides a key for solving the problems stated by Fuchs and Evans & Hartman. To this end we shall leave the group oriented standpoint and exploit the lattice-order of the underlying structure. Moreover for the sake of economy we shall start more generally.

6.6. Definition. Let $\mathfrak{A} := (A, \wedge, \vee, f_i)$ be an algebra such that $\wedge$ and $\vee$ provide a lattice order and the f_i are of arity n_i. Then $\mathfrak{A}$ is called a *lattice-ordered algebra* if each operation is isotone at each place. If each operation even distributes over meet and join at each place we call $\mathfrak{A}$ a distributive lattice-ordered algebra.

Examples of lattice-ordered algebras are the lattice groupoids satisfying the $\cdot/\wedge$- or the $\cdot/\vee$-distributivity laws. Hence lattice quasigroups and thereby lattice loops and lattice groups are lattice-ordered algebras in the above sense. However, there remains an inaccuracy. For example, given a lattice group, what are the fundamental operations? Obviously $^{-1}$ is antitone. On the other hand lattice quasigroups satisfy

$$x\diagdown(a\wedge b) = x\diagdown a\wedge x\diagdown b \quad \& \quad (a\wedge b)/x = a/x\wedge b/x$$

and

$$x\diagdown(a\vee b) = x\diagdown a\vee x\diagdown b \quad \& \quad (a\vee b)/x = a/x\vee b/x.$$

So we may regard lattice quasigroups, lattice loops, and lattice groups as lattice-ordered algebras by defining $l_x(a) := x\diagdown a$ and $r_x(a) := a/x$ and considering $\mathfrak{G}$ as an algebra $(G, \cdot, \wedge, \vee, l_x, r_x)$ $(x\in G)$.

6.7. Definition. Let $\mathfrak{A}$ be a lattice-ordered algebra. A *term* is called *linearly composed* if it is a variable or if it is of the special type $f(x_1, \ldots, q(x, y_1, \ldots, y_m), \ldots, x_n)$ where f is a fundamental operation and $q(x, y_1, \ldots, y_m)$ is (already) linearly composed.

6.7 provides a set of terms with a "*starting variable*" x such that in the case of a distributive lattice-ordered algebra the arising *polynomial functions* $\bar{p}(x)$ of type $p(x, c_1, \ldots, c_n)$ $(c_i \in A)$ satisfy the distributivity laws $\bar{p}(a \wedge b) = \bar{p}(a) \wedge \bar{p}(b)$ and $\bar{p}(a \vee b) = \bar{p}(a) \vee \bar{p}(b)$. To emphasize that $\bar{p}(x)$ stems from a term built up without $\wedge$ and $\vee$ we write also $\tilde{p}(x)$. Now we are ready to show

6.8. Theorem. *A lattice-ordered algebra $\mathfrak{A}$ is representable iff it is distributive and satisfies*

$$(\tilde{0}) \qquad \tilde{p}(a) \wedge \tilde{q}(b) \leqq \tilde{p}(b) \vee \tilde{q}(a),$$

which can be unified to the condition

$$(\bar{0}) \qquad \bar{p}(a) \wedge \bar{q}(b) \leqq \bar{p}(b) \vee \bar{q}(a).$$

Proof. Obviously $(\bar{0})$ is necessary and a fortiori $(\bar{0})$ implies $(\tilde{0})$. Moreover $(\bar{0})$ yields $f(\ldots a \wedge a \ldots) \wedge f(\ldots b \wedge b \ldots) \leqq f(\ldots a \wedge b \ldots) \vee f(\ldots b \wedge a \ldots)$, whence f distributes over meet, and join which is shown similarly. The reader should notice that $(\bar{0})$ follows nearly immediately from $(\tilde{0})$ if $\mathfrak{G}$ is distributive. Hint: write $\bar{p}$ and $\bar{q}$ as meets of joins of $\sim$-functions.

We now prove that distributivity together with $(\tilde{0})$ provides a representation. To this end we may start from $r < s$ in order to construct a totally ordered homomorphic image $\bar{A}$ satisfying $\bar{r} < \bar{s}$. By Zorn's Lemma, we see that there is a maximal lattice ideal M, containing r but avoiding s. Furthermore it is well known that such an M is $\wedge$-prime $(a \wedge b \in M \rightarrow a \in M \vee b \in M)$, since $(A, \wedge, \vee)$ is distributive. (Otherwise there would be a pair u, v with $u \wedge v \in M$ & $u, v \notin M$ which would lead to $U := \{x \mid x \wedge v \in M\}$, $V := \{y \mid u \wedge y \in M\ (\forall u \in U)\}$ with $b \in U \cap V \subseteqq M$.) We define

$$a \equiv b :\Leftrightarrow \tilde{p}(a) \in M \leftrightarrow \tilde{p}(b) \in M.$$

(Obviously we could define this congruence relation also by $V := A - M$ and it is easily checked by the reader that there is a dual proof w.r.t. this prime filter V.) This is a congruence as is easily shown in the groupoid case and analogously proven in the general case. Furthermore we obtain in $\bar{\mathfrak{A}} := \mathfrak{A}/\equiv$

$$\bar{u} \leqq \bar{v} \Leftrightarrow \tilde{p}(v) \in M \rightarrow \tilde{p}(u) \in M$$

since

$$\bar{u} \leqq \bar{v} \Rightarrow \bar{u} = \bar{u} \wedge \bar{v} \Rightarrow \tilde{p}(u) \in M \leftrightarrow \tilde{p}(u) \wedge \tilde{p}(v) \in M \Rightarrow$$

$$\Rightarrow \tilde{p}(v) \in M \rightarrow \tilde{p}(u \wedge v) \in M \rightarrow \tilde{p}(u) \in M$$

and

$$\tilde{p}(v)\in M \to \tilde{p}(u)\in M \Rightarrow \tilde{p}(u\wedge v)\in M \to \tilde{p}(u)\in M \vee \tilde{p}(v)\in M \Rightarrow$$
$$\Rightarrow \tilde{p}(u\wedge v)\in M \to \tilde{p}(u)\in M \Rightarrow \bar{u} \leqq \bar{v}.$$

Hence $\bar{a}$ and $\bar{b}$ are incomparable if and only if there are linearly composed polynomial functions $\tilde{p}(x)$, $\tilde{q}(x)$ satisfying

$$\tilde{p}(a)\notin M, \quad \tilde{p}(b)\in M, \quad \tilde{q}(a)\in M, \quad \tilde{q}(b)\notin M.$$

But this is excluded by $(\tilde{0})$, since otherwise we could infer

$$\tilde{p}(a)\wedge\tilde{q}(b)\notin M \quad \& \quad \tilde{p}(b)\vee\tilde{q}(a)\in M,$$

contradicting $\tilde{p}(a)\wedge\tilde{q}(b)\leqq\tilde{p}(b)\vee\tilde{q}(a)$. Hence $\mathfrak{A}$ is totally ordered.

Theorem 6.8 yields a series of special results.

6.9. Corollary. *An abelian lattice monoid $\mathfrak{M}$ is representable if and only if the underlying lattice is distributive and if furthermore multiplication distributes over meet and join* [30].

Proof. Since $\mathfrak{M}$ is an abelian monoid we may confine ourselves to the proof of $xa\wedge yb\leqq xb\vee ya$ which follows by

$$(xa\wedge yb)\wedge(xb\vee ya) = (xa\wedge yb\wedge xb)\vee(xa\wedge yb\wedge ya) =$$
$$= \big(xa\wedge(y\wedge x)\,b\big)\vee\big((x\wedge y)\,a\wedge yb\big) = xa\wedge(xa\vee yb)\wedge(x\wedge y)(a\vee b)\wedge yb = xa\wedge yb.$$

(Obviously, all we need is a common unit for any pair a, b.)

6.10. Corollary. *A lattice semigroup $\mathfrak{S}=(S,\cdot,\wedge,\vee)$ is representable if and only if the lattice $(S,\wedge,\vee)$ is distributive, multiplication distributes over meet and join and in addition $\mathfrak{S}$ satisfies the inclusion*

(S0) $$xay\wedge ubv \leqq xby\vee uav,$$

for each quadruple x, y, u, v taken from S^1.

Proof. The laws under consideration guarantee $\tilde{p}(a)\wedge\tilde{q}(b)\leqq\tilde{p}(b)\vee\tilde{q}(a)$ as is easily seen.

6.11. Corollary. *A lattice loop $\mathfrak{L}=(L,\cdot,\wedge,\vee)$ is representable if and only if $\mathfrak{L}$ satisfies the equations*

(EH) $$x(a*b)\wedge(b*a)x = x, \quad (a*b)\cdot xy\wedge(b*a)x\cdot y = xy,$$
$$xy\cdot(a*b)\wedge x\cdot y(a*b) = xy \quad [17].$$

Proof. It was already shown in Section 1 that multiplication and join distribute over meet and join. Furthermore the conditions are necessary. So it remains to show that they are sufficient. Obviously this was done already by 6.4. But we wish to give a direct proof of (EH)$\to(\tilde{0})$.

To this end we consider $\mathfrak{L}$ as a lattice-ordered algebra $(L, \cdot, \wedge, \vee, l_s, r_s)$ $(s\in L)$. We have to show

$$\tilde{p}(a)\wedge\tilde{q}(b) \leqq \tilde{p}(b)\vee\tilde{q}(a).$$

Here, by the rules of loop arithmetic we may suppose $\tilde{p}$ to be the identity mapping and furthermore we can transform the general problem to the proof of

$$(a:b)\,u\wedge(b:a)\,\theta \leqq (b:a)\,u\vee(a:b)\,\theta$$

where θ is an inner mapping and u is equal to some $(r(1))^r$. So we may start from $a\perp b, \tilde{p}(x)=xu$ and $\tilde{q}(y)=y\theta$, which leads to

$$au\wedge b\theta = x_a x_u \,\hat{:}\, x_a \leqq a\,\&\,x_u \leqq u,$$

$$au\wedge b\theta = (x_a\wedge x_a x_u)(x_u\vee 1) \leqq 1\vee u \leqq a\theta\vee bu$$

since $a\perp b\theta$ and $a\theta\perp b$. (Recall: if $a\perp b$ implies $a\perp b\theta$ for the generating inner mappings θ then $a\perp b$ implies $a\perp b\theta$ for all inner mappings θ.)

On the grounds of the preceding theorem one can start from (EH) and prove the subdirect decomposition theorem for lattice loops by deducing $(\tilde{0})$ and applying Theorem 6.8. But one has to notice that the proof given above applies the inner mapping theorem which tells that the group of inner mappings is generated by $((*\cdot xy)/y)/x$, $xy\diagdown(x\cdot y*)$ and $(x\cdot *)/x$, see for instance [13].

Furthermore, applying 4.3 (and 1.29) we get as a special result

6.12. Corollary. *Any complete divisibility semiloop* $(L, \cdot, \leqq, 1)$ *is representable, and if moreover the chain condition for closed intervals is satisfied,* $(L, \leqq, 1)$ *is a direct sum of atomic chains* (recall 3.3).

Lattice quasigroups or lattice rings are not lattice-ordered algebras in the sense of Definition 6.6. But sometimes a given structure can be turned to a lattice-ordered algebra as was shown for instance for lattice quasigroups by splitting right and left division into a set of operators. This idea might be fruitful also in other situations. For example, consider a lattice semigroup $\mathfrak{S}$. Then by splitting its multiplication into operators m_x with $m_x(a):=xa$ any left congruence of $\mathfrak{S}$ becomes a congruence of $(S, \wedge, \vee, m_x)$ $(x\in S)$ and vice versa any congruence of $(S, \wedge, \vee, m_x)$ $(x\in S)$ may be considered as a left congruence of $\mathfrak{S}$. This enables us to develop also results based on left congruences, the most important being:

6.13. Corollary. *Any distributive lattice monoid* $\mathfrak{S}$ *is a lattice monoid of chain endomorphisms* [9].

Proof. Consider $\mathfrak{S}$ as a lattice-ordered algebra $(S, \wedge, \vee, m_x)$. This structure satisfies condition $(\tilde{0})$ which is shown by copying the proof of 6.9. Hence there are enough totally ordered residue systems which can be added to a chain C of left classes of $\mathfrak{S}$ on which the elements of S act from the left. Thus $\mathfrak{S}$ can be embedded into the lattice semigroup of all order endomorphisms of C.

As an immediate consequence of 6.13 we get the celebrated theorem of HOLLAND [25]:

6.14. Corollary. *Any lattice group is a lattice group of chain automorphisms* [25].

We now turn to lattice rings. A ring is called partially ordered with respect to $\leqq$ if it satisfies

$$a \leqq b \to x+a \leqq x+b \quad \text{and} \quad 0 \leqq a, b \to 0 \leqq ab.$$

A partially ordered ring is called a lattice ring if $\leqq$ defines a lattice order. Obviously multiplication is not isotone. On the other hand multiplication is completely determined once it is defined on the positive cone. Hence any homomorphic image is completely determined by the image of the cone. So it makes sense to consider a lattice ring $\mathfrak{R}$ as an algebra $(R, +, \wedge, \vee, r_x, l_x)$ where $r_x(a) := ax^+$ and $l_x(a) := x^+a$. Then $\mathfrak{R}$ is a lattice-ordered algebra but $\mathfrak{R}$ need not be distributive since l_x and r_x need not distribute over $\wedge$ and $\vee$. (Consider for instance the ring of 2×2-matrices over the real field with respect to $A \leqq B$ if $a_{ik} \leqq b_{ik}$, $1 \leqq i \leqq 2$, $1 \leqq k \leqq 2$.) To yield this we look for a further condition. Here we succeed by considering the positive cone of $\mathfrak{R}$.

6.15. Lemma. *Let $\mathfrak{R}$ be a lattice ring. Then $(R, +, \wedge, \vee, l_x, r_x)$ is a distributive lattice-ordered algebra in the above sense iff it satisfies*

(L) $$c^+(a*b) \wedge c^+(b*a) = 0 = (a*b)c^+ \wedge (b*a)c^+.$$

Proof. Suppose that (L) is valid and that c is positive. Then we obtain, for example:

$$\begin{aligned} ca \wedge cb &= c((a\wedge b)+a*b) \wedge c((a\wedge b)+b*a) = \\ &= (c(a\wedge b)+c(a*b)) \wedge (c(a\wedge b)+c(b*a)) = \\ &= c(a\wedge b)+(c(a*b)\wedge c(b*a)) = c(a\wedge b) \end{aligned}$$

and thereby

$$\begin{aligned} ca \vee cb &= (ca+cb)-(ca\wedge cb) = c(a+b)-c(a\wedge b) = \\ &= c((a+b)-(a\wedge b)) = c(a\vee b). \end{aligned}$$

Hence, applying Theorem 6.8 we get:

6.16. Corollary. *A lattice ring is a function ring (is representable) iff it satisfies the conditions* (L) *and* $(\tilde{0})$, *briefly* $(\mathrm{L}, \tilde{0})$.

Corollary 6.16 characterizes the function ring along the lines of this paper. This was done by a different condition in a basic paper published by BIRKHOFF and PIERCE [6], and by a further condition in FUCHS [19] where also the equivalence of these two conditions is proved. To this equivalence proof we now add a further one by showing

$$\text{(BP)} \qquad a \perp b \rightarrow c^+ a \perp b \quad \& \quad ac^+ \perp b$$

(Birkhoff—Pierce) and condition $(L, \tilde{0})$ to be equivalent.

6.17. Remark. There is a short direct proof of $(BP) \leftrightarrow (L, \tilde{0})$.

Proof. We shall treat the associative case. However, the reader should notice that associativity is by no means essential, only pleasant for the demonstration.

Let $\mathfrak{R}$ satisfy (BP). Then (L) is obvious. Furthermore it is easy to see that the polynomials in $(\tilde{0})$ are of type $c_1^+ x c_2^+ + s$. Hence, after some simple calculation $(\tilde{0})$ is reduced to

$$(c_1 a c_2 + u) \wedge d_1 b d_2 \leqq (c_1 b c_2 + u) \vee d_1 a d_2$$

for positive elements c_1, c_2, d_1, d_2 and orthogonal pairs a, b. But because of (BP) we may omit $c_1 a c_2$ on the left side (apply Lemma 1.3). Hence condition $(\tilde{0})$ is satisfied, too.

Let now $\mathfrak{R}$ satisfy $(L, \tilde{0})$. Then (BP) follows by

$$c^+ a \wedge b \leqq c^+ b \vee a \rightarrow c^+ a \wedge b = (c^+ a \wedge b \wedge c^+ b) \vee (c^+ a \wedge b \wedge a) =$$

$$= (c^+ (a \wedge b) \wedge b) \vee (c^+ a \wedge 0) = 0.$$

We turn to complementary semigroups $(S, \cdot, *, :)$. Complementary semigroups were introduced in [7] as monoids satisfying $aS = Sa$ in which for any pair a, b there exist uniquely determined elements $a*b$ and $b:a$ such that $b|ax \Leftrightarrow \Leftrightarrow a*b|x$ and $b|xa \Leftrightarrow b:a|x$. Complementary semigroups are partially ordered with respect to $a \leqq b \Leftrightarrow: a|b$, and $a \leqq b$ is equivalent to $b*a = 1$ and to $a:b = 1$ as well. Furthermore $(S, \leqq)$ forms a semilattice under $a \vee b := a(a*b) = (b:a)a$. In addition the following distributivity laws hold:

$$a(b \vee c) = ab \vee ac \quad \& \quad (a \vee b)c = ac \vee bc$$

and

$$a*(b \vee c) = a*b \vee a*c \quad \& \quad (a \vee b):c = a:c \vee b:c.$$

Therefore, defining operators c_x^* and $c_x^:$ by $c_x^*(a) = x*a$ and $c_x^:(a) = a:x$, any complementary semigroup may be considered as a distributive $\vee$-semilattice-ordered algebra $(S, \cdot, c_x^*, c_x^:, \vee)$. However, we have to show that the congruences of $(S, \cdot, c_x^*, c_x^:)$ are congruences of $(S, \cdot, *, :)$ as well. Here we succeed by the formula $a*(b:c)=(a*b):c$ which results from

$$x \geqq a*(b:c) \leftrightarrow ax \geqq b:c \leftrightarrow axc \geqq b \leftrightarrow x \geqq (a*b):c.$$

To see this, let $\equiv$ be a congruence of $(S, \cdot, c_x^*, c_x^{:})$. Then we have

$$a \equiv b \rightarrow a*b \equiv 1 \equiv b*a \quad (\leftrightarrow b:a \equiv 1 \equiv a:b) \rightarrow a \equiv a(a*b) = b(b*a) \equiv b,$$

and thereby

$$a \equiv b \rightarrow a*b \equiv 1 \rightarrow (a*c):(b*c) = a*(c:(b*c)) \equiv 1.$$

Hence, by duality we get $(b*c):(a*c)\equiv 1$ which leads to $a*c\equiv b*c$.

Special complementary semigroups are the lattice group cones under $a*b:=1\vee \vee a^{-1}b$ and $b:a:=1\vee ba^{-1}$ on the one hand, and the brouwerian semilattices on the other hand.

Complementary semigroups need not be $\wedge$-closed, but products of totally ordered complementary semigroups necessarily satisfy $a*b\perp b*a$ which is equivalent to $a:b\perp b:a$ and also to $a:(b*a)\vee b:(a*b)=a\wedge b$. Moreover, in this case further distributivity laws hold, namely:

$$a(b\wedge c) = ab\wedge ac \quad \& \quad (a\wedge b)c = ac\wedge bc,$$

$$a*(b\wedge c) = a*b\wedge a*c \quad \& \quad (a\wedge b):c = a:c\wedge b:c,$$

and

$$a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c).$$

Therefore complementary semigroups with a representation may be regarded as distributive lattice-ordered algebras $(S, \cdot, c_x^*, c_x^{:})$, and we get as an immediate consequence

6.18. Corollary. *A complementary semigroup is representable if and only if the following implication holds:*

(0$^\vee$) $$x \leqq \tilde{p}(a), \tilde{q}(b) \rightarrow x \leqq \tilde{p}(b)\vee\tilde{q}(a).$$

Proof. $x\leqq a*b,\ b*a \rightarrow x\leqq a*a\vee b*b=1$.

This corollary provides as a further characterization

6.18′. Corollary. *A complementary semigroup is representable if and only if it satisfies the equation*

(0^c) $$(a*b)*x\vee(c*(b*a)c\vee c(b*a):c)*x = x \quad [8].$$

Proof. (a) Axiom (0$^\vee$) implies nearly immediately (0$^\perp$) $c*(a*b)c\perp b*a\perp \perp c(a*b):c$. Hence (0^c) can be inferred from

(0$^{c'}$) $$(a*b\wedge(c*(b*a)c\vee c(b*a):c))*x = x.$$

(b) Axiom (0^c) implies $a*b \perp b*a$ whence $(S, \leqq)$ is $\wedge$-closed, and we observe that

(i) $x*yz \leqq (x*y)z$ and $zy:x \leqq z(y:x)$

holds in any case, that

(ii) $ca^{\perp} = a^{\perp}c$

holds according to ($0^{\perp}$), and that

(iii) any $\tilde{p}(a)$ can be extended to some $...x_5((x_3*(x_1a)x_2:x_4)...$.

Hence we may start from a pair $\tilde{p}(a)$, $\tilde{q}(b)$ with $a \perp b$. But, applying (i) and (ii) again and again this leads to $\tilde{p}(a)\wedge\tilde{q}(b) \leqq a^*\tilde{p}(1)\wedge b^*\tilde{q}(1)$ with $a^* \perp b^*$, hence

$$\tilde{p}(a)\wedge\tilde{q}(b) = x_a x_p = x_b x_q: \quad x_a \leqq a^*, \quad x_p \leqq \tilde{p}(1), \quad x_b \leqq b^*, \quad x_q \leqq \tilde{q}(1),$$

which yields

$$\tilde{p}(a)\wedge\tilde{q}(b) = (x_a\wedge x_b)(x_p\vee x_q) \leqq \tilde{p}(1)\vee\tilde{q}(1) \leqq \tilde{p}(b)\vee\tilde{q}(a).$$

The method of proof shows that a lattice group is already representable if $a^{\perp}c \subseteq ca^{\perp}$. To see this look at (S0) in 6.10. Furthermore we see that (0^c) is equivalent to $a \perp b \rightarrow (a*c)*c \perp b$ & $c:(c:a) \perp b$, since $a*bx \leqq (a*b)((b*a)*x)$ & $cb:a \leqq$ $\leqq (c:(a:b))(b:a)$.

As an immediate consequence we get

6.19. Corollary. *An abelian complementary semigroup is representable if and only if it satisfies* $a*b \perp b*a$.

Since 6.19 is a direct consequence no proof is needed. But it should be mentioned that in the commutative case $a*b \perp b*a \rightarrow (0^{\vee})$ has a short proof by the formulas $(a*b)*(a*c) = (b*a)*(b*c)$ and $ab*c = b*(a*c)$.

Next, applying 6.19 to boolean algebras $(B, \vee, *)$ (where $a*b := a'\wedge b$), we can state the celebrated theorem of Stone:

6.20. Corollary. *Any boolean algebra is a subdirect product of 2-element ones, and hence a field of sets* [36].

In a similar manner one shows that normally residuated lattices [12] are distributive lattice-ordered algebras whence 6.8 applies also to these structures. Furthermore one easily sees that dually residuated semigroups [37] may be regarded as extended complementary semigroups by adding $a*b := 0\vee b-a$. Therefore we get

6.21. Corollary. *A dually residuated (commutative) semigroup is representable if and only if it satisfies* $a-b\wedge b-a \leqq 0$ [37].

We consider cone algebras $(C, *, :)$. They were introduced in [11] and turned out to be $*, :$-subalgebras of some lattice group cone $(P, \cdot, *, :)$. Any cone algebra is $\wedge$-closed with respect to $a\wedge b:=a:(b*a)=(b:a)*b$ but a cone algebra need not form a lattice. However $a\vee b$ is contained in C if $\{a, b\}$ is upper bounded, and $ab\in C$ implies that the elements x and y with $xa=ab=by$ are contained in C. So we may apply 6.18 once a prime filter is guaranteed containing b yet not containing a, whenever $a\not\geqq b$. But this is an easy consequence of maximality, since given a filter F maximal with respect to not containing a we get

$$x\vee y\in F \rightarrow x\wedge f_1 \geqq a \,\&\, y\wedge f_2 \geqq a \rightarrow (x\vee y)\wedge(f_1\wedge f_2) \geqq a,$$

a contradiction. Thus we are led to

6.22. Corollary. *A cone algebra is representable if and only if it satisfies*

(C0) $$a*b \perp a:b.$$

(Observe that this condition is equivalent to $(a\wedge b)^2=a^2\wedge b^2$ in lattice group cones and lattice groups as well, and observe furthermore that this equation is equivalent to $aa\wedge bb\leqq ab\vee ab$.)

Proof. Any complementary semigroup satisfies $cb:a=(c:(a:b))(b:a)$, and the method of 6.18 works also in the present case which is shown by cone algebra technique. Hence by the last footnote it suffices to prove the implication $a\perp b\rightarrow a\perp$ $\perp c:(c:b)$. But this can be done as follows: $a\perp b$ implies

$$c:(c:a)*(a\wedge c:(c:b)) \leqq (a\wedge c)\wedge(c:(c:a))*(c:(c:b)) =$$
$$= (c:a)*(c:b)\wedge(c:a):(c:b) = 1,$$

whence

$$a\wedge c:(c:b) = a\wedge c:(c:b)\wedge c:(c:a) \leqq c:(c:(b\wedge a)) = 1.$$

Final remark. Obviously the principle of 6.8 works whenever a partially ordered algebra — this may be an arbitrary algebra with respect to $=$ — has enough *order ideals (order filters)*, i.e. o-ideals (o-filters), M satisfying

(P) $$\tilde{p}(b)\in M \,\&\, \tilde{q}(a)\in M \rightarrow \tilde{p}(a)\in M \wedge \tilde{q}(b)\in M$$

If M is a *prime ideal* in the sense of (P) then $A-M$ is a *prime filter* in the sense of (P) and vice versa, and we see nearly immediately that the set of prime ideals (prime filters) is closed under intersections and unions of chains of prime ideals (prime filters).

Let us suppose now that $\mathfrak{A}$ has enough prime ideals. Then the partially ordered algebra $\mathfrak{A}$ is representable and hence admits an extension to some representable distributive lattice-ordered algebra $\mathfrak{B}$. Therefore we should check how artificial this condition is. To this end we present some applications which lead to well known results.

6.23. Example. Any partially ordered set is a subdirect product of 2-element chains, since any $(a]$ is prime with respect to the identity operator.

6.24. Example. Any $\vee$-semilattice is a subdirect product of 2-element chains, since any $(a]$ is prime in the sense of (P).

6.25. Example. A partially ordered abelian group $\mathfrak{G}$ is representable if and only if it is *semiclosed*, i.e. iff it satisfies, for any $n \in \mathrm{N}$, the implication

(SC) $$a^n \geqq 1 \to a \geqq 1.$$

(The first proof of this result seems to be due to CLIFFORD [15]. Another proof was given by Dieudonné in 1941, cf. [19].)

Proof. Obviously (SC) is necessary. Suppose now that (SC) is satisfied and $a \neq b$. The set N of strictly negative elements is closed under multiplication, and it is easily shown that ab^{-1}, N and $a^{-1}b, N$ cannot both generate a submonoid (with respect to multiplication). Hence there is a maximal subsemigroup $\mathfrak{M}$ containing N and w.l.o.g. ab^{-1} but not containing 1. We show that M is a prime ideal in the sense of (P).

(i) M is an o-ideal, since $u < v \in M$ implies $uv^{-1} < 1$ & $v \in M$ from which it follows that $(uv^{-1})v = u \in M$.

(ii) M is prime, since $ax, by \in M$ and $ay, bx \notin M$ would yield a $k \in \mathrm{N}$ with $a^{-k}y^{-k}, b^{-k}x^{-k} \in M$ whence $a^{-k}b^k$ and a^kb^{-k} would both belong to M, a contradiction.

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Описание скрещенных групповых алгебр над конечными полями

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Пусть группа G содержит бесконечную циклическую подгруппу конечного индекса, K — произвольное поле (с некоторым ограничением на характеристику). В работе [1] показано, что изучение конечнопорожденных KG-модулей сводится к изучению алгебр типа E: скрещенных групповых алгебр над полем K либо бесконечной циклической группы

$$A = \{F, a\}; \quad a\lambda = \lambda^{\varphi} a;$$

либо бесконечной группы диэдра

$$B = \{F, a, b\}; \quad a\lambda = \lambda^{\varphi} a; \quad b\lambda = \lambda^{\psi} b; \quad b^{-1}ab = \gamma a^{-1}; \quad b^2 = \mu,$$

где F — тело, содержащее в своем центре поле K, $\lambda \in F$ — произвольный, γ, $\mu \in F$ — фиксированные элементы, φ и ψ — K-автоморфизмы тела F.

В работе [2] были описаны все алгебры типа E над полем $\mathbf{R}$ вещественных чисел, а в работе [3] — все алгебры типа E над конечным полем K, где F — расширение поля K степени 2. В статье [4] был рассмотрен вопрос об изоморфизме алгебр типа E, описанных в работе [3].

В настоящей работе описываются все алгебры типа E над произвольным конечным полем K по отношению к любому конечному расширению F поля K и выяснен вопрос об изоморфизме этих алгебр.

1.

Лемма 1. *Пусть K — конечное поле характеристики $p(\neq 2)$, F — конечное расширение поля K и задана скрещенная групповая алгебра бесконечной группы диедра*

$$B = \{F, a, b\}; \quad a\lambda = \lambda^{\varphi} a; \quad b\lambda = \lambda^{\psi} b; \quad b^{-1}ab = \gamma a^{-1}; \quad b^2 = \mu,$$

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где $\lambda \in F$ — *произвольный,* γ, $\mu \in F$ — *фиксированные элементы,* φ *и* ψ — *K-автоморфизмы поля* F. *Тогда K-автоморфизмы* φ *и* ψ *могут иметь порядок 2 или являются тождественными.*

Доказательство. Используя определяющие соотношения алгебры B, имеем $b(b\lambda b^{-1})b^{-1} = b\lambda^{\psi}b^{-1} = \lambda^{\psi^2}$. С другой стороны $b(b\lambda b^{-1})b^{-1}b^2\lambda b^{-2} = \mu\lambda\mu^{-1} = \lambda$; значит $\lambda^{\psi^2} = \lambda$, то есть ψ имеет порядок 2 или тождественный автоморфизм.

Рассмотрим автоморфизм φ. С одной стороны $a(b\lambda b^{-1})a^{-1} = a\lambda^{\psi}a^{-1} = \lambda^{\psi\varphi}$, а с другой стороны

$$a(b\lambda b^{-1})a^{-1} = (ab)\lambda(ab)^{-1} = (b\gamma a^{-1})\lambda(b\gamma a^{-1})^{-1} =$$

$$= b\gamma a^{-1}\lambda a\lambda^{-1}b^{-1} = b\gamma\lambda^{\varphi^{-1}}\gamma^{-1}b^{-1} = b\lambda^{\varphi^{-1}}b^{-1} = \lambda^{\varphi^{-1}\psi}.$$

Значит имеем $\lambda^{\psi\cdot\varphi} = \lambda^{\varphi^{-1}\cdot\psi}$. Так как группа автоморфизмов поля F коммутативна, то из последнего равенства получаем $\varphi^2 = 1$. Значит φ либо тождественный автоморфизм поля F, либо имеет порядок 2. Лемма доказана.

Лемма 2. *Имеется 3 основных класса скрещенных групповых алгебр бесконечной группы диедра над K по отношению к полю F:*

(1) $B_1 = \{F, a, b\}$; $a\lambda = \lambda a$; $b\lambda = \lambda b$; $b^{-1}ab = \gamma a^{-1}$; $b^2 = \mu$;

(2) $B_2 = \{F, a, b\}$; $a\lambda = \lambda a$; $b\lambda = \bar{\lambda} b$; $b^{-1}ab = \gamma a^{-1}$; $b^2 = \mu$,

(3) $B_3 = \{F, a, b\}$; $a\lambda = \bar{\lambda} a$; $b\lambda = \bar{\lambda} b$; $b^{-1}ab = \gamma a^{-1}$; $b^2 = \mu$,

где $\lambda \in F$ — *произвольный,* $\gamma, \mu \in F$ — *фиксированные элементы,* $\lambda \to \bar{\lambda}$ — *K-автоморфизм 2-го порядка поля F.*

Доказательство. Из леммы 1 следует, что существует только 4 основные класса скрещенных групповых алгебр бесконечной группы диедра над полем K по отношению к полю F: алгебры B_1, B_2, B_3 и алгебра

(4) $B_3' = \{F, a, b\}$; $a\lambda = \bar{\lambda} a$; $b\lambda = \lambda b$; $b^{-1}ab = \gamma a^{-1}$; $b^2 = \mu$,

однако замена базиса $a_1 = a$; $b_1 = ab$ алгебру B_3' сводит к типу B_3. Действи тельно,

$$b_1\lambda = (ab)\lambda = a\lambda b = \bar{\lambda}(ab) = \bar{\lambda} b_1,$$

$$b_1^{-1}a_1b_1 = (ab)^{-1}a(ab) = b^{-1}ab = \gamma a^{-1} = \gamma a_1^{-1},$$

$$b_1^2 = (ab)^2 = abab = b\gamma a^{-1}ab = \gamma b^2 = \gamma\mu = \mu_1.$$

Лемма доказана.

Пусть порядок поля K равен $|K|=p^m$ и степень расширения $(F:K)=n$. Тогда порядок поля F равен $|F|=p^{nm}$ и все K-автоморфизмы поля F имеют вид $\lambda\to\lambda^{p^{im}}$ $(i=0, 1, \ldots, n-1)$.

Очевидна следующая

Теорема 1. *Все алгебры типа E над полем K по отношению к полю F, являющиеся скрещенными групповыми алгебрами бесконечной циклической группы (a), задаются формулой*

$$E_i=\{F, a\};\quad a\lambda=\lambda^{p^{im}}a\quad (\lambda\in F;\ i=0, 1, \ldots, n-1).$$

Замечание 1. Квадраты элементов мультипликативной группы F^* поля F образуют подгруппу F_1 группы F^* индекса 2, значит группа F^* разлагается в объединение смежных классов

$$F^*=F_1\cup\xi\cdot F_1,$$

где $\xi\in F^*$ — фиксированный квадратный невычет в поле F.

Теорема 2. *Основной класс B_1 алгебр типа E (см.* (1)*) сводится к типам алгебр*

$$A_1=\{F, a, b\};\quad a\lambda=\lambda a;\quad b\lambda=\lambda b;\quad b^{-1}ab=a^{-1};\quad b^2=1,$$

$$A_2=\{F, a, b\};\quad a\lambda=\lambda a;\quad b\lambda=\lambda b;\quad b^{-1}ab=\xi a^{-1};\quad b^2=1;$$

$$A_3=\{F, a, b\};\quad a\lambda=\lambda a;\quad b\lambda=\lambda b;\quad b^{-1}ab=a^{-1};\quad b^2=\xi;$$

где $\lambda\in F$, ξ — фиксированный квадратный невычет в поле F.

Доказательство. В зависимости от того, элемент μ алгебры B_1 лежит в подгруппе F_1 (см. замечание 1) или нет, замена базиса $a_1=a$; $b_1=\sqrt{\mu^{-1}}\cdot b$ или $a_1=a$; $b_1=\sqrt{f^{-1}}\cdot b$, где $\mu=\xi\cdot f$; $f\in F_1$, приводит к соотношениям $b_1^2=1$ или $b_1^2=\xi$ в алгебре B_1, причем остальные соотношения не изменяются.

Теперь, в зависимости от того, элемент γ лежит в подгруппе F_1 или нет, сделаем опять замена базиса $a_1=\sqrt{\gamma^{-1}}\,a$; $b_1=b$, или $a_1=\sqrt{f_1^{-1}}\,a$; $b_1=b$ где $\gamma=\xi\cdot f_1, f_1\in F_1$, что ведет к соотношениям $b_1^{-1}a_1b_1=a_1^{-1}$ или $b_1^{-1}a_1b_1=\xi\cdot a_1^{-1}$.

В конечном счете получаем алгебры типов A_1, A_2, A_3 и алгебру

$$A_2'=\{F, a, b\};\quad a\lambda=\lambda a;\quad b\lambda=\lambda b;\quad b^{-1}ab=\xi a^{-1};\quad b^2=\xi,$$

однако новая замена базиса $a_1=a$; $b_1=\xi^{-1}ab$:

$$(\xi^{-1}ab)^2=\xi^{-2}abab=\xi^{-2}b\xi a^{-1}ab=\xi^{-1}b^2=1$$

приводит эту алгебру к типу A_2. Теорема доказана.

Замечание 2. Если степень расширения основного поля K неченое число: $(F:K)=2k+1$, то все алгебры типа E над полем K по отношению к полю F

осчерпываются алгебрами типов E_i, A_1, A_2, A_3 $(i=0, 1, \ldots, n-1)$, описанных в теоремах 1 и 2.

Доказательство. Так как группа K-автоморфизмов поля F имеет порядок $2k+1$, то K-автоморфизмов второго порядка поле F не имеет. Значит основных классов B_2 и B_3 алгебр типа E в этом случае не существует.

Рассмотрим случай, когда степень расширения поля K — четное число: $(F:K)=2k$. Тогда K-автоморфизм второго порядка поля F имеет вид $\lambda\to\lambda^{p^{km}}$. В дальнейшем будем пользоваться следующей леммой, которая является частным случаем известного результата об автоморфизмах конечного порядка.

Лемма 3. *Пусть элемент $\alpha\in F$ выдерживает K-автоморфизм φ второго порядка поля F. Тогда существует такой элемент $\beta\in F$, для которого выполняется равенство*

$$\alpha=\beta\cdot\beta^{\varphi}. \tag{5}$$

Лемма 4. *Основные классы B_2 и B_3 алгебр типа E* (см. (2) и (3)) *сводятся к основным классам алгебр типа E над K:*

$$B_2'=\{F, a, b\};\quad a\lambda=\lambda a;\quad b\lambda=\lambda^{p^{km}}b;\quad b^{-1}ab=\gamma a^{-1};\quad b^2=1;$$

$$B_3'=\{F, a, b\};\quad a\lambda=\lambda^{p^{km}}a;\quad b\lambda=\lambda^{p^{km}}b;\quad b^{-1}ab=\gamma a^{-1};\quad b^2=1,$$

где $\lambda\in F$ — произвольный, $\gamma\in F$ — фиксированный элемент.

Доказательство. В алгебрах B_2 и B_3 элемент μ выдерживает автоморфизм $\mu\to\mu^{p^{km}}$. Действительно, $\mu^{p^{km}}=b\mu b^{-1}=bb^2b^{-1}=b^2=\mu$. Но тогда и элемент μ^{-1} выдерживает этот автоморфизм, и, согласно Лемме 3, существует такой элемент $\mu_1\in F$, что $\mu^{-1}=\mu_1\cdot\mu_1^{p^{km}}$. Сделаем замену базиса $a_1=a$; $b_1=\mu_1 b$ в обоих алгебрах B_2, B_3:

$$b_1^2=(\mu_1 b)^2=\mu_1 b\mu_1 b=\mu_1\mu_1^{p^{km}}b^2=\mu^{-1}\mu=1,$$

$$b_1^{-1}a_1b_1=(\mu_1 b)^{-1}a(\mu_1 b)=b^{-1}\mu_1^{-1}a\mu_1 b=b^{-1}ab=\gamma a^{-1}=\gamma a_1^{-1}$$

в алгебре B_2, а в алгебре B_3:

$$b_1^{-1}a_1b_1=(\mu_1 b)^{-1}a(\mu_1 b)=b^{-1}\mu_1^{-1}a\mu_1 b=\mu_1^{-p^{km}}\cdot\mu_1 b^{-1}ab=\gamma_1 a_1^{-1},$$

для некоторого $\gamma_1\in F$. Лемма доказана.

Теорема 3. *Общий класс алгебр B_2' (см. лемму 4) сводится к алгебре типа*

$$A_4=\{F, a, b\};\quad a\lambda=\lambda a;\quad b\lambda=\lambda^{p^{km}}b;\quad b^{-1}ab=a^{-1};\quad b^2=1\quad(\lambda\in F).$$

Доказательство. Покажем, что в алгебре B_2' элемент γ выдерживает автоморфизм $\gamma \to \gamma^{p^{km}}$. Действительно, с одной стороны

$$b^{-1}(b^{-1}ab)b = b^{-1}\gamma a^{-1}b = \gamma^{p^{km}}(\gamma a^{-1})^{-1} = \gamma^{p^{km}}a\gamma^{-1} = \gamma^{p^{km}} \cdot \gamma^{-1} \cdot a,$$

а с другой стороны $b^{-1}(b^{-1}ab)b = b^{-2}ab^2 = a$. Значит $\gamma^{p^{km}} \cdot \gamma^{-1} = 1$ то есть $\gamma^{p^{km}} = \gamma$.

Тогда, используя лемму 3, для элемента γ^{-1} существует такой элемент $\gamma_1 \in F$, что $\gamma^{-1} = \gamma_1 \cdot \gamma_1^{p^{km}}$. Сделаем теперь подстановку $a_1 = \gamma_1 a$; $b_1 = b$ и получаем

$$b_1^{-1}a_1b_1 = b^{-1}(\gamma_1 a)b = \gamma_1^{p^{km}} \cdot \gamma^{-1}a = \gamma_1^{p^{km}} \cdot \gamma_1^{-1} \cdot \gamma_1^{-p^{km}}a^{-1} = (\gamma_1 a)^{-1} = a_1^{-1}.$$

Теорема доказана.

Теорема 4. *Общий класс алгебр B_3 (см. Лемму 4) сводится к алгебрам типов*

$$A_5 = \{F, a, b\}; \quad a\lambda = \lambda^{p^{km}} \cdot a; \quad b\lambda = \lambda b; \quad b^{-1}ab = a^{-1}; \quad b^2 = 1,$$

$$A_6 = \{F, a, b\}; \quad a\lambda = \lambda^{p^{km}} \cdot a; \quad b\lambda = \lambda^{p^{km}} \cdot b; \quad b^{-1}ab = \xi a^{-1}; \quad b^2 = 1,$$

где $\lambda \in F$ — произвольный элемент, ξ — фиксированный квадратный невычет в поле F.

Доказательство. Если элемент γ в алгебре B_3' является квадратом в поле F, то подстановка $a_1 = a \cdot \sqrt{\gamma^{-1}}$; $b_1 = b$ алгебру B_3' сводит к алгебре типа

$$A_5' = \{F, a, b\}; \quad a\lambda = \lambda^{p^{km}} \cdot a; \quad b\lambda = \lambda^{p^{km}} \cdot b; \quad b^{-1}ab = a^{-1}; \quad b^2 = 1.$$

Действительно,

$$b_1^{-1}a_1b_1 = b^{-1}(a\sqrt{\gamma^{-1}})b = \gamma a^{-1} \cdot \sqrt{\gamma^{-1}}^{p^{km}} = \gamma \cdot \sqrt{\gamma^{-1}}\, a^{-1} = \left(a\sqrt{\gamma^{-1}}\right)^{-1} = a_1^{-1}.$$

Однако дополнительная замена $a_1 = a$; $b_1 = ab$ алгебру A_5' сводит к алгебре A_5:

$$(ab)\lambda = a\lambda^{p^{km}}b = \lambda(ab); \quad (ab)^{-1}a(ab) = b^{-1}ab = a^{-1}.$$

Если же элемент γ не является квадратом в поле F, то $\gamma = \xi f, f \in F_1$ (см. (4)), и замена базиса $a_1 = a\sqrt{f^{-1}}$; $b_1 = b$ алгебру B_3' сводит к алгебре A_6:

$$b_1^{-1}a_1b_1 = b^{-1}(a\sqrt{f^{-1}})b = \gamma a^{-1}\sqrt{f^{-1}}^{p^{km}} =$$

$$= \gamma\sqrt{f^{-1}}\,a^{-1} = \xi\sqrt{f}\,a^{-1} = \xi\left(a\sqrt{f^{-1}}\right)^{-1} = \xi a_1^{-1}.$$

Теорема доказана.

Следствие 1. *Все алгебры типа E над конечным полем K характеристики $p(\neq 2)$ по отношению к полю F, где $|K| = p^m$, $(F:K) = n$ имеют вид:*

При нечетном n:

$$E_i = \{F, a\}; \quad a\lambda = \lambda^{p^{im}} \cdot a \quad (\lambda \in F;\ i = 0, 1, \dots, n-1),$$

$$A_1 = \{F, a, b\}; \quad a\lambda = \lambda a; \quad b\lambda = \lambda b; \quad b^{-1}ab = a^{-1}; \quad b^2 = 1,$$

$$A_2 = \{F, a, b\}; \quad a\lambda = \lambda a; \quad b\lambda = \lambda b; \quad b^{-1}ab = \xi a^{-1}; \quad b^2 = 1,$$

$$A_3 = \{F, a, b\}; \quad a\lambda = \lambda a; \quad b\lambda = \lambda b; \quad b^{-1}ab = a^{-1}; \quad b^2 = \xi,$$

где $\gamma \in F$ — произвольный элемент, ξ — фиксированный квадратный невычет в поле F. Алгебра E_0 — групповая алгебра бесконечной циклической группы (a) над полем F, E_i $(i=1, 2, \dots, n-1)$ — скрещенные групповые алгебры группы (a) над полем F. Алгебра A_1 — групповая алгебра бесконечной группы диэдра D над полем F; A_2, A_3 — скрещенные групповые алгебры группы D над F.

При четной степени $n=2k$ расширения поля K: Алгебры E_i $(i=0, 1, \dots, n-1)$, A_1, A_2, A_3 и

$$A_4 = \{F, a, b\}; \quad a\lambda = \lambda a; \quad b\lambda = \lambda^{p^{km}} \cdot b; \quad b^{-1}ab = a^{-1}; \quad b^2 = 1,$$

$$A_5 = \{F, a, b\}; \quad a\lambda = \lambda^{p^{km}} \cdot a; \quad b\lambda = \lambda b; \quad b^{-1}ab = a^{-1}; \quad b^2 = 1,$$

$$A_6 = \{F, a, b\}; \quad a\lambda = \lambda^{p^{km}} \cdot a; \quad b\lambda = \lambda^{p^{km}} \cdot b; \quad b^{-1}ab = \xi a^{-1}; \quad b^2 = 1,$$

где $\gamma \in F$ — произвольный элемент, ξ — фиксированный квадратный невычет в поле F. Алгебры A_4, A_5, A_6 — скрещенные групповые алгебры группы D над полем F.

2.

В этом параграфе рассмотрим вопрос об изоморфизме K-алгебр E_i, A_j $(i=0, 1, \dots, n-1;\ j=1, 2, \dots, 6)$. В дальнейшем будем пользоваться следующей леммой, которая доказывается в работе [1].

Лемма 5. *Пусть $A=\{F, a, b\}$; $b^2=1$ — алгебра типа E с делителями нуля. Тогда существует не более четырех попарно неизоморфных A-модулей M, являющихся свободными циклическими $F(a)$-модулями. Если для некоторого элемента $\gamma \in F$ выполняется равенство $(\gamma ab)^2=1$, то A-модуль M изоморфен одному из модулей*

$$I_1 = A(1+b); \quad I_2 = A(1-b); \quad I_3 = A(1+\gamma ab); \quad I_4 = A(1-\gamma ab).$$

В противном случае модуль M изоморфен одному из модулей I_1, I_2. Модули I_1 и I_2 (соответственно I_3 и I_4) не изоморфны тогда и только тогда, когда элемент b (соответственно γab) перестановочен со всеми элементами поля F. Каждый из модулей I_3, I_4 не изоморфен ни одному из модулей I_1, I_2.

Также из работы [1] следует

Лемма 6. *Пусть* $A=\{F, a, b\}$; $b^2=1$ *и* $B=\{F, a, b\}$; $b^2=1$ — *изоморфные алгебры типа* E *над полем* K. *Тогда числа неизоморфных* A-*модулей и* B-*модулей, являющихся свободными циклическими* $F(a)$-*модулями, равны.*

Теорема 6. *Алгебра* E_0 *не* K-*изоморфна ни одной из алгебр* E_i $(i=1, 2, \ldots$ $\ldots, n-1)$, A_j $(j=1, 2, \ldots, 6)$.

Доказательство. Алгебра E_0 коммутативна, а все алгебры E_i, A_j $(i=1, 2, \ldots, n-1;\ j=1, 2, \ldots, 6)$ не коммутативны. Так как при изоморфизме коммутативность алгебр сохраняется, то теорема очевидна.

Теорема 6. *Алгебры* E_i $(i=1, 2, \ldots, n-1)$ *попарно* K-*изоморфны алгебрам* E_j $(j=1, 2, \ldots, n-1)$, $i\neq j$, *тогда и только тогда, когда* $j=n-i$, *и попарно не* K-*изоморфны алгеьрам* A_l $(l=1, \ldots, 6)$.

Доказательство. Сначала докажем первое утверждение теоремы. Пусть

$$E_i = \{F, a_1\};\quad a_1\lambda = \lambda^{p^{im}}\cdot a_1\quad (\lambda\in F),\quad E_j = \{F, a_2\};\quad a_2\lambda = \lambda^{p^{jm}}\cdot a_2\quad (\lambda\in F)$$

для фиксированных $i\neq j$ $(i, j=1, 2, \ldots, n-1)$, и имеет место K-изоморфизм φ: $E_i\to E_j$. Так как множество элементов конечного порядка в обоих алгебрах совпадает с полем F, то ограничение $\varphi|_F$ K-изоморфизма φ на поле F является K-автоморфизмом поля F. Значит на поле F изоморфизм φ задается в виде φ: $\lambda\to\lambda^{p^{sm}}$, где s — фиксированное натуральное число $(1\leqq s\leqq n)$. Все обратимые элементы в E_j имеют вид δa_2^r $(\delta\in F)$. Так как элемент a_1 обратим в E_i, то $\varphi(a_1)=\delta a_2^r$ $(\delta\in F)$. Однако $\varphi(a_1^t)=(\delta a_2^r)^t=\delta_1 a_2^{tr}$. Элементы вида $\sum\limits_{\nu}\delta_\nu a^{\nu r}$ не исчерпывают все элементы алгебры E_j только в случае $r=\pm 1$, поэтому при изоморфизме φ алгебр E_i и E_j должно выполняться $\varphi(a_1)=\delta a_2$, или $\varphi(a_1)=\delta a_2^{-1}$. Рассмотрим первый случай.

С одной стороны для произвольного $\lambda\in F$ имеем

$$\varphi(a_1\lambda a_1^{-1}) = \varphi(a_1)\varphi(\lambda)\varphi(a_1)^{-1} = \delta a_2\lambda^{p^{sm}}a_2^{-1}\delta^{-1} = \delta(\lambda^{p^{sm}})^{p^{jm}}\delta^{-1} = \lambda^{p^{(s+j)m}}.$$

С другой стороны

$$\varphi(a_1\lambda a_1^{-1}) = \varphi(\lambda^{p^{im}}) = (\lambda^{p^{sm}})^{p^{im}} = \lambda^{p^{(s+i)m}}.$$

Следовательно

$$p^{(s+i)m} \equiv p^{(s+j)m}\pmod{p^{nm}-1},$$

что означает $p^{nm}-1|p^{(i-j)m}-1$. Так как $1\leqq i\neq j\leqq n$, то последнее невозможно.

Рассмотрим теперь случай $\varphi(a_1)=\delta a_2^{-1}$. Повторяя рассуждения, сделанные в предыдущем случае, приходим к сравнению

$$p^{(s+i)m} \equiv p^{(s-j)m}\pmod{p^{nm}-1},$$

которое выполняется тогда и только тогда, когда $j \equiv -i \pmod n$. Это значит, что $E_i \cong E_j$ тогда и только тогда, когда $i = n - i$.

Покажем теперь, что каждая алгебра E_i $(i = 1, \ldots, n-1)$ не K-изоморфна ни одной из алгебр $A_1, A_2, \ldots, A_6$. Действительно, алгебра E_i не содержит делителей нуля, значит она не может быть K-изоморфна ни одной из алгебр A_1, A_2, A_4, A_5, A_6, так как все они содержат делителей нуля (напр. $(1+b) \cdot \cdot (1-b) = 0$).

Осталось показать, что алгебры E_i $(i = 1, 2, \ldots, n-1)$ не K-изоморфны алгебре A_3. Действительно, пусть φ: $A_3 \to E_i$ K-изоморфизм алгебры, заданной соотношениями

$$A_3 = \{F, a_1, b_1\}; \quad a_1\lambda = \lambda a_1; \quad b_1\lambda = \lambda b_1; \quad b_1^{-1}a_1b_1 = a_1^{-1}; \quad b_1^2 = \xi$$

на алгебру $E_i = \{F, a\}$; $a\lambda = \lambda^{p^{im}} \cdot a$. Так как элементы a_1 и b_1 обратимы в алгебре A_3, то их образы тоже обратимы в E_i, то есть

$$\varphi(a_1) = \delta a^s; \quad \varphi(b_1) = \delta_1 a^{s_1} \quad (\delta, \delta_1 \in F).$$

Тогда, с одной стороны

$$\varphi(b_1^{-1}a_1b_1) = \varphi(b_1)^{-1}\varphi(a_1)\varphi(b_1) = a^{-s_1}\delta_1^{-1}\delta a^s \delta_1 a^{s_1} = \delta_2 a^s$$

для некоторого элемента $\delta_2 \in F$, а с другой стороны

$$\varphi(b_1^{-1}a_1b_1) = \varphi(a_1^{-1}) = (\delta a^s)^{-1} = a^{-s}\delta^{-1} = \delta_3 a^{-s} \quad (\delta_3 \in F).$$

Сравнивая два равенства, получаем $s = -s$, то есть $s = 0$. Следовательно, $\varphi(a_1) = \delta$. Однако, в поле F элемент δ имеет конечный порядок, когда элемент a_1 — бесконечного порядка в алгебре A_3. Противоречие доказывает неизоморфность алгебр E_i и A_3. Теорема доказана.

Лемма 7. *Число n_i A_i-модулей $(i = 1, 2, 4, 5, 6)$, являющихся свободными циклическими $F(a)$-модулями, задается следующим образом:*

1. $n_1 = 4$, *они изоморфны модулям*

$$I_1^{(1)} = A_1(1+b); \quad I_2^{(1)} = A_1(1-b); \quad I_3^{(1)} = A_1(1+ab); \quad I_4^{(1)} = A_1(1-ab).$$

2. $n_2 = 2$, *они изоморфны модулям*

$$I_1^{(2)} = A_2(1+b); \quad I_2^{(2)} = A_2(1-b).$$

3. $n_4 = 2$, *они изоморфны модулям*

$$I_1^{(4)} = A_4(1+b); \quad I_3^{(4)} = A_4(1+ab),$$

4. $n_5 = 3$, *они изоморфны модулям*

$$I_1^{(5)} = A_5(1+b); \quad I_2^{(5)} = A_5(1-b); \quad I_3^{(5)} = A_5(1+ab).$$

5. $n_6=1$, *он изоморфен модулю*

$$I_1^{(6)} = A_6(1+b).$$

Доказательство. Так как в алгебре A_1 выполняются равенства $b^2=1$, $(ab)^2=1$, кроме того элементы b и ab перестановочны со всеми элементами поля F, то, согласно Лемме 5, утверждение 1 доказано.

Покажем, что в алгебре A_2 нет таких элементов $\gamma\in F$, что $(\gamma ab)^2=1$. Действительно,

$$(\gamma ab)^2 = \gamma^2 abab = \gamma^2 b\xi a^{-1}ab = \gamma^2\xi b^2 = \gamma^2\xi = 1,$$

то есть $\gamma^2\in\xi^{-1}$. Однако последнее равенство противоречит тому, что элемент ξ — квадратный невычет в поле F. Так как в алгебре A_2 имеет место $b^2=1$ и элемент b перестановочен со всеми элементами поля F, то, используя лемму 5, отсюда получаем утверждение 2 леммы.

В алгебре A_4 выполняется $b^2=1$ и $(ab)^2=1$, но ни элемент b, ни элемент ab не перестановочны со всеми элементами поля F. Значит, согласно Лемме 5, имеет место утверждение 3 леммы.

В алгебре A_5 выполняются равенства $b^2=1$ и $(ab)^2=1$, элемент b перестановочен со всеми элементами поля F, однако $(ab)\lambda=a\lambda b=\lambda^{p^{km}}(ab)$, значит из леммы 5 следует утверждение 4 леммы.

Покажем, что в алгебре A_6 нет таких элементов $\gamma\in F$, что $(\gamma ab)^2=1$. Действительно,

$$(\gamma ab)^2 = \gamma ab\gamma ab = \gamma a\gamma^{p^{km}} bab = \gamma\gamma^{p^{2km}} abab = \gamma^2 b\xi a^{-1}ab = \gamma^2\xi^{p^{km}} = 1,$$

то есть $\gamma^2=\xi^{-p^{km}}$. Элемент ξ — квадратный невычет в поле F. Каждый примитивный элемент поля F лежит в смежном классе $\xi\cdot F_1$ (см. замечание 1), значит можно считать, что элемент ξ — примитивный в поле F. Ищем элемент γ в виде $\gamma=\xi^s$. Тогда $\xi^{2s}=\xi^{-p^{km}}$, что ведет к сравнению

$$2s \equiv -p^{km} \pmod{p^{2km}-1}.$$

Так как наибольший общий делитель $(2,p^{2km}-1)=2$, но число p — нечетно, то последнее сравнение неразрешимо.

В алгебре A_6 выполняется равенство $b^2=1$, но элемент b не перестановочен со всеми элементами поля F, поэтому из леммы 5 следует утверждение 5 леммы. Лемма доказана.

Теорема 7. *Алгебры $A_1, A_2, \ldots, A_6$ попарно не K-изоморфны.*

Доказательство. Покажем сначала, что алгебра A не содержит делителей нуля. Так как все алгебры A_1, A_2, A_4, A_5, A_6 содержат делителей нуля, то из этого будет следовать неизоморфность алгебры A_3 с алгебрами A_i $(i=1, 2, 4, 5, 6)$.

Пусть L — поле частных групповой алгебры $F(a)$ бесконечной циклической группы (a) над полем F. Тогда алгебра A_3 погружается в алгебру

$$A = \{L, b\}, \quad b^2 = \xi, \tag{6}$$

которая является скрещенным произведением поля L с автоморфизмом второго порядка, порожденным элементом b.

Согласно общей теории алгебр, A является либо полным матричным кольцом второго порядка (и тогда имеет делителей нуля), либо телом (см. например [5]). Первая возможность имеет место тогда и только тогда, когда элемент ξ (см. (6)) есть норма для некоторого элемента $x \in L$ относительно автоморфизма $\lambda \to \lambda^b$, то есть

$$\xi = x \cdot x^b. \tag{7}$$

Пусть

$$x = \frac{\sum_i \lambda_i a^i}{\sum_j \mu_j a^j} \quad (\lambda_i, \mu_j \in F).$$

Подставим выражение элемента x в формулу (7):

$$\frac{\sum_i \lambda_i a^i}{\sum_j \mu_j a^j} \cdot \frac{\sum_i \lambda_i a^{-i}}{\sum_j \mu_j a^{-j}} = \xi,$$

откуда получаем равенство

$$\sum_i \lambda_i a^i \cdot \sum_i \lambda_i a^{-i} = \xi \cdot \sum_j \mu_j a^j \cdot \sum_j \mu_j a^{-j}, \tag{8}$$

в групповой алгебре $F(a)$. Так как $F(a)$ — кольцо главных идеалов, то элемент $\sum_i \lambda_i a^i$ однозначно (с точностью до единиц кольца) представляется в виде произведения простых элементов

$$\sum_i \lambda_i a^i = \tau \cdot p_1 \cdot \ldots \cdot p_s \quad (p_i \in F(a), \tau \in F). \tag{9}$$

Тогда левая сторона равенства (9) имеет вид $\tau^2 \cdot p_i \cdot \ldots \cdot p_s \cdot p_1^b \cdot \ldots \cdot p_s^b$. Ввиду однозначности разложения элементов кольца $F(a)$ в произведение простых элементов, правая сторона равенства (8) разлагается в произведение тех же простых элементов кольца $F(a)$ причем с точностью до констант из F, что и левая сторона, ибо при

$$\sum_j \mu_j a^j = \delta a^b \cdot p_1 \cdot \ldots \cdot p_s$$

автоморфный образ этого элемента имеет вид

$$\left(\sum_j \mu_j a^j\right)^b = \delta a^{-b} \cdot p_1^b \cdot \ldots \cdot p_s^b,$$

то есть приходим к равенству

$$\tau^2 \cdot p_1 \cdot \ldots \cdot p_s \cdot p_1^b \cdot \ldots \cdot p_s^b = \xi \cdot \delta^2 \cdot p_1 \cdot \ldots \cdot p_s \cdot p_1^b \cdot \ldots \cdot p_s^b.$$

Отсюда следует $\tau^2 = \xi\delta^2$. Так как ξ — квадратный невычет в поле F, то последнее равенство невозможно. Противоречие доказывает, что алгебра не содержит делителей нуля.

Согласно лемме 7, число неизоморфных A_i-модулей $(i=1, 2, 4, 5, 6)$, являющихся свободными циклическими $F(a)$-модулями для этих модулей попарно различается, кроме алгебр A_2 и A_4. Поэтому, согласно лемме 6, среди алгебр A_1, A_2, A_4, A_5, A_6 могут быть K-изоморфны только алгебры A_2 и A_4.

Очевидно, центр алгебры A_2 совпадает с полем F, значит число обратимых элементов центра алгебры A_2 равно $p^{nm}-1$.

В то же время, если θ — примитивный элемент поля F, то группа всех обратимых элементов центра алгебры A_4 порождается элементом $\theta^{p^{km}+1}$, где $n=2k$. Действительно, из равенства

$$b \cdot \theta^x = (\theta^x)^{p^{km}} \cdot b = \theta^x b$$

следует сравнение.

$$x(p^{km}-1) \equiv 0 \pmod{p^{nm}-1}$$

или

$$x \equiv 0 \pmod{p^{km}+1}.$$

Значит, число всех обратимых элементов центра алгебры равно числу $p^{km}=1$.

Так как при изоморфизме центральные элементы переходят в центральные, обратимые в обратимые, то алгебры A_2 и A_4 не K-изоморфны. Теорема доказана.

Следствие 2. *Все не K-изоморфные алгебры типа E над полем K характеристики p $(\neq 2)$ по отношению к полю F, где $|K| = p^m$, $(F\colon K) = n$ задаются алгебрами типов:*

При четном n: алгебры E_i $\left(i=0, 1, \ldots, \left[\frac{n}{2}\right]\right)$ *и A_j* $(j=1, 2, 3, 4, 5, 6)$.

При нечетном n: алгебры E_i $\left(i=0, 1, \ldots, \left[\frac{n}{2}\right]\right)$ *и A_j* $(j=1, 2, 3)$, *где алгебры E_i, A_j заданы в следствии* 1.

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Rare bases for finite intervals of integers

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In this paper we discuss the following finite problem for additive bases: What is the least possible number of elements of a set B, for which all integers in the interval $[1, n]$ can be represented as the sum of two elements of B. (B can be called a basis of order 2 for the interval $[1, n]$). Let us denote this minimal number by c_n.

Clearly, $c_n \geqq \sqrt{2} \cdot \sqrt{n}$ holds, since if there are k elements in B, then we can form at most $\binom{k+1}{2} \sim \frac{k^2}{2}$ sums which have to give at least n different values, hence $n \leqq \frac{k^2}{2}$, i.e. $k \geqq \sqrt{2} \cdot \sqrt{n}$.

On the other hand, a simple construction shows $c_n \leqq 2 \cdot \sqrt{n}$. Let B be the union of two arithmetical progressions; $0, 1, 2, \ldots, \lceil\sqrt{n}\rceil$ and $2 \cdot \lceil\sqrt{n}\rceil$, $3 \cdot \lceil\sqrt{n}\rceil, \ldots, \lceil\sqrt{n}\rceil \cdot \lceil\sqrt{n}\rceil$, where $\lceil a\rceil$ means the least integer $s \geqq a$. These approximately $2 \cdot \sqrt{n}$ elements form a basis of order 2 for the interval $[1, n]$.

Rohrbach conjectured in 1937 that $c_n = 2\sqrt{n} + O(1)$. This was disproved in 1976 by HÄMMERER and HOFMEISTER [2], they showed $c_n \leqq \sqrt{3.6} \cdot \sqrt{n}$. (For further references and related problems see [1], 47.)

The aim of this paper is to improve the result of Hämmerer and Hofmeister:

Theorem. $c_n \leqq \sqrt{3.5} \cdot \sqrt{n} + o(\sqrt{n})$.

The method of proof is different from that of [2], it is completely elementary, and is based on similar simple ideas as the one which gave the obvious upper bound. Some further refinements might yield even better upper bounds for c_n.

Proof. For a clearer exposition of the construction we show first $c_n \leqq \sqrt{3.6} \cdot \sqrt{n} + o(\sqrt{n})$. B will consists of the union of 4 arithmetical progressions:

I. $\alpha_0 = 0$, $\alpha_1 = 1, \ldots, \alpha_t = t$; here the difference of the consecutive terms is $d_1 = 1$.

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II. $\beta_0 = \alpha_t = t, \quad \beta_1 = 2t+1, \ldots, \beta_{3t} = 3t(t+1)+t = 3t^2+4t; \quad d_2 = t+1.$

III. $\gamma_1 = 3t^2+5t+1, \ldots, \gamma_{t+1} = 3t^2+6t+1; \quad d_3 = 1.$

IV. $\delta_1 = 6t^2+12t+3, \ldots, \delta_{t+1} = 7t^2+12t+3; \quad d_4 = t.$

The following inclusions (mostly in form of equalities) are all obvious, except the last but one, which we shall verify below:

$$\{\alpha_i+\alpha_j\} \supseteqq [1, 2t], \quad \{\alpha_i+\beta_j\} \supseteqq [2t+1, 3t^2+5t],$$

$$\{\alpha_i+\gamma_j\} \supseteqq [3t^2+5t+1, 3t^2+7t+1], \quad \{\beta_i+\gamma_j\} \supseteqq [3t^2+7t+2, 6t^2+10t+1],$$

$$\{\gamma_i+\gamma_j\} \supseteqq [6t^2+10t+2, 6t^2+12t+2], \quad \{\alpha_i+\delta_j\} \supseteqq [6t^2+12t+3, 7t^2+13t+3],$$

$$\{\beta_i+\delta_j\} \supseteqq [7t^2+13t+4, 9t^2+17t+3], \quad \{\gamma_i+\delta_j\} \supseteqq [9t^2+17t+4, 10t^2+18t+4].$$

To verify the last but one inclusion we use the following two equalities, which are straightforward from the construction of II and IV:

$$\beta_{i+1}+\delta_{j-1} = \beta_i+\delta_j+1 \quad \text{and} \quad \beta_{i-t+1}+\delta_{j+t} = \beta_i+\delta_j+1.$$

Hence we obtain the consecutive elements of the interval $[7t^2+137+3, 9t^2+17t+3]$ by the following sums:

$$\begin{array}{llll}
\beta_0+\delta_{t+1}, & \beta_1+\delta_t, & \beta_2+\delta_{t-1}, \ldots, & \beta_t+\delta_1, \\
\beta_1+\delta_{t+1}, & \beta_2+\delta_t, & \ldots, & \beta_{t+1}+\delta_1, \\
\beta_2+\delta_{t+1}, & \ldots & & \\
\vdots & & & \\
\beta_{2t}+\delta_{t+1}, & \beta_{2t+1}+\delta_t, & \ldots, & \beta_{3t}+\delta_1, \\
\beta_{2t+1}+\delta_{t+1}, & \beta_{2t+2}+\delta_t, & \ldots, \quad \beta_{3t}+\delta_2. &
\end{array}$$

Summarizing our construction, B contains $k=6t+3$ elements and is a basis for the interval $[1, n]$, where $n=10t^2+18t+4$. This proves $c_n \leqq \sqrt{3.6}\cdot\sqrt{n}+o(\sqrt{n})$ (for all n).

To obtain $c_n \leqq \sqrt{3.5}\cdot\sqrt{n}+o(\sqrt{n})$ we have to add just another arithmetical progression to B:

V. $\varepsilon_1 = 10t^2+18t+5, \ldots, \varepsilon_{t+1} = 11t^2+18t+5; \quad d_5 = t.$

Now

$$\{\alpha_i+\varepsilon_j\} \supseteqq [10t^2+18t+5, 11t^2+19t+5],$$

$$\{\beta_i+\varepsilon_j\} \supseteqq [11t^2+19t+6, 13t^2+23t+5],$$

$$\{\gamma_i+\varepsilon_j\} \supseteqq [13t^2+23t+6, 14t^2+24t+6].$$

Here the first and last inclusions are obvious, and the second one follows exactly the same way as the one for $\{\beta_i+\delta_j\}$.

Hence we have a basis for $[1, n]$ with $n=14t^2+24t+6$, and it consists of $k=7t+4$ elements, i.e. $k \sim \sqrt{3.5}\cdot\sqrt{n}$.

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Wreath product decomposition of categories. I*)

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1. Introduction. In this paper I prove a theorem (Theorem 4.1) giving sufficient conditions for decomposing a functor F: $\mathbf{C}\to\mathbf{Cat}$ into the wreath product of two functors, given a natural transformation λ: $F\to G$. When the functors are discrete (set-valued) the sufficient conditions always hold.

The theorem is a double generalization of the theorem about embedding a group into a wreath product due to KALOUJNINE—KRASNER ([7], stated also in WELLS [13]). To be precise, it generalizes the one-step version of that theorem, although for any action — not just for the regular representation as it is commonly stated in group theory texts.

The generalization is double in the sense that the group is generalized to a category and the action not merely to a set-valued functor (which already gives a new theorem) but to a **Cat**-valued one. The theorem provides a decomposition of *any* Set-valued functor with given quotient, and any **Cat**-valued one provided the fibers of the quotient are split opfibrations. Since the wreath product itself is a split fibration, this brings the theory of fibrations into the picture in two different ways.

Some applications are given in Section 6. One, Proposition 6.4, provides a generalization of a technique used in some proofs of the Krohn—Rhodes Theorem (see KROHN—RHODES [10], WELLS [13]). (A generalization of another of the techniques to **Cat**-valued functors is in WELLS [17].)

My hope is that both techniques might be useful in developing a theory of *state-transition systems* with *structured, typed states.* Any functor F: $\mathbf{C}\to\mathbf{Cat}$ can be thought of as such a system. The objects of **C** are the types of states. For each object c, the objects of Fc are the states of type c. The transitions are the functors Ff: $Fc\to Fd$ for f: $c\to d$ in **C**. The structure on the states of type c is the category structure on Fc (thus having a poset or monoid or group structure as possible special cases).

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Perhaps the theorem of the present paper will also be useful in developing a theory of varieties for categories, in the way the embedding into a wreath product has proved useful in group theory (NEUMANN [12]).

Categorical fibrations and opfibrations are discussed in Section 2, and the wreath product with categorical action in Section 3. The decomposition theorem is stated in Section 4 and proved in Section 5. Some applications are given in Section 6.

Throughout this paper, a set is identified with the category which has the elements of the set as objects and no non-identity arrows. Such a category is called *discrete*.

These results were obtained in part while I was a guest of the Forschungsinstitut für Math., E.T.H. Zürich, for whose support I am grateful. An earlier version, containing errors, called *Wreath product decomposition of categories and functors*, was distributed but never published.

2. Fibrations. In this section, I outline that part of the theory of split fibrations and opfibrations needed for the main theorems. The material is not new, and is scattered through GROTHENDIECK [5], GIRAUD [1], GRAY [2], [3], [4].

Given a functor $P: \mathbf{E} \to \mathbf{C}$ there is an induced functor S from the arrow category $\mathrm{Ar}\,\mathbf{E}$ to the comma category $(\mathbf{C}, P)$ which takes $u: e' \to e$ to (Pu, e). A right adjoint right inverse R for S is called a *cleavage*, and a *left* adjoint right inverse R° to the functor S°: $\mathrm{Ar}\,\mathbf{E} \to (P, \mathbf{C})$ which takes $u: e' \to e$ to (e', Pu) is an *opcleavage*. P, together with a cleavage R, is a *fibration* of $\mathbf{C}$. If R° is an opcleavage, (P, R°) is an *opfibration* of $\mathbf{C}$. Neither a cleavage nor an opcleavage necessarily exists for any given functor P.

Assume $(P: \mathbf{E} \to \mathbf{C}, R)$ is a fibration. Let $f: b \to c$ in $\mathbf{C}$ and $u: e' \to e$ lie over c (i.e. $Pu = 1_c$). Define $\Phi f . e'' = \mathrm{dom}\, R(f, e'')$ for any object e'' over c, and $\Phi f . u$ by requiring $R(1_b, u) = (\Phi f . u, u)$ (the second component is necessarily u). Similarly for an opfibration (P, R°), let $\Phi^\circ f . e'' = \mathrm{cod}\, R^\circ(e'', f)$ for e'' over b, and $R^\circ(u, 1_c) = (u, \Phi^\circ f . u)$. One then has the commutative squares

$$(2.1)\qquad \begin{array}{ccc} \Phi f . e' & \xrightarrow{R(f,e')} & e' \\ {\scriptstyle \Phi f . u}\downarrow & & \downarrow{\scriptstyle u} \\ \Phi f . e & \xrightarrow[R(f,e)]{} & e \end{array} \qquad\qquad \begin{array}{ccc} e' & \xrightarrow{R^\circ(e',f)} & \Phi^\circ f . e' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \Phi^\circ f . u} \\ e & \xrightarrow[R^\circ(e,f)]{} & \Phi^\circ f . e \end{array}$$

By setting $\Phi c = \Phi^\circ c = P^{-1} c$ (the full subcategory of $\mathbf{E}$ lying over 1_c) one has Φ, Φ° both defined on objects and arrows of c. They may not be functors. If they are, they are functors to $\mathbf{Cat}$ and $R(f, —)$ and $R^\circ(—, f)$ are natural transformations for each f. If $P^{-1}c$ is a set (no non-trivial arrows) the fibration or opfibration is called *discrete*.

A fibration (P, R) is *split* if

a) Φ is a functor, and

b) if $f\colon c' \to c$, $g\colon c \to c''$ in **C** and $Pe'' = c''$, $Pe = c$, then

$$R(f, \Phi g . e'') \circ R(g, e'') = R(g \circ f, e''). \tag{2.2}$$

Then Φ is a *splitting*, and I shall refer to the split fibration as $(P\colon \mathbf{E} \to \mathbf{C}, R, \Phi)$.

A split opfibration (P, R°, Φ°) requires

a)° Φ° is a functor, and

b)° if $f\colon c' \to c$, $g\colon c \to c''$ in **C**, $Pe' = c'$, $Pe = c$, then

$$R^\circ(e', g \circ f) = R'(\Phi^\circ f . e', g) \circ R^\circ(e', f). \tag{2.2}^\circ$$

It is easy to see that $(P\colon \mathbf{E} \to \mathbf{C}, R, \Phi)$ is a split fibration if and only if $(P^{\mathrm{op}}\colon \mathbf{E}^{\mathrm{op}} \to \mathbf{C}^{\mathrm{op}}, R^{\mathrm{op}}, \Phi^{\mathrm{op}})$ is a split opfibration.

A morphism of split fibrations is a pair $(U, V)\colon (P\colon \mathbf{E} \to \mathbf{C}, R, \Phi) \to (P'\colon \mathbf{E}' \to \mathbf{C}', R', \varphi')$ where $U\colon \mathbf{C} \to \mathbf{C}'$ and $\mathbf{V}\colon \mathbf{E} \to \mathbf{E}'$ are functors for which

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{V} & \mathbf{E}' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ \mathbf{C} & \xrightarrow[U]{} & \mathbf{C}' \end{array} \tag{2.3}$$

commutes and for $f\colon b \to c$ in **C**, e an object of Φc,

$$V\big(R(f, e)\big) = R'(Uf, Ve). \tag{2.4}$$

Composition of morphisms is componentwise, giving a category **F** of split fibrations.

Morphisms of opfibrations are defined similarly. (2.3)° is the same as (2.3) and (2.4) becomes

$$V\big(R^\circ(e, f)\big) = R'(Ve, Uf) \tag{2.4}^\circ$$

where e is an object of $\Phi^\circ b$. The resulting category is denoted $\mathbf{F}^\circ$.

It follows from (2.4) that

$$V(\Phi f . e) = \Phi'(Uf) . Ve, \tag{2.5}$$

i.e. V respects fibers. A similar statement holds for morphisms of opfibrations.

Now I define another category **Scat** which will turn out to be equivalent to both **F** and $\mathbf{F}^\circ$. The objects of **Scat** are all **Cat**-valued functors from all categories. An arrow $(K, \lambda)\colon F \to G$ has $K\colon \operatorname{dom} F \to \operatorname{dom} G$ a functor and $\lambda\colon F \to G \circ K$ a natural transformation. Composition is given by

$$(L, \mu) \circ (K, \lambda) = (L \circ K, \mu K \circ \lambda). \tag{2.6}$$

All functor categories **Func** (**C**, **Cat**) are subcategories of **Scat**, and so is the comma category (**Cat**, **Cat**), where the second "**Cat**" is an object in the first. **Scat** is the category called $\mathrm{Cat}_0 \circ \mathrm{Cat}_0$ by Kelly [8, § 7].

Given any functor F: $\mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$, let $SD(F)$ be the category defined this way: an object of $SD(F)$ is a pair (c, x) with c an object of $\mathbf{C}$ and x an object of Fc. An arrow (f, u): $(c, x) \to (c', x')$ has f: $c' \to c$ in $\mathbf{C}$ and u: $x \to Ff.x'$ in Fc. If (g, v): $(c', x') \to (c'', x'')$, then

(2.7) $$(g, v) \circ (f, u) = (f \circ g, (Ff.v) \circ u).$$

Likewise, given F: $\mathbf{C} \to \mathbf{Cat}$ define $SD^\circ(F)$ the same way except that for (f, u): $(c, x) \to (c', x')$, f: $c \to c'$ and u: $Ff.x \to x'$, and

(2.7)° $$(g, v) \circ (f, u) = (g \circ f, V \circ (Fg.u)).$$

There are then functors $SN(F)$: $SD(F) \to \mathbf{C}^{\mathrm{op}}$ and $SN^\circ(F)$: $SD(F) \to \mathbf{C}$ taking (f, u) to f.

There are then functors $R_F(R_F)$ and $\bar{F}(\bar{F}^\circ)$ for which $(SN(F), R_F, \bar{F})$ (resp. $(SN^\circ(F), R_F^\circ, \bar{F}^\circ)$) is a split fibration (split opfibration). The definitions are, for (f, u): $(c, x) \to (c', x')$ in $SD(F)$,

(2.8) $$R_F(f, (c', x')) = (f, 1_{Ff.x'}): (c, Ff.x') \to (c', x')$$

and for (f, u): $(c, x) \to (c', x')$ in $SD^\circ(F)$,

(2.8)° $$R_F^\circ((c, x), f) = (1_{Ff.x}, f): (c, x) \to (c', Ff.x).$$

As for $\bar{F}$ and $\bar{F}^\circ$ the definitions are determined by R_F. In particular (because it is used later), for F: $\mathbf{C} \to \mathbf{Cat}$, u an arrow in Fc,

(2.9)° $$\bar{F}^\circ f.(1_c, u) = (1_{c'}, Ff.u).$$

These constructions make SN: $\mathbf{Scat} \to \mathbf{F}$ and SN°: $\mathbf{Scat} \to \mathbf{F}^\circ$ into the object maps of functors.

I will continue the development only for opfibrations, since the constructions for fibrations are not needed. Let F: $\mathbf{C} \to \mathbf{Cat}$, G: $\mathbf{D} \to \mathbf{Cat}$, (K, λ): $F \to G$ in $\mathbf{Scat}$. Let (f, u): $(c, x) \to (c', x')$ in $SD^\circ(F)$. Then define

(2.10)° $$SD^\circ(K, \lambda)(f, u) = (Kf, \lambda c'.u)$$

and

(2.11)° $$SN^\circ(K, \lambda) = (K, SD^\circ(K, \lambda)).$$

Thus SD°: $\mathbf{Scat} \to \mathbf{Cat}$ and SN°: $\mathbf{Scat} \to \mathbf{F}^\circ$ are functors.

SN° is an equivalence of categories. Define the functor Λ°: $\mathbf{F}^\circ \to \mathbf{Scat}$ as follows.

(2.12)° $$\Lambda^\circ(P: \mathbf{E} \to \mathbf{C}, R^\circ, \Phi^\circ) = \Phi^\circ.$$

(2.13)° $$\Lambda^\circ(U, V) = (U, \alpha_V), \quad \text{where}$$

(2.14)° $$\alpha_V.c = V \mid \Phi^\circ c$$

for c an object of $\mathbf{C}$.

There is a natural isomorphism $\varepsilon\colon \mathrm{id}_{\mathbf{Scat}} \to \Lambda^\circ \circ SN^\circ$, whose component at $F\colon \mathbf{C} \to \mathbf{Cat}$ is

$$\varepsilon F = (1_{\mathbf{C}}, \bar{\varepsilon}F)\colon F \to \bar{F}^\circ \tag{2.15)°}$$

(see (2.9)), where for an object c of $\mathbf{C}$, $\bar{\varepsilon}F.c\colon Fc \to \bar{F}^\circ c$ takes an object x to (c, x) and an arrow u over 1_c to $(1_c, u)$.

There is also a natural isomorphism $\eta\colon \mathrm{id}_{\mathbf{F}^\circ} \to SN^\circ \circ \Lambda^\circ$, defined as follows. Given a split opfibration $(P\colon \mathbf{E} \to \mathbf{C}, R^\circ, \Phi^\circ)$, let $I\colon \mathbf{E} \to SD^\circ(\Phi^\circ)$ take an arrow u to (Pu, u). Then the component of η at (P, R°, Φ°) is $(\mathrm{id}_{\mathbf{C}}, I)\colon (P, R^\circ, \Phi^\circ) \to (SN^\circ(\Phi^\circ), R_{\Phi^\circ}, \bar{\Phi}^\circ)$. Thus SN° and Λ° are equivalences.

This Lemma is needed later:

Lemma 2.1. *Let $(U, V), (U, W)\colon (P\colon \mathbf{E} \to \mathbf{C}, R^\circ, \Phi^\circ) \to (P'\colon \mathbf{E}' \to \mathbf{C}', R^{\circ\prime}\, \Phi^{\circ\prime})$ be morphisms of split opfibrations for which for every object c of $\mathbf{C}$, $V|Gc = W|Gc$. Then $V = W$.*

Proof. Let $m\colon e \to e_0$ in E lie over $f\colon b \to c$. It is enough to show that $Vm = Wm$. Since R° is left adjoint to S°, there is a unique morphism of $\mathrm{Ar}\,\mathbf{E}$ from $R^\circ(e, f)$ to m corresponding to the identity arrow in $(P, \mathbf{C})$ from (e, f) to $(e, f) = S^\circ m$. Since R° is left inverse to S°, this arrow must be of the form $(1_e, k)$ where $k\colon \Phi^\circ f.\, e \to e_0$ and k is in $\Phi^\circ c$. Then by definition of morphism in $\mathrm{Ar}\,\mathbf{E}$, $m = k \circ R^\circ(e, f)$. Hence by (2.4)°,

$$Vm = Vk \circ VR^\circ(e, f) = Wk \circ R^{\circ\prime}(Uf, Ve) = Wk \circ R^{\circ\prime}(Uf, We) = Wk \circ WR^{\circ\prime}(f, e) = Wm$$

since k is in $\Phi^\circ C$ and e is in $\Phi^\circ b$.

3. The wreath product of categories. Given categories $\mathbf{B}$ and $\mathbf{C}$ and a functor $G\colon \mathbf{C} \to \mathbf{Cat}$, let $G_{\mathbf{B}} = \mathbf{Func}(G(-), \mathbf{B})\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$. The *wreath product* of $\mathbf{B}$ by $\mathbf{C}$ with action G, denoted $\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$, is $SD(G_{\mathbf{B}})$. Thus via $SN(G_{\mathbf{B}})$ it is a split fibration of $\mathbf{C}$ in a canonical way. Note that $\mathbf{Scat} = \mathbf{Cat}\,\mathrm{wr}^I\,\mathbf{Cat}$ with I being the identity functor.

The concept is due to KELLY [8, § 5], who denotes $\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$ by $[\mathbf{C}, G] \circ B$ and calls it the *composite*. His definition is more general than mine, since for him $\mathbf{B}$ can be any object in a 2-category.

$\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$ is natural in both variables in the sense that functors $U\colon \mathbf{B} \to \mathbf{B}'$ and $V\colon \mathbf{C}' \to \mathbf{C}$ induce a functor $SD(\mathbf{Func}(G(-), U), V)\colon \mathbf{B}\,\mathrm{wr}^{GV}\mathbf{C}' \to \mathbf{B}'\,\mathrm{wr}^G\,\mathbf{C}$ which is natural in both variables.

More important, a functor $F\colon \mathbf{B} \to \mathbf{Cat}$ induces a functor $F\,\mathrm{wr}\,G\colon \mathbf{B}\,\mathrm{wr}^G\mathbf{C} \to \mathbf{Cat}$ which generalizes the concept of the wreath product of two actions. Given F, define $\bar{F}\colon \mathbf{B}\,\mathrm{wr}^G\,\mathbf{C} \to \mathbf{Scat}$ as follows. For an object (c, P) of $\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$ (whence $P\colon GC \to \mathbf{B}$ is a functor), set $\bar{F}(c, P) = F \circ P$. For an arrow $(f, \lambda)\colon (c, P) \to (d, Q)$ (whence $f\colon c \to d$ in $\mathbf{C}$ and $\lambda\colon P \to Q \circ Gf$), set $\bar{F}(f, \lambda) = (Gf, F\lambda)$. Then set $F\,\mathrm{wr}\,G = SD^\circ \circ \bar{F}\colon \mathbf{B}\,\mathrm{wr}^G\,\mathbf{C} \to \mathbf{Cat}$.

Kelly [8, §7] shows that wreathing for categories and for functors is associative up to a 2-natural isomorphism.

If $\mathbf{B}$ and $\mathbf{C}$ are groups regarded as categories and G is discrete (**Set**-valued) then $\mathbf{B}\ \mathrm{wr}^G\ \mathbf{C}$ is the usual wreath product of groups. If G is not discrete then $\mathbf{B}\ \mathrm{wr}^G\ \mathbf{C}$ is a groupoid. If $\mathbf{B}$ is a set regarded as a discrete category, $\mathbf{C}$ is a monoid acting on $\mathbf{B}$ and G is the action, then $\mathbf{B}\ \mathrm{wr}^G\ \mathbf{C}$ is a directed graph with objects which are functions $f\colon \mathbf{B}\to\mathbf{B}$ and edges $f\to fg^{-1}$ where g is an invertible element of $\mathbf{C}$. When $\mathbf{B}$ and $\mathbf{C}$ are groupoids, $\mathbf{B}\ \mathrm{wr}^G\ \mathbf{C}$ has as a special case the untwisted version of the wreath product due to Houghton [6]. Here the functor G is discrete; its value at an object c of $\mathbf{C}$ is the total sieve on c (the set of all arrows into c).

4. Coordinate systems. In the Kaloujnine—Krasner setup a group action is decomposed along a quotient action. The second coordinate is the quotient, and the first coordinate (the one with the most dependencies) is the action on a fiber. One can get away with this because the fibers are all isomorphic — although to get a decomposition you have to specify the isomorphisms.

In the present schema this corresponds to introducing a "typing functor" (defined below), which allows a partial skeletonization of the fibers of the quotient action. To do this we will make the fibers into a category $\mathbf{Fib}(\lambda)$ where λ is the quotient map. A "coordinate system" will then be a category and an action (**Cat**-valued functor) which "includes" $\mathbf{Fib}(\lambda)$ in a certain sense. All this requires that the components of λ be split normal opfibrations, a condition which is vacuous in the discrete case. The main Theorem 4.1 then says that in the presence of a coordinate system the action can be decomposed into the wreath of the action on the (partially skeletonized) fibers and the quotient action.

Let $\mathbf{C}$ be a category, $F\colon \mathbf{C}\to\mathbf{Cat}$ and $G\colon \mathbf{C}\to\mathbf{Cat}$ functors, and $\lambda\colon F\to G$ a natural transformation. Then λ is *split* if for each object c of $\mathbf{C}$, $\lambda c\colon Fc\to Gc$ is a split opfibration with splitting $Lc\colon Gc\to\mathbf{Cat}$, and for each $f\colon c\to d$ in $\mathbf{C}$, the pair (Gf, Ff) is an $\mathbf{F}^{\circ}$-morphism. The latter requirement implies that for each object x of Gc, $Ff|Lc\,.\,x$ has values in $Ld(Gf\,.\,x)$, and for each $u\colon x\to y$ in Gc,

$$\begin{array}{ccc} Lc.x & \xrightarrow{Ff|Lc.x} & Ld(Gf.x) \\ \downarrow{\scriptstyle Lc.u} & & \downarrow{\scriptstyle Ld(Gf.u)} \\ Lc.y & \xrightarrow[Ff|Lc.z]{} & Ld(Gf.y) \end{array} \qquad (4.1)$$

commutes. If F and G are discrete, any natural transformation $\lambda\colon F\to G$ is split.

The fibers of λ, in other words the categories $Lc\,.\,x$ for c an object of $\mathbf{C}$ and x an object of Gc, are objects of a category $\mathbf{Fib}(\lambda)$. The arrows are the functors from $Lc\,.\,x$ to $Ld(Gf\,.\,y)$ given by (4.1) for each $f\colon c\to d$ in $\mathbf{C}$ and each $u\colon x\to y$ in Gc. Thus $\mathbf{Fib}(\lambda)$ is a subcategory of $\mathbf{Cat}$.

A functor T: **Fib**$(\lambda)\to$**Cat** is a *typing functor* if there is a natural isomorphism $\tau: I_\lambda \to T$, where I_λ: **Fib**$(\lambda)\to$**Cat** is inclusion. Extreme cases of typing functors are I_λ and a skeletonizing functor. An intermediate case is actually used in an application in Section 6.

(M, K, T) is a *coordinate system* for a split $\lambda: F\to G$ with splitting L if T is a typing functor for **Fib**(λ), **M** is a category and $K: M\to$ **Cat** a functor for which

CS—1. For each object c of **C** there is a set Φ_c of functors $P: Gc\to$ **M** for each of which $T\circ Lc$ is a subfunctor of $K\circ P$, and

CS—2. If $f: c\to d$ in **C** and $P: Gc\to$ **M** in Φ_c, then there is $Q: Gd\to$ **M** in Φ_d for which for each object x of Gc there is an arrow $m: Px\to Q(Gf.x)$ for which $Km|T(Lc.x)=T(Ff|Lc.x)$.

A transitive group action with a quotient always has a coordinate system. Let **C** be the group, F the action, G the quotient action, λ the quotient map, so the fibers form a system of imprimitivity. T is then a way of identifying all the fibers with one of them, **M** is the isotopy subgroup of that fiber with action K. P is then a constant map. Even a nontransitive group action with quotient has a coordinate system, but then **M** will be a disjoint union of isotopy subgroups regarded as categories.

If F, G: **C**$\to$**Set**, $\lambda: F\to G$ any natural transformation, then λ always has a coordinate system based on **Fib**(λ). This is discussed further in Section 6.

A functor H: **A**$\to$**B** *lifts triangles* if for all arrows f of **A** and h, k of **B** for which $Hf\circ h$ and $k\circ Hf$ are defined, there are arrows u, v of **A** for which $f\circ u$ and $v\circ f$ are defined, and $Hu=h$, $Hv=k$. A decomposition ought to lift triangles, as I explain later. Too bad, because the decomposition is trivial to construct if it needn't lift triangles.

In the following theorem, F: **C**$\to$**Cat**, G: **C**$\to$**Cat** are functors and $\lambda: F\to G$ a natural transformation. $\overline{\mathbf{G}}$ is the image of G in **Cat**, and I_G: $\overline{\mathbf{G}}\to$**Cat** is inclusion.

Theorem 4.1. *If F is faithful and λ is split with coordinate system (M, K, T), then there is a subcategory* $\mathbf{S}\subset\mathbf{M}\,\mathrm{wr}^{IG}\,\overline{G}$ *and a triangle-lifting functor* H: **S**$\to$**C** *for which $F\circ H$ is isomorphic to a subfunctor of the restriction of $K\,\mathrm{wr}\,I_G$ to* **S**.

The proof is given in Section 5, and applications are discussed in Section 6.

If you think of this theorem as giving sufficient conditions for simulating a state-transition system triangularly (in the sense of Krohn, Langer and Rhodes [11]) by a wreath product or cascade of systems, then the simulation has the property that for any state and any transition from that state in the simulat*ed* system, there is at least one state and transition from it in the simulat*ing* system which mimics (functorially) the operation of the simulat*ed* system. Moreover *you can always simulate the next transition from the simulating state you find yourself in.* That is the meaning of triangle-lifting. Clearly it is a necessary property of typed-state simulations.

Note that the system $F\colon \mathbf{C}\to\mathbf{Cat}$ might very well allow a sequence of transitions which begin and end at the same state, but for which the simulation begins and ends at different states, behavior reminiscent of a path in a Riemann surface lying over a loop.

Theorem 4.1 is similar to, but apparently not exactly a generalization of, both Theorem 11.1 of WELLS [13] and the main theorem of WELLS [15].

5. Proof of Theorem 4.1. S is the subcategory of $\mathbf{M}\,\mathrm{wr}^{I_G}\bar{G}$ defined this way: an object of S is any pair (Gc, P) where c is an object of $\mathbf{C}$ and $P\colon Gc\to\mathbf{M}$ is a functor in Φ_c. An arrow $(Gf,\gamma)\colon (Gc,P)\to(Gd,Q)$ has $f\colon c\to d$ in $\mathbf{C}$ and γ any function from the objects of Gc to the arrows of $\mathbf{M}$ with the properties that for each object x of Gc,

$$\gamma x\colon Px\to Q(Gf.x), \tag{5.1}$$

$$T(Lc.x)\subset KPx, \tag{5.2}$$

$$T\big(Ff(Lc.x)\big)\subset KQ(Gf.x), \quad\text{and} \tag{5.3}$$

$$K(\gamma x)|T(Lc.x) = T(Ff|Lc.x). \tag{5.4}$$

There may not be such a γ for a given f, P, and Q as above, but for a given f and P there *is* a Q in Φ_d for which there is at least one such γ. That follows from CS—1 and CS—2.

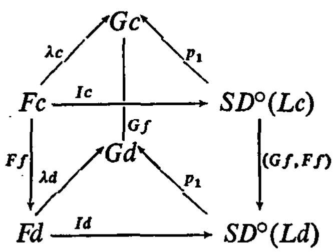
The functor $H\colon \mathbf{S}\to\mathbf{C}$ is defined by

$$H(Gf,\gamma) = f. \tag{5.5}$$

It is necessary to see that H is well-defined. Because $T(Lc\,.\,x)$ is naturally isomorphic to $Lc\,.\,x$, (5.4) says that the arrows which make up γ determine the effect of Ff on the categories $Lc\,.\,x$. Because (Gf, Ff) is a morphism in $\mathbf{F}^{\circ}$, Lemma 2.1 says that γ and Gf determine Ff. That determines f because F is faithful. It is clear that H is triangle lifting.

To show that $F\circ H$ is a subfunctor of the restriction of $K\,\mathrm{wr}\,I_G$ requires several steps. In the first place

(5.6)



commutes, where I_c is the natural isomorphism defined by $\eta_{\lambda c}=(\mathrm{id}_{Gc}, I_c)$ as in Section 2, and p_1 is first projection (representing the elements as ordered pairs as in

Section 2). This follows because (Gf, Ff) is an $\mathbf{F}^\circ$-morphism and $SN^\circ(\Lambda^\circ(Fc)) = SD^\circ(Lc)$ and $SN^\circ(\Lambda^\circ(Gf, Gf)) = (Gf, Gf)$.

Because T is a typing functor, there are natural isomorphisms τc, τd making this diagram of functors and natural transformations commute. The component of τc at x is $\tau(Lc\,.\,x)$, τ as in the definition of typing functor.

(5.7)
$$\begin{array}{ccc} Lc & \xrightarrow{\tau c} & T\circ Lc \\ {\scriptstyle \Lambda^\circ(Gf,Ff)}\downarrow & & \downarrow{\scriptstyle \Lambda^\circ(Gf,TFf)} \\ Ld & \xrightarrow[\tau d]{} & T\circ Ld. \end{array}$$

By (2.13)° and (2.14)°, the left vertical arrow is $\alpha\,.\,Ff$ and the right one is $T(\alpha\,.\,Ff)$. Applying these functors at an object x of Gc and using (2.14)° yields

(5.8)
$$\begin{array}{ccc} Lc.x & \xrightarrow{\tau(Lc.x)} & T(Lc.x) \\ \downarrow{\scriptstyle Ff|Lc.x} & & \downarrow{\scriptstyle T(Ff|Lc.x)} \\ Ld.Gf.x & \xrightarrow[\tau(Ld.Gf.x)]{} & T(Ld.Gf.x) \end{array}$$

(the right arrow is also $TFf|T(Lc\,.\,x)$). The point is not to prove that (5.8) commutes, which is easy, but to see for later use that (5.8) is (5.7) evaluated at x.

By definition of S there is an arrow $(Gf, \gamma): (Gc, P) \to (Gd, Q)$ of S for which by (5.4) the following diagram commutes. The horizontal arrows are the inclusions of (5.2).

(5.9)
$$\begin{array}{ccc} T(Lc.x) & \rightarrowtail & KP.x \\ \downarrow{\scriptstyle T(Ff|Lc.x)} & & \downarrow{\scriptstyle K(\gamma x)} \\ T(Ld.Gf.x) & \rightarrowtail & KQ.Gf.x \end{array}$$

By (2.14)°, $\Lambda^\circ(Gf, Ff) = (Gf, \alpha_{Ff})$ (a **Scat**-morphism from Lc to Ld), where $\alpha_{Ff}: Lc \to Ld\circ Gf$ is a natural transformation whose component at an object x of Lc is $\alpha_{Ff}\,.\,x = Ff|Lc\,.\,x$. Then putting (5.8) and (5.9) together yields a commutative diagram

(5.10)
$$\begin{array}{ccc} Lc & \overset{i_c}{\rightarrowtail} & K\circ P \\ \downarrow{\scriptstyle \alpha_{Ff}} & & \downarrow{\scriptstyle K\gamma} \\ Ld\circ Gf & \underset{i_d Gf}{\rightarrowtail} & K\circ Q\circ Gf \end{array}$$

of functors and natural transformations with i_c, i_d monic. This yields a **Scat**-diagram

(5.11)
$$\begin{array}{ccc} Lc & \overset{(id_{Gc}, i_c)}{\rightarrowtail} & K\circ P \\ \downarrow{\scriptstyle \Lambda^\circ(Gf,Ff)} & & \downarrow{\scriptstyle (Gf,K\gamma)} \\ Ld & \underset{(id_{Gd}, i_d)}{\rightarrowtail} & K\circ Q. \end{array}$$

Applying the functor SD° then yields a diagram of categories and functors whose left vertical arrow is $SD^\circ(\Lambda^\circ(Gf, Ff)) = (Gf, Ff)$: $SD^\circ(Lc) \to SD^\circ(Ld)$, the same as the right vertical arrow in (5.6). Pasting the front face of (5.6) and (5.11) together yields

(5.12)
$$\begin{array}{ccc} Fc & \rightarrowtail & SD^\circ(K \circ P) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle SD^\circ(Gf, K\gamma)} \\ Fd & \rightarrowtail & SD^\circ(K \circ Q). \end{array}$$

Now to complete the proof of Theorem 4.1. By (5.5), the left vertical arrow in (5.12) is $(F \circ H)(Gf, v)$. By the definition of wreathing functors in Section 3 (warning — the G there is I_G here, the f there is Gf), the right vertical arrow is $SD^\circ(Gf, K\gamma) = SD^\circ(\bar{K}(Gf, \gamma)) = K \text{ wr } I_G(Gf, \gamma)$. Thus $F \circ H$ is isomorphic to a subfunctor of the restriction of $K \text{ wr } I_G$ to **S**, as required.

6. Applications of coordinate systems. If the actions in Theorem 4.1 are discrete (F and G are set-valued), there is no requirement on λ except that it be a natural transformation. Then the category **Fib**(λ) has only arrows corresponding to the horizontal arrows in (4.1). In *any* case, if λ is split, **Fib**(λ) itself, with $K = T$ the inclusion of **Fib**(λ) into **Cat**, is a coordinate system; in CS—1, $\Phi_c = \{Lc\}$ where Lc is the splitting, and in CS—2, $m = Ff$. Thus we have the following corollary, in which I_F is the inclusion of **Fib**(λ) in **Cat** and I_G the inclusion of Im G in **Cat**.

Corollary 6.1. *If F: **C**$\to$**Cat** is faithful, G: **C**$\to$**Cat**, and λ: $F \to G$ a split natural transformation, then there is a subcategory* **S** *of* **Fib**(λ) wr$^{I_G}\bar{G}$ *for which F is isomorphic in* **Scat** *to the restriction of I_F wr I_G to* **S**.

Corollary 6.2. *If F: **C**$\to$**Set**, G: **C**$\to$**Set** and λ: $F \to G$ is any natural transformation, then the conclusion to Corollary* 6.1 *holds.*

The preceding corollary, when **C** is a group, could be called the *natural* Kaloujnine—Krasner theorem. It embeds **C** into a groupoid. The Kaloujnine—Krasner embedding into a group is obtained by constructing an *unnatural* typing functor which identifies all the fibers with one by noncanonical isomorphisms.

If F, G are set-valued one can always construct a coordinate system which is minimal (in states) but excessively large in transitions this way: let γ be any set whose cardinality is the supremum of the cardinalities of all the sets $Lc \,.\, x$, and the typing functor T a collection of injections of $Lc \,.\, x$ into Y. Let **M** be Trans Y, the monoid of all transformations of Y, with K its natural action. This yields

Corollary 6.3. *If F, G, λ are as in Corollary* 6.2, *then there is a subcategory* **S** *of* Trans Y wr^{I_G} $\bar{G}$ *and a triangle-lifting functor H: **S**$\to$**C** for which $F \circ H$ is isomorphic to a subfunctor of K wr I_G, where K is the action of* Trans Y *on Y.*

A more complicated construction leads to a decomposition via a subfunctor instead of a quotient functor; nevertheless it is an application of Theorem 4.1.

Some concepts are necessary. A functor $F\colon \mathbf{C}\to\mathbf{Cat}$ is *separated* if for distinct objects c, c' of $\mathbf{C}$, c is not an object of Fc and $Fc\cap Fc'$ is empty. Every functor $F\colon \mathbf{C}\to\mathbf{Cat}$ is isomorphic in $\mathbf{Func}(\mathbf{C}, \mathbf{Cat})$ to a separated one. (In mathematical practice people commonly assume implicitly that set-valued functors are separated.) A *transversal* of a separated functor $F\colon \mathbf{C}\to\mathbf{Cat}$ is a function Y with domain the objects of $\mathbf{C}$ such that Yc is an object of Fc. Any separated functor has a transversal by the axiom of choice.

If $\mathbf{D}$ is a subcategory of $\mathbf{Cat}$, the *constant completion* of $\mathbf{D}$, denoted $\mathbf{D}^c$, is the category whose objects are the objects of $\mathbf{D}$ and whose arrows are the arrows of $\mathbf{D}$ plus all constant functors $K_y^A\colon A\to B$, where A, B are objects of $\mathbf{D}$ and y is an object of B.

Let $F, H\colon \mathbf{C}\to\mathbf{Cat}$ be functors with H a subfunctor of F. H is *isolated* in F if for each object c of $\mathbf{C}$, Hc is the union of one or more connected components of Fc. Thus if $u\colon x\to y$ in Fc and either x or y is an object of Hc then u is an arrow of Hc. Note that if F, H are set valued then H is automatically isolated.

If H is isolated in F and F is separated then $F/H\colon \mathbf{C}\to\mathbf{Cat}$ is the functor defined by

(6.1) $$(F/H)c = (Fc-Hc)\{c\} \quad \text{for } c \text{ an object of } \mathbf{C}$$

(remember $\{c\}$ is the trivial category with object c), and for $f\colon c\to c'$ in $\mathbf{C}$,

(6.2) $$(F/H)f.x=\begin{cases} c' & \text{if } x=c \text{ or } Ff.x \text{ is in } Hc' \\ F.fx & \text{otherwise.}\end{cases}$$

There is a natural transformation $\lambda_H\colon F\to F/H$, easily seen to be split, defined by

(6.3) $$\lambda_H c.y=\begin{cases} c & \text{if } y \text{ is in } Hc \\ y & \text{otherwise.}\end{cases}$$

Proposition 6.4. *Let* $F\colon \mathbf{C}\to\mathbf{Cat}$ *be a separated functor with isolated subfunctor* H. *Then there is a subcategory* $\mathbf{S}$ *of* $(\mathrm{Im}\, H)^c\, \mathrm{wr}^I(F/H)$ *and a triangle-lifting functor* $H\colon \mathbf{S}\to\mathbf{C}$ *for which* $H\circ F$ *is isomorphic to a subfunctor of* J wr I, *where* J *is the inclusion of* $(\mathrm{Im}\, H)^c$ *in* $\mathbf{Cat}$ *and* I *the inclusion of* $\mathrm{Im}\,(F/H)$ *in* $\mathbf{Cat}$.

Proof. The objects of $\mathbf{Fib}(\lambda_H)$ are (a) the categories Hc for object c of $\mathbf{C}$, and (b) the categories $\{x\}$ where x is an object of Fc not in Hc. Arrows are of the form (a) $Hf\colon Hc\to Hd$ for arrows $f\colon c\to d$ in $\mathbf{C}$, and (b) $\{x\}\to\{y\}\to\{Ff.y\}$ where $u\colon x\to y$ is an arrow of Fc not in Hc and $f\colon c\to d$ in $\mathbf{C}$. Arrows of type (a) do not compose with arrows of type (b) in either order. Thus $Lc.c=Hc$, $Lc.x=\{x\}$ for

x an object of $Fc-Hc$, and for $f\colon c\to d$, $(Lc\,.f)c=Hf$, $(Lc\,.f)u=\{Ff\,.\,x\}\to\{Ff\,.\,y\}$ for $u\colon x\to y$ in $Fc-Hc$.

Define a typing functor T as follows. For objects Hc of $\mathbf{Fib}(\lambda_H)$, $T(Hc)=Hc$. For objects $\{x\}$ where x is an object of $Fc-Hc$, $T\{x\}=\{Yc\}$. For arrows $Hf\colon Hc\to Hd$, $T(Hf)=Hf$. For arrows $g\colon \{x\}\to\{y\}\to\{Ff\,.\,y\}$ where $u\colon x\to y$ in $Fc-Hc$ and $f\colon c\to d$ in $\mathbf{C}$, $Hg=\{Yc\}\to\{Yd\}$.

Then $((\operatorname{Im} H)^c, J, T)$ is a coordinate system. For CS—1, let $\Phi_c=\{K_{Hc}^{(F/H).c}\}$. For CS—2, let $f\colon c\to d$ in $\mathbf{C}$ and x be an object of $(F/H)c$. If $x=c$ set $m=Hf\colon Hc\to Hd$. If $x\in Fc-Hc$ and $Ff\,.\,x$ is in $Fd-Hd$, set $m=K_{Ff.x}^{Hc}$. If $Ff\,.\,x$ is in Hd, set $m=K_{Yd}^{Hc}$. It is straightforward to verify that CS—2 holds for this definition. The proposition now follows from Theorem 4.1.

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Wreath product decomposition of categories. II*)

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1. Introduction. In this paper, I prove a theorem which shows how to decompose a functor F: $\mathbf{C} \to \mathbf{Cat}$ into the wreath product of two functors, given a right ideal and a "wide" subcategory of $\mathbf{C}$ which together generate $\mathbf{C}$ (this is made precise in Section 2).

The decomposition is in the sense of Krohn—Rhodes theory: the functor F is not *embedded* in a wreath product, but rather a subfunctor of the wreath product maps *onto* F, like a covering space. This is in contrast to the decomposition theorem of WELLS [4], although of course any embedding is an example of decomposition in the present sense. The theorem in this paper actually generalizes one of the decomposition techniques used in proving the Krohn—Rhodes Theorem (KROHN—RHODES [2], EILENBERG [1], WELLS [3]), although it works just as well for infinite categories. Note that one of the corollaries of the decomposition theorem in WELLS [4] generalizes another of the techniques used in proving the Krohn—Rhodes Theorem.

My hope is that the decomposition techniques described here and in WELLS [4] will be useful in developing a theory of "state-transition systems with structured, typed states". This is discussed in WELLS [4] so I will say no more about it here.

The present paper is self-contained except for the terminology developed in Section 2.3 of WELLS [4].

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2. Statement of the theorem. If $\mathbf{C}$, $\mathbf{D}$ are categories and x an object of $\mathbf{D}$, the *constant functor* $K_x^{\mathbf{C}}$ takes all objects of $\mathbf{C}$ to x and all arrows to 1_x. The *constant completion* of a subcategory $\mathbf{A}$ of $\mathbf{Cat}$ consists of the subcategory of $\mathbf{Cat}$ consisting of everything in $\mathbf{A}$ and all constant functors K_x^a where a is an object of $\mathbf{A}$ and x is an object of some object of $\mathbf{A}$.

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If $\mathbf{C}$ is a small category, the *global* hom *functor* $\mathbf{C}_*: \mathbf{C} \to \mathbf{Cat}$ takes an object c to the set of all arrows into c, and $f: c \to d$ to the function from $\mathbf{C}_* c$ to $\mathbf{C}_* d$ which takes $x: a \to c$ to $f \circ x$. C_* is set valued, regarded as a discrete-category-valued functor.

The *constant completion* of a small category $\mathbf{C}$, denoted $\mathbf{C}^c$, is the constant completion in the sense defined earlier of the image of $\mathbf{C}_*$. $\mathbf{C}_*$ is injective, and I shall identify $\mathbf{C}$ with its image, so that $\mathbf{C}_* f: \mathbf{C}_* c \to \mathbf{C}_* d$ *is* $f: c \to d$. I shall write K^c_x for $K^{\mathbf{C}_* c}_x$. This has the following notational consequences:

a) $K^c_x: c \to d$ where x is an arrow with codomain d. (The notation does not determine dom x.)

b) If $K^c_x: c \to d$ and $g: d \to e$ then $g \circ K^c_x = K^c_{g \circ x}$.

c) If $K^c_x: c \to d$ and $h: b \to c$ then $K^c_x \circ h = K^b_x$.

d) If it is defined, $K^d_y \circ K^c_x = K^c_y$.

The inclusion $\mathbf{C}_*: \mathbf{C}^c \to \mathbf{Cat}$ is denoted $d_{\mathbf{C}}$.

A subclass I of arrows of a category $\mathbf{C}$ is a *right ideal* if for any arrow f of $\mathbf{C}$ and g of I, if $g \circ f$ is defined then it is in I. An example of a right ideal is any Grothendieck topology on $\mathbf{C}$. If I is a right ideal (which need not be a subcategory of $\mathbf{C}$), I^1 denotes the subcategory consisting of all objects and identity arrows of $\mathbf{C}$ and all arrows of I.

A subcategory $\mathbf{D}$ of $\mathbf{C}$ is *wide* if it has the same objects as $\mathbf{C}$. If $\mathbf{C} = \mathbf{D} \circ I$ for some subcategory $\mathbf{D}$ and right ideal I then $\mathbf{C}$ is *generated* by $\mathbf{D}$ and I. A functor $H: \mathbf{A} \to \mathbf{B}$ *lifts triangles* if for all arrows f of $\mathbf{A}$ and h, k of $\mathbf{B}$ for which $Hf \circ h$ and $k \circ Hf$ are defined, there are arrows u, v of $\mathbf{A}$ for which $f \circ u$ and $v \circ f$ are defined and $Hu = h$, $Hv = k$. The motivation for requiring this property in wreath product decompositions is discussed in WELLS [4, § 4].

Theorem. *Let* $\mathbf{C}$ *be a small category and* $G: \mathbf{C} \to \mathbf{Cat}$ *a functor. Let* $\mathbf{D}$ *be a wide subcategory and* I *a right ideal which generate* $\mathbf{C}$. *Then there is a subcategory* $\mathbf{S}$ *of* I^1 wr $\mathbf{D}^c$ *(action by* $J_{\mathbf{D}}$*), a triangle-lifting functor* $H: \mathbf{S} \to \mathbf{C}$ *and a surjective natural transformation*

$$\theta: W \to G \circ H \quad \textit{where} \quad W = [(G|I^1) \text{ wr } J_{\mathbf{D}}]|\mathbf{S}.$$

Note. This theorem cannot be strengthened to make $G \circ H$ a subfunctor of W, even when G is set valued and the categories are all monoids.

3. Proof of the Theorem. For an object c of $\mathbf{C}$, let $\delta^c: \mathbf{D}_* c \to I^1$ be the function taking an arrow to its domain, and $i^c: \mathbf{D}_* c \to I^1$ the function taking an arrow to the identity arrow of its domain.

Define $\mathbf{S}$ as follows. An object of $\mathbf{S}$ is any pair (c, δ^c) for any object c of $\mathbf{C}$. Arrows are of the following two forms.

$$(f, i^b): (b, \delta^b) \to (c, \delta^c) \tag{3.1}$$

for all arrows $f\colon b \to c$ in $\mathbf{D}$, and

$$(K_g^c, \mathbf{C}_* h | \mathbf{D}_* c)\colon (c, \delta^c) \to (e, \delta^e) \tag{3.2}$$

for all $h\colon c \to d$ in I^1 and $g\colon d \to e$ in $\mathbf{D}$.

Let's check that (3.2) makes sense ((3.1) is easier). An arrow of I^1 wr $\mathbf{D}^c$ must by definition be of the form $(f, \lambda)\colon (c, P) \to (d, Q)$ where $f\colon c \to d$, $P\colon \mathbf{D}_* c \to I^1$, $Q\colon \mathbf{D}_* d \to I^1$, and $\lambda\colon P \to Q \circ J_{\mathbf{D}} f$ is a natural transformation (note that $\mathbf{D}_*$ is discrete so there are no commutativity conditions for natural transformations here). Here, $K_g^c\colon \mathbf{D}_* c \to \{g\} \subset \mathbf{D}_* e$. For an object $f\colon b \to c$ of $\mathbf{D}_* c$ the component of the natural transformation must be an arrow from $\delta^c f = b$ to $(\delta^e \circ K_g^c) f = \delta^e g = d$. This works because $\mathbf{C}_* h \,.\, f = h \circ f\colon b \to d$.

Define the functor $H\colon \mathbf{S} \to \mathbf{C}$ by $H(f, i^b) = f$ and $H(K_g^c, \mathbf{C}_* h) = g \circ h$.

We have the following formulas for composition of arrows in $\mathbf{S}$, which prove that H is a functor. H is bijective on objects, so lifts triangles.

$$(g, \delta^c) \circ (f, \delta^b) = (g \circ f, \delta^b) \tag{3.3}$$

for $f\colon b \to c$, $g\colon c \to d$ in $\mathbf{D}$

$$(K_g^c, \mathbf{C}_* h) \circ (f, \delta^b) = (K_g^b, \mathbf{C}_*(h \circ f)) \tag{3.4}$$

for $f\colon b \to c$, $h\colon c \to d$, $g\colon d \to e$ in $\mathbf{D}$.

$$(g, \delta^d) \circ (K_m^b, \mathbf{C}_* k) = (K_{g \circ m}^b, \mathbf{C}_* k) \tag{3.5}$$

for $k\colon b \to c$ in I, $m\colon c \to d$, $g\colon d \to e$ in $\mathbf{D}$.

$$(K_n^e, \mathbf{C}_* m) \circ (K_g^c, \mathbf{C}_* h) = (K_n^c, \mathbf{C}_*(m \circ g \circ h)) \tag{3.6}$$

for $h\colon c \to d$, $m\colon e \to p$ in I, $g\colon d \to e$, $n\colon p \to q$ in $\mathbf{D}$.

To simplify notation in the definition of θ, the component of θ at an object (b, δ^b) of $\mathbf{S}$ will be denoted θb. First note that for each object b, $W(b, \delta^b)$ is the disjoint union of categories Ga indexed by all arrows $f\colon a \to b$ of $\mathbf{D}$. This follows from the definition of the wreath product of functors in WELLS [4, §3]: An object of $W(b, \delta^b)$ is a pair (f, x) with $f\colon a \to b$ (some a) and x an object of Ga. An arrow has to look like $(f, r)\colon (f, x) \to (f, y)$ where $r\colon x \to y$ in Ga, $f\colon a \to b$ in $\mathbf{D}$, since $\mathbf{D}_* b$ is a set (discrete category).

Now, to define the component $\theta b\colon W(b, \delta^b) \to G \circ H(b, \delta^b) = Gb$, set

$$\theta b.(f, r) = Gf.r, \tag{3.7}$$

for $f\colon a \to b$ in $\mathbf{D}$, r an arrow of Ga.

To prove that θ is a natural transformation requires (after applying the definition of H) proving the following diagrams commute.

$$\begin{array}{ccc} W(b, \delta^b) & \xrightarrow{\theta b} & Gb \\ {\scriptstyle W(g, \delta^b)}\downarrow & & \downarrow{\scriptstyle Gg} \\ W(c, \delta^c) & \xrightarrow[\theta c]{} & Gc, \end{array} \tag{3.8}$$

for $g\colon b\to c$ in $\mathbf{D}$, and

$$(3.9)\qquad \begin{array}{ccc} W(b,\delta^b) & \xrightarrow{\theta b} & Gb \\ {\scriptstyle W(K_g^b,\,C_* h)}\downarrow & & \downarrow{\scriptstyle G(g\circ h)} \\ W(c,\delta^c) & \xrightarrow[\theta c]{} & Gc \end{array}$$

for $h\colon b\to c$ in I and $g\colon c\to d$ in $\mathbf{D}$.

These facts follow from an easy application of the definitions. Given $f\colon a\to b$ in $\mathbf{D}$ and starting at the upper left corner of (3.8), the northeast route gives $(f,r)\mapsto (Gf).r\mapsto (Gg\circ Gf).r$ and the southwest route gives $(f,r)\mapsto (g\circ f,r)\mapsto G(g\circ f).r$. For (3.9) the corresponding chases are $(f,r)\mapsto Gf.r\mapsto \big(G(g\circ h)\circ Gf\big).r$ and $(f,r)\mapsto \big(g, G(h\circ f).r\big)\mapsto \big(Gg\circ G(h\circ f)\big).r$.

This proves the Theorem.

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On a geometric problem concerning discs

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Introduction

Let $(X, \|\cdot\|)$ be an n-dimensional real normed linear space, and let d be a metric defined on $B_X(0, 1) \equiv \{x \in X: \|x\| \leqq 1\}$ with the following properties:

(i) d is topologically equivalent with $\|\cdot\|$,

(ii) $d(x_1, x_2) = \|x_1 - x_2\|$ for all $1 = \|x_1\| = \|x_2\|$.

At first glance, one can conjecture that there will exist an $y^* \in B_X(0, 1)$ such that

$$\min_{x \in S_X(0,1)} d(y^*, x) \geqq 1.$$

In case of $n=1$, this is an easy consequence of the triangle inequality.

The aim of this paper is to show that in general, this is not the situation. For arbitrary $n \geqq 2$, we construct an example d and $(X, \|\cdot\|)$ for which

$$\max_{y \in B_X(0,1)} \min_{x \in S_X(0,1)} d(y, x) < 1.$$

On the contrary, we prove that

$$\max_{y \in B_X(0,1)} \min_{x \in S_X(0,1)} d(y, x) \geqq \frac{1}{n}.$$

Results

Example. Let $n \geqq 2$. Then there exists a metric d on the n-dimensional euclidean unit ball $E(0, 1)$ such that d has properties (i) and (ii), and

$$\max_{y \in E(0,1)} \min_{x \in S(0,1)} d(y, x) < 1.$$

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Contsruction of d. Let us recall first that there exists a norm $\|\cdot\|$ on $\mathbf{R}^n \oplus \mathbf{R}$ with the following properties (here $|\cdot|$ denotes the euclidean norm in $\mathbf{R}^n$):

(a) $$\|(x, 0)\| = |x| \quad \text{for all} \quad x \in \mathbf{R}^n,$$

(b) $$\|0, \lambda)\| = |\lambda| \quad \text{for all} \quad \lambda \in \mathbf{R},$$

(c) for any projection P: $\mathbf{R}^n \oplus \mathbf{R} \to \mathbf{R}^n$ onto, there holds

$$\|P\| = \sup\{\|P(x, \lambda)\|: x \in \mathbf{R}^n, \lambda \in \mathbf{R}, \|(x, \lambda)\| \leqq 1\} \geqq 1+\delta_0$$

for some fixed $\delta_0 > 0$.

Several types of such norms can be constructed. For example, the existence of such a norm is a consequence of [1].

We shall define now the metric d_α on the set $E(0, 1)$. For $y_1, y_2 \in E(0, 1)$, let

$$d_\alpha(y_1, y_2) = \|h_\alpha(y_1) - h_\alpha(y_2)\|,$$

where h_α: $\mathbf{R}^n \to \mathbf{R}^n \oplus \mathbf{R}$ is defined by

$$h_\alpha(y) = \begin{cases} \left(y, \dfrac{\alpha}{\alpha-1}(|y|-1)\right) & \text{if} \quad \alpha \leqq |y| \leqq 1 \\ (y, \alpha) & \text{if} \quad |y| \leqq \alpha \end{cases}$$

and the contant $0 < \alpha < 1$ is to be specified later. Clearly, d_α has the desired properties (i) and (ii).

We shall show now that for all $y \in E(0, 1)$

(1) $$\min_{x \in S(0, 1)} d_\alpha(y, x) < 1$$

provided that α is sufficiently small.

Firstly, let $|y| > \alpha$. For $x = \dfrac{y}{|y|}$, there holds $|x| = 1$ and

$$d_\alpha(y, x) = \|h_\alpha(y) - h_\alpha(x)\| = \left\|\left(y, \frac{\alpha}{\alpha-1}(|y|-1)\right) - \left(\frac{y}{|y|}, 0\right)\right\| =$$

$$= \left\|\left(0, \frac{\alpha}{\alpha-1}(1-|y|)\right) + \left(1-\frac{1}{|y|}\right)(y, 0)\right\| \leqq$$

$$\leqq \frac{\alpha}{1-\alpha}(1-|y|) + \left(\frac{1}{|y|} - 1\right)|y| = \frac{1-|y|}{1-\alpha} < 1.$$

Secondly, let $|y| \leqq \alpha$. We have a P: $\mathbf{R}^n \oplus \mathbf{R} \to \mathbf{R}^n$ onto projection for which

$$\operatorname{Ker} P = \{\lambda(y, 1): \lambda \in \mathbf{R}\}.$$

By (c), there exists an $(x, c)\in \mathbf{R}^n \oplus \mathbf{R}$ satisfying

$$\|(x, c)\| = 1, \quad \|P(x, c)\| \geqq 1+\delta_0. \tag{2}$$

Clearly we have $P(x, c)=(x-cy, 0)$, so

$$\|(x-cy, 0)\| = |x-cy| \geqq 1+\delta_0. \tag{3}$$

For $z=-\dfrac{x-cy}{|x-cy|}$ there holds $|z|=1$ and

$$d_\alpha(z, y) = \|h_\alpha(z)-h_\alpha(y)\| = \left\|\left(\frac{cy-x}{|x-cy|}, 0\right)-(y, \alpha)\right\| =$$

$$= \left\|\left(\frac{cy-x}{|x-cy|}, 0\right)-\alpha\left(\frac{y}{\alpha}, 1\right)\right\| =$$

$$= \left\|-\frac{\alpha}{c}(x, c)+\frac{c-|x-cy|\alpha}{c}\cdot\left(\frac{cy-x}{|x-cy|}, 0\right)\right\| \leqq$$

$$\leqq \frac{\alpha}{|c|}+\frac{|c-|x-cy|\alpha|}{|c|}.$$

At the last step, we have used (2).

In case of $0<\alpha<\dfrac{|c|}{|x-cy|}$, we have by (3)

$$\frac{\alpha}{|c|}+\frac{|c-|x-cy|\alpha|}{|c|} = 1-\frac{|x-cy|-1}{|c|}\cdot\alpha < 1-\delta_0\frac{\alpha}{|c|},$$

so

$$d_\alpha\left(\frac{-x+cy}{|x-cy|}, y\right) < 1-\delta_0\frac{\alpha}{|c|}. \tag{4}$$

Pick a $\beta>0$. By elementary compactness arguments, $x\in\mathbf{R}^n$ and $c\in\mathbf{R}$ (satisfying condition (2)) can be chosen so that

$$c_1 < |c| < c_2 \quad \text{and} \quad |x-cy| < c_3, \tag{5}$$

for some fixed $c_1, c_2, c_3>0$ whenever $|y|\leqq\beta$.

It is clear that (5) implies (1) provided that

$$\alpha < \min\{\beta, c_1/c_3\}.$$

Theorem. *Let* $(X, \|\cdot\|)$ *be a real n-dimensional linear space, d a metric on* $B_X(0, 1)$ *with properties* (i) *and* (ii). *Then there exists an* $y^*\in B_X(0, 1)$ *such that*

$$\min_{x\in B_X(0,1)} d(y^*, x) \geqq \frac{1}{n}.$$

We shall need the following two lemmas.

Lemma 1. *Let* $(X, \|\cdot\|)$ *and* $(X_1, \|\cdot\|_1)$ *be n-dimensional real normed linear spaces. Then there exists a* $T\colon X\to X_1$ *linear onto operator such that* $\|T^{-1}\|\leqq 1$, *and* $\|T\|\leqq n$.

Lemma 2. *Let* $(Z, \|\cdot\|_\infty)$ *be the n-dimensional* l_∞ *space,* $Y_1\subset B_X(0, 1)$ *and let us assume we have a nonexpansive mapping* $g\colon Y_1\to\{z\in Z\colon \|z\|\leqq r\}$ ($r>0$ *arbitrary*). *Then there exists a nonexpansive*

$$\tilde g\colon B_X(0, 1)\to\{z\in Z\colon \|z\|\leqq r\}\quad \textit{with}\quad \tilde g|_{B_X(0,1)} = g.$$

(A special case of [2] p. 48. Theorem 11.2.)

Now, let us prove the theorem. First, by Lemma 1, there exists a $T\colon X\to Z$ linear onto mapping such that $\|T\|\leqq n$, $\|T^{-1}\|\leqq 1$. Let us introduce now the metric d^* on $B_X(0, 1)$ as follows:

$$d^*(y_1, y_2) = n\cdot d(y_1, y_2). \tag{6}$$

Clearly $T=g$ restricted to the set $S_X(0, 1)$ is nonexpansive from (Y_1, d^*) to $Z_1= =\{z\in Z\colon 1\leqq\|z\|_\infty\leqq n\}$. So, using Lemma 2, we have a

$$\tilde g\colon B_X(0, 1)\to\{z\in Z\colon \|z\|_\infty\leqq n\}$$

nonexpansive extension of g.

Since $T^{-1}\tilde g$ maps $B_X(0, 1)$ into itself and $T^{-1}\tilde g$ restricted to $S_X(0, 1)$ is the identity, it follows from Borsuk's nonrecractibility theorem that $0_X\in T^{-1}\tilde g\ (B_X(0, 1))$. Consequently, $0_Z\in\tilde g(Y)$. Clearly,

$$\min_{z_1\in Z_1}\|0_Z-z_1\|_\infty\geqq 1,$$

so, for arbitrary element y^* of $\tilde g^{-1}(0_Z)$, there holds

$$\min_{z_1\in Z_1} d^*(y^*, \tilde g^{-1}(z_1))\geqq 1,$$

and this implies

$$\min_{x\in S_X(0,1)} d^*(y^*, z)\geqq 1.$$

Using (6), we obtain the desired result.

Remark 1. Instead of $1/n$ we can write 1 in the Theorem provided that $(X, \|\cdot\|)=(Z, \|\cdot\|_\infty)$.

Remark 2. Considerations similar to the ones used in the paper play an interesting role in the theory of Liapunov functions [3], and of metrics of Liapunov type [4].

Remark 3. The infinite dimensional analog of the Theorem does not hold. There exist examples d with

$$\inf_{x \in S_X(0,1)} \sup_{y^* \in Y} d(y^*, x) = 0,$$

for arbitrary $(X, \|\cdot\|)$ real, infinite dimensional, normed linear space, where d has properties (i) and (ii).

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A note on D_p spaces

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In a recent paper [3] the author has introduced a new class of topological spaces, called D_p spaces. The purpose of this paper is to obtain some new characterizations of D_p spaces.

1. Preliminaries. Throughout the present paper, spaces will always mean topological spaces on which no separation axioms are assumed unless explicitely stated.

Definition 1.1. A space X is *paracompact* iff every open covering of X has an open locally finite refinement, [1].

Definition 1.2. Let X be a space and A a subset of X. The set A is *α-paracompact* iff every X-open cover of A has an X-open X-locally finite refinement which covers A, [8].

Definition 1.3. A subset A of a space X is *α-regular* iff for any point $a\in A$ and any X-open set containing a there exists an X-open set V such that $a\in V\subset\bar{V}\subset U$, [4].

Definition 1.4. A space X is D_p iff there exists an α-paracompact subset A such that $\bar{A}=X$, [3].

Theorem 1.1. ([3]) *Let X be a D_p space such that there exists a dense α-regular α-paracompact subset A. Then, every open covering of the set A has a closed locally finite refinement, hence every open covering of X has a locally finite closed refinement.*

Theorem 1.2. ([3]) *Let X be a space such that there is a dense α-regular subset D. If every X-open covering of D has an X-locally finite refinement which covers D, then every X-open covering of D has a closed (in X) X-locally finite refinement.*

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Theorem 1.3. ([3]) *Let X be a space such that there exists a dense α-regular subset D. Then if every X-open covering of D has an X-locally finite refinement which covers D, then D is α-paracompact, i.e. X is paracompact.*

Definition 1.5. An open cover $\mathscr{U}$ is *even* iff there exists a neighbourhood V of diagonal in $X \times X$ such that for each $x \in X$, $V(x) \subset U$ $(V(x) = \{y: (x, y) \in V\})$ for some $U \in \mathscr{U}$, [2].

Theorem 1.4. ([2]) *If the open covering $\mathscr{U}$ has a closed locally finite refinement, then $\mathscr{U}$ is even.*

Theorem 1.5. ([2]) *Let X be a space such that each open cover is even and let $\mathscr{A}$ be a locally finite (or a discrete) family of subsets of X. Then, there is an open neighbourhood V of the diagonal in $X \times X$ such that the family of all sets $V(A)$ $(V(A) = \cup \{V(x): x \in A\})$ for A in $\mathscr{A}$ is locally finite (respectively discrete).*

Theorem 1.6. ([2]) *If every open covering of a space X is even, then any open cover of X has an open σ-discrete refinement.*

Definition 1.6. Let $\mathscr{A}$ be a family of subsets of a space X. The *star* of a point $x \in X$ in $\mathscr{A}$ is defined to be the union of all members of $\mathscr{A}$ which contain x. A family $\mathscr{A}$ of subsets of a space X is said to be *star refinement* of another family $\mathscr{B}$ of subsets of X iff the family of all stars of points of X in $\mathscr{A}$ forms a covering of X which refines $\mathscr{B}$.

Theorem 1.7. ([2]) *Every open covering of a space X is even iff every open covering has an open star refinement.*

Definition 1.7. A family $\mathscr{A}$ of subsets of a space X is called *closure preserving* iff for every subfamily $\mathscr{A}'$ of $\mathscr{A}$ we have $\cup \{\bar{A}: A \in \mathscr{A}'\} = \overline{\cup \{A: A \in \mathscr{A}'\}}$, [5].

Theorem 1.8. ([6]) *Let X be a space such that every open covering of X has a closure preserving closed refinement. Then:*

a) *X is normal;*

b) *Every open covering of X has a σ-discrete open refinement.*

Theorem 1.9. ([2]) *If every open covering of a space X has a σ-locally finite open refinement then, every open covering of X has a locally finite refinement.*

2. Main results.

Lemma 2.1. *Let D be any dense α-regular subset of a space X. If every X-open covering of D has an X-locally finite refinement which covers D, then every open covering of X has an open σ-discrete refinement.*

Proof. By assumption it follows that every open covering of D is open covering of X, hence by Theorem 1.4 it follows that every open covering of X is even. The result follows from Theorem 1.6.

Lemma 2.2. *Let D be any dense α-regular subset of a space X such that every open covering of D is open covering of X. Then, if every open covering of X has a σ-locally finite open refinement, then every open covering of X has a locally finite refinement, hence D is α-paracompact and X is paracompact.*

Proof. The result follows from Theorems 1.9 and 1.3.

Theorem 2.1. *Let D be any dense α-regular subset of a space X such that every open covering of D is open covering of X. Then, the following are equivalent:*

a) *X is paracompact;*
b) *D is α-paracompact;*
c) *every open covering of X has a locally finite closed refinement;*
d) *every open covering of X has a locally finite refinement;*
e) *every open covering of X is even;*
f) *every open covering of X has an open star refinement;*
g) *every open covering of X has a σ-discrete open refinement;*
h) *every open covering of X has a σ-locally finite open refinement.*

Proof. a)$\Leftrightarrow$b): Obvious.
b)$\Rightarrow$c): It follows from Theorem 1.1.
c)$\Rightarrow$d): Obvious.
d)$\Rightarrow$c): It follows from Theorem 1.2.
d)$\Rightarrow$a): It follows from Theorem 1.3.
c)$\Rightarrow$e): It follows from Theorem 1.4.
e)$\Leftrightarrow$f): It follows from Theorem 1.7.
e)$\Rightarrow$g): It follows from Lemma 2.1.
g)$\Rightarrow$h): Obvious.
h)$\Rightarrow$a): It follows from Lemma 2.2.

Corollary 2.1. *For a regular space, the following are equivalent:*

a) *X is paracompact;*
b) *every open covering of X has a locally finite closed refinement;*
c) *every open covering of X has locally finite refinement;*
d) *every open covering of X is even;*
e) *every open covering of X has an open star refinement;*
f) *every open covering of X has a σ-discrete open refinement;*
g) *every open covering of X has a σ-locally finite open refinement.*

The assumption "Every open covering of D is open covering of X" in Theorem 2.1 can not be dropped as can be seen from the following example.

Example 2.1. Let $X=\{a, b, a_i, b_i: i=1, 2, \dots\}$. Let each point a_i be isolated. Let the fundamental system of neighbourhoods of a be the set

$$\{V^n(a): n=1, 2, \dots\} \quad \text{where} \quad V^n(a)=\{a, a_i: i \geqq n\}.$$

Let the fundamental system of neighbourhoods of b be the set

$$\{\{b\} \cup V^n(a): n=1, 2, \dots\}.$$

Let the fundamental system of neighbourhoods of b_i be the set

$$\{U^n(b_i): n=1, 2, \dots\} \quad \text{where} \quad U^n(b_i)=\{b_i, a_j: j \geqq n\}.$$

Let $D=\{a_i: i=1, 2, \dots\}$; D is α-regular. X is not regular at a, hence X is not regular. The subset D is not α-paracompact. X is not paracompact, since the family consisting of the sets

$$V^n(a), \ \{b\} \cup V^n(a), \ U^i(b_i) \quad \text{for all } i \text{ and all } \{a_i\}$$

is open covering of X which admits of no locally finite open refinement. The family consisting of the sets $\{a_i\}$ for all i is an X-open covering of D, but it is not open covering of X. Let

$$\mathcal{U}=\{U_i: i \in I\}$$

be any open covering of X. There exists n such that

$$\{b\} \cup V^n(a) \subset U_i$$

for some $U_i \in \mathcal{U}$. Let $\mathcal{V}_1$ be the family consisting of the sets

$$\{b\} \cup V^n(a), \ \{a_1\}, \{a_2\}, \dots, \{a_{n-1}\}.$$

For any b_i, there exists $n(b_i)$ such that $U^{n(b_i)} \subset U_{i(b_i)}$ for some $i(b_i) \in I$.

Let

$$\mathcal{V}=\{\mathcal{V}_1, U^{n(b_i)}: i=1, 2, \dots\}.$$

$\mathcal{V}$ is σ-locally finite open refinement of $\mathcal{U}$, but X is not paracompact i.e. D is not α-paracompact.

Lemma 2.3. *Let X be a space and D be a dense α-regular subset of X. If for every X-open covering $\mathcal{U}$ of D there exists a closure preserving family $\mathcal{V}$ which refines $\mathcal{U}$ and covers D, then for every X-open covering $\mathcal{A}$ of D there exists a closed closure preserving family $\mathcal{B}$ which refines $\mathcal{A}$ and covers D.*

Proof. Let $\mathcal{U}=\{U_i: i \in I\}$ be any X-open covering of D. Since D is α-regular, for each point $x \in D$, there exists an open set V_x such that $x \in V_x \subset \bar{V}_x \subset U_{i(x)}$ for some $i(x)\in I$. Let $\mathscr{V}=\{V_x\colon x\in D\}$. By assumption, there exists a closure preserving family

$$\mathscr{H}=\{H_j\colon j\in J\},$$

which refines $\mathscr{V}$ and covers D. Then $\{\overline{H}_j\colon j\in I\}$ is a closure preserving closed family which refines $\mathscr{U}$ and covers D.

From this lemma it follows that every open covering of D is an open covering of the space X.

Definition 2.1. A subset A of a space X is T_1 iff every point of A is closed in X.

Lemma 2.4. *Let D be a dense T_1 subset of a space X such that every X-open covering of D has a closed closure preserving refinement. Then, D is α-paracompact i.e. X is paracompact.*

Proof. From Theorem 1.8. it follows that X is normal i.e. D is α-regular. From Theorem 1.8 it follows that every open covering of X (by assumption it follows that every open covering of D is open covering of X) has a σ-discrete open refinement. Now, the result follows from Theorem 2.1.

Theorem 2.2. *Let D be a dense α-regular subset of a space X. Then, the following are equivalent:*

a) *D is α-paracompact;*
b) *every open covering of D has a closure preserving open refinement;*
c) *every open covering of D has a closure preserving refinement;*
d) *every open covering of D has a closure preserving closed refinement.*

Proof. a)⇒b): Every locally finite family is closure preserving.
b)⇒c): Obvious.
c)⇒d): It follows from Lemma 2.3.
d)⇒a): It follows from Lemma 2.4.

Corollary 2.2. ([6]) *For a regular space X, the following are equivalent:*

a) *X is paracompact;*
b) *every open covering of X has a closure preserving open refinement;*
c) *every open covering of X has a closure preserving refinement;*
d) *every open covering of X has a closure preserving closed refinement.*

Corollary 2.3. *Let D be a dense α-regular α-paracompact subset of X. Then, X is normal.*

Proof. From Theorem 2.2 it follows that every open covering of D (hence of X) has a closure preserving closed refinement, hence by Theorem 1.8 it follows that X is normal.

There exists a space with the properties as in Theorem 2.2 which is not regular. The following example will serve the purpose.

Example 2.2. Let $X=\{a, b, a_i: i=1, 2, ...\}$. Let each point a_i be isolated. Let $\{V^n(a): n=1, 2, ...\}$ be the fundamental system of neighbourhoods of a where $V^n(a)=\{a, a_i: i\geqq n\}$.

Let $\{U^n(b): n=1, 2, ...\}$ be the fundamental system of neighbourhoods of b where

$$U^n(b) = \{b, a, a_i: i \geqq n\}.$$

Let $D=\{b, a_i: i=1, 2, ...\}$; D is a dense T_1 (α-regular) α-paracompact subset of X. X is normal, X is not T_1. X is not regular at a, hence X is not regular.

Theorem 2.3. *Let D be a dense T_1 α-paracompact subset of a normal space X. If f is a closed and continuous mapping of the space X onto a space Y, then Y is paracompact.*

Proof. Let D be a dense T_1 α-paracompact subset of a normal space X. Y is normal. Since $f(\bar{D})=\overline{f(D)}=Y$, it follows that $f(D)$ is the dense T_1 (hence α-regular) subset of the normal space Y. Let $\mathscr{U}=\{U_i: i\in I\}$ be any open covering of $f(D)$. Let $\mathscr{W}=\{f^{-1}(U_i): U_i\in\mathscr{U}\}$, it is the open covering of D (hence it is open covering of X). It follows that every open covering of $f(D)$ is an open covering of Y. $\mathscr{W}$ has a closure preserving closed refinement $\mathscr{A}=\{A_j: j\in J\}$.

Then, $f(\mathscr{A})=\{f(A_j): j\in J\}$ is the closure preserving closed refinement of $\mathscr{U}$, hence Y is paracompact.

Corollary 2.4. ([6]) *The image of a Hausdorff paracompact space, under a continuous closed mapping, must be paracompact.*

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Interval filling sequences and additive functions

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1. Introduction. Interval filling sequences have been defined in our paper [1]. Let Λ denote the set of all real sequences, for which the conditions $\lambda_n > \lambda_{n+1} > 0$ $(n \in \mathbf{N})$ and $L := \sum_{n=1}^{\infty} \lambda_n < \infty$ hold.

Definition 1.1. We call the sequence $\{\lambda_n\} \in \Lambda$ *interval filling*, if for any $x \in [0, L]$ there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n \in \{0, 1\}$ $(n \in \mathbf{N})$, such that

$$x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n. \tag{1.1}$$

We have the following result ([1]):

Theorem 1.2. *The sequence $\{\lambda_n\} \in \Lambda$ is interval filling if and only if*

$$\lambda_n \leqq \sum_{i=n+1}^{\infty} \lambda_i \tag{1.2}$$

for any $n \in \mathbf{N}$.

Let $\{\lambda_n\} \in \Lambda$ be an interval filling sequence. For $x \in [0, L\}$ we define by induction on n

$$\varepsilon_n(x) := \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n \leqq x, \\ 0 & \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n > x. \end{cases} \tag{1.3}$$

It is known ([1]) that

$$x = \sum_{n=1}^{\infty} \varepsilon_n(x)\lambda_n. \tag{1.4}$$

We call the representation (1.4) of the number x the *regular* expansion of x.

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Definition 1.3. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence and $a_n\in\mathbf{C}$ such that $\sum_{n=1}^{\infty}|a_n|<\infty$. Then we call the function

$$F(x):=\sum_{n=1}^{\infty}\varepsilon_n(x)\,a_n \quad (x\in[0, L]) \tag{1.5}$$

additive (with respect to the interval filling sequence $\{\lambda_n\}\in\Lambda$), where $\varepsilon_n(x)$ denotes the digits (0, 1) determined by algorithm (1.3).

In this paper we give an exact description of the set of those points in which an additive function is *continuous*. Following this, with the help of quasiregular expansions we give a criterium for the continuity in $[0, L]$ of additive functions. Thus we generalize our results obtained in [2] which referred to special interval filling sequences

$$\lambda_n:=\frac{1}{q^n} \quad (1<q\leqq 2).$$

As to further properties of continuous additive functions, we refer to our result in [2], according which there exist an interval filling sequence and a function F continuous and additive with respect to it, such that this function is nowhere differentiable in $[0, L]$.

In this paper $\{\lambda_n\}\in\Lambda$ will denote an arbitrary but fixed interval filling sequence, even if we do not emphasize it explicitely.

2. Finite numbers. Finite numbers will play a fundamental role in the sequel.

Definition 2.1. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence. We call the number $x\in[0, L]$ *finite*, if there exists $N\in\mathbf{N}$ such that $\varepsilon_n(x)=0$ for $n>N$. If x is finite and $\varepsilon_m(x)=1$ moreover $\varepsilon_n(x)=0$ for $n>m$, then we say that x has *length* m, and write $h(x)=m$. We define $h(0)=0$, i.e. $x=0$ is also a finite number.

Let $N\in\mathbf{N}$ and

$$V_N:=\{t\mid t\in[0, L],\, h(t)\leqq N\} \tag{2.1}$$

the set of finite numbers having length not greater than N. For $0<x\leqq L$ we put

$$b_N(x):=\max\{t\mid t\in V_N,\, t<x\} \tag{2.2}$$

and call this number the *left neighbour* of x in V_n.

Lemma 2.1. *Let* $0<x\leqq L$ *be arbitrary. Then for any* $b_N(x)<y<x$ *we have*

$$\varepsilon_n(y)=\varepsilon_n[b_N(x)] \quad \textit{if} \quad n\leqq N. \tag{2.3}$$

Proof. If $b_N(x)<y<x$ then let

$$y=\sum_{n=1}^{N}\varepsilon_n(y)\lambda_n+\sum_{n=N+1}^{\infty}\varepsilon_n(y)\lambda_n.$$

Clearly

$$S_N(y) := \sum_{n=1}^{N} \varepsilon_n(y)\lambda_n \in V_N.$$

The inequality $b_N(x) < S_N(y) \leqq y < x$ is impossible by the definition of $b_N(x)$. Thus $S_N(y) \leqq b_N(x)$. Now $S_N(y) < b_N(x)$ implies the existence of a first index $k \in \{1, 2, \dots, N\}$ such that $\varepsilon_k(y) = 0$ and $\varepsilon_k[b_N(x)] = 1$. From this, by algorithm (1.3),

$$b_N(x) \geqq \sum_{n=1}^{k-1} \varepsilon_n[b_N(x)] + \lambda_k = \sum_{n=1}^{k-1} \varepsilon_n(y)\lambda_n + \lambda_k > y$$

follows, a contradiction. Thus $S_N(y) = b_N(x)$, and this implies (2.3).

3. Additive functions.

Theorem 3.1. *Let F: $[0, L] \to \mathbf{C}$ be an additive function. Then F is continuous at every nonfinite point x.*

Proof. Let $0 < x < L$ be a nonfinite number. Let $\varepsilon > 0$. Then there exists $N_0 \in \mathbf{N}$ such that

$$2 \sum_{n=N_0+1}^{\infty} |a_n| < \varepsilon.$$

Let $N > N_0$ be such that $x < \sum_{n=1}^{N} \lambda_n \in V_N$ and put

$$j_N(x) := \min \{t \mid t \in V_N, x < t\}.$$

Then $x < j_N(x)$. We assert that

$$b_N[j_N(x)] < x < j_N(x). \tag{3.1}$$

As a matter of fact, $b_N[j_N(x)] \neq x$ because x is nonfinite, and $x < b_N[j_N(x)]$ would contradict the definition of $j_N(x)$.

If $b_N[j_N(x)] < y < j_N(x)$ (i.e. if y is in the neighbourhood (3.1) of x), then by Lemma 2.1

$$\varepsilon_n(y) = \varepsilon_n\{b_N[j_N(x)]\} = \varepsilon_n(x) \quad \text{for} \quad n \leqq N,$$

whence

$$|F(x) - F(y)| = \left|\sum_{n=1}^{\infty} \varepsilon_n(x)a_n - \sum_{n=1}^{\infty} \varepsilon_n(y)a_n\right| =$$

$$= \left|\sum_{n=N+1}^{\infty} [\varepsilon_n(x) - \varepsilon_n(y)\, a_n\right| \leqq 2 \sum_{n=N+1}^{\infty} |a_n| < \varepsilon,$$

i.e. F is continuous at x.

We still have to consider the case $x=L$ (L is a nonfinite number). Here we must prove continuity from the left. Now

$$b_N(L) = \max\{t \mid t\in V_N,\ t < L = x\} = \sum_{n=1}^{N} \lambda_n .$$

Hence, if $b_N(L)<y<L$ then by Lemma 2.1 $\varepsilon_n(y)=1$ for $n\leqq N$. This implies

$$|F(L)-F(y)| = \Big|\sum_{n=N+1}^{\infty} [1-\varepsilon_n(y)]\, a_n\Big| \leqq 2 \sum_{n=N+1}^{\infty} |a_n| < \varepsilon$$

for $N>N_0$, i.e. F is left continuous in $x=L$.

Theorem 3.2. *Let F: $[0, L]\to \mathbf{C}$ be an additive function. Then F is right continuous at every finite point $x\in[0, L]$.*

Proof. Let x be finite and $m=h(x)$. Then for any $\varepsilon>0$ there exists $N>m$ such that

$$\sum_{n=N+1}^{\infty} |a_n| < \varepsilon.$$

Now $x\in V_N$. We have by definitions $b_N[j_N(x)]=x$. Hence by Lemma 2.1 for any

$$x = b_N[j_N(x)] < y < j_N(x)$$

the relation

$$\varepsilon_n(y) = \varepsilon_n\{b_N[j_N(x)]\} = \varepsilon_n(x) \quad (n \leqq N)$$

holds. Hence

$$|F(x)-F(y)| = \Big|\sum_{n=1}^{N} \varepsilon_n(x)\, a_n - \sum_{n=1}^{\infty} \varepsilon_n(y)\, a_n\Big| =$$

$$= \Big|\sum_{n=N+1}^{\infty} \varepsilon_n(y)\, a_n\Big| \leqq \sum_{n=N+1}^{\infty} |a_n| < \varepsilon,$$

i.e. F is right continuous in x.

4. Examples.

Example 4.1. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence. Let moreover $a_1=a_2=1$ and $a_n=0$ for $n>2$. The additive function determined by the sequence a_n is

$$F(x) = \begin{cases} 0 & \text{for} \quad 0 \leqq x < \lambda_2, \\ 1 & \text{for} \quad \lambda_2 \leqq x < \lambda_1+\lambda_2, \\ 2 & \text{for} \quad \lambda_1+\lambda_2 \leqq x \leqq L. \end{cases}$$

Clearly, this function is not continuous at the finite points λ_2, $\lambda_1+\lambda_2$. On the basis of this the question arises, how exact are Theorems 3.1 and 3.2. The answer is given by the following example.

Example 4.2. *There exists with respect to the interval filling sequence* $\left\{\lambda_n := \frac{1}{2^n}\right\} \in \Lambda$ *an additive function* F *which is noncontinuous at every finite point* $x > 0$.

Proof. We have $L := \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ and the algorithm (1.3) yields the unique dyadic respresentation of the numbers $x \in [0, 1]$. The numbers $\frac{l}{2^n}$ $(0 \leqq l < 2^n)$ and only these are finite, any other number is nonfinite. Let $a_n := \frac{1}{n^2}$ for which $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, and let

$$F(x) := \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{n^2}$$

for any $x \in [0, 1]$. Let still $x \in]0, 1[$ be finite and $h(x) = m \geqq 1$. Then

$$x = \sum_{n=1}^{m-1} \frac{\varepsilon_n(x)}{2^n} + \frac{1}{2^m}.$$

Let $N > m$ and

(4.1) $$x_N := \sum_{n=1}^{m-1} \frac{\varepsilon_n(x)}{2^n} + \frac{0}{2^m} + \frac{1}{2^{m+1}} + \dots + \frac{1}{2^N}.$$

Since the right hand side of (4.1) is a regular expansion of x_N, we get

(4.2) $$F(x_N) = \sum_{n=1}^{m-1} \frac{\varepsilon_n(x)}{n^2} + \frac{1}{(m+1)^2} + \dots + \frac{1}{N^2}$$

If F were continuous in x, then $x_N \to x$ would imply $F(x_N) \to F(x)$ $(N \to \infty)$. However from (4.2) we get

$$\lim_{N \to \infty} F(x_N) = \sum_{n=1}^{m-1} \frac{\varepsilon_n(x)}{n^2} + \frac{\pi^2}{6} - \frac{1}{1^2} - \dots - \frac{1}{m^2},$$

and this would imply

$$F(x) = \sum_{n=1}^{m-1} \frac{\varepsilon_n(x)}{n^2} + \frac{1}{m^2} = \lim_{N \to \infty} F(x_N),$$

i.e.

$$\frac{1}{m^2} = \frac{\pi^2}{6} - \frac{1}{1^2} - \dots - \frac{1}{m^2},$$

which is a contradiction, because π^2 is not rational.

5. Quasiregular expansions. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence. For $x\in[0, L]$, by induction on n, let

$$\varepsilon_n^*(x) := \begin{cases} 1 & \text{for} \quad \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i + \lambda_n < x, \\ 0 & \text{for} \quad \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i + \lambda_n \geqq x. \end{cases} \tag{5.1}$$

Theorem 5.1. *For any* $x\in[0, L]$ *we have*

$$x = \sum_{n=1}^{\infty} \varepsilon_n^*(x)\lambda_n. \tag{5.2}$$

Proof. (i): For $x=0$ and $x=L$ (5.2) is trivially valid. (ii): If $0<x<L$ and $\varepsilon_n^*(x)=0$ for infinitely many values of n, then $N_0 := \{n \mid n\in N,\ \varepsilon_n^*(x)=0\}$ is an infinite set. If $n\in N_0$ then

$$0 \leqq x - \sum_{i=1}^{\infty} \varepsilon_i^*(x)\lambda_i \leqq x - \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i \leqq \lambda_n$$

whence by $\lambda_n\to 0$ ($n\in N_0$, $n\to\infty$) (5.2) follows. (iii): If $0<x<L$ and $\varepsilon_n^*(x)=0$ holds only for finitely many values of n, then let N be the greatest index, for which $\varepsilon_N^*(x)=0$ (i.e. $\varepsilon_n^*(x)=1$ if $n>N$). Then

$$x - \sum_{i=1}^{N-1} \varepsilon_i^*(x)\lambda_i \leqq \lambda_N \leqq \sum_{i=N+1}^{\infty} \lambda_i = \sum_{i=N+1}^{\infty} \varepsilon_i^*(x)\lambda_i$$

whence

$$x \leqq \sum_{i=1}^{\infty} \varepsilon_i^*(x)\lambda_i,$$

i.e. (5.2) holds.

Definition 5.2. We call the representation (5.2) the *quasiregular* expansion of x.

Lemma 5.3. *If* $0<x\leqq L$ *then* $\varepsilon_N^*(x)=1$ *for infinitely many values of* n.

Proof. Suppose the contrary, and let N be the largest index with $\varepsilon_N^*(x)=1$. Then

$$x = \sum_{i=1}^{N} \varepsilon_i^*(x)\lambda_i = \sum_{i=1}^{N-1} \varepsilon_i^*(x)\lambda_i + \lambda_N$$

and so by (5.1) $\varepsilon_N^*(x)=0$, a contradiction.

Lemma 5.4. *If* $0<x\leqq L$ *is a nonfinite number, then* $\varepsilon_n(x)=\varepsilon_n^*(x)$ *for every* $n\in N$, *i.e. the regular and quasiregular expansions coincide.*

Proof. Suppose the contrary, and let k be the first index for which $\varepsilon_k(x) \neq \varepsilon_k^*(x)$. By the definitions of $\varepsilon_k(x)$ and $\varepsilon_k^*(x)$ then we have $\varepsilon_k(x)=1$ and $\varepsilon_k^*(x)=0$. Hence

$$\sum_{i=1}^{k-1} \varepsilon_i(x)\lambda_i+\lambda_k \leqq x$$

and

$$\sum_{i=1}^{k-1} \varepsilon_i^*(x)\lambda_i+\lambda_k \geqq x.$$

Now $\varepsilon_i(x)=\varepsilon_i^*(x)$ for $i=1, 2, \ldots, k-1$; hence the previous inequalities yield

$$x = \sum_{i=1}^{k-1} \varepsilon_i(x)\lambda_i+\lambda_k,$$

i.e. x is finite, a contradiction.

Quasiregular expansions make it possible to determine for a number $0<x\leqq L$ its left neighbour $b_N(x)$ (see Definition 2.1), and to describe exactly the regular expansion of the latter. This we formulate in the following statement.

Theorem 5.5. *If* $0<x\leqq L$ *then*

$$b_N(x) = \sum_{n=1}^{N} \varepsilon_n^*(x)\lambda_n, \tag{5.3}$$

where the right hand side is the regular expansion of $b_N(x)$, *i.e.*

$$\varepsilon_n[b_N(x)] = \varepsilon_n^*(x) \quad \textit{for} \quad n = 1, 2, \ldots, N. \tag{5.4}$$

Proof. Suppose that, contradicting our assertion, there exists $z\in V_N$ such that $b_N(x)<z<x$.

(i) If x is *nonfinite*, then its regular and quasiregular expansions coincide. Let $x=\sum_{n=1}^{\infty}\varepsilon_n(x)\lambda_n$. Then $b_N(x)=\sum_{n=1}^{N}\varepsilon_n(x)\lambda_n$. Let $z=\sum_{n=1}^{N}\varepsilon_n(z)\lambda_n$. Since $b_N(x)<z$, there exists a first index $k\in\{1, 2, \ldots, N\}$ such that $\varepsilon_k(x)\neq\varepsilon_k(z)$. This is only possible if $\varepsilon_k(z)=1$ and $\varepsilon_k(x)=0$. Hence

$$z = \sum_{i=1}^{k-1} \varepsilon_i(z)\lambda_i+\lambda_k+\sum_{i=k+1}^{N} \varepsilon_i(z)\lambda_i \geqq$$

$$\geqq \sum_{i=1}^{k-1} \varepsilon_i(z)\lambda_i+\lambda_k = \sum_{i=1}^{k-1} \varepsilon_i(x)\lambda_i+\lambda_k > x,$$

a contradiction.

(ii) If x is *finite*, then let $h(x)=m\geqq 1$, i.e.

$$x = \sum_{n=1}^{m-1} \varepsilon_n(x)\lambda_n+\lambda_m.$$

Then

$$\lambda_m = \sum_{i=1}^{\infty} \varepsilon_i^*(\lambda_m)\lambda_i = \sum_{i=m+1}^{\infty} \varepsilon_i^*(\lambda_m)\lambda_i$$

because $\varepsilon_i^*(\lambda_m)=0$ for $i=1, 2, \ldots, m$. Hence

$$x = \sum_{n=1}^{m-1} \varepsilon_n(x)\lambda_n + \sum_{i=m+1}^{\infty} \varepsilon_i^*(\lambda_m)\lambda_i. \tag{5.5}$$

Clearly, the right hand side of (5.5) is the *quasiregular* expansion of x, i.e.

$$\varepsilon_n^*(x) = \begin{cases} \varepsilon_n(x) & \text{for} \quad n = 1, 2, \ldots, m-1, \\ 0 & \text{for} \quad n = m, \\ \varepsilon_n^*(\lambda_m) & \text{for} \quad n = m+1, m+2, \ldots. \end{cases} \tag{5.6}$$

If $m \geqq N$ then the proof is the same as in (i). If $m<N$, then let $z=\sum_{n=1}^{N} \varepsilon_n(z)\lambda_n$. Now by $b_N(x)<z$ there exists a first index $m<k\leqq N$ such that $\varepsilon_k(z)=1$ and $\varepsilon_k^*(x)=0$. Hence

$$z = \sum_{i=1}^{k-1} \varepsilon_i(z)\lambda_i + \lambda_k + \sum_{i=k+1}^{N} \varepsilon_i(z)\lambda_i \geqq$$

$$\geqq \sum_{i=1}^{k-1} \varepsilon_i(z)\lambda_i + \lambda_k = \sum_{i=1}^{k-1} \varepsilon_i^*(x)\lambda_i + \lambda_k \geqq x,$$

and this contradicts the condition $z<x$.

6. Quasiadditive functions. The notion of quasiadditive function will be defined in analogy to that of additive function.

Definition 6.1. Let $a_n \in \mathbf{C}$ and $\sum_{n=1}^{\infty} |a_n| < \infty$. The function $F\colon [0, L] \to \mathbf{C}$ is said to be *quasiadditive* if

$$F(x) = \sum_{n=1}^{\infty} \varepsilon_n^*(x)\,a_n \tag{6.1}$$

for any $x\in[0, L]$, where $\varepsilon_n^*(x)$ denotes the digits 0, 1 determined by algorithm (5.1).

Remark. If $a_n \in \mathbf{C}$ $\left(\sum_{n=1}^{\infty} |a_n| < \infty\right)$ then this sequence determines an *additive* function (say F_1), and a *quasiadditive* function (say F_2). By Lemma 5.4. $F_1(x)= =F_2(x)$ holds for any *nonfinite* $x\in[0, L]$, and trivially also for $x=0$ and $x=L$. Hence, in general, the two functions differ only at the *finite* points $0<x<L$.

Definition 6.2. We call the function $F\colon [0, L] \to \mathbf{C}$ *biadditive*, if it is both additive and quasiadditive.

Lemma 6.3. *The additive function* F: $[0, L]\to\mathbf{C}$ *determined by the sequence* $a_n\in\mathbf{C}$ $\left(\sum_{n=1}^{\infty}|a_n|<\infty\right)$ *is biadditive if and only if*

(6.2) $$a_n = \sum_{i=n+1}^{\infty} \varepsilon_i^*(\lambda_n)\, a_i$$

is satisfied for every $n\in\mathbf{N}$.

Proof. (i): If F is also quasiadditive, then

$$\lambda_n = \sum_{i=n+1}^{\infty} \varepsilon_i^*(\lambda_n)\,\lambda_i$$

implies

$$a_n = F(\lambda_n) = \sum_{i=n+1}^{\infty} \varepsilon_i^*(\lambda_n)\, a_i,$$

i.e. (6.2) holds. (ii): If (6.2) is valid, then by the foregoing it suffices to show that (6.1) holds for every *finite* number $0<x<L$. Let $h(x)=m\geqq 1$ and

$$x = \sum_{n=1}^{m-1} \varepsilon_n(x)\,\lambda_n+\lambda_m = \sum_{n=1}^{m-1} \varepsilon_n(x)\,\lambda_n+ \sum_{i=m+1}^{\infty} \varepsilon_i^*(\lambda_m)\,\lambda_i.$$

Then by (5.6) we know the quasiregular representation of x, hence using (6.2) we get

$$F(x) = \sum_{n=1}^{m-1} \varepsilon_n(x)\, a_n+a_m =$$

$$= \sum_{n=1}^{m-1} \varepsilon_n(x)\, a_n+ \sum_{i=m+1}^{\infty} \varepsilon_i^*(\lambda_m)\, a_i = \sum_{n=1}^{\infty} \varepsilon_n^*(x)\, a_n,$$

i.e. (6.1) holds.

Lemma 6.4. *If* F: $[0, L]\to\mathbf{C}$ *is additive and continuous in* $[0, L]$, *then* F *is quasiadditive (i.e.* F *is biadditive).*

Proof. The function F is left continuous at every λ_n, where

$$\lambda_n = \sum_{i=n+1}^{\infty} \varepsilon_i^*(\lambda_n)\,\lambda_i.$$

Let $N>n$ and

(6.3) $$b_N(\lambda_n) = \sum_{i=n+1}^{N} \varepsilon_i^*(\lambda_n)\,\lambda_i.$$

Then by Theorem 5.5 the right hand side of (6.3) is a regular expansion and $b_N(\lambda_n)\to\lambda_n$ (for $N\to\infty$), hence by continuity

$$a_n = F(\lambda_n) = \lim_{\substack{N\to\infty\\ N>n}} F[b_N(\lambda_n)] =$$

$$= \lim_{\substack{N\to\infty\\ N>n}} \sum_{i=n+1}^{N} \varepsilon_i^*(\lambda_n)\, a_i = \sum_{i=n+1}^{\infty} \varepsilon_i^*(\lambda_n)\, a_i$$

for every $n\in\mathbf{N}$, i.e. (6.2) holds. From Lemma 6.3. it follows immediately that F is quasiadditive (i.e. biadditive).

Remark. By Lemma 6.4 quasiadditivity is a *necessary* condition for the continuity of an additive function F; also, by Lemma 6.3 it is necessary that for the sequence $a_n\in\mathbf{C}$ $\left(\sum_{n=1}^{\infty}|a_n|<\infty\right)$ the difference equations (6.2) $(n=1, 2, \ldots)$ should be valid.

7. Continuous additive functions.

Theorem 7.1. *An additive function* $F\colon [0, L]\to\mathbf{C}$ *is continuous in* $[0, L]$ *if and only if it is quasiadditive (i.e. biadditive).*

Proof. By Theorems 3.1—3.2 and Lemma 6.4. it will be sufficient to show that if F is also quasiadditive then it is left continuous at every *finite* point $0<x<L$.

For the sequence $a_n\in\mathbf{C}$ determining the additive function F it is clearly true that for any $\varepsilon>0$ there exists N_0 such that $N>N_0$ implies

$$2\sum_{n=N+1}^{\infty}|a_n| < \varepsilon.$$

Let x be finite and $h(x)=m\geqq 1$, i.e.

$$x = \sum_{i=1}^{m-1} \varepsilon_i(x)\,\lambda_i+\lambda_m.$$

If $N>m$ then

$$b_N(x) = \sum_{i=1}^{m-1} \varepsilon_i(x)\,\lambda_i+\sum_{i=m+1}^{N} \varepsilon_i^*(\lambda_m)\,\lambda_i$$

is a regular expansion (Theorem 5.5), and in case $b_N(x)<y<x$ we have by Lemma 2.1

$$\varepsilon_n(y) = \varepsilon_n[b_N(x)] \qquad (n = 1, 2, \ldots, N).$$

Hence by the quasiadditivity of F we get from (6.2)

$$|F(x)-F(y)| = \Big|\sum_{i=1}^{m-1} \varepsilon_i(x)\, a_i + a_m -$$

$$-\sum_{i=1}^{m-1} \varepsilon_i(x)\, a_i - \sum_{i=m+1}^{N} \varepsilon_i^*(\lambda_m)\, a_i - \sum_{i=N+1}^{\infty} \varepsilon_i(y)\, a_i\Big| =$$

$$= \Big|\sum_{i=N+1}^{\infty} [\varepsilon_i^*(\lambda_m)-\varepsilon_i(y)]\, a_i\Big| \leqq 2 \sum_{i=N+1}^{\infty} |a_i| < \varepsilon,$$

i.e. F is left continuous at x.

Corollary. *Let* $a_n \in \mathbf{C}$ $\left(\sum_{n=1}^{\infty} |a_n| < \infty\right)$ *and* $F\colon [0, L] \to \mathbf{C}$ *the additive function determined by the sequence* a_n. *Then for the continuity of* F *in* $[0, L]$ *it is necessary and sufficient that the difference equations* (6.2) *should be valid for every* $n \in \mathbf{N}$.

Remark. For $1<q<2$ and $\lambda_n := 1/q^n$ the previous statement has been proved in [2].

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Absolute summability of double orthogonal series

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Dedicated to Professor B. Sz.-Nagy on his 75th birthday

1. Introduction: Summability of numerical series

We consider a quadruply infinite matrix

$$T = \{t_{ik}^{mn}\colon\ i, k, m, n = 0, 1, \ldots\}$$

of real numbers such that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} |t_{ik}^{mn}| < \infty \qquad (m, n = 0, 1, \ldots). \tag{1.1}$$

Condition (1.1) is trivially satisfied if the matrix T is such that for each m and n there exists an integer $\varkappa_{mn}$ with the property that $t_{ik}^{mn}=0$ whenever $\max(i, k)>\varkappa_{mn}$. In this case T is called *generalized triangular*. In particular, T is called *triangular* if for each m and n we have $t_{ik}^{mn}=0$ whenever at least one of the relations $i>m$ and $k>n$ is satisfied.

With every double series

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} u_{ik} \tag{1.2}$$

of real numbers, we associate a double sequence $\{\sigma_{mn}\}$ given by

$$\sigma_{mn} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} t_{ik}^{mn} u_{ik} \qquad (m, n = 0, 1, \ldots), \tag{1.3}$$

provided the double series on the right converges in the sense of Pringsheim. This is the case if (1.1) is satisfied and the terms u_{ik} of series (1.2) are bounded. We note that in this case the series on the right (1.3) is even absolutely convergent.

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If σ_{mn} tends to a finite limit s as $\min(m, n)\to\infty$ we say that series (1.2) is *T-summable* to the sum s. The σ_{mn} are called the *T-means* of (1.2).

We introduce the following notation:

$$\Delta_{mn} = \sigma_{mn} - \sigma_{m-1,n} - \sigma_{m,n-1} + \sigma_{m-1,n-1} \tag{1.4}$$

with the agreement that

$$\sigma_{-1,n} = \sigma_{m,-1} = \sigma_{-1,-1} = 0 \qquad (m, n = 0, 1, \ldots). \tag{1.5}$$

We say that series (1.2) is *absolutely T-summable* (shortly: $|T|$-*summable*) if

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |\Delta_{mn}| < \infty. \tag{1.6}$$

Clearly, $|T|$-summability implies T-summability. In addition, $|T|$-summability also implies that σ_{mn} converges as $n\to\infty$ for each $m=0, 1, \ldots$ and that σ_{mn} converges as $m\to\infty$ for each $n=0, 1, \ldots$.

2. Main results: Summability of orthogonal series

Let $\varphi=\{\varphi_{ik}(x)\colon\ i, k=0, 1, \ldots\}$ be a real-valued *orthonormal system* (in abbreviation: ONS) defined on a positive measure space $(X, \mathcal{F}, \mu)$. We consider the double *orthogonal series*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}\varphi_{ik}(x), \tag{2.1}$$

where $\{a_{ik}\colon i, k=0, 1, \ldots\}$ is a double sequence of real numbers such that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2 < \infty. \tag{2.2}$$

The T-means of series (2.1) are defined according to (1.3):

$$\sigma_{mn}(x) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} t_{ik}^{mn} a_{ik}\varphi_{ik}(x) \qquad (m, n = 0, 1, \ldots).$$

If conditions (1.1) and (2.2) are satisfied, then $\sigma_{mn}(x)$ is well defined μ-a.s. for each m and n. In fact, it follows from (2.2), via B. Levi's theorem, that

$$\lim_{\max(i,k)\to\infty} a_{ik}\varphi_{ik}(x) = 0 \qquad \mu\text{-a.s.},$$

and, a foriori, the terms $a_{ik}\varphi_{ik}(x)$ are bounded μ-a.s.

We introduce the following notation:

$$\tau_{ik}^{mn} = t_{ik}^{mn} - t_{ik}^{m-1,n} - t_{ik}^{m,n-1} + t_{ik}^{m-1,n-1} \tag{2.3}$$

with the agreement that

(2.4) $$t_{ik}^{-1,n} = t_{ik}^{m,-1} = t_{ik}^{-1,-1} = 0 \qquad (i, k, m, n = 0, 1, \ldots).$$

Theorem 1. *If conditions* (1.1), (2.2) *are satisfied and*

(2.5) $$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\Big\{\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}[\tau_{ik}^{mn}]^2 a_{ik}^2\Big\}^{1/2} < \infty,$$

then series (2.1) *is* $|T|$*-summable* μ*-a.e. on* X.

The surprising fact is that condition (2.5), under a mild assumption on T, is not only sufficient but also necessary for the μ-a.e. $|T|$-summability of series (2.1) if all ONS φ are taken into consideration.

To be more specific, let $(X, \mathcal{F}, \mu)$ be the familiar unit square

$$U = \{x = (x_1, x_2)\colon 0 \leqq x_j \leqq 1 \quad \text{for} \quad j = 1, 2\}$$

with the Borel measurable subsets as $\mathcal{F}$ and with the planar Lebesgue measure as μ. We remind that the ordinary one-dimensional *Rademacher system* $\{r_i(x_1)\}$ is defined as follows

$$r_i(x_1) = \operatorname{sign}\sin(2^i \pi x_1) \qquad (i = 0, 1, \ldots;\ 0 \leqq x_1 \leqq 1)$$

(see, e.g. [1, p. 51] or [15, p. 212]).

Theorem 2. *Assume that conditions* (1.1), (2.2), *are satisfied and*

(2.6) $$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\tau_{ik}^{mn}| < \infty \qquad (i, k = 0, 1, \ldots).$$

If condition (2.5) *is not satisfied, then the two-dimensional Rademacher series*

(2.7) $$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik} r_i(x_1) r_k(x_2)$$

is not $|T|$*-summable a.e. on* U.

Putting Theorems 1 and 2 together, we obtain the following

Corollary 1. *Assume that conditions* (1.1), (2.2), *and* (2.6) *are satisfied. Then series* (2.1) *is* $|T|$*-summable a.e. for every double ONS* φ *defined on* U *if and only if condition* (2.5) *is satisfied.*

The corresponding results for single ONS defined on the unit interval $I = \{x_1\colon 0 \leqq x_1 \leqq 1\}$ were proved by Leindler and Tandori [8].

As an application, we will conclude a number of results on $|C, \alpha, \beta|$-summability of double orthogonal series for $\alpha > -1$ and $\beta > -1$. As is known, (C, α, β)-summability is defined by means of the triangular matrix $T=\{t_{ik}^{mn}\}$:

$$(2.8)\qquad t_{ik}^{mn}=\begin{cases}\dfrac{A_{m-i}^{\alpha}}{A_m^{\alpha}}\dfrac{A_{n-k}^{\beta}}{A_n^{\beta}}, & \text{for } i=0,1,\dots,m;\ k=0,1,\dots,n;\ m,n=0,1,\dots;\\ 0, & \text{otherwise.}\end{cases}$$

Here

$$A_m^{\alpha}=\binom{\alpha+m}{m}=\frac{(\alpha+1)(\alpha+2)\dots(\alpha+m)}{m!}\qquad (m=0,1,\dots;\ \alpha>-1)$$

is the binomial coefficient.

3. Proofs of Theorems 1 and 2

Similarly to (1.4) and (1.5), we set

$$(3.1)\qquad \Delta_{mn}(x)=\sigma_{mn}(x)-\sigma_{m-1,n}(x)-\sigma_{m,n-1}(x)+\sigma_{m-1,n-1}(x)$$

with the agreement that

$$\sigma_{-1,n}(x)=\sigma_{m,-1}(x)=\sigma_{-1,-1}(x)=0\quad (m,n=0,1,\dots)$$

for every x in X.

Proof of Theorem 1. By Minkowski's inequality, orthogonality, and (2.5), we get in turn that

$$\Big\{\int_X\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\Delta_{mn}(x)|\Big]^2 d\mu(x)\Big\}^{1/2}\le\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\int_X\Delta_{mn}^2(x)\,d\mu(x)\Big\}^{1/2}=$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}[\tau_{ik}^{mn}]^2a_{ik}^2\Big\}^{1/2}<\infty.$$

This means that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\Delta_{mn}(x)|\in L^2(X,\mathcal{F},\mu),$$

and, in particular, series (2.1) is $|T|$-summable μ-a.e.

The proof of Theorem 1 is complete.

In the proof of Theorem 2 we need the following auxiliary result proved in [9].

Theorem A. *Given any measurable set E $(\subset U)$ of positive measure, then there exist an integer n_0 and a constant $C_1>0$ such that for every finite sum*

$$P(x_1,x_2)=\sum_{i=m}^{M}\sum_{k=n}^{N}a_{ik}r_i(x_1)r_k(x_2)$$

with $\max(m, n) \geqq n_0$, $M \geqq m \geqq 0$ *and* $N \geqq n \geqq 0$ *we have*

$$\iint_E |P(x_1, x_2)|\, dx_1\, dx_2 \geqq C_1 \{ \sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik}^2 \}^{1/2}.$$

We note that this is an extension of a result due to ORLICZ [10] from the one-dimensional Rademacher system to the two-dimensional one.

Proof of Theorem 2. We will prove that if series (2.7) is $|T|$-summable on a subset of U with positive measure, then condition (2.5) necessarily holds.

To realize this goal, then by Egorov's theorem there exist a constant C_2 and a subset E $(\subset U)$ of positive measure such that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{mn}(x_1, x_2)| \leqq C_2 \quad \text{for} \quad (x_1, x_2) \in E, \tag{3.2}$$

where this time $\Delta_{mn}(x_1, x_2)$ is defined by (3.1) in the case of the two-dimensional Rademacher functions and $x=(x_1, x_2)$.

We are going to apply Theorem A formulated above. To this effect, we must get rid of the functions $r_i(x_1)$, $r_k(x_2)$ in the definition of $\Delta_{mn}(x_1, x_2)$ for which $\max(i, k) < n_0$. Therefore, we set

$$\tilde{a}_{ik} = \begin{cases} a_{ik} & \text{if} \quad \max(i, k) \geqq n_0, \\ 0 & \text{if} \quad \max(i, k) < n_0; \end{cases}$$

and denote by $\tilde{\Delta}_{mn}(x_1, x_2)$ the corresponding difference of the T-means for the "truncated" double series

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_{ik} r_i(x_1) r_k(x_2). \tag{3.3}$$

Since $|r_i(x_1) r_k(x_2)| \leqq 1$ for every x_1, x_2, an elementary estimation shows that

$$\left| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{mn}(x_1, x_2)| - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\tilde{\Delta}_{mn}(x_1, x_2)| \right| \leqq$$

$$\leqq \sum\sum_{\max(m,n) \geqq n_0} \sum_{i=0}^{\min(m, n_0-1)} \sum_{k=0}^{\min(n, n_0-1)} |\tau_{ik}^{mn} a_{ik}| =$$

$$= \sum_{i=0}^{n_0-1} \sum_{k=0}^{n_0-1} |a_{ik}| \{ \sum_{m=i}^{\infty} \sum_{n=n_0}^{\infty} + \sum_{m=n_0}^{\infty} \sum_{n=k}^{n_0-1} + \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \} |\tau_{ik}^{mn}| \leqq$$

$$\leqq 3 \sum_{i=0}^{n_0-1} \sum_{k=0}^{n_0-1} |a_{ik}| \sum_{m=i}^{\infty} \sum_{n=k}^{\infty} |\tau_{ik}^{mn}| < \infty,$$

the last inequality is due to (2.6). Consequently, the $|T|$-summability of series (2.7) and (3.3) are equivalent for every x_1, x_2.

So, we may assume without loss of generality that $a_{ik}=0$ in (2.7) for $i, k = 0, 1, \ldots, n_0-1$, and use the notations a_{ik} and $\Delta_{mn}(x_1, x_2)$ rather than $\tilde{a}_{ik}$ and $\tilde{\Delta}_{mn}(x_1, x_2)$. On the one hand, by (3.2),

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \iint_E |\Delta_{mn}(x_1, x_2)|\, dx_1 dx_2 \leqq C_2 \mu(E), \tag{3.4}$$

μ being the plane Lebesgue measure here. On the other hand, applying Theorem A yields

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \iint_E |\Delta_{mn}(x_1, x_2)|\, dx_1 dx_2 \geqq \tag{3.5}$$

$$\geqq C_1 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big\{ \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} [\tau_{ik}^{mn}]^2 a_{ik}^2 \Big\}^{1/2}.$$

Combining inequalities (3.4) and (3.5) results in (2.5) to be proved.

4. Application of Theorem 1: Sufficient conditions for $|C, \alpha, \beta|$-summability of orthogonal series

The next seven theorems will be consequences of Theorem 1. We make the following convention: by 2^{-1} we mean 0 in this paper.

Theorem B. *If* $\alpha > 1/2$, $\beta > 1/2$, *and*

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \Big\{ \sum_{i=2^{p-1}}^{2^p-1} \sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2 \Big\}^{1/2} < \infty, \tag{4.1}$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

This theorem was proved in [9] by the first named author, extending the relevant results of Tandori [14] ($\alpha=1$) and Leindler [5] ($\alpha>1/2$) from single to double orthogonal series. The proving method in [9] is a direct one. Nevertheless, it is instructive to present here how Theorem B can be deduced from Theorem 1. Since the same technique will be used in the proofs of Theorems 3—8 below, we enter into full details.

Proof of Theorem B. We will prove that condition (4.1) implies (2.5), and a fortiori, Theorem 1 implies Theorem B.

To this end, we introduce the notations

$$n_q = \begin{cases} 2^{q-1} & \text{if} \quad q = 1, 2, \ldots, \\ 0 & \text{if} \quad q = 0; \end{cases} \tag{4.2}$$

and

$$\mathscr{A}_{mn} = \Big\{ \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} [\tau_{ik}^{mn}]^2 a_{ik}^2 \Big\}^{1/2} \quad (m, n = 0, 1, \ldots). \tag{4.3}$$

Thus, the left-hand side of (2.5) can be rewritten as follows

(4.4) $$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}[\tau_{ik}^{mn}]^2 a_{ik}^2\Big\}^{1/2} = \mathscr{A}_{00}+\sum_{n=1}^{\infty}\mathscr{A}_{0n}+\sum_{m=1}^{\infty}\mathscr{A}_{m0}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathscr{A}_{mn}.$$

According to this, the proof is divided into four parts.

Part 1. By (2.3), (2.4) and (2.8)

(4.5) $$\tau_{00}^{00}=1 \quad\text{and}\quad \tau_{ik}^{00}=0 \quad\text{otherwise,}$$

whence

(4.6) $$\mathscr{A}_{00}=|a_{00}|.$$

Part 2. By definition, for $n=1,2,\ldots$

$$\tau_{0k}^{0n}=\begin{cases}\dfrac{A_{n-k}^{\beta}}{A_n^{\beta}}-\dfrac{A_{n-k-1}^{\beta}}{A_{n-1}^{\beta}} & \text{if} \quad k=0,1,\ldots,n-1;\\[2mm] \dfrac{1}{A_n^{\beta}} & \text{if} \quad k=n;\end{cases}$$

and $\tau_{ik}^{0n}=0$ if $i>0$ or $k>n$. Using the relevant estimates in [5], we have, for $\beta>-1$,

(4.7) $$\tau_{ik}^{0n}=\begin{cases}O_\beta\big(kn^{-\beta-1}(n+1-k)^{\beta-1}\big) & \text{if} \quad k=0,1,\ldots,n;\\ 0 & \text{if} \quad i>0 \quad\text{or}\quad k>n \quad (n=1,2,\ldots).\end{cases}$$

By the Cauchy inequality,

$$\sum_{n=1}^{\infty}\mathscr{A}_{0n}=\sum_{n=1}^{\infty}\Big\{\sum_{k=0}^{n}[\tau_{0k}^{0n}]^2a_{0k}^2\Big\}^{1/2}=\sum_{q=0}^{\infty}\sum_{n=n_q+1}^{n_{q+1}}\Big\{\sum_{k=0}^{n}[\tau_{0k}^{0n}]^2a_{0k}^2\Big\}^{1/2}\leqq$$

$$\leqq\sum_{q=0}^{\infty}\Big\{(n_{q+1}-n_q)\sum_{n=n_q+1}^{n_{q+1}}\sum_{k=0}^{n}[\tau_{0k}^{0n}]^2a_{0k}^2\Big\}^{1/2}=$$

$$=O(1)\sum_{q=1}^{\infty}\Big\{(n_{q+1}-n_q)\sum_{n=n_q+1}^{n_{q+1}}\sum_{k=0}^{n-1}k^2n^{-2\beta-2}(n-k)^{2\beta-2}a_{0k}^2\Big\}^{1/2}+$$

$$+O(1)\sum_{q=0}^{\infty}\Big\{(n_{q+1}-n_q)\sum_{n=n_q+1}^{n_{q+1}}n^{-2\beta}a_{0k}^2\Big\}^{1/2}=O(1)(\Sigma_1+\Sigma_2),\quad\text{say.}$$

Since

(4.8) $$n_{q+1}-n_q=n_q \quad (q=1,2,\ldots),$$

it immediately follows from (4.1) that $\Sigma_2<\infty$.

Now we turn to Σ_1. A simple computation gives that

$$\Sigma_1 = \sum_{q=1}^{\infty} \Big\{(n_{q+1}-n_q) \sum_{n=n_q+1}^{n_{q+1}} \sum_{r=0}^{q} \sum_{k=n_r}^{\min(n_{r+1},n)-1} k^2 n^{-2\beta-2}(n-k)^{2\beta-2} a_{0k}^2\Big\}^{1/2} =$$

$$= \sum_{q=1}^{\infty} \Big\{(n_{q+1}-n_q) \sum_{r=0}^{q} \sum_{k=n_r}^{n_{r+1}-1} \sum_{n=\max(n_q,k)+1}^{n_{q+1}} k^2 n^{-2\beta-2}(n-k)^{2\beta-2} a_{0k}^2\Big\}^{1/2} \leqq$$

$$\leqq \sum_{q=2}^{\infty} \Big\{(n_{q+1}-n_q) \sum_{r=0}^{q-2} \sum_{k=n_r}^{n_{r+1}-1} \sum_{n=n_q+1}^{n_{q+1}} k^2 n^{-2\beta-2}(n-k)^{2\beta-2} a_{0k}^2\Big\}^{1/2} +$$

$$+ \sum_{q=1}^{\infty} (n_{q+1}-n_q)^{1/2} n_q^{\beta-1} \sum_{r=q-1}^{q} \Big\{ \sum_{k=n_r}^{n_{r+1}-1} k^2 a_{0k}^2 \sum_{n=\max(n_q,k)+1}^{n_{q+1}} (n-k)^{2\beta-2}\Big\}^{1/2} =$$

$$= \Sigma_{11} + \Sigma_{12}, \quad \text{say.}$$

It is easy to see that

$$\sum_{n=\max(n_q,k)+1}^{n_{q+1}} (n-k)^{2\beta-2} = O(n_q^{2\beta-1}) \tag{4.9}$$

$$\text{if} \quad n_{q-1} \leqq k < n_{q+1} \quad \left(q = 1, 2, \dots;\ \beta > \frac{1}{2}\right).$$

Consequently, (4.1) and (4.8) yield $\Sigma_{12} < \infty$.

Now we treat Σ_{11}. It is not hard to check that

$$(n-k)^{2\beta-2} \leqq 4(n_q - n_{r+1})^{2\beta-2} \tag{4.10}$$

$$\text{if} \quad n_q < n \leqq n_{q+1};\ n_r \leqq k < n_{r+1};$$

$$r = 0, 1, \dots, q-2;\ q = 2, 3, \dots;\ \beta > \frac{1}{2}.$$

Using this inequality together with

$$(u+v+\dots)^{1/2} \leqq u^{1/2} + v^{1/2} + \dots \quad (u \geqq 0,\ v \geqq 0, \dots),$$

we find that

$$\Sigma_{11} = \sum_{q=2}^{\infty} \Big\{(n_{q+1}-n_q) \sum_{n=n_q+1}^{n_{q+1}} \sum_{r=0}^{q-2} \sum_{n=n_r}^{n_{r+1}-1} k^2 n^{-2\beta-2}(n-k)^{2\beta-2} a_{0k}^2\Big\}^{1/2} \leqq$$

$$\leqq \sum_{q=2}^{\infty} (n_{q+1}-n_q)^{1/2} n_q^{-\beta-1} \Big\{(n_{q+1}-n_q) \sum_{r=0}^{q-2} (n_q - n_{r+1})^{2\beta-2} \sum_{k=n_r}^{n_{r+1}-1} k^2 a_{0k}^2\Big\}^{1/2} =$$

$$= O(1) \sum_{q=2}^{\infty} (n_{q+1}-n_q)\, n_q^{-\beta-1} \sum_{r=0}^{q-2} n_r (n_q - n_{r+1})^{\beta-1} \Big\{\sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\Big\}^{1/2} =$$

$$= O(1) \sum_{r=0}^{\infty} n_r \Big\{\sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=r+2}^{\infty} (n_{q+1}-n_q) n_q^{-\beta-1} (n_q - n_{r+1})^{\beta-1} = \Sigma, \quad \text{say.}$$

It is easy to see that

(4.11) $$(n_q - n_{r+1})^{\beta-1} = O(n_q^{\beta-1})$$

$$\text{if} \quad q \geqq r+2;\ r = 0, 1, \ldots;\ \beta > \frac{1}{2}.$$

Using this, (4.1) and (4.8) we can conclude that

(4.12) $$\Sigma = O(1) \sum_{r=0}^{\infty} n_r \Big\{\sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=r+2}^{\infty} n_q^{-1} = O(1) \sum_{r=0}^{\infty} \Big\{\sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\Big\}^{1/2} < \infty.$$

Consequently, $\Sigma_{11} < \infty$, $\Sigma_1 < \infty$, and

(4.13) $$\sum_{n=1}^{\infty} \mathscr{A}_{0n} < \infty.$$

Remark. A careful examination of the method used just above shows that if $\{C_k\colon k=0, 1, \ldots\}$ is a sequence of nonnegative numbers, then

(4.14) $$\sum_{n=1}^{\infty} \Big\{\sum_{k=0}^{n} [\tau_{0k}^{0n}]^2 C_k\Big\}^{1/2} = O_\beta(1) \sum_{r=0}^{\infty} \Big\{\sum_{k=n_r}^{n_{r+1}-1} C_k\Big\}^{1/2},$$

where $O_\beta(1)$ does not depend on $\{C_k\}$ and as before $n_r = 2^{r-1}$.

In a similar way, we can obtain that for every sequence $\{B_i\colon i=0, 1, \ldots\}$ of nonnegative numbers we have

(4.15) $$\sum_{m=1}^{\infty} \Big\{\sum_{i=0}^{m} [\tau_{i0}^{m0}]^2 B_i\Big\}^{1/2} = O_\alpha(1) \sum_{r=0}^{\infty} \Big\{\sum_{i=n_r}^{n_{r+1}-1} B_i\Big\}^{1/2}.$$

Part 3. According to (4.15),

(4.16) $$\sum_{m=1}^{\infty} \mathscr{A}_{m0} < \infty.$$

Part 4. It remains to prove that

(4.17) $$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mathscr{A}_{mn} < \infty.$$

To this end, first we observe that

(4.18) $$\tau_{ik}^{mn} = \tau_{i0}^{m0} \tau_{0k}^{0n} \quad (i, k = 0, 1, \ldots;\ m, n = 1, 2, \ldots).$$

In particular, this implies that

$$\tau_{ik}^{mn} = 0 \quad \text{if} \quad i > m \quad \text{or} \quad k > n.$$

Then setting

(4.19) $$C_k = \sum_{i=0}^{m} [\tau_{i0}^{m0}]^2 a_{ik}^2 \quad (k = 0, 1, \ldots)$$

and

$$B_i = \sum_{k=n_r}^{n_{r+1}-1} a_{ik}^2 \quad (i = 0, 1, \ldots), \tag{4.20}$$

we can proceed as follows

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathscr{A}_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\{\sum_{i=0}^{m}\sum_{k=0}^{n}[\tau_{i0}^{m0}\tau_{0k}^{0n}]^2 a_{ik}^2\Big\}^{1/2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\{\sum_{k=0}^{n}[\tau_{0k}^{0n}]^2 C_k\Big\}^{1/2} =$$

$$= O_\beta(1)\sum_{m=1}^{\infty}\sum_{r=0}^{\infty}\Big\{\sum_{k=n_r}^{n_{r+1}-1}\sum_{i=0}^{m}[\tau_{i0}^{m0}]^2 a_{ik}^2\Big\}^{1/2} = O_\beta(1)\sum_{m=1}^{\infty}\sum_{r=0}^{\infty}\Big\{\sum_{i=0}^{m}[\tau_{i0}^{m0}]^2 B_i\Big\}^{1/2} =$$

$$= O_\beta(1)O_\alpha(1)\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\Big\{\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=n_r}^{n_{r+1}-1} a_{ik}^2\Big\}^{1/2} < \infty,$$

the last inequality being (4.1). This proves (4.17).

Combining (4.4), (4.6), (4.13), (4.16) and (4.17) completes the proof of Theorem B.

Now we introduce the following notations:

$$m_q = \begin{cases} 2^{\sqrt{q-1}} & \text{if } q = 1, 2, \ldots, \\ 0 & \text{if } q = 0; \end{cases} \tag{4.21}$$

$$i_p = p^{1/(1-2\alpha)} \quad \text{if} \quad p = 0, 1, \ldots; \tag{4.22}$$

$$k_q = q^{1/(1-2\beta)} \quad \text{if} \quad q = 0, 1, \ldots. \tag{4.23}$$

We agree that if u and v are real numbers, $u \leqq v$ then by $\sum_{n=u}^{v}$ we mean the sum extended for all integers n such that $u \leqq n \leqq v$.

Theorem 3. *If*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{\sum_{i=m_p}^{m_{p+1}-1}\sum_{k=m_q}^{m_{q+1}-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.24}$$

then series (2.1) *is* $|C, 1/2, 1/2|$*-summable* μ*-a.e.*

Theorem 4. *If* $0 \leqq \alpha < 1/2$, $0 \leqq \beta < 1/2$, *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{\sum_{i=i_p}^{i_{p+1}-1}\sum_{k=k_q}^{k_{q+1}-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.25}$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Theorems 3 and 4 are the extensions of the corresponding theorems of Leindler and Schwinn [7] from single to double orthogonal series.

Conditions (4.26) and (4.27) below imply the fulfilment of conditions (4.24) and (4.25), respectively, through an appropriate grouping and the Cauchy inequality (cf. [6]). In this way we obtain the following two corollaries.

Corollary 2. *If*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{(p+1)(q+1)\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1}a_{ik}^2\}^{1/2}<\infty, \tag{4.26}$$

then series (2.1) *is* $|C, 1/2, 1/2|$*-summable* μ*-a.e.*

Corollary 3. *If* $0\leqq\alpha<1/2$, $0\leqq\beta<1/2$, *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{2^{p(1-2\alpha)}2^{q(1-2\beta)}\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1}a_{ik}^2\}^{1/2}<\infty, \tag{4.27}$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Corollaries 2 and 3 as well as Theorem 5 below are the extensions of the corresponding theorems of LEINDLER [5] from single to double orthogonal series.

Theorem 5. *If* $-1<\alpha<0$, $-1<\beta<0$, *and condition* (4.27) *is satisfied, then series* (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Proofs of Theorems 3 and 4. We follow the scheme of the proof of Theorem B, changing it only at the reference numbers indicated by * or * *. Instead of (4.1), (4.2), (4.8)—(4.12) we have to take (4.24), (4.21), (4.8*)—(4.12*) and (4.25), (4.22)—(4.23), (4.8**)—(4.12**), respectively, and the proofs run along the same line as the proof of Theorem B. The * estimates below are valid for $\beta=1/2$, while the * * estimates are valid for $0\leqq\beta<1/2$, but some of them remain valid for $\beta>-1$ too.

The appropriate estimates are the following:

$$m_{q+1}-m_q = O\left(\frac{m_q}{\log m_q}\right) \tag{4.8*}$$

and

$$k_{q+1}-k_q = O_\beta(k_q^{2\beta}) \tag{4.8**}$$

(this latter estimate holds true for $\beta>-1$);

$$\sum_{n=\max(m_q,k)+1}^{m_{q+1}}(n-k)^{-1} = O(\log m_q) \tag{4.9*}$$

and

$$\sum_{n=\max(k_q,k)+1}^{k_{q+1}}(n-k)^{2\beta-2} = O_\beta(1); \tag{4.9**}$$

$$(n-k)^{-1}\leqq(m_q-m_{r+1})^{-1} \tag{4.10*}$$

and

(4.10**) $$(n-k)^{2\beta-2} \leqq (k_q-k_{r+1})^{2\beta-2};$$

(4.11*) $$(m_q-m_{r+1})^{-1/2} \leqq \begin{cases} 2r^{1/4}(q-1-r)^{-1/2} m_{r+1}^{-1/2} & \text{if} \quad r+2 \leqq q \leqq r+r^{1/2}, \\ 2m_q^{-1/2} & \text{if} \quad r+r^{1/2} < q; \end{cases}$$

and

(4.11**) $$(k_q-k_{r+1})^{\beta-1} = \begin{cases} O_\beta(1)\,(q-1-r)^{\beta-1} k_{r+1}^{2\beta(\beta-1)} & \text{if} \quad r+2 \leqq q \leqq 2r+1, \\ O_\beta(1)\, k_q^{\beta-1} & \text{if} \quad 2r+1 < q; \end{cases}$$

finally, for $\beta=1/2$,

(4.12*) $$\Sigma = O(1) \sum_{r=4}^{\infty} m_r \Big\{ \sum_{k=m_r}^{m_{r+1}-1} a_{0k}^2 \Big\}^{1/2} \sum_{q=r+2}^{\infty} m_q^{-1/2} (m_q-m_{r+1})^{-1/2} \log^{-1} m_q =$$

$$= O(1) \sum_{r=4}^{\infty} r^{1/4} m_r^{1/2} \Big\{ \sum_{k=m_r}^{m_{r+1}-1} a_{0k}^2 \Big\}^{1/2} \sum_{q=r+2}^{r+r^{1/2}} m_q^{-1/2} (q-1-r)^{-1/2} \log^{-1} m_q +$$

$$+ O(1) \sum_{r=4}^{\infty} m_r \Big\{ \sum_{k=m_r}^{m_{r+1}-1} a_{0k}^2 \Big\}^{1/2} \sum_{q=r+r^{1/2}+1}^{\infty} m_q^{-1} \log^{-1} m_q =$$

$$= O(1) \sum_{r=4}^{\infty} r^{-1/4} \Big\{ \sum_{k=m_r}^{m_{r+1}-1} a_{0k}^2 \Big\}^{1/2} \sum_{q=1}^{r^{1/2}} q^{-1/2} +$$

$$+ O(1) \sum_{r=4}^{\infty} m_r \Big\{ \sum_{k=m_r}^{m_{r+1}-1} a_{0k}^2 \Big\}^{1/2} \sum_{q=r+1}^{\infty} m_q^{-1} \log^{-1} m_q < \infty,$$

while for $0<\beta<1/2$,

(4.12**) $$\Sigma = O_\beta(1) \sum_{r=1}^{\infty} k_r \Big\{ \sum_{k=k_r}^{k_{r+1}-1} a_{0k}^2 \Big\}^{1/2} \sum_{q=r+2}^{\infty} k_q^{\beta-1} (k_q-k_{r+1})^{\beta-1} =$$

$$= O_\beta(1) \sum_{r=1}^{\infty} k_r^{\beta(2\beta-1)} \Big\{ \sum_{k=k_r}^{k_{r+1}-1} a_{0k}^2 \Big\}^{1/2} \sum_{q=r+2}^{2r+1} (q-1-r)^{\beta-1} +$$

$$+ O_\beta(1) \sum_{r=1}^{\infty} k_r \Big\{ \sum_{k=k_r}^{k_{r+1}-1} a_{0k}^2 \Big\}^{1/2} \sum_{q=2r+2}^{\infty} k_q^{2\beta-2} < \infty,$$

and for $\beta=0$,

(4.13**) $$\sum_{n=1}^{\infty} \mathscr{A}_{0n} = \sum_{n=1}^{\infty} \Big\{ \sum_{k=0}^{n} [\tau_{0k}^{0n}]^2 a_{0k}^2 \Big\}^{1/2} = \sum_{n=1}^{\infty} |a_{0n}| < \infty.$$

These inequalities completes the proof of Theorems 3 and 4.

Proof of Theorem 5. We use notation (4.2) and follow the pattern of the proof of Theorem B again. By (4.8) and (4.27),

$$\Sigma_2 = \sum_{q=0}^{\infty} \Big\{ (n_{q+1}-n_q) \sum_{n=n_q+1}^{n_{q+1}} n^{-2\beta} a_{0n}^2 \Big\}^{1/2} \leqq \sum_{q=0}^{\infty} \Big\{ (n_{q+1}-n_q)\, n_{q+1}^{-2\beta} \sum_{n=n_q+1}^{n_{q+1}} a_{0n}^2 \Big\}^{1/2} =$$

$$= O_\beta(1) \sum_{q=0}^{\infty} \Big\{ 2^{q(1-2\beta)} \sum_{n=2^{q-1}+1}^{2^q} a_{0n}^2 \Big\}^{1/2} < \infty$$

and

$$\Sigma_1 = \sum_{q=1}^{\infty} \{(n_{q+1}-n_q) \sum_{n=n_q+1}^{n_{q+1}} \sum_{r=0}^{q} \sum_{k=n_r}^{\min(n_{r+1},n)-1} k^2 n^{-2\beta-2}(n-k)^{2\beta-2} a_{0k}^2\}^{1/2} =$$

$$= O_\beta(1) \sum_{q=1}^{\infty} \{2^{-q(1+2\beta)} \sum_{r=0}^{q} \sum_{k=n_r}^{n_{r+1}-1} k^2 a_{0k}^2 \sum_{n=\max(n_q,k)+1}^{n_{q+1}} (n-k)^{2\beta-2}\}^{1/2} =$$

$$= O_\beta(1) \sum_{q=2}^{\infty} \{2^{-q(1+2\beta)} \sum_{r=0}^{q-2} 2^{2r} \sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2 \sum_{n=2^{q-1}+1}^{2^q} (n-2^r)^{2\beta-2}\}^{1/2} +$$

$$+ O_\beta(1) \sum_{q=1}^{\infty} \{2^{-q(1+2\beta)} \sum_{r=q-1}^{q} 2^{2q} \sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\}^{1/2} =$$

$$= O_\beta(1)(1 + \sum_{q=2}^{\infty} \{2^{-q(1+2\beta)} \sum_{r=0}^{q-2} 2^{2r} \sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2 2^{q(2\beta-1)}\}^{1/2}) =$$

$$= O_\beta(1)(1 + \sum_{r=0}^{\infty} 2^r \{\sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\}^{1/2} \sum_{q=r}^{\infty} 2^{-q}) < \infty.$$

These calculations show that (4.13) is satisfied.

In the above manner (cf. Remark in the proof of Theorem B), we can conclude that if $\{C_k\colon k=0, 1, \dots\}$ is a sequence of nonnegative numbers, then

$$\sum_{n=1}^{\infty} \{\sum_{k=0}^{n} [\tau_{0k}^{0n}]^2 C_k\}^{1/2} = O_\beta(1) \sum_{r=0}^{\infty} \{2^{r(1-2\beta)} \sum_{k=n_r}^{n_{r+1}-1} C_k\}^{1/2} \tag{4.14*}$$

and if $\{B_i\colon i=0, 1, \dots\}$ is a sequence of nonnegative numbers, then

$$\sum_{m=1}^{\infty} \{\sum_{i=0}^{m} [\tau_{i0}^{m0}]^2 B_i\}^{1/2} = O_\alpha(1) \sum_{r=0}^{\infty} \{2^{r(1-2\alpha)} \sum_{i=n_r}^{n_{r+1}-1} B_i\}^{1/2}. \tag{4.15*}$$

The latter inequality implies the fulfilment of (4.16).

As to the fulfilment of (4.17), we use notation (4.19) and set

$$B_i = \sum_{k=n_r}^{n_{r+1}-1} k^{1-2\beta} a_{ik}^2 \quad (i = 0, 1, \dots). \tag{4.20*}$$

We proceed as follows (cf. (4.18))

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mathscr{A}_{mn} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{\sum_{k=0}^{n} [\tau_{0k}^{0n}]^2 \sum_{i=0}^{m} [\tau_{i0}^{m0}]^2 a_{ik}^2\}^{1/2} =$$

$$= O_\beta(1) \sum_{m=1}^{\infty} \sum_{q=0}^{\infty} \{2^{q(1-2\beta)} \sum_{k=n_q}^{n_{q+1}-1} \sum_{i+0}^{m} [\tau_{i0}^{m0}]^2 a_{ik}^2\}^{1/2} =$$

$$= O_\beta(1) \sum_{m=1}^{\infty} \sum_{q=0}^{\infty} \{\sum_{i=0}^{m} [\tau_{i0}^{m0}]^2 \sum_{k=n_q}^{n_{q+1}-1} k^{1-2\beta} a_{ik}^2\}^{1/2} =$$

$$= O_\beta(1)O_\alpha(1)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{2^{p(1-2\alpha)}\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=n_q}^{n_{q+1}-1} k^{1-2\beta}a_{ik}^2\}^{1/2} =$$

$$= O_\beta(1)O_\alpha(1)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{2^{p(1-2\alpha)}2^{q(1-2\beta)}\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=n_q}^{n_{q+1}-1} a_{ik}^2\}^{1/2} < \infty,$$

completing the proof of Theorem 5.

The following three theorems cover the so-called "mixed" cases. We remind notations (4.2), (4.21)—(4.23).

Theorem 6. *If* $\alpha>1/2$, $\beta=1/2$ *and*

(4.28) $$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=m_q}^{m_{q+1}-1} a_{ik}^2\}^{1/2} < \infty,$$

or if $\alpha>1/2$, $0\leqq\beta<1/2$ *and*

(4.29) $$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=k_q}^{k_{q+1}-1} a_{ik}^2\}^{1/2} < \infty,$$

or if $\alpha>1/2$, $-1<\beta<0$ *and*

(4.30) $$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{2^{q(1-2\beta)}\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=n_q}^{n_{q+1}-1} a_{ik}^2\}^{1/2} < \infty,$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Theorem 7. *If* $\alpha=1/2$, $0\leqq\beta<1/2$ *and*

(4.31) $$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{\sum_{i=m_p}^{m_{p+1}-1}\sum_{k=k_q}^{k_{q+1}-1} a_{ik}^2\}^{1/2} < \infty,$$

or if $\alpha=1/2$, $-1<\beta<0$ *and*

(4.32) $$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{2^{q(1-2\beta)}\sum_{i=m_p}^{m_{p+1}-1}\sum_{k=n_q}^{n_{q+1}-1} a_{ik}^2\}^{1/2} < \infty,$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Theorem 8. *If* $0\leqq\alpha<1/2$; $-1<\beta<0$ *and*

(4.33) $$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{2^{q(1-2\beta)}\sum_{i=i_p}^{i_{p+1}-1}\sum_{k=n_q}^{n_{q+1}-1} a_{ik}^2\}^{1/2} < \infty,$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Combining the proofs of Theorem B and Theorems 3—5 yields Theorem 6, combining those of Theorems 3 and 4 yields Theorem 7, while combining those of Theorems 4 and 5 yields Theorem 8.

As an example, we sketch the proof for the case $\alpha>1/2$ and $\beta=1/2$. Similarly to (4.14), for any sequence $\{C_k\colon k=0, 1, \ldots\}$ of nonnegative numbers we have

$$\sum_{n=1}^{\infty}\Big\{\sum_{k=0}^{n}[\tau_{0k}^{0n}]^2 C_k\Big\}^{1/2} = O_\beta(1)\Big\{\sum_{k=m_q}^{m_{q+1}-1} C_k\Big\}^{1/2}. \tag{4.14**}$$

Furthemore, we have (4.15).

Assume (4.28) is satisfied. First, setting $C_k=a_{0k}^2$ and $B_i=a_{i0}^2$ we can derive (4.13) and (4.16). Second, using notation (4.19) and setting

$$B_i = \sum_{k=m_q}^{m_{q+1}-1} a_{ik}^2 \tag{4.20**}$$

we can conclude (4.17). So, applying Theorem 1 provides the first statement in Theorem 6.

The next two corollaries of Theorems 6 and 7 can be deduced via the Cauchy inequality.

Corollary 4. *If* $\alpha>1/2$, $\beta=1/2$ *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{(q+1)\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.34}$$

or if $\alpha>1/2$, $-1<\beta<1/2$ *and condition* (4.30) *is satisfied, then series* (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Corollary 5. *If* $\alpha=1/2$, $-1<\beta<1/2$ *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{(p+1)2^{q(1-2\beta)}\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.35}$$

or if $-1<\alpha<1/2$, $-1<\beta<1/2$ *and condition* (4.27) *is satisfied, then series* (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Corollaries 4 and 5 as well as Corollaries 2 and 3 were proved by Ponomarenko and Timan [11] for the two-dimensional trigonometric system.

We remind that a double sequence $\{\lambda_{ik}\colon i, k=0, 1, \ldots\}$ of numbers is said to be nondecreasing if

$$\lambda_{ik} \leqq \min\{\lambda_{i+1,k}, \lambda_{i,k+1}\}$$

and to be nonincreasing if

$$\lambda_{ik} \geqq \max\{\lambda_{i+1,k}, \lambda_{i,k+1}\} \quad (i, k = 0, 1, \ldots).$$

In Corollaries 6 and 7 below, let $\{\lambda_{ik}\}$ be a nondecreasing sequence of positive numbers such that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(i+1)(k+1)\lambda_{ik}} < \infty, \tag{4.36}$$

or equivalently,

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{\lambda_{2^p,2^q}}<\infty.$$

Applying the Cauchy inequality to series (4.1), (4.26), (4.27) and then to series (4.34), (4.30) and (4.35) results in the following two corollaries.

Corollary 6. *If* $\alpha>1/2$, $\beta>1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}<\infty,$$

or if $\alpha=1/2$, $\beta=1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}\log(i+2)\log(k+2)<\infty,$$

or if $-1<\alpha<1/2$, $-1<\beta<1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}(i+1)^{1-2\alpha}(k+1)^{1-2\beta}<\infty,$$

then series (2.1) *is* $|C,\alpha,\beta|$*-summable* μ*-a.e.*

Corollary 7. *If* $\alpha>1/2$, $\beta=1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}\log(k+2)<\infty,$$

or if $\alpha>1/2$, $-1<\beta<1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}(k+1)^{1-2\beta}<\infty,$$

or if $\alpha=1/2$, $-1<\beta<1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}(k+1)^{1-2\beta}\log(i+2)<\infty,$$

then series (2.1) *is* $|C,\alpha,\beta|$*-summable* μ*-a.e.*

Corollary 6 is the extension of the corresponding results of UL'JANOV [15, pp. 46—37 and 51—52] from single to double orthogonal series.

5. Application of Theorem 2: Necessary conditions for $|C, \alpha, \beta|$-summability of orthogonal series

The sufficient conditions (4.24), (4.25) and (4.27)—(4.32) are the best possible. To see this, we consider the special case where the double sequence $\{|a_{ik}|: i, k=0, 1, \ldots\}$ is nonincreasing. Then (4.24) is equivalent to (4.26), and both are equivalent to the condition

$$S_1 = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (p+1)^{1/2}(q+1)^{1/2} 2^{p/2} 2^{q/2} |a_{2^p, 2^q}| < \infty; \tag{5.1}$$

while (4.25), (4.27) and (4.33) are also equivalent to each other, and each of them is equivalent to the condition

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} 2^{p(1-\alpha)} 2^{q(1-\beta)} |a_{2^p, 2^q}| < \infty \quad (-1 < \alpha, \beta < 1/2). \tag{5.2}$$

Similarly, in the special case where $\{|a_{ik}|\}$ is nonincreasing in k for each fixed i both (4.28) and (4.34) are equivalent to the condition

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (q+1)^{1/2} 2^{q/2} \Big\{ \sum_{i=2^{p-1}}^{2^p-1} a_{i, 2^q}^2 \Big\}^{1/2} < \infty; \tag{5.3}$$

while both (4.29) and (4.30) are equivalent to the condition

$$S_4 = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} 2^{q(1-\beta)} \Big\{ \sum_{i=2^{p-1}}^{2^p-1} a_{i, 2^q}^2 \Big\}^{1/2} < \infty \quad (-1 < \beta < 1/2). \tag{5.4}$$

Furthermore, in the special case where again the double sequence $\{|a_{ik}|\}$ is nonincreasing, each of the conditions (4.31), (4.32) and (4.35) is equivalent to

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (p+1)^{1/2} 2^{p/2} 2^{q(1-\beta)} |a_{2^p, 2^q}| < \infty \quad (-1 < \beta < 1/2). \tag{5.5}$$

As an illustration, we show the equivalence in two cases.

Case 1. The equivalence of (4.24), (4.26) and (5.1). We remind notation (4.21).

First, we show that (4.26) implies (4.24) without any restriction. By the Cauchy inequality,

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \Big\{ \sum_{i=m_p}^{m_{p+1}-1} \sum_{k=m_q}^{m_{q+1}-1} a_{ik}^2 \Big\}^{1/2} \leqq 2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1)^{1/2}(n+1)^{1/2} \times$$

$$\times \Big\{ \sum_{p: 2^{m-1} \leqq m_p < 2^m} \; \sum_{q: 2^{n-1} \leqq n_q < 2^n} \; \sum_{i=m_p}^{m_{p+1}-1} \sum_{k=m_q}^{m_{q+1}-1} a_{ik}^2 \Big\}^{1/2},$$

since for every $m=1, 2, \ldots$ the number of those integers for which $2^{m-1} \leqq m_p < 2^m$ is less than $2m$. Taking into account that the quadruple sum in the last square root does not exceed the double sum

$$\sum_{i=2^{m-1}}^{2^{m+1}-1} \sum_{k=2^{n-1}}^{2^{n+1}-1} a_{ik}^2,$$

we get implication (4.26)⇒(4.24).

Second, if we use the monotonicity of $\{|a_{ij}|\}$ we can immediately see that

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \{(p+1)(q+1) \sum_{i=2^{p-1}}^{2^p-1} \sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2\}^{1/2} \leqq$$

$$\leqq \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (p+1)^{1/2}(q+1)^{1/2} 2^{p/2} 2^{q/2} |a_{2^{p-1}, 2^{q-1}}|,$$

which shows implication (5.1)⇒(4.26).

Third, we show implication (4.24)⇒(5.1) in the monotonic case. Again by the Cauchy inequality,

$$S_1 = O(1) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=2^{p-1}}^{2^p-1} \sum_{n=2^{q-1}}^{2^q-1} |a_{mn}| \times$$

$$\times (m+1)^{-1/2}(n+1)^{-1/2} \log^{1/2}(m+2) \log^{1/2}(n+2) =$$

$$= O(1) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |a_{mn}| (m+1)^{-1/2}(n+1)^{-1/2} \log^{1/2}(m+2) \log^{1/2}(n+2) =$$

$$= O(1) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=m_p}^{m_{p+1}-1} \sum_{n=m_q}^{m_{q+1}-1} |a_{mn}| \times$$

$$\times (m+1)^{-1/2}(n+1)^{-1/2} \log^{1/2}(m+2) \log^{1/2}(n+2) =$$

$$= O(1) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \{\sum_{m=m_p}^{m_{p+1}-1} \sum_{n=m_q}^{m_{q+1}-1} a_{mn}^2\}^{1/2} R_{pq},$$

where by (4.8*),

$$R_{pq} = \{\sum_{m=m_p}^{m_{p+1}-1} \sum_{n=m_q}^{m_{q+1}-1} (m+1)^{-1}(n+1)^{-1} \log(m+2) \log(n+2)\}^{1/2} \leqq$$

$$\leqq \{(m_{p+1}-m_p)(m_{q+1}-m_q)(m_p+1)^{-1}(m_q+1)^{-1} p^{1/2} q^{1/2}\}^{1/2} = O(1).$$

This proves implication (4.24)⇒(5.1).

Case 2. The equivalence of (4.29), (4.30), and (5.4). This time we use notation (4.23).

First, we show that (4.30) implies (4.29) without any restriction. By the Cauchy inequality,

$$S_4=\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q:2^{n-1}\le k_q<2^n}\Big\{\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=k_q}^{k_{q+1}-1}a_{ik}^2\Big\}^{1/2}\le$$

$$\le\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\sum_{q:2^{n-1}\le k_q<2^n}\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=k_q}^{k_{q+1}-1}a_{ik}^2\Big\}^{1/2}\times\Big\{\sum_{q:2^{n-1}\le k_q<2^n}1\Big\}^{1/2}.$$

Since the number of those integers q for which $2^{n-1}\le k_q<2^n$ is $O_\beta(2^{n(1-2\beta)})$ thus

$$S_4=O_\beta(1)\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\Big\{2^{n(1-2\beta)}\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{n-1}}^{2^{n+1}-1}a_{ik}^2\Big\}^{1/2}.$$

This proves implication (4.30)⇒(4.29).

Second, using the monotonicity of $\{|a_{ik}|\}$ we can easily get implication (5.4)⇒ (4.30) as follows

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{2^{q(1-2\beta)}\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1}a_{ik}^2\Big\}^{1/2}\le\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{q(1-\beta)}\Big\{\sum_{i=2^{p-1}}^{2^p-1}a_{i,2^{q-1}}^2\Big\}^{1/2}.$$

Third, we show implication (4.29)⇒(5.4) in the monotonic case. By the Cauchy inequality again,

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{q(1-\beta)}\Big\{\sum_{i=2^{p-1}}^{2^p-1}a_{i,2^q}^2\Big\}^{1/2}=O_\beta(1)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=2^{q-1}}^{2^q-1}k^{-\beta}\Big\{\sum_{i=2^{p-1}}^{2^p-1}a_{ik}^2\Big\}^{1/2}=$$

$$=O_\beta(1)\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}k^{-\beta}\Big\{\sum_{i=2^{p-1}}^{2^p-1}a_{ik}^2\Big\}^{1/2}=O_\beta(1)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=k_q}^{k_{q+1}-1}k^{-\beta}\Big\{\sum_{i=2^{p-1}}^{2^p-1}a_{ik}^2\Big\}^{1/2}=$$

$$=O_\beta(1)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{\sum_{k=k_q}^{k_{q+1}-1}k^{-2\beta}\Big\}^{1/2}\Big\{\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=k_q}^{k_{q+1}-1}a_{ik}^2\Big\}^{1/2}.$$

Since (4.8**) holds true for $\beta>-1$ we have

$$\sum_{k=k_q}^{k_{q+1}-1}k^{-2\beta}\le(k_{q+1}-k_q)k_q^{-2\beta}=O_\beta(1),$$

proving implication (4.29)⇒(5.4).

After these preliminaries, the point is that if $\{|a_{ik}|\}$ is nonincreasing in a certain sense indicated above, then conditions (5.1)—(5.5) are not only sufficient, but also necessary for the a.e. $|C,\alpha,\beta|$-summability of series (2.1), for a fixed pair of α and β in the appropriate domain, if all ONS φ are considered.

To go into details, the case $\min(\alpha,\beta)>1/2$ was studied in [9] without any additional restriction on $\{|a_{ik}|\}$. Theorem C obtained there extends the corresponding results of Billard [2] ($\alpha=1$) and Grepachevskaja [4] ($\alpha>1/2$) from single to double orthogonal series.

Theorem C. *If* $\alpha>1/2$, $\beta>1/2$ *and condition* (4.1) *is not satisfied, then the two-dimensional Rademacher series* (2.7) *is not* $|C, \alpha, \beta|$*-summable a.e.*

The following theorems cover various cases in the domain $-1<\min(\alpha, \beta)\leq 1/2$.

Theorem 9. *If the double sequence* $\{|a_{ik}|\}$ *is nonincreasing and condition* (5.1) *is not satisfied, then series* (2.7) *is not* $|C, 1/2, 1/2|$*-summable a.e.*

Theorem 10. *If* $-1<\alpha<1/2$, $-1<\beta<1/2$, *the double sequence* $\{|a_{ik}|\}$ *is nonincreasing, and condition* (5.2) *is not satisfied, then series* (2.7) *is not* $|C, \alpha, \beta|$*-summable a.e.*

Theorems 9 and 10 are the extensions of the corresponding results of GREPACHEVSKAJA [4] from the one-dimensional Rademacher system to the two-dimensional one. Theorem 10 for two-dimensional trigonometric series was proved by PONOMARENKO and TIMAN [11], assuming that $\{a_{ik}\}$ is a nonincreasing sequence of nonnegative numbers.

Serving as a pattern, we present here the proof of Theorem 9. In this case, t_{ik}^{mn} is defined by (2.8) for $\alpha=\beta=1/2$.

First, we check that condition (2.6) is satisfied. This is simple by the means of estimates (4.18), (4.7), and the corresponding estimate on τ_{ik}^{m0} all applied in the case $\alpha=\beta=1/2$.

Second, we verify that condition (2.5) is not satisfied. Thus, we can apply Theorem 2 and conclude the statement of Theorem 9. In fact, again by (4.18), (4.7) and its symmetric counterpart as well as by the monotonicity of $\{|a_{ik}|\}$,

$$(5.6) \qquad S_{11} = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} 2^{p/2}2^{q/2}p^{1/2}q^{1/2}|a_{2^p,2^q}| =$$

$$= O(1)\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=2^{p-1}}^{2^p-1}\sum_{n=2^{q-1}}^{2^q-1} |a_{mn}|\, m^{-1/2}n^{-1/2}\log^{1/2}(m+1)\log^{1/2}(n+1) =$$

$$= O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{mn}|\, m^{-1/2}n^{-1/2}\log^{1/2}(m+1)\log^{1/2}(n+1) =$$

$$= O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{mn}|\, m^{-3/2}n^{-3/2}\Big\{\sum_{i=m/2}^{m} i^2(m+1-i)^{-1}\sum_{k=n/2}^{n} k^2(n+1-k)^{-1}\Big\}^{1/2} =$$

$$= O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\{\sum_{i=0}^{m}\sum_{k=0}^{n} a_{ik}^2 i^2 m^{-3}(m+1-i)^{-1}\times$$

$$\times k^2 n^{-3}(n+1-k)^{-1}\Big\}^{1/2} = O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathcal{A}_{mn}$$

(cf. notation (4.3)).

Similarly, we can obtain that

(5.7) $$S_{01} = \sum_{q=1}^{\infty} 2^{q/2} q^{1/2} |a_{0,2^q}| = O(1) \sum_{n=1}^{\infty} \mathscr{A}_{0n}$$

and

(5.8) $$S_{10} = \sum_{p=1}^{\infty} 2^{p/2} p^{1/2} |a_{2^p,0}| = O(1) \sum_{m=1}^{\infty} \mathscr{A}_{m0}.$$

Collecting (5.6)—(5.8) we find that

$$S_1 = |a_{00}| + S_{01} + S_{10} + S_{11} = O(1) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathscr{A}_{mn}$$

(see also (4.6)). Since, by assumption $S_1 = \infty$ condition (2.5) cannot be satisfied either. Applying Theorem 2 gives the statement of Theorem 9.

The last two theorems in this Section are concerned with the "mixed" cases.

Theorem 11. *Assume that the sequence* $\{|a_{ik}|\}$ *is nonincreasing in k for each fixed i. If* $\alpha > 1/2$, $\beta = 1/2$ *and condition* (5.3) *is not satisfied, or if* $\alpha > 1/2$, $-1 < \beta < 1/2$ *and condition* (5.4) *is not satisfied, then series* (2.7) *is not* $|C, \alpha, \beta|$*-summable a.e.*

Theorem 12. *If* $\alpha = 1/2$, $-1 < \beta < 1/2$, *the sequence* $\{|a_{ik}|\}$ *is nonincreasing, and condition* (5.5) *is not satisfied, then series* (2.7) *is not* $|C, \alpha, \beta|$*-summable a.e.*

Theorems 10—12 can be proved in a similar fashion to as Theorem 9 is proved above on the basis of Theorem 2.

6. Generalized $|C, \alpha, \beta|_l$-summability of orthogonal series

Let $l \geqq 1$ be a real number. Following Flett [3], series (2.1) is said to be $|C, \alpha, \beta|_l$*-summable* at x if

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1)^{l-1} (n+1)^{l-1} |\Delta_{mn}^{\alpha\beta}(x)|^l < \infty,$$

where $\Delta_{mn}^{\alpha\beta}(x)$ is defined in (3.1) with the matrix given by (2.8). The case $l=1$ gives back the ordinary $|C, \alpha, \beta|$-summability. Using the same techniques which occur in the proofs of Theorems 3—12 and Corollaries 2—7, we can derive both necessary and sufficient conditions on the a.e. $|C, \alpha, \beta|_l$-summability of series (2.1). Here we present only three samples of these extensions. We use the notation $m_p = 2^{(p-1)^{1-1/l}}$.

Theorem 3*. *If* $1 \leqq l \leqq 2$ *and*

(6.1) $$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \Big\{ \sum_{i=m_p}^{m_{p+1}-1} \sum_{k=m_q}^{m_{q+1}-1} a_{ik}^2 \Big\}^{l/2} < \infty,$$

then series (2.1) *is* $|C, 1/2, 1/2|_l$*-summable* μ*-a.e.*

Corollary 6*. *Let* $1 \leqq l \leqq 2$ *and* $\{\lambda_{ik}\}$ *be a nondecreasing sequence of positive numbers satisfying the condition*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(i+1)(k+1)\lambda_{ik}^{l}} < \infty. \tag{6.2}$$

If $\alpha > 1/2$, $\beta > 1/2$ *and*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^{2} \lambda_{ik}^{2-l} < \infty;$$

or if $\alpha = 1/2$, $\beta = 1/2$ *and*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^{2} \lambda_{ik}^{2-l} \log(i+2) \log(k+2) < \infty;$$

or if $-1 < \alpha < 1/2$, $-1 < \beta < 1/2$ *and*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^{2} \lambda_{ik}^{2-l} (i+1)^{1-2\alpha} (k+1)^{1-2\beta} < \infty,$$

then series (2.1) *is* $|C, \alpha, \beta|_l$*-summable* μ*-a.e.*

We note that in case $l=2$ condition (6.2) can be dropped.

Theorem 9*. *Let* $1 \leqq l \leqq 2$. *If the sequence* $\{|a_{ik}|\}$ *is nonincreasing and condition* (6.1) *is not satisfied, then series* (2.7) *is not* $|C, 1/2, 1/2|$*-summable a.e.*

Theorems 3*, 9* and Corollary 6* are the extensions of the corresponding theorems of the second named author [12] and Spevakov [13], respectively, from single orthogonal series to double ones.

On closing, we mention that our results can be extended in a natural way to d-multiple orthogonal series with $d \geqq 3$, too.

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О подсистемах сходимости произвольной ортонормированной системы

Г. А. КАРАГУЛЯН

Известна следующая теорема, доказанная в 1936 г. независимо Д. Е. Меньшовым и И. Марцинкевичем:

Теорема А. (см. [1], [2]). *Для любой ортонормированной системы (ОНС)* $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x\in(0, 1)$ *существуют номера* $n_1<n_2<\dots$ *такие, что подсистема* $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ *является системой сходимости (ОНС* $\{\psi_n(x)\}_{n=1}^{\infty}$ *называется системой сходимости, если всякий ряд*

$$\sum_{n=1}^{\infty} a_n\psi_n(x), \quad \sum_{n=1}^{\infty} a_n^2<\infty$$

сходится почти всюду).

По этому поводу в работе [3] Г. Беннетом был поставлен следующий вопрос: существует ли последовательность чисел $\{r_k\}_{k=1}^{\infty}$ такая, что из лыбой ОНС $\{\varphi_n(x)\}_{n=1}^{\infty}$ можно извлечь подсистему сходимости $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$, для которой $\lim\limits_{k\to\infty}\frac{n_k}{r_k}=0$?

В работе [4] Б. С. Кашиным дан положительный ответ на этот вопрос. В ней сформулирована следующая.

Теорема В. *Из произвольной ОНС* $\{\varphi_n(x)\}_{n=1}^{\infty}$ *можно извлечь подсистему сходимости* $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ *с* $n_k<R_k$ $(k=1, 2, \dots)$, *где* $R_1=3$, $R_{k+1}=(R_k)!$ $(k=1, 2, \dots)$.

В той же работе ([4]) Б. С. Кашин поставил следующий вопрос: можно ли в формулировке теоремы В условие $n_k<R_k$ заменить на $n_k<k^{1+\varepsilon}$ $(k=1, 2, \dots)$ для любого $\varepsilon>0$?

В настоящей работе усилен результат теоремы В. Точнее, доказывается следующая

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Теорема. *Для любой ОНС* $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x\in(0,1)$ *и любого* $\beta>0$ *существует подсистема сходимости* $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ *с условием*

$$n_1=1,\ 2^{(2+\beta)(k-1)\log_2(k-1)}<n_k\leqq 2^{(2+\beta)k\log_2 k}\quad (k=2,3,\dots).$$

Лемма 1. *Пусть* $\{\varphi_k(x)\}_{k=1}^{n}$, $x\in(0,1)$ *конечная ОНС и* $\{E_i\}_{i=1}^{m}$ *семейство измеримых множеств из* $(0,1)$. *Тогда существует целое число* $1\leqq k\leqq n$ *такое, что*

$$\frac{1}{\mu(E_p)}\left|\int_{E_p}\varphi_k(t)\,dt\right|\leqq\sqrt{\frac{m}{n\cdot\min\limits_{1\leqq i\leqq m}\mu(E_i)}},\quad p=1,2,\dots,m$$

(μ*-мера Лебега на* $(0,1)$).

Доказательство. Предположим обратное, что для любого $1\leqq k\leqq n$ существует, зависящее от k, число $1\leqq p(k)\leqq m$ такое, что

$$\text{(1)}\qquad \frac{1}{\mu(E_{p(k)})}\left|\int_{E_{p(k)}}\varphi_k(t)\,dt\right|>\sqrt{\frac{m}{n\cdot\min\limits_{1\leqq i\leqq m}\mu(E_i)}}\quad(1\leqq k\leqq n).$$

Тогда легко убедится, что для некоторого $1\leqq q\leqq m$ равенство $p(k)=q$ выполняется при некоторых различных $k=k_1,k_2,\dots,k_l$, где $l\geqq\dfrac{n}{m}$. И следовательно, для этого q имеем (см. (1))

$$\text{(2)}\qquad \sum_{i=1}^{l}\left(\frac{1}{\mu(E_q)}\int_{E_q}\varphi_{k_i}(t)\,dt\right)^2=\sum_{i=1}^{l}\left(\frac{1}{\mu(E_{p(k_i)})}\int_{E_{p(k_i)}}\varphi_{k_i}(t)\,dt\right)^2>$$

$$>l\left(\sqrt{\frac{m}{n\cdot\min\limits_{1\leqq i\leqq m}\mu(E_i)}}\right)^2=l\cdot\frac{m}{n}\cdot\frac{1}{\min\limits_{1\leqq i\leqq m}\mu(E_i)}\geqq\frac{1}{\mu(E_q)}.$$

С другой стороны, используя неравенство Бесселя для ОНС $\{\varphi_k(x)\}_{k=1}^{n}$, имеем

$$\text{(3)}\qquad \sum_{i=1}^{l}\left(\frac{1}{\mu(E_q)}\int_{E_q}\varphi_{k_i}(t)\,dt\right)^2\leqq\frac{1}{[\mu(E_q)]^2}\|\chi_{E_q}\|^2_{L^2(0,1)}=\frac{1}{\mu(E_q)}.$$

Из (2) и (3) получится противоречие. Следовательно, наше предположение неверно. Лемма 1 доказана.

Пусть $\{E_i^{(m)},\ i\in Q^{(m)}\}$, $m=1,2,\dots$, где $Q^{(m)}$ $(m\geqq1)$ конечное или счётное множество индексов i, семейства измеримых множеств из $(0,1)$, уловлетворяющих следующим условиям:

(4) 1) $\mu\Big(\bigcup\limits_{i\in Q^{(m)}}E_i^{(m)}\Big)=1$, $\mu(E_i^{(m)})>0$, $i\in Q^{(m)}$, $m=1,2,\dots,$

(5) 2) $E_i^{(m)}\cap E_j^{(m)}=\emptyset$ при $i\neq j$ $(i,j\in Q^{(m)})$,

(6) 3) если $E_i^{(n)}\cap E_j^{(m)}\neq\emptyset$ $(n\geqq m)$, то $E_i^{(n)}\subset E_j^{(m)}$.

Тогда имеем, что множество

$$E = \bigcap_{m=1}^{\infty} \Big(\bigcup_{i \in Q^{(m)}} E_i^{(m)} \Big)$$

имеет полную меру на (0, 1). Обазначая $\tilde{E}_i^{(m)} = E_i^{(m)} \cap E$, легко убедится, что выполняются чледующие условия:

1°) $\bigcup_{i \in Q^{(m)}} \tilde{E}_i^{(m)} = E, \quad \mu(\tilde{E}_i^{(m)}) > 0, \quad i \in Q^{(m)}, \quad m = 1, 2, \ldots,$

2°) $\tilde{E}_i^{(m)} \cap \tilde{E}_j^{(m)} = \emptyset$ при $i \neq j \quad (i, j \in Q^{(m)})$,

3°) если $\tilde{E}_i^{(n)} \cap \tilde{E}_j^{(m)} \neq \emptyset \quad (n \geqq m)$, то $\tilde{E}_i^{(n)} \subset \tilde{E}_j^{(m)}$.

Пусть T_m $(m \geqq 1)$ — σ-алгебра порождённая из множеств $E_i^{(m)}$, $i \in Q^{(m)}$. Тогда имеем $T_m \subset T_{m+1}$ $(m = 1, 2, \ldots)$. Очевидно, что если $g \in L^1(0, 1)$, то последовательность функций

$$g_m(x) = \frac{1}{\mu(\tilde{E}_{i^{(m)}(x)}^{(m)})} \int_{\tilde{E}_{i^{(m)}(x)}^{(m)}} g(t)\,dt = \frac{1}{\mu(E_{i^{(m)}(x)}^{(m)})} \int_{E_{i^{(m)}(x)}^{(m)}} g(t)\,dt, \quad x \in E,$$

где $\tilde{E}_{i^{(m)}(x)}^{(m)}$ есть множество из семейства $\{\tilde{E}_i^{(m)}, i \in Q^{(m)}\}$ $(m \geqq 1)$ соодержащее в себе точку $x \in E$ (в силу условий 1° и 2°, очевидно, что для любого $x \in E$ такое множество существует и единственно), образует мартингал относительно семейства σ-алгебр T_m $(m \geqq 1)$ и удовлетворяет условию $\sup_{1 \leqq m < \infty} \int_E |g_m(t)|\,dt < \infty$ (определение мартингала см. напр. [5] стр. 103).

Тогда, используя известный факт (см. [5] стр. 112) о том, что любой мартингал $\{f_m(x)\}_{m=1}^{\infty}$ (относительно некоторого семейства σ-алгебр $T_m (T_m \subset T_{m+1})$), удовлетворяющий условию $\sup_{1 \leqq m < \infty} \int |f_m(t)|\,dt < \infty$, сходится почти всюду, имеем, что существует предел

$$\lim_{m \to \infty} g_m(x) = \lim_{m \to \infty} \frac{1}{\mu(E_{i^{(m)}(x)}^{(m)})} \int_{E_{i^{(m)}(x)}^{(m)}} g(t)\,dt = g_\infty(x), \quad \big(g(x) \in L^1(0, 1)\big) \tag{7}$$

п.в. на E, и следовательно, п.в. на (0, 1) $\big(g_\infty(x)$ — некоторая п.в. конечная функция на (0, 1)$\big)$.

Используя этот факт, докажем следующую лемму:

Лемма 2. *Пусть* $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in (0, 1)$ *ортонормированная система и* $\{E_i^{(m)}, i \in Q^{(m)}\}$, $m = 1, 2, \ldots$, *где* $Q^{(m)}$ $(m \geqq 1)$ *есть конечное или счётное множество индексов* i, *семейства множеств удовлетворяющих условиям* (4), (5) *и* (6). *Предположим, что справедливы следующие соотношения:*

1) *Для любого* $m=1, 2, \dots$ *существует подмножество индексов* $G^{(m)}\subset Q^{(m)}$ *такое, что*

$$\sum_{m=1}^{\infty}\left[1-\mu\Big(\bigcup_{i\in G^{(m)}} E_i^{(m)}\Big)\right]<\infty \tag{8}$$

и

$$\frac{1}{\mu(E_i^{(k)})}\left|\int_{E_i^{(k)}}\varphi_n(t)\,dt\right|<\gamma_n \tag{9}$$

при

$$1\leqq k\leqq n-1,\quad i\in G^{(k)}\quad (n\geqq 2),$$

где γ_n $(n\geqq 2)$ *такие числа, что*

$$\sum_{n=2}^{\infty}\gamma_n<\infty; \tag{10}$$

2) *Для любого* $k=1, 2, \dots$ *справедливо равенство*

$$\lim_{m\to\infty}\left|\frac{1}{\mu(E^{(m)}_{i^{(m)}(x)})}\int_{E^{(m)}_{i^{(m)}(x)}}\varphi_k(t)\,dt-\varphi_k(x)\right|=0 \quad \text{для п.в.}\quad x\in(0, 1), \tag{11}$$

3) *Для любой точки x из множества*

$$E=\bigcup_{k\geqq 1}\bigcap_{m\geqq k}\Big[\bigcup_{i\in G^{(m)}} E_i^{(m)}\Big] \tag{12}$$

существует, зависящее от x постоянное $c(x)$ *такое, что*

$$\sum_{k=1}^{m}\left|\frac{1}{\mu(E^{(m)}_{i^{(m)}(x)})}\int_{E^{(m)}_{i^{(m)}(x)}}\varphi_k(t)\,dt-\varphi_k(x)\right|^2<c(x),\quad m=1, 2, \dots. \tag{13}$$

Тогда $\{\varphi_n(x)\}_{n=1}^{\infty}$ *является системой сходимости.*

Доказательство. Предполагая

$$\sum_{k=1}^{\infty}a_k^2<\infty, \tag{14}$$

имеем, что ряд

$$\sum_{k=1}^{\infty}a_k\varphi_k(x) \tag{15}$$

сходится в метрике $L^2(0, 1)$ к некоторой функции $f(x)\in L^2(0, 1)$. Тогда для любого измеримого множества $F\subset(0, 1)$, произведя пределный переход под знаком интеграла, имеем

$$\int_F f(t)\,dt=\int_F\sum_{k=1}^{\infty}a_k\varphi_k(t)\,dt=\sum_{k=1}^{\infty}a_k\int_F\varphi_k(t). \tag{16}$$

В силу (7), справедливо равенство

$$\lim_{m\to\infty}\frac{1}{\mu(E^{(m)}_{i(m)(x)})}\int\limits_{E^{(m)}_{i(m)(x)}} f(t)\,dt=f_\infty(x)\quad \text{для п.в.}\quad x\in(0,1) \tag{17}$$

для некоторой п.в. конечной функции $f_\infty(x)$.

Пусть A и B множества таких x, для которых выполняются, соответственно, равенства (17) и (11). Тогда они имеют полные меры. Из (8) следует, что множество E (см. (12)) тоже имеет полную меру. Следовательно, обозначив

$$D=A\cap B\cap E, \tag{18}$$

имеем $\mu(D)=1$. Тогда, для доказательства сходимости п.в. ряда (15), достаточно доказать её сходимость на множестве D.

Итак, пусть точка $x\in D$ фиксирована. Из (16) для любого $m=1,2,\dots$ имеем

$$\sum_{k=1}^{\infty}\frac{a_k}{\mu(E^{(m)}_{i(m)(x)})}\int\limits_{E^{(m)}_{i(m)(x)}}\varphi_k(t)\,dt=\frac{1}{\mu(E^{(m)}_{i(m)(x)})}\int\limits_{E^{(m)}_{i(m)(x)}} f(t)\,dt\quad(m\geqq 1). \tag{19}$$

Пусть $\varepsilon>0$ некоторое число. В силу (14) существует число M такое, что

$$\Big(\sum_{k=M+1}^{\infty}a_k^2\Big)^{1/2}<\varepsilon/\big(2\sqrt{c(x)}\big). \tag{20}$$

Используя (10), (11), а также включение $x\in D\subset B$ (см. (18)), где B определено выше, для этого M найдётся число $N>M$ такое, что одновременно выполнялись неравенства

$$\sum_{k=N+1}^{\infty}\max_{1\leqq l<\infty}|a_l|\cdot\gamma_k<\varepsilon/4 \tag{21}$$

и

$$\sum_{k=1}^{M}|a_k|\left|\frac{1}{\mu(E^{(m)}_{i(m)(x)})}\int\limits_{E^{(m)}_{i(m)(x)}}\varphi_k(t)\,dt-\varphi_k(x)\right|<\varepsilon/4\quad\text{при}\quad m>N. \tag{22}$$

Тогда, используя неравенство Гёльдера и включение $x\in D\subset E$ (см. (18)), при $m>N>M$ имеем (см. (13), (20), (21), (22))

$$\left|\sum_{k=1}^{m}a_k\varphi_k(x)-\sum_{k=1}^{\infty}a_k\frac{1}{\mu(E^{(m)}_{i(m)(x)})}\int\limits_{E^{(m)}_{i(m)(x)}}\varphi_k(t)\,dt\right|=$$

$$=\left|\sum_{k=1}^{M}a_k\left(\varphi_k(x)-\frac{1}{\mu(E^{(m)}_{i(m)(x)})}\int\limits_{E^{(m)}_{i(m)(x)}}\varphi_k(t)\,dt\right)+\right.$$

$$+\sum_{k=M+1}^{m} a_k\left(\varphi_k(x)-\frac{1}{\mu(E^{(m)}_{i^{(m)}(x)})}\int\limits_{E^{(m)}_{i^{(m)}(x)}}\varphi_k(t)\,dt\right)-$$

$$-\sum_{k=m+1}^{\infty}\frac{a_k}{\mu(E^{(m)}_{i^{(m)}(x)})}\int\limits_{E^{(m)}_{i^{(m)}(x)}}\varphi_k(t)\,dt\Bigg|\leqq$$

$$\leqq\sum_{k=1}^{M}|a_k|\left|\frac{1}{\mu(E^{(m)}_{i^{(m)}(x)})}\int\limits_{E^{(m)}_{i^{(m)}(x)}}\varphi_k(t)\,dt-\varphi_k(x)\right|+$$

$$+\Big(\sum_{k=M+1}^{m}a_k^2\Big)^{1/2}\left[\sum_{k=M+1}^{m}\left(\frac{1}{\mu(E^{(m)}_{i^{(m)}(x)})}\int\limits_{E^{(m)}_{i^{(m)}(x)}}\varphi_k(t)\,dt-\varphi_k(x)\right)^2\right]^{1/2}+$$

$$+\sum_{k=m+1}^{\infty}\max_{1\leqq l<\infty}|a_l|\cdot\gamma_k\leqq\varepsilon/4+\varepsilon\sqrt{c(x)}/(2\sqrt{c(x)})+\varepsilon/4=\varepsilon.$$

И следовательно, получаем

$$\lim_{m\to\infty}\left(\sum_{k=1}^{m}a_k\varphi_k(x)-\sum_{k=1}^{\infty}\frac{a_k}{\mu(E^{(m)}_{i^{(m)}(x)})}\int\limits_{E^{(m)}_{i^{(m)}(x)}}\varphi_k(t)\,dt\right)=0. \tag{23}$$

Из (17), (19) и (23), имея ввиду включение $x\in D\subset A$ (см. (18)), где A определено выше, имеем $\lim\limits_{m\to\infty}\sum\limits_{k=1}^{m}a_k\varphi_k(x)=f_\infty(x)$.

Лемма 2 доказана.

Лемма 3. *Для любых* $m=1,2,\ldots,$ $k\leqq m$ *и* $\alpha>1/2$ *существует семейство полуоткрытых интервалов* $\{\Delta_i^{(m)}(k),\ i\in\mathbf{Z}\}$ *(*$\mathbf{Z}$*-множество целых чисел), удовлетворяющих следующим условиям:*

1) $\bigcup\limits_{i\in\mathbf{Z}}\Delta_i^{(m)}(k)=R^1$ *при любых* $m\geqq k\geqq 1$ $(R^1=(-\infty,+\infty))$;

2) $\Delta_i^{(m)}(k)\cap\Delta_j^{(m)}(k)=\emptyset$ *при* $i\neq j$ $(m\geqq k\geqq 1)$;

3) *если* $m\geqq n\geqq k$ *и* $\Delta_i^{(m)}(k)\cap\Delta_j^{(n)}(k)\neq\emptyset$, *то* $\Delta_i^{(m)}(k)\subset\Delta_j^{(n)}(k)$;

4) *для любого* $k=1,2,\ldots$ *справедливо равенство*

$$\lim_{\substack{m\to\infty\\ m\geqq k}}\big[\sup_{i\in\mathbf{Z}}d(\Delta_i^{(m)}(k))\big]=0, \tag{24}$$

где $d(\Delta_i^{(m)}(k))$ *— длина интервала* $\Delta_i^{(m)}(k)$;

5) *при любых* $m\geqq k\geqq 1$ *существуют конечные подмножества целых чисел* $G^{(m)}(k)$ *такие, что*

$$[-[k^\alpha],[k^\alpha])=\bigcup_{i\in G^{(m)}(k)}\Delta_i^{(m)}(k)\quad(m\geqq k\geqq 1), \tag{25}$$

$$\max_{i\in G^{(m)}(k)}d(\Delta_i^{(m)}(k))\leqq\frac{2}{\sqrt{m}}\quad(m\geqq k\geqq 1), \tag{26}$$

где $[k^\alpha]$ *— целая часть числа* k^α;

6) *для любых* $m \geqq k \geqq 1$ *существуют конечные подмножества целых чисел* $L^{(m)}(k)$ *такие, что*

(27) $$[-[m^{1/2+\alpha}], [m^{1/2+\alpha}]) = \bigcup_{i \in L^{(m)}(k)} \Delta_i^{(m)}(k) \quad (m \geqq k \geqq 1),$$

(28) $$|L^{(m)}(k)| \leqq 4m^{2\alpha} \quad (m \geqq k \geqq 1),$$

(29) $$L^{(m)}(k) \supset G^{(m)}(k) \quad (m \geqq k \geqq 1),$$

где через $|L^{(m)}(k)|$ *обозначается количество элементов множества* $L^{(m)}(k)$.

Доказательство. Для определения множеств $\Delta_i^{(m)}(k)$, $i \in \mathbf{Z}$ $(m \geqq k \geqq 1)$ обозначим

(30) $$r_m = \left[\left(\alpha - \frac{1}{2}\right)\log_2 m\right] \quad \text{при} \quad m \geqq 1,$$

(31) $$l_k^{(m)} = \left[\frac{1}{2}\log_2 m\right] \quad \text{при} \quad m \geqq k \geqq 1.$$

Теперь, используя эти обозначения, для фиксированных $m \geqq k \geqq 1$ определим множество $\Delta_i^{(m)}(k)$ $(i \in \mathbf{Z})$ равным

(32) $$[i/2^{l_k^{(m)}}, (i+1)/2^{l_k^{(m)}}) \quad \text{при} \quad -[k^\alpha]\cdot 2^{l_k^{(m)}} \leqq i < [k^\alpha]\cdot 2^{l_k^{(m)}},$$

(33) $$\left[\frac{i-[k^\alpha](2^{l_k^{(m)}}-2^{r_m})}{2^{r_m}}, \frac{i+1-[k^\alpha](2^{l_k^{(m)}}-2^{r_m})}{2^{r_m}}\right) \quad \text{при} \quad i \geqq [k^\alpha]\cdot 2^{l_k^{(m)}},$$

(34) $$\left[\frac{i+[k^\alpha](2^{l_k^{(m)}}-2^{r_m})}{2^{r_m}}, \frac{i+1+[k^\alpha](2^{l_k^{(m)}}-2^{r_m})}{2^{r_m}}\right) \quad \text{при} \quad i < -[k^\alpha]\cdot 2^{l_k^{(m)}}.$$

Для дальнейших рассуждений заметим, что группа (32) интервалов $\Delta_i^{(m)}(k)$ (при фиксированных $m \geqq k \geqq 1$) представляет собою разбиение множества $[-[k^\alpha], [k^\alpha])$ на непересекающиеся полуоткрытые интервалы длинами $\dfrac{1}{2^{l_k^{(m)}}}$, а группы (33) и (34) представляют собою соотственно разбиение множеств $[[k^\alpha], +\infty)$ и $(-\infty, -[k^\alpha])$ на непересекающиеся полуоткрытые интервалы длинами $\dfrac{1}{2^{r_m}}$.

Тогда, очевидна выполнимость условий 1) и 2). Очевидно также 3), если заметить, что при фиксированном $k \geqq 1$, $l_k^{(m)}$ и r_m неубывают относительно m (см. (30), (31)).

Из того же факта следует

$$\sup_{i \in \mathbf{Z}} d(\Delta_i^{(m)}(k)) = \max\{1/2^{l_k^{(m)}}, 1/2^{r_m}\} \quad (m \geqq k \geqq 1),$$

и следовательно, имея ввиду (30) и (31), имеем (24).

Обозначим

$$(35)\qquad G^{(m)}(k)=\{i\in\mathbf{Z},\ -[k^\alpha]\cdot 2^{l_k^{(m)}}\leqq i<[k^\alpha]\cdot 2^{l_k^{(m)}}\}\quad (m\geqq k\geqq 1).$$

Тогда, очевидно, что множества $\Delta_i^{(m)}(k)$, $i\in G^{(m)}(k)$ представляют интервалы из группы (32), и следовательно, в силу замечания сделанного выше, имеем (25). Выполнимость условия (26) тоже очевидно. Действительно, имеем (см. (31), (32), (35))

$$\max_{i\in G^{(m)}(k)} d(\Delta_i^{(m)}(k))=1/2^{l_k^{(m)}}=1/2^{\left[\frac{1}{2}\log_2 m\right]}\leqq 2/2^{\frac{1}{2}\log_2 m}=2/\sqrt{m}.$$

Обозначим

$$(36)\qquad L^{(m)}(k)=\{i\in\mathbf{Z};\ -[m^{1/2+\alpha}]\cdot 2^{r_m}-[k^\alpha](2^{l_k^{(m)}}-2^{r_m})\leqq$$
$$\leqq i<[m^{1/2+\alpha}]\cdot 2^{r_m}+[k^\alpha](2^{l_k^{(m)}}-2^{r_m})\}.$$

Тогда, из (35) и (36), с учётом неравенства $m\geqq k$, легко получить (29). Легко убедиться, также, в справедливости соотношения (27). Покажем выполнимость неравенства (28). Используя неравенства $m\geqq k$ и $\alpha>1/2$ имеем (см. (30), (31), (36))

$$|L^{(m)}(k)|=[m^{1/2+\alpha}]\cdot 2^{r_m}+[k^\alpha](2^{l_k^{(m)}}-2^{r_m})-$$
$$-\left(-[m^{1/2+\alpha}]2^{r_m}-[k^\alpha](2^{l_k^{(m)}}-2^{r_m})\right)=$$
$$=2\left([m^{1/2+\alpha}]\cdot 2^{r_m}+[k^\alpha](2^{l_k^{(m)}}-2^{r_m})\right)\leqq$$
$$\leqq 2\left(m^{1/2+\alpha}\cdot 2^{\left(\alpha-\frac{1}{2}\right)\log_2 m}+k^\alpha\cdot 2^{\frac{1}{2}\log_2 m}\right)\leqq$$
$$\leqq 2(m^{2\alpha}+m^{\alpha+1/2})<4m^{2\alpha}.$$

Лемма 3 доказана.

Доказательство Теоремы. Прежде, чем выделить подсистему $\{\varphi_{n_k}(x)\}_{k=1}^\infty$ удовлетворяющую требованиям теоремы, предположим её известным и введём некоторые обозначения. Обозначим через $A^{(m)}$ $(m\geqq 1)$ множество всевозможных мультииндексов $\bar{i}=(i_1,i_2,\dots,i_m)$, где $i_1,i_2,\dots,i_m$ целые числа. Для каждого $m=1,2,\dots$ обозначим

$$(37)\qquad E_{\bar{i}}^{(m)}=E_{i_1i_2\dots i_m}^{(m)}=\bigcap_{k=1}^{m}\{x;\ \varphi_{n_k}(x)\in\Delta_{i_k}^{(m)}(k)\}\quad\text{при}\quad \bar{i}\in A^{(m)},$$

где $\Delta_i^{(m)}(k)$ $(i\in\mathbf{Z},\ m\geqq k\geqq 1)$ интервалы удовлетворяющие всем условиям Леммы 3 при $\alpha=\dfrac{1}{2}+\dfrac{\beta}{5}$ (β — заданное в теореме число). Пусть $Q^{(m)}\subset A^{(m)}$ $(m\geqq 1)$ множество тех мультииндексов, для которых имеем

$$(38)\qquad \mu(E_{\bar{i}}^{(m)})>0\quad\text{при}\quad \bar{i}\in Q^{(m)}\quad (m\geqq 1).$$

Обозначим

(39)

$$B^{(m)} = \{\bar{i} = (i_1, i_2, \dots, i_m);\ i_1 \in L^{(m)}(1),\ i_2 \in L^{(m)}(2), \dots, i_{m-1} \in L^{(m)}(m-1),\ i_m \in G^{(m)}(m)\},$$

где $L^{(m)}(k)$ и $G^{(m)}(k)$ $(m \geqq k \geqq 1)$ — множества из Леммы 3. Очевидно, что (см. (28), (29))

$$(40) \qquad |B^{(m)}| = |L^{(m)}(1)| \cdot |L^{(m)}(2)| \cdot \ldots \cdot |L^{(m)}(m-1)| \cdot |G^{(m)}(m)| \leqq 4^m \cdot m^{2\alpha m}.$$

Через $G^{(m)}$ $(m \geqq 1)$ обозначим множество тех мультииндексов из $B^{(m)}$, для которых имеем

$$(41) \qquad \mu(E_{\bar{i}}^{(m)}) \geqq 1/(m^2 \cdot 4^m \cdot 2^{2\alpha m \log_2 m}) \quad \text{при} \quad \bar{i} \in G^{(m)} \subset B^{(m)}.$$

Тогда из соотношений $G^{(m)} \subset B^{(m)}$ и (40), следует

$$(42) \qquad |G^{(m)}| \leqq |B^{(m)}| \leqq 4^m \cdot m^{2\alpha m},$$

и, следовательно, имеем

$$(43) \qquad \sum_{k=1}^{m} |G^{(k)}| \leqq m \cdot 4^m \cdot m^{2\alpha m}.$$

Теперь приступим к построению номеров n_k $(k=1, 2, \dots)$. Построим их такими, чтобы выполнялись следующие условия:

$$(44) \qquad n_1 = 1,\ 2^{(2+\beta)(k-1)\log_2(k-1)} < n_k \leqq 2^{(2+\beta)k \log_2 k} \quad \text{при} \quad k \geqq 2,$$

$$(45) \qquad \frac{1}{\mu(E_{\bar{i}}^{(m)})} \Big| \int_{E_{\bar{i}}^{(m)}} \varphi_{n_k}(t)\, dt \Big| < ((k-1)^3 \cdot 16^{k-1} / 2^{((k-1)\log_2(k-1))/5})^{1/2}$$

$$\text{при} \quad \bar{i} \in G^{(m)}, \quad m = 1, 2, \dots, k-1 \quad (k \geqq 2),$$

где $G^{(m)}$ $(m \geqq 1)$ определны выше (см. (41)).

Сделаем это методом математической индукции. Определим $n_1 = 1$. Предположим, что определены номера $n_1, n_2, \dots, n_p$ такие, что справедливы условия (44) и (45) при $k = 1, 2, \dots, p$. Определим число n_{p+1}. Используя Лемму 1 для множеств $E_{\bar{i}}^{(k)}$, $\bar{i} \in G^{(k)}$, $k = 1, 2, \dots, p$ (количество которых равно $\sum_{k=1}^{p} |G^{(k)}|$) и функций $\varphi_j(x)$, $2^{(2+\beta) p \log_2 p} < j \leqq 2^{(2+\beta)(p+1)\log_2(p+1)}$, найдём натуральное число, которое обозначим через n_{p+1}, такое, что

$$(46) \qquad 2^{(2+\beta) p \log_2 p} < n_{p+1} \leqq 2^{(2+\beta)(p+1)\log_2(p+1)},$$

и выполнялось неравенство

$$\text{(47)}\qquad \frac{1}{\mu(E_l^{(k)})}\Big|\int\limits_{E_l^{(k)}}\varphi_{n_{p+1}}(t)\,dt\Big| <$$

$$<\Big(\sum_{m=1}^{p}|G^{(m)}|/(2^{(2+\beta)(p+1)\log_2(p+1)}-2^{(2+\beta)p\log_2 p}\Big)\min_{\substack{l\in G^{(m)}\\1\le m\le p}}\mu(E_l^{(m)})\Big)^{1/2},$$

$$l\in G^{(k)},\quad k=1,2,\dots,p.$$

С другой стороны, используя очевидное неравенство

$$\text{(48)}\qquad 2^{(2+\beta)(p+1)\log_2(p+1)}-2^{(2+\beta)p\log_2 p}\ge 2^{(2+\beta)p\log_2 p}\quad (p\ge 1)$$

и равенство $\alpha=\frac{1}{2}+\frac{\beta}{5}$, имеем (см. (41), (43), (48))

$$\text{(49)}\qquad \Big(\sum_{m=1}^{p}|G^{(m)}|/(2^{(2+\beta)(p+1)\log_2(p+1)}-2^{(2+\beta)p\log_2 p}\Big)\min_{\substack{l\in G^{(m)}\\1\le m\le p}}\mu(E_l^{(m)})\Big)^{1/2}\le$$

$$\le\big((p\cdot 4^p\cdot p^{2\alpha p})/2^{(2+\beta)p\log_2 p}(1/(p^2\cdot 4^p\cdot p^{2\alpha p}))\big)^{1/2}=\big((p^3\cdot 16^p\cdot 2^{4\alpha p\log_2 p})/2^{(2+\beta)p\log_2 p}\big)^{1/2}=$$

$$=(p^3\cdot 16^p/2^{(p\log_2 p)/5})^{1/2}.$$

Из (46), (47) и (49) следуют неравенства (44) и (45) при $k=p+1$.

Итак, мы построили подсистему $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ удовлетворяющую условиям (44) и (45) при $k\ge 1$.

Докажем, что она есть система сходимости. Для этого достаточно доказать, что она удовлетворяет всем тербованиям Леммы 2 вместе с выше определенными множествами $E_l^{(m)}$ ($l\in Q^{(m)}$) и $G^{(m)}$ ($m=1,2,\dots$). Выполнимость условий (4), (5) и (6) непосредственно следует из определения $E_l^{(m)}$ ($l\in Q^{(m)}$, $m=1,2,\dots$) (см. (37), (38)), если учитывать условия 1)—3) Леммы 3. Обозначив

$$\gamma_k=\big((k-1)^3\cdot 16^{k-1}/2^{((k-1)\log_2(k-1))/5}\big)^{1/2},$$

имеем (10). Тогда из (45) непосредственно следует неравенство (9) для системы $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$. Теперь докажем (8). Так как $G^{(m)}\subset B^{(m)}$ (см. (41)), то имеем

$$\text{(50)}\qquad \mu\Big(\bigcup_{l\in G^{(m)}}E_l^{(m)}\Big)=\mu\Big(\bigcup_{l\in B^{(m)}}E_l^{(m)}\Big)-\mu\Big(\bigcup_{l\in B^{(m)}\setminus G^{(m)}}E_l^{(m)}\Big)\quad (m\ge 1).$$

Из определения $G^{(m)}$ (см. (41)) следует, что

$$\text{(51)}\qquad \mu(E_l^{(m)})<\frac{1}{m^2\cdot 4^m\cdot m^{2\alpha m}}\quad\text{при}\quad l\in B^{(m)}\setminus G^{(m)}\quad (m\ge 1).$$

И, следовательно, имеем (см. (40))

$$\text{(52)}\qquad \mu\Big(\bigcup_{l\in B^{(m)}\setminus G^{(m)}}E_l^{(m)}\Big)<\frac{|B^{(m)}\setminus G^{(m)}|}{m^2\cdot 4^m\cdot m^{2\alpha m}}\le\frac{4^m\cdot m^{2\alpha m}}{m^2\cdot 4^m\cdot m^{2\alpha m}}=\frac{1}{m^2}.$$

Покажем следующее равенство:

(53) $$\bigcup_{\bar{i}\in B^{(m)}} E_{\bar{i}}^{(m)} = \{x;\ \varphi_{n_k}(x)\in[-[m^{1/2+\alpha}],[m^{1/2+\alpha}])$$
$$\text{при}\quad k=1,2,\dots,m-1,\quad \varphi_{n_m}(x)\in[-[m^\alpha],[m^\alpha])\}.$$

Для этого покажем эквивалентность следующих предложений:

(i) $x\in \bigcup_{\bar{i}\in B^{(m)}} E_{\bar{i}}^{(m)}$;

(ii) $x\in E_{\bar{i}^{(m)}(x)}^{(m)}$, где $\bar{i}^{(m)}(x)=(i_1^{(m)}(x), i_2^{(m)}(x),\dots,i_m^{(m)}(x))\in B^{(m)}$;

(iii) $\varphi_{n_k}(x)\in\Delta_{i_k(x)}^{(m)}(k)$, где $i_k(x)\in L^{(m)}(k)$, при $k=1,2,\dots,m-1$ и $i_m(x)\in G^{(m)}(m)$;

(iv) $\varphi_{n_k}(x)\in[-[m^{1/2+\alpha}],[m^{1/2+\alpha}])$ при $k=1,2,\dots,m-1$ и $\varphi_{n_m}(x)\in[-[m^\alpha],[m^\alpha])$;

(v) $x\in\{x;\ \varphi_{n_k}(x)\in[-[m^{1/2+\alpha}],[m^{1/2+\alpha}])$ при $k=1,2,\dots,m-1$ и $\varphi_{n_m}(x)\in[-[m^\alpha],[m^\alpha])\}$.

Эквивалентность условий (i) и (ii) очевидно. Эквивалентность (ii) и (iii) легко следует из определений $E_{\bar{i}}^{(m)}$ и $B^{(m)}$ $(m\geqq 1)$ (см. (37), (39)). Из (25) и (27) следует эквивалентность (iii) и (iv). Эквиалентность (iv) и (v) тоже очевидно. Итак, имеем, что условия (i) и (v) эквивалентны, откуда следует (53).

Обозначая

$$D_k=\{x;\ \varphi_{n_k}(x)\in[-[m^{\alpha+1/2}],[m^{\alpha+1/2}])\}\quad\text{при}\quad k=1,2,\dots,m-1,$$
$$D_m=\{x;\ \varphi_{n_k}(x)\in[-[m^\alpha],[m^\alpha])\}$$

и, используя (53), легко убедится, что

(54) $$\bigcup_{\bar{i}\in B^{(m)}} E_{\bar{i}}^{(m)} = \bigcap_{k=1}^{m} D_k \quad (m\geqq 1).$$

Воспользуясь неравенством Чебишева и ортонормированностью в $L^2(0,1)$ функций $\{\varphi_{n_k}(x)\}_{k=1}^\infty$, имеем

$$\mu(D_k^c)\leqq\frac{1}{[m^{\alpha+1/2}]^2}\quad\text{при}\quad k=1,2,\dots,m-1\quad\text{и}\quad \mu(D_m^c)<\frac{1}{[m^\alpha]^2},$$

где $D_k^c=(0,1)\diagdown D_k$, и, следовательно

$$\mu\Big(\bigcap_{k=1}^m D_k\Big)=1-\mu\Big[\Big(\bigcap_{k=1}^m D_k\Big)^c\Big]=1-\mu\Big(\bigcup_{k=1}^m D_k^c\Big)>$$
$$>1-\frac{m-1}{[m^{\alpha+1/2}]^2}-\frac{1}{[m^\alpha]^2}.$$

Отсюда, используя (54), имеем

(55) $$\mu\Big(\bigcup_{\mathbf{l}\in B^{(m)}} E_{\mathbf{l}}^{(m)}\Big) > 1-\frac{m-1}{[m^{1/2+\alpha}]^2}-\frac{1}{[m^\alpha]^2}.$$

Применив (50), (52) и (55) получаем

(56) $$\mu\Big(\bigcup_{\mathbf{l}\in G^{(m)}} E_{\mathbf{l}}^{(m)}\Big) > 1-\frac{m-1}{[m^{1/2+\alpha}]^2}-\frac{1}{[m^\alpha]^2}-\frac{1}{m^2}.$$

Имея ввиду неравенство $\alpha>\frac{1}{2}$, легко убедиться, что

(57) $$\sum_{m=1}^{\infty}\left(\frac{m-1}{[m^{1/2+\alpha}]^2}+\frac{1}{[m^\alpha]^2}+\frac{1}{m^2}\right)<\infty.$$

Из (56) и (57) следует (8).

Докажем равенство (11) для системы $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$. Для этого, используя определение множеств $E_{\mathbf{l}}^{(m)}$ (см. (37)), заметим, что если $x, t\in E_{\mathbf{l}}^{(m)}$ для некоторого $\mathbf{l}=(i_1, i_2, \dots, i_m)$ $(m\geqq 1)$, то

$$\varphi_{n_k}(t),\ \varphi_{n_k}(x)\in\Delta_{i_k}^{(m)}(k)\quad (k\leqq m),$$

и следовательно, получится

$$|\varphi_{n_k}(x)-\varphi_{n_k}(t)|\leqq d\big(\Delta_{i_k}^{(m)}(k)\big)\leqq\sup_{j\in\mathbf{Z}} d\big(\Delta_j^{(m)}(k)\big)\quad (m\geqq k\geqq 1).$$

Используя последнее, имеем

$$\left|\frac{1}{\mu(E_{\mathbf{l}^{(m)}(x)}^{(m)})}\int_{E_{\mathbf{l}^{(m)}(x)}^{(m)}}\varphi_{n_k}(t)\,dt-\varphi_{n_k}(x)\right| = \left|\frac{1}{\mu(E_{\mathbf{l}^{(m)}(x)}^{(m)})}\int_{E_{\mathbf{l}^{(m)}(x)}^{(m)}}(\varphi_{n_k}(t)-\varphi_{n_k}(x))\,dt\right|\leqq$$
$$\leqq\sup_{j\in\mathbf{Z}} d\big(\Delta_j^{(m)}(k)\big).$$

Из последнего, с учётом (24), получаем (11) для системы $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$. Осталось доказать неравенство (13) для системы $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$. Пусть фиксировано $x\in E= =\bigcup_{k\geqq 1}\bigcap_{m\geqq k}\Big[\bigcup_{\mathbf{l}\in G^{(m)}} E_{\mathbf{l}}^{(m)}\Big]$. Тогда существует целое число $M(x)$ такое, что

$$x\in\bigcup_{\mathbf{l}\in G^{(m)}} E_{\mathbf{l}}^{(m)}\quad\text{при}\quad m>M(x).$$

Последнее означает, что

(58) $$\mathbf{l}^{(m)}(x)=\big(i_1^{(m)}(x), i_2^{(m)}(x), \dots, i_m^{(m)}(x)\big)\in G^{(m)}\quad\text{при}\quad m>M(x)$$

где $\mathbf{l}^{(m)}(x)$ — индекс из $Q^{(m)}$ для которого $E_{\mathbf{l}^{(m)}(x)}^{(m)}$ содержит в себе точку x. Теперь пусть $p\geqq k>M(x)$ фиксированы. Тогда, используя определение множеств

$E_{\bar{i}}^{(m)}$ (см. (37)), имеем

(59) $$\varphi_{n_k}(E_{\bar{i}^{(k)}(x)}^{(k)}) = \{\varphi_{n_k}(x);\ x \in E_{\bar{i}^{(k)}(x)}^{(k)}\} \subset \Delta_{i_k^{(k)}(x)}^{(k)}(k).$$

Используя (58) при $m=k$ $(k>M(x))$ получаем $\bar{i}^{(k)}(x)\in G^{(k)}$. Тогда из $G^{(k)}\subset B^{(k)}$ (см. (41)) следует $\bar{i}^{(k)}(x)\in B^{(k)}$. Следовательно, ичпользуя определение множества $B^{(k)}$ (см. (39)), имеем $i_k^{(k)}(x)\in G^{(k)}(k)$. Отсюда, применив также (59) и (25) при $m=k$ получаем

(60) $$\varphi_{n_k}(E_{\bar{i}^{(k)}(x)}^{(k)}) \subset [-[k^\alpha], [k^\alpha]).$$

Из $p \geqq k$, ввиду того, что множества $E_{\bar{i}^{(p)}(x)}^{(p)}$ и $E_{\bar{i}^{(k)}(x)}^{(k)}$ имеют общую точку x, имеем $E_{\bar{i}^{(p)}(x)}^{(p)}\subset E_{\bar{i}^{(k)}(x)}^{(k)}$ (см. (6)), и следовательно, получим (см. (60))

(61) $$\varphi_{n_k}(E_{\bar{i}^{(p)}(x)}^{(p)}) \subset [-[k^\alpha], [k^\alpha])$$

В силу (37) имеем

(62) $$\varphi_{n_k}(E_{\bar{i}^{(p)}(x)}^{(p)}) \subset \Delta_{i_k^{(p)}(x)}^{(p)}(k).$$

Тогда, применив (61) и (62), получаем, что множества $\Delta_{i_k^{(p)}(x)}^{(p)}(k)$ и $[-[k^\alpha], [k^\alpha])$ имеют общие точки. Следовательно, используя учловие 2, Леммы 3, а также равенство (25) (при $m=p$)· имеем

$$\Delta_{\bar{i}_k^{(p)}(x)}^{(p)}(k) \subset [-[k^\alpha], [k^\alpha]).$$

Отсюда, из (25) следует

(63) $$i_k^{(p)}(x)\in G^{(p)}(k).$$

Из (26), (62) и (63) получаем

$$|\varphi_{n_k}(t)-\varphi_{n_k}(x)| \leqq 2/\sqrt{p} \quad \text{при} \quad x, t\in E_{\bar{i}^{(p)}(x)}^{(p)}.$$

И следовательно, имеем

(64) $$\left|\frac{1}{\mu(E_{\bar{i}^{(p)}(x)}^{(p)})} \int\limits_{E_{\bar{i}^{(p)}(x)}^{(p)}} \varphi_{n_k}(t)\,dt - \varphi_{n_k}(x)\right| \leqq$$

$$\leqq \frac{1}{\mu(E_{\bar{i}^{(p)}(x)}^{(p)})} \int\limits_{E_{\bar{i}^{(p)}(x)}^{(p)}} |\varphi_{n_k}(t)-\varphi_{n_k}(x)|\,dt \leqq 2/\sqrt{p} \quad (p \geqq k \geqq m(x)).$$

С другой стороны, из (11) (для системы $\{\varphi_{n_k}(x)\}_{k=1}^\infty$) (выполнимость этого условия уже доказана) следует

(65) $$\sum_{k=1}^{m(x)} \left|\frac{1}{\mu(E_{\bar{i}^{(p)}(x)}^{(p)})} \int\limits_{E_{\bar{i}^{(p)}(x)}^{(p)}} \varphi_{n_k}(t)\,dt - \varphi_{n_k}(x)\right|^2 \leqq C_1(x),$$

где $C_1(x)$ зависит только от x. Используя (64) и (65), имеем $(p > M(x))$

$$\sum_{k=1}^{p}\left|\frac{1}{\mu(E_{i(p)}^{(p)}(x))}\int\limits_{E_{i(p)}^{(p)}(x)}\varphi_{n_k}(t)\,dt-\varphi_{n_k}(x)\right|^2=$$

$$=\sum_{k=1}^{M(x)}\left|\frac{1}{\mu(E_{i(p)}^{(p)}(x))}\int\limits_{E_{i(p)}^{(p)}(x)}\varphi_{n_k}(t)\,dt-\varphi_{n_k}(x)\right|^2+$$

$$+\sum_{k=M(x)+1}^{p}\left|\frac{1}{\mu(E_{i(p)}^{(p)}(x))}\int\limits_{E_{i(p)}^{(p)}(x)}\varphi_{n_k}(t)\,dt-\varphi_{n_k}(x)\right|^2\leqq$$

$$\leqq C_1(x)+\sum_{k=M(x)+1}^{p}\left(\frac{2}{\sqrt{p}}\right)^2\leqq C_1(x)+\frac{4p}{p}=C(x).$$

Итак, выполнимость условия (13) для системы $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ тоже доказана. Следовательно, используя Лемму 2, имеем, что $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ является системой сходимости. Соединяя этот факт с (44), получаем утверждение теоремы.

Теорема доказана.

В заключение выражаю благодарность профессору А. А. Талаляну, под руководством которого выполнена настоящая работа.

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On some inequalities for Walsh—Fourier series

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Let $f(x)$ be a distribution on [0, 1] and its Walsh—Fourier series be $\sum_{n=0}^{\infty} \hat{f}(n) W_n(x)$, $\hat{f}(n)=(f, W_n)$. The Littlewood—Paley function $g(f)(x)$ is defined by

$$\{\sum_{n=1}^{\infty} n(\sigma_{n+1} f(x) - \sigma_n f(x))^2 a_n\}^{1/2},$$

where $\sigma_n f(x) = \sum_{k=0}^{n-1} \left(1-\frac{k}{n}\right) \hat{f}(k) W_k(x)$ and $\{a_n\}$ is a sequence of positive constants satisfying some conditions. The Marcinkiewicz multiplier operator M is given by

$$Mf(x) \sim \sum_{k=0}^{\infty} \lambda(k) \hat{f}(k) W_k(x),$$

where $\{\lambda(k)\}$ is bounded and varies boundedly over each dyadic block.

We shall show some inequalities for $g(f)(x)$ and $Mf(x)$ using Zygmund's inequalities.

Let $r_0(x) = \text{sgn} \sin 2\pi x$, and $r_n(x) = r_0(2^n x)$. The Walsh—Paley functions are defined as follows:

$$w_0(x) = 1; \quad w_n(x) = r_{n_1}(x) \ldots r_{n_k}(x), \quad \text{if} \quad n = 2^{n_1} + \ldots + 2^{n_k}, \quad n_1 > n_2 > \ldots > n_k \geqq 0.$$

The collection $\{w_n(x);\ n=0, 1, 2, \ldots\}$ forms a complete orthonormal system for L^2 over the unit interval $0 \leqq x \leqq 1$.

Let S be the collection of Walsh polynomials, and S' be the space of distributions on $0 \leqq x \leqq 1$. If $f \in S'$, the Fourier coefficients $\{\hat{f}(n)\}_{n=0}^{\infty}$ are given by $\hat{f}(n) = (f, w_n)$, where (f, w_n) denotes the action of $f \in S'$ on $w_n \in S$. The Fourier series is

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given by $f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n) w_n(x)$. Write

$$S_n f(x) = \sum_{k=0}^{n-1} \hat{f}(k) w_k(x), \quad \sigma_n f(x) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \hat{f}(k) w_k(x),$$

$$d_n f(x) = S_{2^n} f(x) - S_{2^{n-1}} f(x), \quad d_0 f(x) = \hat{f}(0), \quad \text{and} \quad Sf(x) = \Big(\sum_{n=0}^{\infty} |d_n f(x)|^2\Big)^{1/2}.$$

For $0 < p < \infty$, let H^p be the space of $f \in S'$ whose $Sf(x) \in L^p$ with the H^p-norm $\|Sf\|_{L^p}$.

Let $f \in S'$, and the Littlewood-Paley function be

$$g(f)(x) = \Big\{\sum_{n=1}^{\infty} n\big(\sigma_{n+1} f(x) - \sigma_n f(x)\big)^2 a_n\Big\}^{1/2},$$

where $\{a_n\}$ is a sequence of positive constants satisfying $c \cdot n \leqq \sum_{k=1}^{n} a_k \leqq C \cdot n$ and $\sum_{k=2^n}^{2^{n+1}-1} (a_k)^{-1} \leqq C \cdot 2^n$ for all n and some positive constans c and C. As special cases, if we take $a_n = 1$ for all n, then

$$g(f)(x) = \Big\{\sum_{n=1}^{\infty} (\sigma_{n+1} f(x) - S_{n+1} f(x))^2 / n\Big\}^{1/2}$$

and if we take $a_n = n$ $(n = 2^k - 1)$, 0 (otherwise), then

$$g(f)(x) = \Big\{\sum_{n=0}^{\infty} (S_{2^n} f(x) - \sigma_{2^n} f(x))^2\Big\}^{1/2}.$$

By c, C and C_p we always denote a positive constant that may be different on various occasions. We can prove the following theorem.

Theorem 1. (1) *If* $f \in H^1$ *and* $\lambda > 0$, *then*

$$m(\{x \in [0, 1]: g(f)(x) > \lambda\}) \leqq C \|f\|_{H^1} / \lambda$$

(2) *If* $f \in H^1$, *then* $\|g(f)\|_{L^p} \leqq C_p \|f\|_{H^1}$ $(0 < p < 1)$.

(3) *If* $Sf \in L^1 \log^+ L^1$, *then* $\|g(f)\|_{L^1} \leqq C \|Sf\|_{L^1 \log^+ L^1} + C$.

(4) *If* $\hat{f}(0) = 0$ *and* $g(f) \in L^1$, *then* $\|f\|_{H^p} \leqq C \|g(f)\|_{L^1}$ $(0 < p < 1)$.

To prove Theorem 1 we need the following. Let $F(x) = \{f_1(x), f_2(x), \ldots\}$ be a l^2-valued function of $x \in [0, 1]$ and measurable in the Bochner sense. Write $|F(x)| = \big(\sum_{k=1}^{\infty} |f_k(x)|^2\big)^{1/2}$. Similarly let $N = \{n(k)\}$ be an arbitrary sequence of integer-valued function of k, and write $S_N(F)(x) = \{S_{n(1)}(f_1)(x), S_{n(2)}(f_2)(x), \ldots\}$. Using this notation, analogue of Zygmund's inequalities can then be stated as

Lemma. (1) *For any* $\lambda>0$,

$$m(\{x\in[0,1]:\ |S_N(F)(x)|>\lambda\})\leqq\int|F(x)|\,dx/\lambda,$$

(2)
$$\int|S_N(F)(x)|^p\,dx\leqq C_p\Big(\int|F(x)|\,dx\Big)^p,\quad (0<p<1),$$

(3)
$$\int|S_N(F)(x)|\,dx\leqq C\int|F(x)|\log^+|F(x)|\,dx+c.$$

(1) is due to G. Sunouchi [3]. (2) is due to W. R. Wade [4]. Proof of (3) is the same as (2), using the inclusion $L^1\log^+L^1\subset H^1\subset L^1$, Khinchin's inequality and Paley's decomposition.

Proof of Theorem 1. By an identity due to E. M. Stein [2, p. 114],

$$\sigma_{n+1}f(x)-\sigma_nf(x)=$$
$$=\frac{1}{n(n+1)}\Big[\sum_{j=0}^{[\log_2 n]}(2^j-1)\,d_jf(x)+nS_{n+1}(d_{1+[\log_2 n]}f)(x)-\sum_{k=1}^{n}e_kS_k(d_{j(k)}f)(x)\Big],$$

where $j(k)$ is an appropriate integer-valued function of k and e_k is 0 or 1. Then

$$g(f)(x)\leqq\Big(\sum_{n=1}^{\infty}a_nn^{-2}(n+1)^{-1}\Big|\sum_{j=0}^{[\log_2 n]}(2^j-1)\,d_jf(x)\Big|^2\Big)^{1/2}+$$
$$+\Big(\sum_{n=1}^{\infty}a_n(n+1)^{-1}|S_{n+1}(d_{1+[\log_2 n]}f)(x)|^2\Big)^{1/2}+$$
$$+\Big(\sum_{n=1}^{\infty}a_nn^{-2}(n+1)^{-1}\Big|\sum_{k=1}^{n}e_kS_k(d_{j(k)}f)(x)\Big|^2\Big)^{1/2}=A(x)+B(x)+C(x).$$

For $0<p<2$, by Khinchin's inequality,

$$\|A\|_{L^p}^p\leqq C\Big\|\int\Big|\sum_{n=1}^{\infty}(a_n\cdot n^{-3})^{1/2}\Big(\sum_{j=0}^{[\log_2 n]}(2^j-1)\,d_jf\Big)r_n(t)\Big|^p\,dt\Big\|_1.$$

From $\|f\|_{L^p}\leqq\|f\|_{H^p}$, Hölder's inequality and the condition on $\{a_n\}$,

$$\|A\|_{L^p}^p\leqq C\int dt\int\Big\{\sum_{j=0}^{\infty}2^{2j}\Big(\sum_{n=2^j}^{\infty}(a_n\cdot n^{-3})^{1/2}r_n(t)\Big)^2(d_jf(x))^2\Big\}^{p/2}dx\leqq$$
$$\leqq C\int dx\Big\{\int\sum_{j=0}^{\infty}2^{2j}(d_jf(x))^2\Big(\sum_{n=2^j}^{\infty}(a_n\cdot n^{-3})^{1/2}r_n(t)\Big)^2dt\Big\}^{p/2}\leqq$$
$$=C\int\Big\{\sum_{j=0}^{\infty}2^{2j}(d_jf(x))^2\sum_{n=2^j}^{\infty}(a_n\cdot n^{-3})\Big\}^{p/2}dx=C\,\|Sf\|_{L^p}^p.$$

Thus, $\|A\|_{L^p}\leqq C\|f\|_{H^p}$. By the condition on $\{a_n\}$, we have

$$B(x)\leqq C\Big(\sum_{n=1}^{\infty}|S_{n+1}(d_{1+[\log_2 n]}f)|^2\Big)^{1/2},$$

and

$$C(x) \leqq \Big(\sum_{k=1}^{\infty} |S_k(d_{j(k)} f)(x)|^2 \sum_{n=k}^{\infty} a_n \cdot n^{-2}\Big)^{1/2} \leqq C\Big(\sum_{k=1}^{\infty} |S_k(d_{J(k)} f)(x)|^2\Big)^{1/2}.$$

We can now prove part 1 of Theorem 1, by using the above estimates for A, B and C, and Lemma (1). For if $f \in H^1$, then we have

$$m(\{x \in [0, 1]:\ g(f)(x) > \lambda\}) \leqq$$

$$\leqq m(\{x:\ A(x) > \lambda/3\}) + m(\{x:\ B(x) > \lambda/3\}) + m(\{x:\ C(x) > \lambda/3\}) \leqq$$

$$\leqq C\|f\|_{H^1}/\lambda + m\Big(\Big\{x:\ \Big(\sum_{n=1}^{\infty} |S_{n+1}(d_{1+[\log_2 n]} f)(x)|^2\Big)^{1/2} > C \cdot \lambda/3\Big\}\Big) +$$

$$+ m\Big(\Big\{x:\ \Big(\sum_{n=1}^{\infty} |S_n(d_{j(n)} f)(x)|^2\Big)^{1/2} > \lambda/3\Big\}\Big) \leqq C\|f\|_{H^1}/\lambda.$$

Similarly we can easily prove part 2 and 3 of Theorem 1. To prove part 4 of Theorem 1, write

$$Sf(x) = \Big(\sum_{n=0}^{\infty} |S_{2^{n+1}} f(x) - S_{2^n} f(x)|^2\Big)^{1/2} \leqq$$

$$\leqq 2\Big(\sum_{n=0}^{\infty} |S_{2^n} f(x) - \sigma_{2^n} f(x)|^2\Big)^{1/2} + \Big(\sum_{n=0}^{\infty} |\sigma_{2^{n+1}} f(x) - \sigma_{2^n} f(x)|^2\Big)^{1/2}.$$

By the condition on $\{a_n\}$ and Schwartz's inequality,

$$|\sigma_{2^{n+1}} f(x) - \sigma_{2^n} f(x)| \leqq \sum_{k=2^n}^{2^{n+1}-1} |\sigma_{k+1} f(x) - \sigma_k f(x)| \leqq$$

$$\leqq \Big\{\sum_{k=2^n}^{2^{n+1}-1} k\big(\sigma_{k+1} f(x) - \sigma_k f(x)\big)^2 a_k\Big\}^{1/2}.$$

Hence

$$\Big(\sum_{n=1}^{\infty} |\sigma_{2^{n+1}} f(x) - \sigma_{2^n} f(x)|^2\Big)^{1/2} \leqq g(f)(x).$$

On the other hand, it is evident to see

$$\Big(\sum_{n=0}^{\infty} |S_{2^n} f(x) - \sigma_{2^n} f(x)|^2\Big)^{1/2} = \Big(\sum_{n=0}^{\infty} 2^{-2n}\Big|\sum_{j=0}^{2^n-1} j\hat{f}(j) w_j(x)\Big|^2\Big)^{1/2} \leqq$$

$$\leqq C\Big(\sum_{n=0}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} a_k k^{-1}(k+1)^{-2}\Big|S_{2^n}\Big(\sum_{j=0}^{k} j\hat{f}(j) w_j(x)\Big)\Big|^2\Big)^{1/2}.$$

Thus, by Lemma (2), for $0<p<1$; $\|Sf\|_{L^p}^p \leqq C\|g(f)\|_{L^1}^p$. This completes the proof of Theorem 1.

Remark. It is easily verified that $f \to g(f)$ is strong type (2.2). Therefore, by Theorem 1 (1) and interpolation argument, we have $\|g(f)\|_{L^p} \leqq C_p \|f\|_{L^p}$ $(1<p<\infty)$. On the other hand $f \to g(f)$ is not weak type (1, 1) for L^1. See S. IGARI [1].

Next we study Marcinkiewicz multiplier theorem. Let $\{\lambda(k)\}$ be a sequence of constants such that

$$\sup_k |\lambda(k)| \leqq C, \quad \sup_j \sum_{k=2^j+1}^{2^{j+1}-1} k|\Delta\lambda(k)|^2 \leqq C, \quad \text{where} \quad \Delta\lambda(k) = \lambda(k-1)-\lambda(k),$$

and consider the linear transformation M, defined by

$$Mf(x) \sim \sum \lambda(k) \hat{f}(k) w_k(x) \quad \text{for} \quad f(x) \sim \sum \hat{f}(k) w_k(x).$$

Theorem 2. *Under the assumption made above,*

(1) $$m(\{x \in [0, 1]: S(Mf)(x) > \lambda\}) \leqq C \|f\|_{H^1}/\lambda,$$

(2) $$\|Mf\|_{H^p} \leqq C_p \|f\|_{H^1} \quad (0<p<1),$$

(3) $$\|Mf\|_{H^1} \leqq C \|Sf\|_{L^1 \log^+ L^1} + c.$$

Proof. By summation by part,

$$d_j(Mf)(x) = \sum_{k=2^{j-1}}^{2^j-1} \lambda(k) \hat{f}(k) w_k(x) =$$

$$= \sum_{k=2^{j-1}+1}^{2^j-1} \Delta\lambda(k) S_k(d_j f)(x) + \lambda(2^j-1)\, d_j f(x).$$

Then, by Schwartz's inequality and assumption of $\{\lambda(k)\}$,

$$S(Mf)(x) = (\sum |d_j(Mf)(x)|^2)^{1/2} \leqq$$

$$\leqq \{\sum_j (\sum_{k=2^{j-1}+1}^{2^j-1} k|\Delta\lambda(k)|^2)(\sum_{k=2^{j-1}+1}^{2^j-1} k^{-1}|S_k(d_j f)(x)|^2\}^{1/2} + C(\sum_j |d_j f(x)|^2)^{1/2} \leqq$$

$$\leqq C(\sum_{j,k} |S_k(d_j f)(x)|^2)^{1/2} + CSf(x).$$

Thus Theorem 2 is proved by the Lemma.

Remark. The same argument works for double Walsh—Fourier series and Vilenkin—Fourier series.

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On the integral of fundamental polynomials of Lagrange interpolation

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1. Introduction. Let $X=\{x_{kn}\}$, $n=1, 2, \ldots;\ 1\leqq k\leqq n$, be a triangular interpolatory matrix in $[-1, 1]$, i.e.

$$(1.1)\qquad -1 \leqq x_{nn} < x_{n-1,n} < \ldots < x_{1n} \leqq 1, \quad n = 1, 2, \ldots.$$

If, sometimes omitting the superfluous notation,

$$\omega(x) = \omega_n(X, x) = \prod_{k=1}^{n} (x-x_k), \quad n = 1, 2, \ldots,$$

then

$$(1.2)\qquad l_k(x) = l_{kn}(X, x) = \frac{\omega(x)}{\omega'(x_k)(x-x_k)}, \quad k = 1, 2, \ldots, n,$$

are the corresponding fundamental polynomials of Lagrange interpolation. It is well known that the so called Lebesgue function and Lebesgue constant

$$(1.3)\qquad \lambda_n(x) := \lambda_n(X, x) := \sum_{k=1}^{n} |l_k(x)|, \quad \Lambda_n := \Lambda_n(X) := \max_{-1\leqq x\leqq 1} \lambda_n(x)$$

are of fundamental importance considering the convergence and divergence properties of the Lagrange interpolation. Many important properties can be found in [1]—[7] and in their references.

One of them is as follows.

There exists a constant $c_1>0$ *such that we have for arbitrary X.*

$$(1.4)\qquad \int_{-1}^{1} \sum_{k=1}^{n} |l_{kn}(X, x)|\, dx > c_1 \log n.$$

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This statement, proved by P. ERDŐS and J. SZABADOS [3][1]), was explicitly formulated, perhaps first, in P. Erdős [2, p. 242], where he also stated (without proof) that

To every $\varepsilon>0$ there exists a $\delta>0$ so that the number of indices k, $1\leqq k\leqq n$, for which

$$\int_{-1}^{1}|l_k(x)|\,dx<\frac{\delta\log n}{n} \tag{1.5}$$

is less than εn, and the number of k's for which

$$\int_{-1}^{1}|l_k(x)|\,dx<\frac{c_2}{n}\quad \text{is less than}\quad \frac{c_3\log n}{n}.$$

2. Results.

2.1. From (1.5) one could easily obtain (1.4). The first result in this paper gives another statement by which we can get again (1.4).

Let $x_{0n}=1$, $x_{nn}=-1$, $l_{0n}(x)=l_{n+1,n}(x)=0$,

$$J_{kn}=[x_{k+1,n},x_{kn}],\quad 0\leqq k\leqq n. \tag{2.1}$$

First a remark. *If for a fixed k, $0\leqq k\leqq n$, $|J_{kn}|>\delta_n:=\dfrac{75\log n}{n}$, then* $\int_{-1}^{1}\lambda_n(x)\,dx\geqq 4n$ $(n\geqq n_1)$ which is even stronger than (1.4) (see [3, case 1] and [3, (5)]; the last formula shows that $|J_{kn}|\leqq 25\log\Lambda_n/n$ if $k\neq 0, n$; but it can easily be proved for J_0 and J_n, too).

I.e. the real problem is to settle those so called "*short*" intervals J_{kn}, for which $|J_{kn}|\leqq\delta_n$.

The short interval J_{kn} is said to be *exceptional* iff for a given sequence $\varepsilon=\{\varepsilon_n\}_{n=1}^{\infty}$, $0<\varepsilon_n\leqq 2$,

$$\frac{1}{|J_{kn}|}\int_{-1}^{1}\big(|l_k(x)|+|l_{k+1}(x)|\big)\,dx<c\varepsilon_n\log n \tag{2.2}$$

(where c can be taken as 71680). Further, let $k\in K_n$ iff J_{kn} is exceptional. We prove

Theorem 2.1. *If $\varepsilon=\{\varepsilon_n\}$ is given then for any fixed n the total measure of intervals for which* (2.2) *is valid, could not exceed ε_n, or which is the same,*

$$\sum_{k\in K_n}|J_{kn}|\leqq\varepsilon,\quad n=1,2,\ldots. \tag{2.3}$$

[1]) (1.4) is an easy consequence of another statement in [1, Theorem 2] which was proved by P. Erdős and P. Vértesi (cf. [6] and [7]).

Now let us suppose that for a fixed n all the intervals J_{kn} are short. Then, using Theorem 2.1 with $\varepsilon_n=1$, we can write

$$\int_{-1}^{1} \lambda_n(x)\,dx \geqq \frac{1}{2} \int_{-1}^{1} \sum_{k=1}^{n} (|l_k(x)|+|l_{k+1}(x)|)\,dx \geqq$$

$$\geqq \frac{1}{2} \sum_{k\notin K_n} \int_{-1}^{1} (|l_k(x)|+|l_{k+1}(x)|)\,dx \geqq \frac{c}{2}\log n \sum_{k\notin K_n} |J_{kn}| \geqq \frac{c}{2}\log n,$$

i.e. we obtain (1.4).

2.2. The next theorem gives information on both short and long intervals. The interval J_{kn} is *bad* iff for a given $\varepsilon>0$

$$\int_{-1}^{1} (|l_k(x)|+|l_{k+1}(x)|)\,dx < \eta(\varepsilon)\,\frac{\log n}{n}, \quad n \geqq n_0(\varepsilon), \tag{2.4}$$

where $\eta(\varepsilon)$ can be choosen as $(10^2\cdot 14336)^{-1}\varepsilon^2$.[2] Further, let $k\in T_n$ iff J_{kn} is bad. Then we prove

Theorem 2.2. *By the previous notations*

$$\sum_{k\in T_n} |J_{kn}| \leqq \varepsilon \quad if \quad n \geqq n_0(\varepsilon). \tag{2.5}$$

2.3. Finally we remark that analogous results can be proved for a fixed interval $[a, b]\subset[-1, 1]$. We omit the details.

3. Proof.

3.1. Proof of Theorem 2.1. If for a fixed n, $0<\varepsilon_n<(c\log n)^{-1}$, then by

$$|l_k(x)|+|l_{k+1}(x)| \geqq 1 \quad \text{if} \quad x\in J_{k},\quad k=0, 1, \dots, n, \quad n\geqq 1, \tag{3.1}$$

(cf. [4, Lemma 4] for $k\neq 0, n$; if $k=0$ (or n), (3.1) comes from $l_1(x)\geqq 1$, $x\geqq x_1$ (or $l_n(x)\geqq 1$, $x\leqq x_n$)) we get

$$\int_{-1}^{1} (|l_k(x)|+|l_{k+1}(x)|)\,dx \geqq \int_{J_k} (\dots) \geqq |J_k| > |J_k|\,c\varepsilon_n \log n, \quad \text{i.e.}$$

there is no exceptional interval. That means from now on we can suppose

$$\varepsilon_n \geqq \frac{1}{c\log n}, \quad n = 2, 3, \dots. \tag{3.2}$$

[2]) Instead of ε, we can choose a sequence $\{\varepsilon_n\}$ which would give $\eta(\varepsilon_n)$ in (2.3). I hint with a finer argument the relation $\eta(\varepsilon_n)=c\varepsilon_n$ can be proved.

We introduce the following notations

(3.3) $$J_k(q) = J_{kn}(q) = [x_{k+1}+q|J_k|, x_k-q|J_k|] \quad (0 \leqq k < n),$$

where $0 \leqq q \leqq 1/2$. Let $z_k = z_{kn}(q)$ be defined by

(3.4) $$|\omega_n(z_k)| = \min_{x \in J_k(q)} |\omega_n(x)|, \quad k = 0, 1, ..., n,$$

finally let

$$|J_i, J_k| = \max(|x_{i+1}-x_k|, |x_{k+1}-x_i|) \quad (0 \leqq i, k \leqq n).$$

In [5, Lemma 4.2] we proved

Lemma 3.1. *If* $1 \leqq k, r < n$ *then for arbitrary* $0 < q \leqq 1/2$

(3.5) $$|l_k(x)|+|l_{k+1}(x)| \geqq q^2 \frac{|\omega_n(z_r)|}{|\omega_n(z_k)|} \frac{|J_k|}{|J_r, J_k|} \quad \textit{if} \quad x \in J_r(q).$$

Later we shall also [6, Lemma 3.2]:

Lemma 3.2. *Let* $I_k=[a_k, b_k]$, $1 \leqq k \leqq t$, $t \geqq 2$, *be any* t *intervals in* $[-1, 1]$ *with* $|I_k \cap I_j| = 0$, $(k \neq j)$, $|I_k| \leqq \varrho$ $(1 \leqq k \leqq t)$, $\sum_{k=1}^{t} |I_k| = \mu$. *Supposing that for certain integer* $R \geqq 2$ *we have* $\mu \geqq 2^R \varrho$, *there exists the index* s, $1 \leqq s \leqq t$, *such that*

(3.6) $$S := \sum_{k=1}^{t} \frac{|I_k|}{|I_s, I_k|} \geqq \frac{R}{8} \mu.$$

I_s *will be called accumilation interval of* $\{I_k\}_{k=1}^{t}$.

(Here and later mutatis mutandis we apply the previous notations for arbitrary intervals.)

Note that we do not require $b_k \leqq a_{k+1}$.

Let $\sum_{k \in K_n'} |J_k| := \mu_n$, where $K_n' := K_n \setminus \{0, n\}$. If for a fixed $n \geqq n_0(\varepsilon_n)$, $\mu_n \leqq \varepsilon_n/2$, (2.3) holds true. So we investigate those $n \geqq n_0(\varepsilon)$ $\mu_n \geqq \varepsilon_n/10$, say.

We now apply Lemma 3.2 for the exceptional J_{kn}'s with $\mu=\mu_n$, $\varrho=\delta_n$ and $R=[\overset{2}{\log} n^{1/7}]+1$, $n \in N$, $n \geqq n_0(\varepsilon)$ (shortly $n \in N_1$).

Denote by $M_1=M_{1n}$ the accumulation interval. Dropping M_1, we apply Lemma 3.2 again for the remaining exceptional intervals with $\mu=\mu_n-|M_1|>\mu_n/2$ and the above ϱ and R, supposing $\mu_n \geqq \varrho^{R+1}$ whenever $n \in N_1$. We denote the accumulation interval by M_2. At the i-th step $(2 \leqq i \leqq \psi_n)$ we drop $M_1, M_2, ..., M_{i-1}$ and apply Lemma 3.2 for the remaining exceptional intervals with $\mu=\mu_n-\sum_{j=1}^{i-1}|M_j|$ using the same ϱ and R.

Here ψ_n is the first index for which

$$\sum_{i=1}^{\psi_n-1} |M_i| \leqq \frac{\mu_n}{2} \quad \text{but} \quad \sum_{i=1}^{\psi_n} |M_i| > \frac{\mu_n}{2}, \quad n\in N_1. \tag{3.7}$$

If we denote by $M_{\psi_n+1}, M_{\psi_n+2}, \dots, M_{\varphi_n}$ the remaining (i.e. not accumulation) exceptional intervals, by (3.6) we can write

$$\sum_{k=r}^{\varphi_n} \frac{|M_k|}{|M_r, M_k|} \geqq \frac{\mu_n \log n}{112} \quad \text{if} \quad 1 \leqq r \leqq \psi_n \quad (n\in N_1). \tag{3.8}$$

Now we have

$$\sum_{k\in K_n} \int_{-1}^{1} (|l_k(x)|+|l_{k+1}(x)|)\,dx \geqq \sum_{k=1}^{\varphi_n}\sum_{r=1}^{\varphi_n} \int_{M_r} (|l_k(x)|+|l_{k+1}(x)|)\,dx \geqq ^{3)} \tag{3.9}$$

$$\geqq \sum_{k=1}^{\varphi_n}\sum_{r=1}^{\varphi_n} (1-2q)|M_r|\,q^2 \left|\frac{\omega(\bar z_r)}{\omega(\bar z_k)}\right| \frac{|M_k|}{|M_r, M_k|} \geqq$$

$$\geqq \frac{q^2(1-2q)}{2} \sum_{k=1}^{\varphi_n}\sum_{r=1}^{\varphi_n} \left(\left|\frac{\omega(\bar z_r)}{\omega(\bar z_k)}\right| + \left|\frac{\omega(\bar z_k)}{\omega(\bar z_r)}\right|\right) \frac{|M_r|\,|M_k|}{|M_r, M_k|} \geqq$$

$$\geqq q^2(1-2q) \sum_{k=1}^{\psi_n} |M_k| \sum_{r=k}^{\varphi_n} \frac{|M_r|}{|M_r, M_k|} > \frac{\mu_n^2 \log n}{16\cdot 2\cdot 2\cdot 112} \quad \text{if} \quad q=\frac{1}{2}$$

(see (3.5), (3.7) and (3.8); we used that $x+x^{-1}\geqq 2$).

On the other hand, by (2.2)

$$\sum_{k\in K_n} \int_{-1}^{1} (|l_k(x)|+|l_{k+1}(x)|)\,dx < c\varepsilon_n \log n \sum_{k\in K_n} |J_k| = c\varepsilon_n \mu_n \log n$$

i.e. $\mu_n^2 \log n < 7168\, c\varepsilon_n \mu_n \log n$, from where by $\mu_n \geqq \varepsilon_n/10$ $1 < 71\,680c$, a contradiction if $c=(71\,680)^{-1}$ and $n\geqq n_0$.

If $n\leqq n_0$, by (3.1) we have for arbitrary k, $0\leqq k\leqq n$,

$$\int_{-1}^{1} (|l_k(x)|+|l_{k+1}(x)|)\,dx \geqq \int_{J_k} (\dots) \geqq |J_k| \geqq |J_k|\,2c\log n_0 \geqq |J_k|\,c\varepsilon_n \log n$$

whenever $2c\log n_0 \leqq 1$. Considering, that if $n_0=10^{420}$, $2c\log n_0\leqq 1$, indeed. But then for $n\leqq n_0(\varepsilon)$, $K_n=\emptyset$, which gives the statement for arbitrary $n\geqq 2$.

3.2. Proof of Theorem 2.2. If $|J_{kn}|\geqq \delta_n$, then by (3.1)

$$\int_{-1}^{1} (|l_k(x)|+|l_{k+1}(x)|)\,dx \geqq \int_{J_k} (\dots)\,dx \geqq |J_k| \geqq \frac{75 \log n}{n},$$

[3]) We denote the fundamental polynomials corresponding to M_k, by $l_k(x)$ and $l_{k+1}(x)$, the corresponding minimums are $|\omega(\bar z_k)|$.

i.e. a long interval could not be bad. Considering the short intervals, again we suppose that $\mu_n := \sum_{k\in T_n} |J_{kn}| \geqq \varepsilon/10$ to get a contradiction. Then, as above, we obtain that for $n \geqq n_0(\varepsilon)$

$$\frac{\mu_n^2 \log n}{7168} < \sum_{k\in T_n} \int_{-1}^{1} (|l_k(x)|+|l_{k+1}(x)|)\,dx := P.$$

By (2.3), $P < |T_n|\eta(\varepsilon)\frac{\log n}{n} \leqq 2\eta(\varepsilon)\log n$, i.e.

$$\frac{\varepsilon^2 \log n}{10^2 \cdot 7168} \leqq \frac{\mu_n^2 \log n}{7168} < P \leqq 2\eta(\varepsilon)\log n,$$

a contradiction, if $\eta(\varepsilon) = (10^2 \cdot 14\,336)^{-1}\varepsilon^2$, $n \geqq n_0(\varepsilon)$.

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Comparison theorems and convergence properties for functional differential equations with infinite delay

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Dedicated to Lajos Pintér on his 60th birthday

1. Introduction

In the general area of stability theory for functional differential equations, Lyapunov functions (Lyapunov—Razumikhin or — Krasovskiĭ functions) often are employed instead of Lyapunov functionals [8, 12]. The derivative of such a function with respect to the equation under investigation is estimated from above on some appropriately chosen subset of the underlying solution (phase) space. The method requires a comparison theorem (or theorems) since the Lyapunov function in question usually is compared to a solution of a certain ordinary differential equation.

The technique of comparison theorems has been thoroughly investigated for functional differential equations with finite delay. (See, for example, [2, 6, 9].) For infinite delay cases DRIVER [1] obtained the first results, and his technique has been generalized in several directions and applied to examine various notions of stability. For instance, KATO [7] and ZHICHENG [13] have obtained results for general "admissible" phase spaces, while PARROTT [11] developed her work in terms of certain (exponentially weighted) C_γ spaces. In a recent paper of the authors [3], this method was applied for general C_g spaces, but the comparison differential equation was only a trivial one.

In the present paper we examine the technique of comparison results from several points of view. In Section 2 we formulate general comparison theorems in terms of arbitrary real functions and then apply the theorems (in Section 3) to obtain various convergence results for these functions. Among the consequences of Section 3 there is a generalization of the main convergence result of [4] for semigroups on a special function space.

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As may be surmised from the title, one of our primary motivations has been to generate convergence theorems for solutions of functional differential equations with infinite delay. This is accomplished in Section 4 with the aid of the work in Sections 2 and 3. The main thrust in Section 4 is to compare convergence properties of certain functionals $W(=W(t, x_t))$ to corresponding properties of related Lyapunov functions $V(=V(t, x(t)))$.

The paper is concluded with several examples given in Section 5.

2. Comparison theorems

Let ω: $R^+\times R^+\to R^+$ be a continuous function, $t_0, u_0\in R^+$ and let $u(t)$ be the maximal solution of

(1) $$\begin{cases} u'(t) = \omega(t, u(t)) & (t \geqq t_0) \\ u(t_0) = u_0 \end{cases}$$

on an interval $[t_0, a)$ $(t_0<a\leqq\infty)$. Let f: $R^+\to R^+$ g: $R\to R^+$, and let g be continuous on $[t_0, \infty)$.

Theorem 1. *If for all* $t\in[t_0, a)$ *the inequalities*

(A₁) $$g(t) \leqq f(t),$$

(B₁) $$f(t) \leqq \max\{\max_{-r\leqq s\leqq 0} g(t+s), f(t-r)\} \quad (r\in[0, t-t_0]),$$

are fulfilled and if for $t\in[t_0, a)$

(C₁) $$0 < g(t) = f(t)$$

implies

(D₁) $$D^+g(t) \leqq \omega(t, g(t)),$$

then $f(t_0)\leqq u_0$ *implies* $f(t)\leqq u(t)$ $(t\in[t_0, a))$.

Proof. First we remark that (A₁), (B₁) imply

(2) $$\liminf_{h\to 0+} f(t-h) \geqq f(t) \quad (t\in(t_0, a)),$$

(3) $$\limsup_{h\to 0+} f(t+h) \leqq f(t) \quad (t\in[t_0, a)).$$

Let $\varepsilon>0$ and define the function

$$F(t) = \max\{\sup_{t_0\leqq s\leqq t} f(s), \varepsilon\} \quad (t \geqq t_0).$$

Clearly F is monotone nondecreasing. So, (2) and (3) imply F is continuous. Obviously

(4) $$g(t) \leqq f(t) \leqq F(t) \quad (t \geqq t_0)$$

and

(5) $$F(t) = \max\{\sup_{t-r\leqq s\leqq t} f(s), F(t-r)\} \leqq$$

$$\leqq \max\{\sup_{t-r\leqq s\leqq t} \max\{\max_{t-r-s\leqq u\leqq 0} g(s+u), f(t-r)\}, F(t-r)\} \leqq$$

$$\leqq \max\{\max_{-r\leqq s\leqq 0} g(t+s), F(t-r)\} \quad (t \geqq t_0,\ r\in[0, t-t_0]).$$

If $g(t)<F(t)$, then by the continuity of g there is a $\delta>0$ so that $\max_{0\leqq s\leqq \delta} g(t+s)< <F(t)$. Hence by using (5)

$$F(t+h) \leqq \max\{\max_{0\leqq s\leqq \delta} g(t+s), F(t)\} \leqq F(t)$$

whenever $0<h\leqq\delta$. So, $g(t)<F(t)$ implies $D^+F(t)\leqq 0$.

Assume $g(t)=F(t)$ and $D^+F(t)>0$. Then there exists a sequence $\{\delta_n\}$ such that $\delta_n>0$, $\delta_n\to 0$ as $n\to\infty$, $F(t+\delta_n)>F(t)$ and

$$D^+F(t) = \lim_{n\to\infty} \frac{F(t+\delta_n)-F(t)}{\delta_n}.$$

From (5) it follows that for any n there is a γ_n, $0<\gamma_n\leqq\delta_n$, such that

$$g(t+\gamma_n) \geqq F(t+\delta_n).$$

Using (4) and (D_1) we have

$$D^+F(t) = \lim_{n\to\infty} \frac{F(t+\delta_n)-F(t)}{\delta_n} \leqq \limsup_{n\to\infty} \frac{g(t+\gamma_n)-g(t)}{\gamma_n} \leqq$$

$$\leqq D^+g(t) \leqq \omega(t, g(t)) = \omega(t, f(t)) = \omega(t, F(t)).$$

Since ω is a nonnegative function, we obtain

$$D^+F(t) \leqq \omega(t, F(t)) \quad (t\in[t_0, a)).$$

By using this inequality, the continuity of F, $F(t_0)\leqq\max\{u(t_0), \varepsilon\}$ and a well-known differential inequality [9, vol. 1, pp. 15] we get

$$f(t) \leqq F(t) \leqq u_\varepsilon(t) \quad \text{on} \quad [t_0, a_\varepsilon),$$

where $u_\varepsilon(t)$ is the maximal solution of

$$\begin{cases} u_\varepsilon'(t) = \omega(t, u_\varepsilon(t)) & (t \geqq t_0) \\ u_\varepsilon(t_0) = \max\{u_0, \varepsilon\} \end{cases}$$

on $[t_0, a_\varepsilon)$. If $\varepsilon\to 0+$, then $a_\varepsilon\to a$ and $u_\varepsilon(t)\to u(t)$ uniformly on every compact interval of $[t_0, a)$. This completes the proof.

Corollary 1. *Let* (A_1), (B_1) *hold and suppose that* (C_1) *implies*

(D_2) $$D^+g(t) \leqq 0.$$

Then $f(t)$ *is a monotone non-increasing function on* $[t_0, a)$.

Theorem 2. *Suppose that* $a=\infty$, (A_1), (B_1) *are satisfied and* (C_1) *implies* (D_1), *moreover* $\omega(t, u)$ *is nondecreasing in* u *and the solutions of equation* (1) *are bounded on* $[t_0, \infty)$ *for every* u_0. *Then* $\lim_{t\to\infty} f(t)$ *exists.*

Proof. Since f is bounded below, it is enough to prove that $V^+f<\infty$, where V^+f denotes the positive variation of f on $[t_0, \infty)$. Let $\tilde{u}(t)$ be the maximal solution of (1) on $[t_0, \infty)$ with $\tilde{u}(t_0)=f(t_0)$. Theorem 1 implies $f(t)\leqq\tilde{u}(t)$ for $t\geqq t_0$. From $\omega(t, u)\geqq 0$ and the boundedness of $\tilde{u}(t)$ it follows that $\tilde{u}'\in L^1([t_0, \infty))$. If $0<f(t)=$ $=g(t)$, then

$$D^+g(t) \leqq \omega(t, g(t)) = \omega(t, f(t)) \leqq \omega(t, \tilde{u}(t)) = \tilde{u}'(t) \quad (t \geqq t_0).$$

That is Theorem 1 is applicable with $\omega(t, u)=\tilde{u}'(t)$.

Obviously the maximal solution of

$$\begin{cases} u'(t) = \tilde{u}'(t), & t \geqq t_1 \\ u(t_1) = f(t_1) \end{cases}$$

is $u(t)=f(t_1)+\int_{t_1}^{t} \tilde{u}'(s)\,ds=f(t_1)+\tilde{u}(t)-\tilde{u}(t_1)$. Replace t_0 by t_1 and apply Theorem 1 to get

$$f(t) \leqq f(t_1)+\tilde{u}(t)-\tilde{u}(t_1) \quad \text{for all} \quad t_0 \leqq t_1 \leqq t.$$

Using that $\tilde{u}(t)$ is nondecreasing on $[t_0, \infty)$, this inequality gives $V^+f<\infty$. This completes the proof.

Remark 1. Theorem 1 is an extension of Driver's result [1, Lemma 1]. He examined the case $f(t)=\sup_{\alpha\leqq s\leqq t} g(s)$, $-\infty\leqq\alpha\leqq t_0$ and g is continuous on $[\alpha, a)$.

Remark 2. Theorem 2 may be false if $\omega(t, u)$ is decreasing in u. For example, let

$$\omega(t, u) = \begin{cases} 3-u & \text{if} \quad u \leqq 3 \\ 0 & \text{if} \quad u > 3, \end{cases}$$

and put $f(t)=g(t)=\sin t$. Then all the assumptions of Theorem 2 are satisfied except the monotonicity condition on $\omega(t, u)$ and $\lim_{t\to\infty} f(t)$ does not exist.

Further on, we need a sharper version of Theorem 1. Namely, inequality (D_1) will be required only on a subset of the set of the points of $[t_0, a)$ where (C_1) is satisfied. In order to give this subset we introduce the following notation.

Let us suppose $a(t,r)$, $p(t,r)$, $h(t,r)$ are continuous functions on $[\tau,\infty)\times R^+$, where $\tau\geqq 0$ is a constant, $p(t,r)$ is nondecreasing in r, $a(t,r)<r$ for all $r>0$, $t\geqq 0$. Suppose that $\tau\leqq h(t,r)$, $p(t,r)\leqq t$ for all $r>0$, $t\geqq\tau$. Let $\sigma(t,r)=$ $=\sup\{s\colon p(s,r)\leqq t\}$. It is not difficult to see that $\sigma(t,r)$ is nonincreasing in r, $\sigma(t,r)\geqq t$ and if f is a locally bounded function on $[\tau,\infty)$, $\sigma(\tau,r)<\infty$ for all $r>0$, then there is $0<u_0(=u_0(f,\tau))$ such that $f(t)\leqq u_0$ on $[\tau,\sigma(\tau,u_0)]$. For $r>0$, $0\leqq z\leqq s\leqq t$ define the function

$$g^*(z,s,t,r)=\begin{cases}D^+g(s) & \text{if}\quad a(t,r)<g(v),\ f(v)\leqq r\quad\text{for all}\quad v\in[z,s]\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 3. *Suppose g is continuously differentiable on $[\tau,\infty)$, (A_1), (B_1) are satisfied on $[\tau,\infty)$ and that*

$$(E_1)\qquad \int_z^t g^*(z,s,t,r)\,ds<r-a(t,r)$$

for all $r>0$, $t\geqq\sigma(\tau,r)$ $t>z\geqq h(t,r)$. Moreover, if the inequalities

$$(C_2)\qquad \begin{cases}0<g(t)=f(t),\quad p(t,f(t))\geqq\tau,\\ a(t,f(t))<g(v)\leqq f(v)\leqq f(t)\quad\textit{for all}\quad v\in[h(t,f(t)),t]\end{cases}$$

imply (D_1), then

$$f(v)\leqq u_0\quad\textit{for all}\quad v\in[\tau,\sigma(\tau,u_0)]$$

implies

$$f(t)\leqq u(t)\quad(t\in[\sigma(\tau,u_0),a)),$$

where $u(t)$ is the maximal solution of (1) on $[t_0,a)$ with $t_0=\sigma(\tau,u_0)$.

Proof. Define $t_0=\sigma(\tau,u_0)$ and for $t\geqq t_0$

$$G(t)=\max(g(t),u_0),\quad F(t)=\sup_{t_0\leqq s\leqq t}\max(f(s),u_0).$$

Then in the same way as in the proof of Theorem 1 we can see that

$$G(t)\leqq F(t)\quad(t\geqq t_0),$$

$$F(t)\leqq\max\{\max_{-r\leqq s\leqq 0}G(t+s),F(t-r)\}\quad(t\geqq t_0,\ r\in[0,t-t_0]),$$

$G(t)<F(t)$ implies $D^+F(t)\leqq 0$, and if $G(t)=F(t)$, $D^+F(t)>0$ then $D^+F(t)\leqq$ $\leqq D^+G(t)$. It is easy to see that in the case $t\geqq t_0$, $G(t)=F(t)$, $D^+F(t)>0$ the following relations are true: $F(t)=f(t)=G(t)=g(t)\geqq u_0$, $\frac{d}{dt}g(t)=D^+G(t)$. We want to show that in this case $D^+G(t)\leqq\omega(t,G(t))$ is fulfilled, too. This would be sufficient to the completeness of the proof by using Theorem 1.

Since $F(t)=f(t)$ implies $f(v)\leqq f(t)$ for all $v\in[h(t,f(t)),t]$, by the conditions of Theorem 3 it is enough to prove that $a(t,f(t))<g(v)$ for all $v\in[h(t,f(t)),t]$. Suppose the contrary, that is there exists a $z\in[h(t,f(t)),t]$ such that $a(t,f(t))<g(v)$ for all $v\in(z,t]$, $a(t,f(t))=g(z)$. Then $g^*(z,s,t,f(t))=D^+g(s)$ for all $s\in(z,t)$. Therefore, by inequality (E_1) one gets

$$f(t)-a(t,f(t))=g(t)-g(z)=\int_z^t g^*(z,s,t,f(t))\,ds<f(t)-a(t,f(t)),$$

which is a contradiction, thereby completing the proof.

We can extend Theorem 2 in a similar way:

Theorem 4. *Suppose that* $a=\infty$, (A_1), (B_1), (E_1) *are satisfied and* (C_2) *implies* (D_1), *moreover* $\omega(t,u)$ *is nondecreasing in* u *and the solutions of equation* (1) *are bounded on* $[t_0,\infty)$ *for every* u_0. *Then* $\lim_{t\to\infty} f(t)$ *exists.*

If we analyse the proof of Theorem 3 we can find that the differentiability property of function $g(t)$ is used only in relation $g(t)-g(z)=\int_z^t g^*(z,s,t,f(t))\,ds$, where $z\in[h(t,f(t)),t]$. So, if $h(t,r)\equiv t$, then it is sufficient for g to be continuous. Therefore, a J. Kato and W. Zhicheng type comparison theorem [7, 13] can be deduced from Theorem 1. We shall formulate it in the next

Corollary 2. *Assume* $\tau\geqq 0$, $g\colon[\tau,\infty)\to R^+$ *is a continuous function and*

$$p(t,g(t))\geqq\tau,\quad 0<g(t)=\max_{p(t,g(t))\leqq s\leqq t} g(s)$$

imply

$$D^+g(t)\leqq\omega(t,g(t)).$$

If there is $u_0>0$ *such that* $\sigma(\tau,u_0)<\infty$, $g(t)\leqq u_0$ *on* $[\tau,\sigma(\tau,u_0)]$, *then* $g(t)\leqq u(t)$ *for all* $t\in[\sigma(\tau,u_0),a)$, *where* $u(t)$ *is the maximal solution of* (1) *on* $[t_0,a)$ *with* $t_0= =\sigma(\tau,u_0)$.

Proof. Define $h(t,r)\equiv t$, and $f(t)=\max_{\tau\leqq s\leqq t} g(s)$ for $t\geqq t_0$. If $p(t,f(t))\geqq\tau$, $0<g(t)=f(t)$, then $g(t)=\max_{\tau\leqq s\leqq t} g(s)$, consequently $g(t)=\max_{p(t,f(t))\leqq s\leqq t} g(s)$, therefore ($D_1$) is fulfilled, and the assertion follows from Theorem 3.

Z. Mikolajska [10] used a comparison result analogous with the special case $p(t,r)=t_0$. This case is stated in the following corollary. The proof is omitted because it is similar to that of Corollary 2.

Corollary 3. *Suppose* $\tau\leqq t_0$, $g\colon[\tau,\infty)\to R^+$ *is continuously differentiable,* (E_1) *is satisfied for all* $r>0$, $t\geqq t_0$, $t>z\geqq h(t,r)$. *If* $h(t,r)\geqq\tau$ *for all* $r>0$, $t\geqq t_0$, *and if* $t\geqq t_0$,

$$a(t,g(t))<\min_{h(t,g(t))\leqq s\leqq t} g(s)\leqq\max_{h(t,g(t))\leqq s\leqq t} g(s)=g(t)$$

imply (D_1), *then* $\max_{\tau\leqq s\leqq t_0} g(s)\leqq u_0$ *implies* $g(t)\leqq u(t)$ *for all* $t\geqq t_0$.

3. Convergence properties of real functions

In the previous chapter sufficient conditions on functions f and g were given to guarantee the existence of the limit of f as $t\to\infty$. Now, we show that it is possible to modify condition (B_1) such that the existence of $\lim\limits_{t\to\infty} f(t)$ implies that of $\lim\limits_{t\to\infty} g(t)$.

Lemma 1. *Suppose* (A_1) *for* $t\geqq t_0$ *and that there exists a function* $h\colon R^+\times R^+\to R^+$ *such that*

$$(F_1)\qquad \lim_{t\to\infty} h(t-r, t) = 0 \quad (r>0),$$

$$(B_2)\qquad f(t)\leqq \max_{-r\leqq s\leqq 0} g(t+s)+h(r, t) \quad (t\geqq t_0,\ r\in[0, t-t_0]).$$

Then $\limsup\limits_{t\to\infty} g(t)=\limsup\limits_{t\to\infty} f(t)$.

Proof. (A_1) implies $\limsup\limits_{t\to\infty} g(t)\leqq\limsup\limits_{t\to\infty} f(t)$. On the other hand, if $c=\limsup\limits_{t\to\infty} g(t)<\infty$, then for all $\varepsilon>0$ there is a $T=T(\varepsilon)\geqq t_0$ such that $g(t)\leqq c+\varepsilon$ for $t\geqq T$. By (B_2) we have $f(t)\leqq c+\varepsilon+h(t-T, t)$ for all $t\geqq T$. Using (F_1), we obtain $\limsup\limits_{t\to\infty} f(t)\leqq c+\varepsilon$. Since $\varepsilon>0$ is arbitrary, the theorem is proved.

Theorem 5. *Suppose g is uniformly continuous on* $[t_0, \infty)$, (A_1) *is satisfied for* $t\geqq t_0$ *and there exist functions* $h, k_1, k_2\colon R^+\times R^+\to R^+$ *such that* (F_1) *is fulfilled,* $k_1(r, u)$, $k_2(r, u)$ *are monotone nondecreasing and continuous in u for all* $r\in R^+$,

$$k_1(0, u)=\lim_{r\to 0+} k_1(r, u)=u \quad (u>0),$$

$$k_2(r, u)<u \quad \textit{for all} \quad r, u>0, \quad k_2(0, u)\leqq u \quad \textit{and}$$

$$(B_3)\qquad f(t)\leqq \max\Big\{k_1\big(r, \max_{-r\leqq s\leqq 0} g(t+s)\big),\ k_2\big(r, \max_{-\tau\leqq s\leqq -r} g(t+s)\big)\Big\}+h(\tau, t)$$

$$(t\geqq t_0,\ \tau\in[0, t-t_0],\ r\in[0, \tau]).$$

Then $\lim\limits_{t\to\infty} g(t)=c$ *if and only if* $\lim\limits_{t\to\infty} f(t)=c$.

Proof. If $\lim\limits_{t\to\infty} g(t)=c$, then according to ($A_1$), ($B_3$) with $r=0$ and Lemma 1

$$c=\liminf_{t\to\infty} g(t)\leqq \liminf_{t\to\infty} f(t)\leqq \limsup_{t\to\infty} f(t)=\limsup_{t\to\infty} g(t)=c,$$

i.e. $\lim\limits_{t\to\infty} f(t)=c$.

Now, assume $\lim\limits_{t\to\infty} f(t)=c$. It is enough to prove that $\liminf\limits_{t\to\infty} g(t)\geqq c$. Suppose the contrary, i.e. $\liminf\limits_{t\to\infty} g(t)<c$. Let $c_1\in(\liminf\limits_{t\to\infty} g(t), c)$. From the uniform continuity of g there is a $\delta>0$ such that $t_1, t_2\geqq t_0$, $|t_2-t_1|<\delta$ imply $|g(t_1)-g(t_2)|<$

$<(c-c_1)/4$. Define a sequence $\{t_n\}$ such that $t_n\to\infty$ as $n\to\infty$ and $g(t_n)\le c_1$ for $n=1, 2, \dots$. Then

$$\max_{-\delta\le s\le 0} g(t_n+s) \le \max_{-\delta\le s\le 0}\big(g(t_n+s)-g(t_n)\big)+g(t_n) \le$$

$$\le \frac{c-c_1}{4}+c_1 = \frac{c+3c_1}{4}.$$

Let $r\in(0, \delta)$ be chosen such that $k_1\big(r, (c+3c_1)/4\big)\le(c+c_1)/2$. Choose $\varepsilon>0$, $T=T(\varepsilon)\ge t_0$ such that $k_2(r, c+\varepsilon)<c$ and $g(t)\le c+\varepsilon$ for $t\ge T$. From (B_3) we obtain

$$f(t_n) \le \max\Big\{k_1\big(r, \max_{-r\le s\le 0} g(t_n+s)\big), k_2\big(r, \max_{T-t_n\le s\le -r} g(t_n+s)\big)\Big\}+$$

$$+h(t_n-T, t_n) \le \max\left\{\frac{c+c_1}{2}, k_2(r, c+\varepsilon)\right\}+h(t_n-T, t_n)$$

for $t_n\ge T$. Using $\lim_{t\to\infty} h(t_n-T, t_n)=0$ we get the contradiction

$$c = \limsup_{n\to\infty} f(t_n) \le \max\{(c+c_1)/2, k_2(r, c+\varepsilon)\} < c.$$

This completes the proof.

4. Applications for functional differential equations

Let X be a Banach space with the norm $\|\,.\,\|_X$ and let B be a space of functions mapping R^- into X with a semi-norm $\|\,.\,\|_B$. For a function $x\colon (-\infty, a)\to X$ and for $t\in(-\infty, a)$ define x_t as a function from R^- into X by $x_t(s)=x(t+s)$, $s\in R^-$. For $\tau\in R^+$ define B_τ as the set of $\varphi\in B$ such that $\varphi_t\in B$ for each $t\in[-\tau, 0]$ and $\varphi(s)$ is continuous on $[-\tau, 0]$. Let $D\subset B$ and let $f\colon R^+\times D\to X$ be a given function. Consider the functional differential equation

$$\dot x(t) = f(t, x_t). \tag{6}$$

A solution of equation (6) on $[t_0, a)$, $t_0<a\le\infty$ is a function $x\colon (-\infty, a)\to X$ such that $x_t\in D$ for $t\in[t_0, a)$, $x(t)$ is continuous on $[t_0, a)$, differentiable on (t_0, a) and $\dot x(t)=f(t, x_t)$ on (t_0, a).

Let $V\colon R\times X\to R^+$ be a locally Lipschitzian function.

Suppose that there exists a function $W\colon R^+\times D\to R^+$ such that

$$V\big(t, \varphi(0)\big) \le W(t, \varphi) \quad (t\in R^+, \varphi\in D) \tag{AV}$$

and

$$W(t, \varphi) \le \max\Big\{\max_{-r\le s\le 0} V\big(t+s, \varphi(s)\big), W(t-r, \varphi_{-r})\Big\} \tag{BV$_1$}$$

$$(t\in R^+, r\in[0, t], \varphi\in B_r).$$

If $x(t)$ is a solution of (6), then $g(t)=V(t, x(t))$ and $f(t)=W(t, x_t)$ satisfy conditions (A_1) and (B_1). So, we may apply Theorem 1, when the derivative of $V(t, x(t))$ has an appropriate estimate on the set $V(t, \varphi(0))=W(t, \varphi)$.

If $W(t, \varphi)=\sup_{-\tau\leq s\leq 0} V(t+s, \varphi(s))$, $\tau\in R^+$, then we get a RAZUMIKHIN type comparison result [6, 12]. One may put

(7) $$W(t, \varphi)=\sup_{s\in R^-} V(t+s, \varphi(s))$$

or

(8) $$W(t, \varphi)=\sup_{s\in R^-} l(s, V(t+s, \varphi(s))),$$

where $l\colon R^-\times R^+\to R^+$ is a continuous function such that $l(s_1, v_1)<l(s_2, v_2)<v_2$ for all $s_1<s_2<0$, $0\leq v_1<v_2$ and supposing that the supremums on the right-hand side of (7) and (8) exist for all $\varphi\in D$. If $l(s, v)=e^{\gamma s}v$ for a $\gamma>0$, then we obtain the case examined by M. PARROTT [11].

Let $k\colon R^-\to R^+$ be a measurable function such that $k(s_0)=0$ implies $k(s)=0$ for all $s\leq s_0$, for each $r\geq 0$

(9) $$\operatorname*{ess\,sup}_{s\in R^-,\, k(s)>0} \frac{k(s-r)}{k(s)}+\int_{-r}^{0} k(s)\,ds\leq 1$$

holds and $\int_{-\infty}^{0} k(s)V(t+s, \varphi(s))ds$ exists for all $t\geq 0$, $\varphi\in D$. Then one can choose

(10) $$W(t, \varphi)=\max\Big\{V(t, \varphi(0)), \int_{-\infty}^{0} k(s)V(t+s, \varphi(s))\,ds.$$

We remark if k is continuous then (9) implies $k(s)\leq Me^{\gamma s}$ for all $s\in(-\infty, 0]$ where $M, \gamma>0$. On the other hand, (9) is true if $k(s)=Me^{\gamma s}$ such that $\gamma\geq M>0$.

Our comparison results are useful to prove stability, uniqueness and continuous dependence of the solutions (see e.g. [1]). In this paper we deal with the convergence properties of solutions as $t\to\infty$. From Theorems 2 and 4 we get the following results. The derivative of V with respect to (6) is defined by

$$\dot V(t, \varphi)=\limsup_{h\to 0+}\big(V(t+h, \varphi(0)+hf(t, \varphi))-V(t, \varphi(0))\big)h^{-1}.$$

Corollary 4. *Suppose* (AV), (BV_1) *and*

(DV) $$\dot V(t, \varphi)\leq \omega(t, V(t, \varphi(0)))$$

whenever

(CV_1) $$0<V(t, \varphi(0))=W(t, \varphi)$$

for $t\in R^+$, $\varphi\in D$, where $\omega\colon R^+\times R^+\to R^+$ is continuous, nondecreasing in its second variable and the solutions of the equation $\dot u(t)=\omega(t, u(t))$ are defined and bounded *on R^+. Then for each solution $x(t)$ of (6) defined on $[0, \infty)$ the limit $\lim_{t\to\infty} W(t, x_t)$ exists.*

Corollary 5. *Let $a(t, r)$, $p(t, r)$, $h(t, r)$ be the same functions as in Theorem 3 and for $r>0$, $0 \leqq z \leqq s \leqq t$ define*

$$g^*(z, s, t, r) = \sup\{\dot{V}(s, \varphi):\ a(t, r) < V(v, \varphi(v-s)),$$
$$W(v, \varphi_{v-s}) \leqq r \text{ for all } v\in[z, s]\}.$$

Suppose $V(t, x)$ has continuous partial derivatives, (AV), (BV$_1$), (E$_1$) *are fulfilled and* (DV) *is true whenever* (CV$_1$), $p(t, V(t, \varphi(0)))>0$ *and for all* $z\in[h(t, V(t, \varphi(0))), t]$ *the inequality*

$$a(t, V(t, \varphi(0)) < V(z, \varphi(z-t)) \leqq W(z, \varphi_{z-t}) \leqq W(t, \varphi))$$

is satisfied. Then for each solution $x(t)$ of (6) that is defined on $[0, \infty)$, the limit $\lim_{t\to\infty} W(t, x_t)$ exists.

Generally, the existence of the limit $\lim_{t\to\infty} W(t, x_t)$ gives little information about the asymptotic behavior of solutions. For example, if $W(t, \varphi) = \sup_{s\in R^-} V(t+s, \varphi(s))$, then the existence of $\lim_{t\to\infty} W(t, x_t)$ means the boundedness of $V(t, x(t))$ on $[t_0, \infty)$ only. Using Theorem 5 we may obtain conditions for $W(t, \varphi)$ to guarantee the existence of $\lim_{t\to\infty} V(t, x(t))$, which gives much more information about $x(t)$.

Corollary 6. *Suppose that all conditions of Corollary 4 (or 5) are satisfied and there exist functions k_1, k_2: $R^+\times R^+\to R^+$ and h: $R^+\times R^+\times D\to R^+$ such that $k_1(r, u)$, $k_2(r, u)$ are monotone nondecreasing and continuous in u for all $r\in R^+$, $\lim_{r\to 0+} k_1(r, u)=u$ for all $u>0$, $k_2(r, u)<u$ for all $r, u>0$, $k_2(0, u)\leqq u$ for all $u\geqq 0$, $h(t-r, t, \varphi)\to 0$ as $t\to\infty$ for all $r>0$, $\varphi\in D$, moreover*

(BV$_2$)
$$W(t, \varphi) \leqq \max\{k_1(r, \max_{-r\leqq s\leqq 0} V(t+s, \varphi(s))),$$
$$k_2(r, \max_{-\tau\leqq s\leqq -r} V(t+s, \varphi(s)))\}+h(\tau, t, \varphi_{-\tau})$$

for all $t\in R^+$, $\tau\in[0, t]$, $r\in[0, \tau]$, $\varphi\in B_{-\tau}\cap D$. Then $\lim_{t\to\infty} V(t, x(t))$ exists for every solution $x(t)$ of (7) which is defined on $[0, \infty)$ and for which $V(t, x(t))$ is uniformly continuous on $[0, \infty)$.

If $W(t, \varphi)$ is defined by (8), where $l(s, v)\to 0$ as $s\to -\infty$ for every $v>0$, $V(t+s, \varphi(s))$ is bounded on R^-, then (BV$_2$) is true with $k_1(r, u)=u$, $k_2(r, u)=$ $=l(-r, u)$ and $h(r, t, \varphi)=\sup_{s\leqq -r} l(s, V(t+s, \varphi(s)))$. If $W(t, \varphi)$ is defined by (10), $\int_{-\infty}^{0} k(s)V(t+s, \varphi(s))\, ds<\infty$ for all $\varphi\in D$ and $t\in R^+$, $k(s)$ is nondecreasing,

$\int_{-\infty}^{0} k(s)ds=1$, then (BV_2) is true with

$$k_1(r, u) = u\Big(1+\Big(\int_{-r}^{0} k(s)\,ds\Big)^2-\int_{-r}^{0} k(s)\,ds\Big)^{-1},$$

$$k_2(r, u) = u\Big(1+\Big(\int_{-r}^{0} k(s)\,ds\Big)^2\Big)^{-1},$$

$$h(\tau, t, \varphi) = \int_{-\infty}^{0} k(s-\tau)V\big(t+s-\tau, \varphi(s)\big)\,ds.$$

We get an important special case if

(11) $$D = B,\quad V(t, x) = \|x\|_X,\quad W(t, \varphi) = \|\varphi\|_B.$$

Then (AV), (BV_1) and (BV_2) are axioms for these norms as it is used generally in functional differential equations with infinite delay.

These axioms resemble axioms of admissible phase spaces in which the estimation

(12) $$\mu\|\varphi(0)\|_X \leqq \|\varphi\|_B \leqq K(r) \sup_{-r\leqq s\leqq 0} \|\varphi(s)\|_X+M(r)\|\varphi_{-\tau}\|_B$$

is true with $\mu>0$ and some continuous functions K, M: $R^+\to R^+$ [7]. If $\mu=1$ and $K(r)+M(r)\leqq 1$ then (12) implies (AV) and (BV_1) in the case (11). So (AV) and (BV_1) are true in special admissible phase spaces. In case (11) property (BV_2) cannot be compared to (12).

In case of several phase spaces used in theory of functional differential equations with infinite delay we may define a norm such that (AV), (BV_1) and (BV_2) are fulfilled. So, in the special case (11), if

a) $B=BC$ is the space of bounded continuous functions on $(-\infty, 0]$ into X with norm

$$\|\varphi\|_{BC} = \sup_{s\in R^-} \|\varphi(s)\|_X$$

then (AV) and (BV_1) are fulfilled but (BV_2) is not statisfied. If we put

$$\|\varphi\|_{BC} = \sup_{s\in R^-} p(s)\|\varphi(s)\|_X,$$

where p: $R^-\to R^+$, $p(s_1)<p(s_2)<1$ for all $s_1<s_2<0$, $p(0)=1$ and $\lim_{s\to-\infty} p(s)=0$, then (AV), (BV_1) and (BV_2) are fulfilled.

b) $B=C_\gamma$ $(\gamma\in R^+)$ is the space of continuous functions φ on $(-\infty, 0]$ such that $\lim_{s\to-\infty} e^{\gamma s}\|\varphi(s)\|_X$ exists and

$$\|\varphi\|_{C_\gamma} = \sup_{s\in R^-} e^{\gamma s}\|\varphi(s)\|_X,$$

then for $\gamma>0$ (AV), (BV_1) and (BV_2) are fulfilled. For $\gamma=0$ (AV), (BV_1) hold, but (BV_2) does not.

c) $B=L_k^p$, $p\geqq 1$ is the space of measurable functions on R^- such that

$$\int_{-\infty}^{0} k(s)\|\varphi(s)\|^p\,ds<\infty,$$

where $k\colon R^-\to R^+$ is measurable, $\int_{-\infty}^{0} k(s)ds=1$ and $\int_{-r}^{0} k(s)ds>0$ for all $r>0$ then (AV) and (BV_2) are true with the norm

$$\|\varphi\|_{L_k^p}=\max\Big(\|\varphi(0)\|_X,\ \Big(\int_{-\infty}^{0} k(s)\|\varphi(s)\|_X^p\,ds\Big)^{1/p}\Big).$$

If (9) is valid for all $r>0$, then (BV_1) is fulfilled, too.

5. Examples

1. Consider the equation

$$\dot{x}(t)=H\Big(t,x(t)-\int_{-\infty}^{0} k(s)x(t+s)\,ds\Big). \tag{13}$$

Here $H\colon R^+\times R\to R$ is continuous, $H(t,u)u\leqq 0$ for all $t\in R^+$, $u\in R$; $\sup_{t\in R^+,u\in K}|H(t,u)|<\infty$ for every compact set $K\subset R$; $k\colon R^-+R^+$ is nondecreasing, measurable, $\int_{-\infty}^{0} k(s)ds=1$. So, for each constant c, $x(t)\equiv c$ is a solution of equation (13). Let us choose L_k^1 as a phase space for (13). Then the existence and continuity of a solution through every $\varphi\in L_k^1$ is insured, further, if a solution $x(t)$ is bounded, then it can be continued as $t\to\infty$.

Assertion. *If* (9) *is fulfilled then every noncontinuable solution of* (13) *has a finite limit as* $t\to\infty$.

In order to prove this assertion, we define the following functions for $t\in R^+$, $\varphi\in L_k^1$. $V(t,\varphi(0))=|\varphi(0)|$,

$$W(t,\varphi)=\max\Big(|\varphi(0)|,\ \int_{-\infty}^{0} k(s)|\varphi(s)|\,ds\Big).$$

If $x(t)$ is a noncontinuable solution of (13) on $[t_0,a)$ through φ, then $g(t)=V(t,x(t))$ and $W(t,x_t)$ satisfy the assumptions of Corollary 1 with $\omega(t,u)\equiv 0$. So, we have

$$|x(t)|\leqq\max\Big(|x(0)|,\ \int_{-\infty}^{0} k(s)|x(t+s)|\,ds\Big)$$

for $t \in [t_0, a)$. Consequently, $x(t), \dot{x}(t)$ are bounded, and $a = \infty$, therefore we may apply Corollaries 4 and 6 with $V(t, \varphi(0))$, $W(t, \varphi)$, which implies the assertion.

Assertion. *If $k(s)$ is differentiable and $k'(s) \geqq k(0)k(s)$ for $s \in R^-$, then every bounded solution of equation* (13) *has a finite limit as $t \to \infty$.*

Indeed, let $x(t)$ be a bounded solution of (13) on $[t_0, \infty)$ and put $g(t) = V(t, x(t))$, $f(t) = W(t, x_t)$ where V and W are defined above.

We want to estimate the derivative $D^+ f(t)$. We have three cases:

a)
$$|x(t)| \geqq \int_{-\infty}^{0} k(s)|x(t+s)|\,ds.$$

Then $f(t) = g(t)$ and (13) implies $\frac{d}{dt}|x(t)| \leqq 0$.

b)
$$|x(t)| < \int_{-\infty}^{0} k(s)|x(t+s)|\,ds.$$

In this case
$$f(t) = \int_{-\infty}^{0} k(s)|x(t+s)|\,ds = \int_{-\infty}^{t} k(s-t)|x(s)|\,ds,$$

so using the inequality

$$\frac{d}{dt}\int_{-\infty}^{0} k(s)|x(t+s)|\,ds = k(0)|x(t)| - \int_{-\infty}^{t} k'(s-t)|x(s)|\,ds \leqq \tag{14}$$
$$\leqq k(0)|x(t)| - \int_{-\infty}^{0} k'(s)|x(t+s)|\,ds \leqq$$
$$\leqq k(0)\left(|x(t)| - \int_{-\infty}^{0} k(s)|x(t+s)|\,ds\right)$$

we get $\frac{d}{dt} f(t) \leqq 0$.

c)
$$|x(t)| = \int_{-\infty}^{0} k(s)|x(t+s)|\,ds.$$

Then using the case a) and inequality (14) we have

$$D^+ f(t) \leqq D^+|x(t)| + D^+ \int_{-\infty}^{0} k(s)|x(t+s)|\,ds \leqq$$
$$\leqq D^+|x(t)| + k(0)\left(|x(t)| - \int_{-\infty}^{0} k(s)|x(t+s)|\,ds\right) \leqq 0.$$

Therefore $D^+f(t)\leqq 0$ for all $t\in[t_0, \infty)$, so $\lim_{t\to\infty} f(t)$ exists. Consequently, Theorem 5 implies our assertion.

2. These results may be extended to the equation

$$\dot{x}(t) = H(t, x(t), h(t, x_t)), \tag{15}$$

where

$$H: R^+\times R^n\times R^n \to R^n, \quad h: R^+\times L_k^1 \to R^n,$$

$$\|h(t, \varphi)\| \leqq \int_{-\infty}^{0} k(s)\|\varphi(s)\|\, ds, \quad \sup_{u, v\in K, t\in R^+} \|H(t, u, v)\| < \infty$$

for every compact set $K\subset R^n$ and

$$\sup_{\|v\|\leqq\|u\|} (H(t, u, v), u) \leqq p(t)\|u\|^2$$

where $(.\,,.)$ means the inner product in R^n, and $p\colon R^+\to R^+$, $\int_0^\infty p(s)ds<\infty$. We may put $V(t, x)=\|x\|=(x, x)^{1/2}$ and

$$W(t, \varphi) = \max\left\{\|\varphi(0)\|, \int_{-\infty}^{0} k(s)\|\varphi(s)\|\, ds\right\},$$

and we assert that $\lim_{t\to\infty} \|x(t)\|$ exists for every solution of (15), if k satisfies the same properties as in Example 1.

3. Let us examine the equation

$$\dot{x}(t) = -p(t)x(t)+q(t)x(t-\varrho(t)). \tag{16}$$

Let $p, q, \varrho\colon R^+\to R$ be continuous, bounded functions, $\varrho(t)\geqq 0$ for $t\in R^+$. Choose BC as a phase space for (15).

Put $V(t, x)=x^2$, $W(t, \varphi)=\sup_{s\in R^-} e^{2\gamma s}|\varphi(s)|^2$, where $\gamma>0$ is a constant. Then

$$\dot{V}(t, \varphi) = -2p(t)\varphi^2(0)+2q(t)\varphi(0)\varphi(-\varrho(t)),$$

therefore, if $W(t, \varphi)=V(t, \varphi(0))$, i.e.

$$e^{-2\gamma\varrho(t)}\left|\varphi(-\varrho(t))\right|^2 \leqq \varphi^2(0), \quad \text{and} \quad |q(t)|e^{\gamma\varrho(t)} \leqq p^+(t),$$

then

$$\dot{V}(t, \varphi) \leqq -2p(t)\varphi^2(0)+2|q(t)|e^{\gamma\varrho(t)}\varphi^2(0) \leqq 2p^-(t)V(t, \varphi(0)),$$

where and in the sequel, for any $a\in R$, a^+, a^- are defined by $a^+=\max\{0, a\}$, $a^-=\max\{0, -a\}$, respectively. Similarly to Example 1, the existence of solutions for all large t and their boundedness together with the derivative can be proved. Therefore, Corollary 6 gives:

Assertion. *If* $p^-\in L^1$ *and there exists* $\gamma>0$ *such that* $|q(t)|\, e^{\gamma\varrho(t)}\leqq p^+(t)$ *for all* $t\in R^+$, *then* $x(t)\to$constant *as* $t\to\infty$ *for every solution of* (16).

4. Consider the equation

$$\dot{x}(t) = q(t)x(t-\varrho(t)), \tag{17}$$

where $q, \varrho\colon R^+\to R$ are continuous, q is bounded, $\varrho(t)\geqq 0$ for $t\in R^+$, and there exists a $T>0$ such that $t-\varrho(t)\geqq 0$ for all $t\geqq T$. Choose BC as a phase space.

Assertion. *Suppose that there exists a strictly increasing continuous function* $g(s)$ *on* R^- *such that* $\lim\limits_{s\to-\infty} g(s)=0$,

$$\int\limits_{t-\varrho(t)}^{t} |q(t)|/g(-\varrho(s))\,ds < 1$$

for all $t\geqq T$ *and*

$$\int\limits_{T}^{\infty} q^+(t)/g(-\varrho(t))\,dt < \infty.$$

Then for every solution $x(t)$ *of equation* (17) *the limit* $\lim\limits_{t\to\infty} x(t)$ *exists.*

Put $V(t,x)=|x|$, $W(t,\varphi)=\sup\limits_{s\in R^-} g(s)|\varphi(s)|$, $p(t,r)\equiv 0$, $h(t,r)=(t-\varrho(t))^+$,

$$a(t,r) = \frac{r}{2}\Big(1-\int\limits_{t-\varrho(t)}^{t} |q(s)|/g(-\varrho(s))\,ds\Big).$$

Then

$$\dot{V}(s,\varphi) = q(s)\varphi(-\varrho(s))\operatorname{sgn}\varphi(0)$$

for all $\varphi\in BC$, so we have

$$q^*(z,s,t,r) \leqq r|q(z)|/g(-\varrho(z)),$$

therefore (E_1) is fulfilled for $t\geqq T$. If $t\geqq T$, $0<|\varphi(0)|=\sup g(s)|\varphi(s)|$ and

$$a(t,|\varphi(0)|) < |\varphi(z)| \leqq \sup_{s\in R^-} g(s+z)|\varphi(s+z)| \leqq \sup_{s\in R^-} g(s)|\varphi(s)|$$

for all $z\in[-(t-\varrho(t))^+, 0]$, then $\operatorname{sgn}\varphi(0)=\operatorname{sgn}\varphi(-\varrho(t))$ and therefore

$$\dot{V}(t,\varphi) \leqq q^+(t)V(t,\varphi(0))/g(-\varrho(t)).$$

The boundedness of solutions and their derivatives can be proved similarly to Example 1. So, we can apply Corollary 6.

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Banach—Steinhaus theorems of locally convex spaces based on sequential equicontinuity and essentially uniform boundedness

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The main purpose of this paper is to construct locally convex spaces which satisfy the Banach—Steinhaus theorems for sequentially continuous or essentially bounded, linear functionals and maps as naturally as barrelled spaces satisfy these theorems for continuous, linear functionals and maps. For this purpose, we consider the sequential and bornological extensions of a locally convex space and consider the equicontinuous subsets of the duals of these extensions. Such equicontinuous sets are called sequentially equicontinuous and essentially uniformly bounded subsets, respectively. We prove some basic properties of these sets. The required spaces mentioned above are called strictly sequentially barrelled spaces and locally convex spaces satisfying the strict condition of ess-uniform boundedness. The basic properties and Banach—Steinhaus theorems of these spaces are also proved.

I. Introduction

The generalization of Banach—Steinhaus theorems from normed linear spaces to locally convex spaces (abbrev. by LCS) has been thoroughly discussed (e.g. [1] and [10]). These theorems of LCS are originated in the discussion of barrelled spaces and the main notion in the proofs is equicontinuity of linear functionals or maps. In this paper, we will construct LCS satisfying Banach—Steinhaus theorems for sequentially continuous (or essentially bounded) linear functionals and maps as naturally as barrelled spaces satisfy these theorems for continuous, linear functionals and maps. The notions for the proofs are sequential equicontinuity (or essentially uniform boundedness) of linear functionals and maps. Some notations will be introduced in this section.

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Let L be a linear space over $\mathbf{K}$ ($\mathbf{K}=\mathbf{R}$ or $\mathbf{K}=\mathbf{C}$) and L^* be its algebraic dual. If p is a semi-norm on L, then $L_{(p)}$ is the semi-normed linear space defined by p. For any collection $P\neq\emptyset$ of semi-norms on L, let $L_{\mathbf{P}}$ be the projective limit of $\{L_{(p)}: p\in P\}$ in the sense that $L_{\mathbf{P}}$ is with the weakest, locally convex topology for any $p\in\mathbf{P}$ to be continuous. Thus the collection $\left\{\bigcap_{i=1}^{n} V_{p_i}(0,\varepsilon)\ \left(\text{or } \bigcap_{i=1}^{n} \overline{V}_{p_i}(0,\varepsilon)\right): n\in\mathbf{Z}_+, p_i\in\mathbf{P} \text{ for } i=1,2,\dots,n \text{ and } \varepsilon>0\right\}$ is an open (or closed) base of neighborhoods of $0\in L_{\mathbf{P}}$ (Examples 2 and 4, [3]), where $V_p(0,\varepsilon)$ (or $\overline{V}_p(0,\varepsilon)$) is the set $\{x\in L: p(x)<\varepsilon$ (or $p(x)\leqq\varepsilon)\}$. If L is a LCS and L' is its topological dual, then the space $L_w=(L,\sigma(L,L'))$ is the projective limit of $\{L_{(p_\varphi)}: \varphi\in L'\}$, where $p_\varphi=|\varphi|$ on L.

If L_1, L_2 are LCS over $\mathbf{K}$ and $\mathscr{L}(L_1,L_2)$ is the linear space of all linear maps from L_1 into L_2, then we have the following linear subspaces of $\mathscr{L}(L_1,L_2)$.

(i) $\mathscr{B}(L_1,L_2)=\{\varphi\in\mathscr{L}(L_1,L_2): \varphi$ is continuous on $L_1\}$.

(ii) If $\varphi\in(L_2)^{L_1}$ and $\{\varphi(x_n): n=1,2,\dots\}$ converges to $\varphi(x)$ for any sequence $\{x_n: n=1,2,\dots\}$ converging to x, then φ is called sequentially continuous (abbrev. by s-continuous) at x. Let $\mathscr{B}^+(L_1,L_2)$ be the linear space of all s-continuous, linear maps from L_1 into L_2.

(iii) If $\varphi\in(L_2)^{L_1}$ and $\varphi(A)$ is bounded in L_2 for any bounded subset A of L_1, then φ is called essentially bounded (abbrev. by ess-bounded) on L_1. Let $\mathscr{B}^b(L_1,L_2)$ be the linear space of all ess-bounded, linear maps from L_1 into L_2. Hence $\mathscr{B}(L_1,L_2)\subseteqq\mathscr{B}^+(L_1,L_2)\subseteqq\mathscr{B}^b(L_1,L_2)$. If A is a bounded subset of L_1 and p is a continuous seminorm on L_2, then the function $T_{A,p}$: $\mathscr{B}^b(L_1,L_2)\to[0,+\infty)$ defined by $\varphi\mapsto\sup\{p(\varphi(x)): x\in A\}$ is a semi-norm (since $p^{-1}([0,1])$ is a neighborhood of $0\in L_2$). If $\omega\neq\emptyset$ is a collection of bounded subsets of L_1 and $\mathbf{P}_\omega=\{T_{A,p}: A\in\omega$ and p is a continuous semi-norm on $L_2\}$, then we let $\mathscr{B}^b_\omega(L_1,L_2)$ be the projective limit of $\{\mathscr{B}^b_{(T_{A,p})}(L_1,L_2): T_{A,p}\in P_\omega\}$. In particular, if $\omega=\{\{x\}: x\in L_1\}$ (or $\omega=\{A: A$ is a bounded subset of $L_1\}$), then we write $\mathscr{B}^b_\sigma(L_1,L_2)$ (or $B^b_\beta(L_1,L_2)$) for $\mathscr{B}^b_\omega(L_1,L_2)$. If $\omega=\{A: A$ is a precompact subset of $L_1\}$, then we write $\mathscr{B}^b_\lambda(L_1,L_2)$ for $\mathscr{B}^b_\omega(L_1,L_2)$. If L is a LCS, then we write L^+ and L^b for $\mathscr{B}^+(L,\mathbf{K})$ and $\mathscr{B}^b(L,\mathbf{K})$, respectively.

The following general result will be frequently applied later.

Theorem 1. *Let L_1, L_2 be LCS over $\mathbf{K}$, $\omega\neq\emptyset$ be a collection of bounded subsets of L_1 and $\Lambda\subseteqq\mathscr{B}(L_1,L_2)$.*

(i) *Λ is equicontinuous on L_1 iff $A_1'={}^t\Lambda(A_2')$ is an equicontinuous subset of L_1' for any equicontinuous subset A_2' of L_2'.*

(ii) *Λ is bounded in $\mathscr{B}_\omega(L_1,L_2)$ iff $A_1'={}^t\Lambda(A_2')$ is bounded in $(L_1)'_\omega$ for any equicontinuous subset A_2' of L_2'.*

We note that ${}^t\Lambda(A_2')=\{{}^t\varphi(\psi)\in L_1': \varphi\in\Lambda$ and $\psi\in A_2'\}$, where ${}^t\varphi$ is the continuous transpose of φ on L_2' (Definition on p. 254, [1]).

Proof. (i) If Λ is equicontinuous on L_1 and $A_2' \subseteq L_2'$ is equicontinuouson on L_2, then $A_2' \subseteq W^\circ$ for some closed, convex, balanced neighborhood W of $0 \in L_2$ (Proposition 3.4.5, [1]), where W° is the polar of W in L_2'. By assumption, there is a neighborhood V of $0 \in L_1$ with $\varphi(V) \subseteq W$ for $\varphi \in \Lambda$. We can check $A_1' = {}^t\Lambda(A_2') \subseteq V^\circ$. Hence A_1' is equicontinuous on L_1. Conversely, if W is a closed, convex, balanced neighborhood of $0 \in L_2$, then ${}^t\Lambda(W^\circ) \subseteq V^\circ$ for some neighborhood V of $0 \in L_1$. Hence if $x \in V$ and $\varphi \in \Lambda$, then $|\psi(\varphi(x))| = |{}^t\varphi(\psi(x))| \leqq 1$ for any $\psi \in W^\circ$. This implies $\varphi(x) \in {}^\circ(W^\circ) = W$ (Proposition 35.3, [10]) for $x \in V$ and $\varphi \in \Lambda$. Hence Λ is equicontinuous on L_1.

(ii) If Λ is bounded in $\mathscr{B}_\omega(L_1, L_2)$ and A_2' and W are defined in (i), then the Minkowski functional $p(y)$ associated with W on L_2 is a continuous semi-norm and $W = p^{-1}([0, 1])$ (Lemma 5.1, [8]). If $B \in \omega$, then we can check $A_1' = {}^t\Lambda(A_2') \subseteq \alpha B^\circ$ for some $\alpha > 0$. Hence A_1' is bounded in $(L_1)_\omega'$. Conversely, if $T_{B,p} \in \mathbf{P}_\omega$, then $W = p^{-1}([0, 1])$ is a closed, convex, balanced neighborhood of $0 \in L_2$, and ${}^t\Lambda(W^\circ)$ is bounded in $(L_1)_\omega'$. Thus ${}^t\Lambda(W^\circ) \subseteq \alpha B^\circ$ for some $\alpha > 0$. This implies $\Lambda \subseteq \bar{V}_{T_{B,p}}(0, \alpha)$, and Λ is bounded in $\mathscr{B}_\omega(L_1, L_2)$.

Lemma 1. *Let L be a linear space and M^* be a linear subspace of L^*. If $L_w = (L, \sigma(L, M^*))$, then $(L_w)' = M^*$.*

II. Sequential and bornological extensions of locally convex spaces

By applying the projective limit construction of LCS, we can have various extensions of a given LCS. Before presenting two typical examples of such extensions, we consider another extension of a Hausdorff, linear topological space which is totally irrelevant to the construction in I.

If X is a Hausdorff, topological space, then the collection $\eta_{x,s}$ of all s-neighborhoods of any $x \in X$ (Definition in § II, [4]) is a filter on X (Definition on p. 75, [1]) and $\tau_s = \{V \subseteq X \colon x \in V \Rightarrow V \in \eta_{x,s}\}$ is a topology of X which contains all open subsets of X, and has a base of neighborhoods of any $x \in X$ consisting of members of $\eta_{x,s}$ (Propositions 1 and 2, [4]). Hence if X_{τ_s} is X with the topology τ_s, then X_{τ_s} is called the sequential extension of X. X and X_{τ_s} have the same convergent sequences. X is called sequential if $X_{\tau_s} = X$. Hence $X = X_{\tau_s}$ iff any Hausdorff topology of X which has the same convergent sequences as the original topology of X is contained in the original topology.

Lemma 2. *Let X be a topological space and $\emptyset \neq A \subseteq X$, then A is closed in X_{τ_s} iff the limit of any convergent sequence (in X) of points in A is in A (Proposition 3, [4]).*

Corollary 1. *If X is a first countable, topological space, then X is sequential* (*Corollary* 3, [4]), *and has the same neighborhoods and s-neighborhoods.*

Lemma 3. *Let X_1 and X_2 be topological spaces.*

(i) *Let f: $X_1 \to X_2$ be a map, then f is s-continuous on X_1 iff f: $(X_1)_{\tau_s} \to (X_2)_{\tau_s}$ is continuous* (*Theorem* 2 (i), [4]).

(ii) *If $(X_i)_{\tau_s}$ $(i=1, 2)$ has the filter of all neighborhoods of any $x_i \in X_i$ consisting of all s-neighborhoods of x_i* (*Corollary* 1 *and Remark* (ii), [4]), *then* $(X_1 \times X_2)_{\tau_s} = (X_1)_{\tau_s} \times (X_2)_{\tau_s}$, *where both sides of the identity are with the product topology.*

Proof. (ii) We note $(X_1)_{\tau_s} \times (X_2)_{\tau_s} \subseteqq (X_1 \times X_2)_{\tau_s}$ (Theorem 1 (ii), [4]), where "$\subseteqq$" means the set-containment between two topologies of $X_1 \times X_2$. Conversely, we note that W is a s-neighborhood of $(x_1, x_2) \in X_1 \times X_2$ iff $W = U \times V$ for some s-neighborhoods U of x_1 and V of x_2.

For the topological background of this paper, we refer [7].

If L is a linear topological space and $\eta_{0,s}$ is the filter of all s-neighborhoods of $0 \in L$, then $\eta_{x,s} = x + \eta_{0,s}$ for any $x \in L$.

Theorem 2. *If L is a linear topological space over* $\mathbf{K}$ *such that $\eta_{0,s}$ is the filter of all neighborhoods of $0 \in L_{\tau_s}$, then L_{τ_s} is linear topological space and $(L_{\tau_s})' = L^+$.*

Proof. Since $L \times L$ with the product topology is a linear topological space (p. 118, [1]), we have $(L \times L)_{\tau_s} = L_{\tau_s} \times L_{\tau_s}$ by Lemma 3 (ii) and $(\mathbf{K} \times L)_{\tau_s} = \mathbf{K} \times L_{\tau_s}$ by Corollary 1. The vector addition $L \times L \to L$ defined by $(x, y) \mapsto x + y$ is continuous, and hence s-continuous. Thus $L_{\tau_s} \times L_{\tau_s} \to L_{\tau_s}$ defined by $(x, y) \mapsto x + y$ is continuous, i.e. the vector addition of L is continuous w.r.t. τ_s. This is also the case for scalar multiplication of L. Hence L_{τ_s} is a linear topological space. The identity $(L_{\tau_s})' = L^+$ is clear.

Corollary 2. *If L_1, L_2 are linear topological spaces over* $\mathbf{K}$ *with the condition of Theorem* 2 *satisfied, then* $\mathscr{B}^+(L_1, L_2) = \mathscr{B}((L_1)_{\tau_s}, (L_2)_{\tau_s}) = \mathscr{B}((L_1)_{\tau_s}, L_2)$. *Hence A is a sequentially equicontinuous subset of $\mathscr{B}^+(L_1, L_2)$* (*Definition* 1 *in the following*) *iff A is an equicontinuous subset of $\mathscr{B}((L_1)_{\tau_s}, L_2)$.*

Proof. These results are clear since $A \subseteqq \mathscr{L}(L_1, L_2)$ is s-equicontinuous on L_1 iff $\bigcap \{\varphi^{-1}(W) : \varphi \in A\}$ is a s-neighborhood of $0 \in L_1$ for any neighborhood (or s-neighborhood) W of $0 \in L_2$.

Corollary 3. *If L is a linear topological space with the condition of Theorem* 2 *satisfied, then any s-equicontinuous subset of L^+ is relatively compact* ((d), p. 144, [1]) *in* $L_\sigma^+ = (L^+, \sigma(L^+, L))$ (*by Corollary* 2 *and Theorem* 3.4.1, [1]).

Since the sequential extension L_{τ_s} of a linear topological space L is not necessarily a linear topological space (Remark (iv), [4]), we now consider a special kind of sequential extension of LCS such that the resulting spaces are indeed LCS.

If L is a LCS, and $\mathbf{P}_s$ and $\eta_{0,cs}$ are the collections of all s-continuous seminorms on L and all convex, balanced s-neighborhoods of $0\in L$, respectively, then $p=p_V\in\mathbf{P}_s$ and $p^{-1}([0,1))\subseteqq V\subseteqq p^{-1}([0,1])$ for any $V\in\eta_{0,cs}$ (Lemma 5.1, [8]), where $p_V(x)$ is the Minkowski functional associated with V on L. Conversely, $V=p^{-1}([0,1))\in\eta_{0,cs}$ for any $p\in\mathbf{P}_s$. We can check that the map $\mathbf{P}_s\to\eta_{0,cs}$ defined by $p\mapsto V_p$ is injective, and $\mathbf{P}_s=\mathbf{P}\eta_{0,cs}=\{p_V\colon V\in\eta_{0,cs}\}$.

Since $\eta_{0,cs}$ is closed under finite intersection and positive multiple, $\eta_{0,cs}$ can be a base of neighborhoods of $0\in L$ w.r.t. some locally convex topology of L (Proposition 2.4.5, [1]). L with this unique topology is denoted by $L_{\tau_{cs}}$. Thus $L_{\tau_{cs}}=L_{\mathbf{P}_s}$ and $(L_{\tau_{cs}})'=L^+$.

We can check that $\mathbf{P}_s$ (or $\eta_{0,cs}$) is the collection of all continuous seminorms on $L_{\tau_{cs}}$ (or all convex, balanced neighborhoods of $0\in L_{\tau_{cs}}$). (Cf. Example (ii) after Thm. 7 (or Corollary 7), [3]).

For any LCS L, we have the set-containments $L\subseteqq L_{\tau_{cs}}\subseteqq L_{\tau_s}$. Hence $L_{\tau_{cs}}$ is called the c-sequential extension of L, and has the same convergent sequences as L. If L is first countable, then $L=L_{\tau_s}=L_{\tau_{cs}}$, and L must be metrizable (Theorem 2.6.1, [1]).

We now give the definition of sequential equicontinuity (abbrev. by s-equicontinuity).

Definition 1. Let L_1, L_2 be LCS over $\mathbf{K}$ and $A\subseteqq(L_2)^{L_1}$. If $\{\varphi(x_n)\colon n=1,2,\ldots\}$ converges to $\varphi(x)$ uniformly in $\varphi\in A$ for any sequence $\{x_n\colon n=1,2,\ldots\}$ converging to $x\in L_1$, then A is called s-equicontinuous at x. If A is s-equicontinuous on L_1 and $A\subseteqq\mathscr{L}(L_1,L_2)$, then $A\subseteqq\mathscr{B}^+(L_1,L_2)$. For a LCS L, $A^*\subseteqq L^*$ is s-equicontinuous on L iff $\lim_{n\to\infty}\sup\{|\varphi(x_n)|\colon\varphi\in A^*\}=0$ for any null sequence $\{x_n\colon n=1,2,\ldots\}$ in L. This notion was introduced by A. Grothendieck and was called limited subsets of L^* in [11].

Hence equicontinuity of linear maps implies s-equicontinuity. The reason for this terminology is clear from the proof of Corollary 2. Other characterizations of this terminology are the following: (i) for any neighborhood (or s-neighborhood) W of $0\in L_2$, $U\{\varphi(V)\colon\varphi\in A\}\subseteqq W$ for some s-neighborhood V of $0\in L_1$; and (ii) for any neighborhood W of $0\in L_2$, $U\{\varphi(V)\colon\varphi\in A\}\subseteqq W$ for some convex, balanced s-neighborhood V of $0\in L_1$.

If L is a LCS and $A\subseteqq L$ such that, for any convex, balanced s-neighborhood V of $0\in L$, there exist $x_1, x_2, \ldots, x_n\in A$ with $A\subseteqq\bigcup_{i=1}^{n}(x_i+V)$, then A is called s-precompact in L. Hence A is precompact and bounded in L (Proposition 2.10.7, [1]). If L_1, L_2 are LCS over $\mathbf{K}$, then we let $\mathscr{B}^+_{\lambda_s}(L_1,L_2)$ be the projective limit of $\{\mathscr{B}^+_{(T_A,p)}(L_1,L_2)\colon A$ is a s-precompact subset of L_1 and p is a continuous semi-norm on $L_2\}$.

Lemma 4. *If L_1, L_2 are LCS over* **K**, *then* $\mathscr{B}^+(L_1, L_2)=\mathscr{B}((L_1)_{\tau_{cs}}, L_2)=$ $=\mathscr{B}((L_1)_{\tau_{cs}}, (L_2)_{\tau_{cs}})$. *Hence Λ is a s-equicontinuous subset of $\mathscr{B}^+(L_1, L_2)$ iff Λ is an equicontinuous subset of $\mathscr{B}((L_1)_{\tau_{cs}}, L_2)$.*

By Lemma 4, we can easily prove properties of s-equicontinuity of linear maps.

(i) Let L be a LCS, then $A^*\subseteq L^*$ is s-equicontinuous on L iff A^* is equicontinuous on $L_{\tau_{cs}}$ iff $A^*\subseteq V^\circ=\{\varphi\in(L_{\tau_{cs}})'\colon |\varphi(x)|\leq 1$ for $x\in V\}$ for some neighborhood V of $0\in L_{\tau_{cs}}$ (i.e. a s-neighborhood of $0\in L$), Proposition 3.4.6, [1], where V° is the polar of V in $(L_{\tau_{cs}})'=L^+$. Hence $V^\circ=V^{\circ+}$, the polar of V in L^+. In this case, A^* is relatively compact in $((L_{\tau_{cs}})', \sigma((L_{\tau_{cs}})', L))=(L^+, \sigma(L^+, L))=L^+_\sigma$ (Theorem 3.4.1, [1]) and bounded in $L^+_\beta=(L^+, \beta(L^+, L))$ (and so is in L^+_ω for any collection ω of bounded subsets of L).

(ii) Let L_1, L_2 be LCS over **K** and $\Lambda\subseteq\mathscr{L}(L_1, L_2)$ be s-equicontinuous on L_1. Hence Λ is an equicontinuous subset of $\mathscr{B}((L_1)_{\tau_{cs}}, L_2)$ and bounded in $\mathscr{B}_\beta((L_1)_{\tau_{cs}}, L_2)=\mathscr{B}^+_\beta(L_1, L_2)$ (since $(L_1)_{\tau_{cs}}$ and L_1 have the same bounded subsets). Also, the closure of Λ in $((L_2)^{L_1})_\sigma$ is an equicontinuous subset of $\mathscr{B}((L_1)_{\tau_{cs}}, L_2)$ by Proposition 32.4, [10], and hence a s-equicontinuous subset of $\mathscr{B}^+(L_1, L_2)$, where $((L_2)^{L_1})_\sigma$ is the projective limit of $\{L_{(T_{x,p})}\colon x\in L_1$ and p is a continuous semi-norm on $L_2\}$ and $T_{x,p}\colon L=(L_2)^{L_1}\to[0, +\infty)$ is defined by $\varphi\mapsto p(\varphi(x))$ (p. 117—119, [1], Theorem 5, [3] and p. 280, [7]).

(iii) If L_1, L_2 are LCS over **K**, then Λ is a s-equicontinuous subset of $\mathscr{B}^+(L_1, L_2)$ iff Λ is an equicontinuous subset of $\mathscr{B}((L_1)_{\tau_{cs}}, L_2)$ iff $A_1^+={}^t\Lambda(A_2')=\{{}^t\varphi(\psi)\in((L_1)_{\tau_{cs}})'\colon$ $\varphi\in\Lambda$ and $\psi\in A_2'\}$ is an equicontinuous subset of $((L_1)_{\tau_{cs}})'$ for any equicontinuous subset A_2' of L_2' iff A_1^+ is a s-equicontinuous subset of L_1^+ (by Theorem 1). ${}^t\varphi$ is called the s-continuous transpose of φ on L_2^+. Similarly, for a given collection ω of bounded subsets of L_1, Λ is a bounded subset of $\mathscr{B}^+_\omega(L_1, L_2)$ iff $A_1^+={}^t\Lambda(A_2')$ is bounded in $(L_1)^+_\omega$ for any equicontinuous subset A_2' of L_2'.

(iv) Let L_1, L_2 be LCS over **K**, then $\mathscr{B}^+(L_1, L_2)=\mathscr{B}((L_1)_{\tau_{cs}}, L_2)$. If Λ is a s-equicontinuous subset of $\mathscr{B}^+(L_1, L_2)$, then Λ is an equicontinuous subset of $\mathscr{B}((L_1)_{\tau_{cs}}, L_2)$. Thus the relative topologies of Λ induced by $\mathscr{B}^+_\sigma(L_1, L_2)=\mathscr{B}_\sigma((L_1)_{\tau_{cs}}, L_2)$ and $\mathscr{B}^+_{\lambda_s}(L_1, L_2)=\mathscr{B}_\lambda((L_1)_{\tau_{cs}}, L_2)$ are identical (Proposition 32.5, [10]).

A LCS L is called c-sequential if $L_{\tau_{cs}}=L$. Hence $L=L_{\tau_{cs}}$ iff convex s-neighborhoods of $0\in L$ are neighborhoods of $0\in L$ (Theorem 1, [9]). In this case, we have $L'=(L_{\tau_{cs}})'=L^+$.

Let L be a LCS, then barrels (or quasibarrels) of $L_{\tau_{cs}}$ are called c-sequential barrels (or quasibarrels) of L. If c-sequential barrels (or quasibarrels) of L are neighborhoods of $0\in L$, then L is called c-sequentially barrelled (or quasibarrelled). Hence L is c-sequentially barrelled (or quasibarrelled) iff L is c-sequential and barrelled (or quasibarrelled) in the sense of Definition 3.6.1 (or 3.6.2), [1].

Example 1. If L is a complete, metrizable LCS, then L is first countable and barrelled. Thus $L=L_{\tau_s}=L_{\tau_{cs}}$, and L is c-sequentially barrelled (and quasibarrelled since L is also quasibarrelled).

We now consider the external construction of $L_{\tau_{cs}}$, and another characterization of c-sequential LCS.

If L is a LCS and A^+ is a s-equicontinuous subset of L^+, then $T_{A^+}: L \to [0, +\infty)$ defined by $x \mapsto \sup\{|\varphi(x)|: \varphi \in A^+\}$ is a semi-norm. Let $L_{\tau_{cs^+}}$ be the projective limit of $\{L_{(T_{A^+})}$: A^+ is a s-equicontinuous subset of $L^+\}$. Since the collection of all these subsets A^+ of L^+ is closed under finite union and positive multiple, the collection $\{{}^\circ(A^+)$: A^+ is a s-equicontinuous subset of $L^+\}$ is a base of neighborhoods of $0 \in L_{\tau_{cs^+}}$, where ${}^\circ(A^+)$ is the pre-polar of A^+ in L. By Lemma 4 and Proposition 3.4.7, [1], $L_{\tau_{cs}}=L_{\tau_{cs^+}}$.

An analogy of $L_{\tau_{cs^+}}$ is the following: If L is a LCS and $L_{\tau_{cs+}}$ is the projective limit of $\{L_{(T_{A'})}$: A' is a s-equicontinuous subset of $L'\}$, where $T_{A'}: L \to [0, +\infty)$ is the semi-norm defined by $x \mapsto \sup\{|\varphi(x)|: \varphi \in A'\}$, then the collection $\{{}^\circ(A')$: A' is a s-equicontinuous subset of $L'\}$ is a base of neighborhoods of $0 \in L_{\tau_{cs+}}$. Since V° is a s-equicontinuous subset of L' and $V={}^\circ(V^\circ)$ for any closed, convex, balanced neighborhood V of $0 \in L$, we have $L \subseteqq L_{\tau_{cs+}}$.

Theorem 3. *A LCS L is c-sequential iff* $L'=L^+$ *and* $L=L_{\tau_{cs+}}$.

Proof. If L is c-sequential, then $L=L_{\tau_{cs}}=L_{\tau_{cs^+}}$ and $L'=L^+$. Hence $L_{\tau_{cs^+}}=L_{\tau_{cs+}}$. Conversely, $L'=L^+$ implies $L_{\tau_{cs+}}=L_{\tau_{cs^+}}$. Hence $L=L_{\tau_{cs+}}= =L_{\tau_{cs^+}}=L_{\tau_{cs}}$.

An interesting question is to find $(L_{\tau_{cs+}})'$. Since equicontinuity implies s-equicontinuity, we have $L' \subseteqq (L_{\tau_{cs+}})'$. However, $L_{\tau_{cs+}} \subseteqq L_{\tau_{cs^+}}$ implies $(L_{\tau_{cs+}})' \subseteqq L^+$. But the "$=$" sign is generally not true, otherwise we will be led to the following paradox which can be considered as a consequence of Theorem 3.

Corollary 4. *A LCS L is c-sequential iff* $L=L_{\tau_{cs+}}$. *Hence* $L_{\tau_{cs^+}}=L_{\tau_{cs+}}$, *and* L' *and* L^+ *have the same s-equicontinuous subest of for any LCS L.*

Proof. If L is c-sequential, then $L=L_{\tau_{cs+}}$. Conversely, $L=L_{\tau_{cs+}}$ implies $L'=(L_{\tau_{cs+}})'=L^+$ and L is c-sequential. Hence $L_{\tau_{cs+}}$ is the projective limit of $\{L_{(T_{A'})}$: A' is an equicontinuous subset of $(L_{\tau_{cs^+}})'=L^+=(L_{\tau_{cs+}})'\}$ which is the space $L_{\tau_{cs+}}$ (Proposition 3.4.7, [1]).

At the end of this section, we consider another extension of LCS.

If L is a LCS, and $\mathbf{P}_b$ and η_{cb} are the collections of all ess-bounded seminorms of L and all convex, balanced bornivores of L, respectively, then the similar properties as those between $\mathbf{P}_s$ and $\eta_{0,cs}$ can be obtained for $\mathbf{P}_b$ and η_{cb}. Hence η_{cb} can be a base of neighborhoods of $0 \in L$ w.r.t. some locally convex topology of L which is denoted by τ_b. Thus $L_{\tau_b}=L_{\mathbf{P}_b}$ and $(L_{\tau_b})'=L^b$.

For any LCS L, the set-containments $L \subseteqq L_{\tau_{cs}} \subseteqq L_{\tau_b}$ are clear. Hence L_{τ_b} is called the bornological extension of L, and has the same bounded subsets as I.

Definition 2. Let L_1, L_2 be LCS over $\mathbf{K}$ and $\Lambda \in (L_2)^{L_1}$ such that $\varphi(A)$ is uniformly bounded (in L_2) in $\varphi \in \Lambda$ for any bounded subset A of L_1, then Λ is called essentially uniformly bounded (abbrev. by ess-uniformly bounded) on L_1. If $\Lambda \subseteqq \mathscr{L}(L_1, L_2)$, then $\Lambda \subseteqq \mathscr{B}^b(L_1, L_2)$. For a LCS L, $A^* \subseteqq L^*$ is ess-uniformly bounded iff $\sup \{|\varphi(x)|: x \in B \text{ and } \varphi \in A^*\} < +\infty$ for any bounded subset B of L.

Hence equicontinuity of linear maps implies ess-uniform boundedness. If $\Lambda \subseteqq \mathscr{L}(L_1, L_2)$, then the characterizations of ess-uniform boundedness of Λ can be obtained by replacing s-neighborhoods of $0 \in L_1$ and $0 \in L_2$ in those of s-equicontinuity of Λ which are stated previously with bornivores of L_1 and L_2.

Lemma 5. *If L_1, L_2 are LCS over* $\mathbf{K}$, *then* $\mathscr{B}^b(L_1, L_2) = \mathscr{B}((L_1)_{\tau_b}, L_2) = \mathscr{B}((L_1)_{\tau_b}, (L_2)_{\tau_b})$. *Hence Λ is an ess-uniformly bounded subset of $\mathscr{B}^b(L_1, L_2)$ iff Λ is an equicontinuous subset of $\mathscr{B}((L_1)_{\tau_b}, L_2)$.*

We can similarly define b-precompact subsets of a LCS L, i. e. precompact subset of L_{τ_b}. Hence, for LCS L_1 and L_2 over $\mathbf{K}$, we have $\mathscr{B}^b_{\lambda_b}(L_1, L_2) = \mathscr{B}_\lambda((L_1)_{\tau_b}, L_2)$, where $\mathscr{B}^b_{\lambda_b}(L_1, L_2)$ is the projective limit of $\{\mathscr{B}_{(T_{A}, p)}(L_1, L_2): A$ is a precompact subset of L_1 and p is a continuous semigroup on $L_2\}$.

Similarly, we can prove the following properties of ess-uniform boundedness of linear maps.

(i) Let L be a LCS, then $A^* \subseteqq L^*$ is ess-uniformly bounded on L iff $A^* \subseteqq V^{ob}$ for some bornivore V of L, where V^{ob} is the polar of V in L^b. Thus A^* is relatively compact in $L^b_\sigma = (L^b, \sigma(L^b, L))$ and bounded in $L^b_\beta = (L^b, \beta(L^b, L))$.

(ii) Let L_1, L_2 be LCS over $\mathbf{K}$ and $\Lambda \subseteqq \mathscr{L}(L_1, L_2)$ be ess-uniformly bounded on L_1, then Λ is bounded in $\mathscr{B}^b_\beta(L_1, L_2) = \mathscr{B}_\beta((L_1)_{\tau_b}, L_2)$ since L_1 and $(L_1)_{\tau_b}$ have the same bounded subsets. Also, the closure of Λ in $((L_2)^{L_1})_\sigma$ is an ess-uniformly bounded subset of $\mathscr{B}^b(L_1, L_2)$.

(iii) Let L_1, L_2 be LCS over $\mathbf{K}$, then Λ is an ess-uniformly bounded subset of $\mathscr{B}^b(L_1, L_2)$ iff $A^b_1 = {}^t\Lambda(A'_2) = \{{}^t\varphi(\psi) \in L^b_1: \varphi \in \Lambda \text{ and } \psi \in A'_2\}$ is ess-uniformly bounded on L_1, where ${}^t\varphi$ is called the ess-bounded transpose of φ on L^b_2. Similarly, for any collection ω of bounded subsets of L_1. Λ is bounded in $\mathscr{B}^b_\omega(L_1, L_2)$ iff $A^b_1 = {}^t\Lambda(A'_2)$ is bounded in $(L_1)^b_\omega$ for any equicontinuous subset A'_2 of L'_2.

(iv) If L_1, L_2 are LCS over $\mathbf{K}$ and Λ is an ess-uniformly bounded subset of $\mathscr{B}^b(L_1, L_2)$, then the relative topologies of Λ induced by $\mathscr{B}^b_\sigma(L_1, L_2)$ and $\mathscr{B}^b_{\lambda_b}(L_1, L_2)$ are identical.

A LCS L is bornological if $L = L_{\tau_b}$. Hence $L = L_{\tau_b}$ iff convex bornivores of L containing $0 \in L$ are neighborhoods of $0 \in L$. In this case, $L' = L^b$. In particular, L_{τ_b} is bornological.

A bornological barrel of a LCS L is a barrel of L_{τ_b}. L is called bornologically barrelled if any bornological barrel of L is a neighborhood of $0 \in L$. Hence L is bornologically barrelled iff L is bornological and barrelled.

Example 2. If L is a complete, metrizable LCS, then L is barrelled and bornological (Proposition 3.7.3, [1]). Hence L is bornologically barrelled.

At the end of this section, we will give another characterization of bornological spaces, and the external construction of L_{τ_b} for a LCS L.

First, it is clear that L_{τ_b} is the projective limit of $\{L_{(T_{A^b})}$: A^b is an ess-uniformly bounded subset of $L^b\}$, where T_{A^b}: $L \to [0, +\infty)$ is the semi-norm defined by $x \mapsto \sup\{|\varphi(x)|: \varphi \in A^b\}$, by Lemma 5 and Proposition 3.4.7, [1]. Let L_{τ_m} be the Mackey extension of L which is the projective limit of $\{L_{(T_{A'})}$: A' is a (closed) convex, balanced, compact subset of $L'_\sigma\}$, then a Hausdorff LCS L is bornological iff $L' = L^b$ and $L = L_{\tau_m}$ (Proposition 3.7.3, [1]).

An analogy of L_{τ_m} is the following: If L is a LCS and L_{τ^b} is the projective limit of $\{L_{(T_{A^b})}$: A^b is a convex, balanced, compact subset of $L^b_\sigma\}$, then the collection $\{\bigcap_{i=1}^{n} {}^\circ(A^b_i)$: A^b_i is a convex, balanced, compact subset of L^b_σ for $i = 1, 2, \ldots, n\}$ is a base of neighborhoods of $0 \in L_{\tau^b}$. Since compact subsets of L'_σ are compact in L^b_σ, we have $L_{\tau_m} \subseteqq L_{\tau^b}$.

Theorem 4. *A Hausdorff LCS L is bornological iff* $L = L_{\tau^b}$.

Proof. If L is bornological, then $L' = L^b$ and $L = L_{\tau_m} = L_{\tau^b}$. Conversely, since $(L_{\tau^b})' = (L_w)' = L^b$ by Lemma 1, where $L_w = (L, \sigma(L, L^b))$, we have $L' = L^b$. Also, $L = L_{\tau^b} \supseteqq L_{\tau_m} \supseteqq L$ implies $L = L_{\tau_m}$. Hence L is bornological.

Corollary 5. *For any Hausdorff LCS L,* $L_{\tau_b} = L_{\tau^b}$.

Proof. Since L_{τ_b} is bornological, we have $(L_{\tau_b})_{\tau^b} = L_{\tau_b}$. We can easily check $(L_{\tau_b})^b = L^b$ and $(L_{\tau_b})^b_\sigma = L^b_\sigma$ since L and L_{τ_b} have the same bounded subsets. Hence $(L_{\tau_b})_{\tau^b} = L_{\tau^b}$.

For the next application of Theorem 4, we need a technical lemma.

Lemma 6. *Let L be a linear space, M^* be a linear subspace of L^* and $M^*_\sigma = (M^*, \sigma(M^*, L))$. If A^* is bounded in M^*_σ, then $A = {}^\circ(A^*)$ is convex, balanced, absorbing in L and $P_A(x) = T_{A^*}(x)$ for any $x \in L$, where $p_A(x)$ is the Minkowski functional associated with A on L and T_{A^*}: $L \to [0, +\infty)$ is the semi-norm defined by $x \mapsto \sup\{|\varphi(x)|: \varphi \in A^*\}$.*

Corollary 6. *Let L be a Hausdorff LCS and M^* be a linear subspace of L^b.*

(i) *If A^b is a convex, balanced, compact subset of L^b_σ, then ${}^\circ(A^b)$ is a bornivore of L.*

(ii) *If A^b* (or A^*) *is a convex, balanced, relatively compact subset of L^b_σ* (or M^*_σ), *then ${}^\circ(A^b)$ (or ${}^\circ(A^*)$) is a bornivore of L.*

Proof. (i) Since A^b is bounded in L^b_σ, $A={}^\circ(A^b)$ is barrel of $(L, \sigma(L, L^b))$, and $T_{A^b}(x)=p_A(x)$ for $x\in L$. T_{A^b}, and so is p_A, is a continuous seminorm on $L_\tau=L_{\tau_b}$. Hence p_A is ess-bounded on L. Thus $p_A^{-1}([0, 1]))$ is a bornivore of L, and so is A.

(ii) The closure $\overline{A^b}$ of A^b in L^b_σ is convex, balanced, compact in L^b_σ. Thus ${}^\circ(\overline{A^b})$ is a bornivore of L, and so is ${}^\circ(A^b)$. If $\overline{A^*}$ is the closure of A^* is M^*_σ, then $\overline{A^*}$ is convex, balanced, compact in M^*_σ, and so is in L^b_σ. The case of ${}^\circ(A^*)$ is also proved.

Corollary 7. *Let L be a Hausdorff LCS and M^* be a linear subspace of L^b.*
(i) *If A^b is a convex, balanced, compact subset of L^b_σ, then A^b is bounded in L^b_β.*
(ii) *If A^b (or A^*) is a convex, balanced, relatively compact subset of L^b_σ (or M^*_σ), then A^b (or A^*) is bounded in L^b_β (or $M^*_\beta=(M^*, \beta(M^*, L))$).*

If L is a LCS and $V\in\eta_{cb}$, then $W\subseteqq V$ for some closed $W\in\eta_{cb}$ may not be true for the following reason: If this were true and L is quasibarrelled, then $L_{\tau_b}\subseteqq \subseteqq L_{\tau_{c_b}}=L$ implies $L_{\tau_b}=L$, where $L_{\tau_{c_b}}$ is the quasibarrelled extension of L (Definition after Lemma 7). Thus bornological spaces and quasibarrelled spaces are identical. This is a contradiction. Similarly, it is not true that any $V\in\eta_{0,cs}$ satisfies $W\subseteqq V$ for some closed $W\in\eta_{0,cs}$.

III. Special subclasses of c-sequential locally convex spaces and bornological spaces

In this section, we will consider some special subclasses of c-sequential LCS and bornological spaces, which have the properties as nice as those of barrelled spaces and quasibarrelled spaces. The generalizations of these classes will also be considered.

If L is a LCS and $L_w=(L, \sigma(L, L^+))$, then the collection $\{{}^\circ(A^+)$: A^+ is a finite subset of $L^+\}$ is a base of neighborhoods of $0\in L_w$ and $(L_w)'=L^+$ by Lemma 1. A barrel (or quasibarrel) of this L_w is called strictly sequential barrel (or quasibarrel), abbrev. by strict s-barrel (or s-quasibarrel), of L. Since $\sigma(L, L')\subseteqq\sigma(L, L^+)$ on L, and $(L, \sigma(L, L'))$ and L have the same closed, convex subsets and bounded subsets (Proposition 3.4.3, [1] and Theorem 36.2, [10]), bounded subsets of $(L, \sigma(L, L^+))$ are bounded in L, and barrels (or quasibarrels) of L are strict s-barrels (or s-quasibarrels).

Lemma 7. *Let L be a LCS.*
(i) *L and $(L, \sigma(L, L^+))$ have the same bounded subsets.*
(ii) *$L_{\tau_{cs}}$ and $(L, \sigma(L, L^+))$ have the same barrels and quasibarrels.*
(iii) *The same conclusions as* (i) *and* (ii) *can be obtained if $L_{\tau_{cs}}$ is replaced by L_{τ_b} and $(L, \sigma(L, L^+))$ by $(L, \sigma(L, L^b))$.*

Proof. All the results follow directly from $(L_{\tau_{cs}})_w=(L, \sigma(L, (L_{\tau_{cs}})'))=(L, \sigma(L, L^+))$ and $(L_{\tau_b})_w=(L, \sigma(L, L^b))$.

A Hausdorff LCS L is called strictly s-barrelled (or s-quasibarrelled) if strict s-barrels (or s-quasibarrels) of L are s-neighborhoods of $0\in L$. In this case, barrels (or quasibarrels) of L are s-neighborhoods of $0\in L$. Also, strictly s-barrelled spaces are strictly s-quasibarrelled. We note that L is strictly s-barrelled (or s-quasibarrelled) iff $L_{\tau_{cs}}$ is barrelled (or quasibarrelled).

If L is a LCS, then we let $L_{\tau c_e}$ (or $L_{\tau c_b}$) be the projective limit of $\{L_{(p_A)}$: A is a barrel (or quasibarrel) of $L\}$ which is called the barrelled (or quasibarrelled) extension of L since $L\subseteq L_{\tau c_e}$ (or $L\subseteq L_{\tau c_b}$) and L is barrelled (or quasibarrelled) iff $L=L_{\tau c_e}$ (or $L=L_{\tau c_b}$). Also, a semi-norm p on L is called c-sequentially lower semi-continuous if $\{x\in L: p(x)\leqq\alpha\}$ is closed in $L_{\tau_{cs}}$ for any $\alpha\in\mathbf{R}$. The following characterizations can be obtained.

Theorem 5. *Let L be a LCS, then the following statements* (i)$\sim$(iv) *are equivalent (by applying Lemma* 7 (i) *and* (ii)*):*

(i) *L is strictly s-barrelled (or s-quasibarrelled);*

(ii) *bounded subsets of L_σ^+ (or L_β^+) are s-equicontinuous on L;*

(iii) $(L_{\tau_{cs}})_{\tau c_e}=L_{\tau_{cs}}$ (or $(L_{\tau_{cs}})_{\tau c_b}=L_{\tau_{cs}}$);

(iv) *(ess-bounded) c-sequentially lower semi-continuous semi-norms on L are s-continuous on L.*

If L_1 is a LCS over $\mathbf{K}$, *then we have the following:*

(v) *L_1 is strictly s-barrelled (or s-quasibarrelled) iff for any LCS L_2 over* $\mathbf{K}$, *bounded subsets of $\mathcal{B}_\sigma^+(L_1, L_2)$* (or *$\mathcal{B}_\beta^+(L_1, L_2)$*) *are s-equicontinuous on L_1 (by applying* (ii) *and Theorem* 1*).*

For more refined characterization, we need the following lemma.

Lemma 8. *Let L be a LCS, $\emptyset\neq A\subseteq L$ and $A^{\circ+}$ (or $A^{\circ b}$) be the polar of A in* L^+ (or L^b).

(i) *If M^{**} is a linear space with $(L_\sigma^+)'\subseteq M^{**}\subseteq(L_\sigma^+)^*$ and $(A^{\circ+})^{\circ*}$ is the polar of $A^{\circ+}$ in M^{**}, then ${}^\circ((A^{\circ+})^{\circ*})=A^{\circ+}$, where the pre-polar is taken in L^+.*

(ii) *If M^{**} is a linear space with $(L_\beta^+)'\subseteq M^{**}\subseteq(L_\beta^+)^*$, then the above identity is also true.*

(iii) *If M^{**} is a linear space with $(L_\sigma^b)'\subseteq M^{**}\subseteq(L_\sigma^b)^*$, then ${}^\circ((A^{\circ b})^{\circ*})=A^{\circ b}$, where the pre-polar is taken in L^b.*

Proof. For any $x\in L$, the evaluation map $\hat{x}$: $L^+\to\mathbf{K}$ defined by $\varphi\mapsto\varphi(x)$ is in $(L_\sigma^+)'$ (and hence in $(L_\beta^+)'$). Since $(L_\sigma^+)'\subseteq M^{**}$, $A^{\circ+}$ is closed, convex, balanced in $(L^+, \sigma(L^+, M^{**}))$. The conclusion of (i) follows from Theorem 3.3.1, [1].

Proposition 1. *A LCS L is strictly s-barrelled (or s-quasibarrelled) iff for any linear space M^{**} with $(L_\sigma^+)' \subseteqq M^{**} \subseteqq (L_\sigma^+)^b$ (or $(L_\beta^+)' \subseteqq M^{**} \subseteqq (L_\beta^+)^b$), the identity $(M^{**}, e^+(M^{**}, L^+)) = (M^{**}, \beta(M^{**}, L_\sigma^+))$ (or $(M^{**}, e^+(M^{**}, L^+)) = (M^{**}, \beta(M^{**}, L_\beta^+)))$ is true, where $(M^{**}, e^+(M^{**}, L^+))$ $((M^{**}, \beta(M^{**}, L_\sigma^+))$ or $(M^{**}, \beta(M^{**}, L_\beta^+)))$ is the projective limit of $\{M^{**}_{(T_{A^+})}$: A^+ is a s-equicontinuous (or bounded) subset of L^+ (L_σ^+ or L_β^+)$\}$ and T_{A^+}: $M^{**} \to [0, +\infty)$ is the semi-norm defined by $\psi \mapsto \sup \{|\psi(\varphi)|: \varphi \in A^+\}$.*

Proof. Since the indicated collections of subsets of L^+ are closed under finite union and positive multiple, $\{(A^+)^{\circ *}$: A^+ is a s-equicontinuous (or bounded) subset of L^+ (L_σ^+ or L_β^+)$\}$ is a base of neighborhoods of $0 \in M^{**}$ w.r.t. $e^+(M^{**}, L^+)$ ($\beta(M^{**}, L_\sigma^+)$ or $\beta(M^{**}, L_\beta^+)$), where $(A^+)^{\circ *}$ is the polar of A^+ in M^{**}. The set-containment $e^+(M^{**}, L^+) \subseteqq \beta(M^{**}, L_\sigma^+)$ (or $e^+(M^{**}, L^+) \subseteqq \beta(M^{**}, L_\beta^+)$) on on M^{**} is clear. If L is strictly s-barrelled and A^+ is bounded in L_σ^+, then A^+ is s-equicontinuous on L. Hence $\beta(M^{**}, L_\sigma^+) \subseteqq e^+(M^{**}, L^+)$ on M^{**}. This proves $e^+(M^{**}, L^+) = \beta(M^{**}, L_\sigma^+)$ on M^{**}. Conversely, if A^+ is bounded in L_σ^+, then $(A^+)^{\circ *}$ is a neighborhood of $0 \in M^{**}$ w.r.t. $\beta(M^{**}, L_\sigma^+) = e^+(M^{**}, L^+)$, and $(B^+)^{\circ *} \subseteqq \subseteqq (A^+)^{\circ *}$ for some s-equicontinuous $B^+ \subseteqq L^+$. If $V = {}^\circ(B^+)$ is the pre-polar of B^+ in L, then $V \in \eta_{0,cs}$ and $B^+ \subseteqq ({}^\circ(B^+))^{\circ +} = V^{\circ +}$. Hence $A^+ \subseteqq {}^\circ((A^+)^{\circ *}) \subseteqq {}^\circ((B^+)^{\circ *}) \subseteqq \subseteqq {}^\circ((V^{\circ +})^{\circ *}) = V^{\circ +}$ by Lemma 8 (i), and A^+ is s-equicontinuous on L.

The generalizations of the above classes are now given.

Remarks. (i) For a LCS L, c-sequential barrels and strict s-barrels of L are identical. Thus L is c-sequentially barrelled (or strictly s-barrelled) iff barrels of $(L, \sigma(L, L^+))$ are neighborhoods (or s-neighborhoods) of $0 \in L$ — the usages are intended to be consistent with the conventional definitions of various classes of barrelled spaces, e.g. L is barrelled (or quasibarrelled) iff barrels (or quasibarrels) of L are neighborhoods of $0 \in L$.

(ii) Let L be a LCS and ω be a collection of bounded subsets of L covering L. L is called strictly s_ω-barrelled if bounded subsets of L_ω^+ are s-equicontinuous on L w.r.t. the original topology of L. This class can be characterized by directly modifying those of strictly s-barrelled spaces except that (iii) and (iv) are replaced by "$L_{\tau_{cs}}$ is ω-barrelled (Definition 2.2 (i), [6])" and "the semi-norm p satisfies $\sup \{p(x): x \in A\} < +\infty$ for $A \in \omega$", respectively. The classes in Theorem 5 are those w.r.t. $\omega = \{\{x\}: x \in L\}$ and $\omega = \{A$: A is a bounded subset of $L\}$, respectively.

Example 3. (i) A complete, metrizable LCS L must be strictly s-barrelled and s-quasibarrelled by Example 1.

(ii) If L and $L_w = (L, \sigma(L, L'))$ have the same convergent sequences, then $(L_w)_{\tau_{cs}} = L_{\tau_{cs}}$. If L is also strictly s-barrelled (or s-quasibarrelled), then so is L_w.

(iii) If L is a c-sequential, Montel space (Definition 3.9.1, [1]), then L is strictly

s-barrelled and s-quasibarrelled, and so is $L_w=(L, \sigma(L, L'))$ since L, L_w have the same convergent sequences (Corollary 3.9.2, [1]).

If L is a LCS and $L_w=(L, \sigma(L, L^b))$, then $\{^\circ(A^b)\colon A^b\subseteq L^b$ is finite$\}$ is a base of neighborhoods of $0\in L_w$ and $(L_w)'=L^b$. Hence bounded subsets of $(L, \sigma(L, L^b))$ are bounded in L, and barrels (or quasibarrels) of L are barrels (or quasibarrels) of $(L, \sigma(L, L^b))$.

A LCS L is said to satisfy the strict condition of ess-uniform boundedness if barrels of $(L, \sigma(L, L^b))$ are bornivores of L. Hence barrels of L are bornivorous in L. The counterpart of this class of spaces is void since quasibarrels of $(L, \sigma(L, L^b))$ are always bornivorous in L. Hence L satisfies the strict condition of ess-uniform boundedness iff L_{τ_b} is barrelled.

Let L be a LCS, then a semi-norm p on L is called bornologically lower semicontinuous if $\{x\in L\colon p(x)\leqq\alpha\}$ is closed in L_{τ_b} for any $\alpha\in\mathbf{R}$.

Theorem 6. (i) *The characterizations of LCS satisfying the strict condition of ess-uniform boundedness can be obtained from those of strictly s-barrelled spaces by replacing s-equicontinuity with ess-uniform boundedness, c-sequentially with bornologically, L^+ with L^b, and $L_{\tau_{cs}}$ with L_{τ_b}.*

(ii) *A LCS L_1 over* $\mathbf{K}$ *satisfies the strict condition of ess-uniform boundedness iff for any LCS L_2 over* $\mathbf{K}$, *bounded subsets of $\mathcal{B}^b_\sigma(L_1, L_2)$ are ess-uniformly bounded on L_1.*

Proposition 2. *The characterization of a LCS L satisfying the strict condition of ess-uniform boundedness can be obtained from Proposition* 1 *by replacing L^+ with L^b, $(M^{**}, \beta(M^{**}, L^+_\sigma))$ with $(M^{**}, \beta(M^{**}, L^b_\sigma))$, and $(M^{**}, e^+(M^{**}, L^+))$ with $(M^{**}, \varrho^b(M^{**}, L^b))$ which is the projective limit of $\{M^{**}_{(T_{A^b})}\colon A^b$ is an ess-uniformly bounded subset of $L^b\}$.*

The similar remarks as Remark (i) can be made for this class of spaces.

Example 4. (i) A complete, metrizable LCS must satisfy the strict condition of ess-uniform boundedness.

(ii) If L is a bornological, barrelled space, then L and $L_w=(L, \sigma(L, L'))$ have the same bounded subsets and satisfy the strict condition of ess-uniform boundedness.

We now consider other properties of the space in this section.

Corollary 8. *Let L_1, L_2 be LCS over* $\mathbf{K}$.

(i) *If L_1 is strictly s-barrelled, then all the s-equicontinuous subsets of $\mathcal{B}^+(L_1, L_2)$, all bounded subsets of $\mathcal{B}^+_\sigma(L_1, L_2)$ and all bounded subsets of $\mathcal{B}^+_\beta(L_1, L_2)$ are identical.*

(ii) *If L_1 satisfies the strict condition of ess-uniform boundedness, then the conclusion in* (i) *is true if s-equicontinuity is replaced with ess-uniform boundedness, and all $\mathcal{B}^+(L_1, L_2)$ with $\mathcal{B}^b(L_1, L_2)$.*

More stringent results are in the following.

Corollary 9. *Let L be a LCS.*

(i) *Let L be strictly s-barrelled, then A^+ is a s-equicontinuous (or bounded) subset of L^+ (L^+_σ or L^+_β) iff A^+ is relatively compact in L^+_σ.*

(ii) *Let L satisfy the strict condition of ess-uniform boundedness, then the conclusions in* (i) *are true if s-equicontinuity is replaced with ess-uniform boundedness, all A^+ with A^b, and all L^+ with L^b.*

Proof. (i) If $A^+ \subseteqq L^+$ is s-equicontinuous on L, then A^+ is relatively compact in L^+_σ. The converse is clear since relative compactness implies boundedness.

Corollary 10. *If L is a strictly s-barrelled space (or LCS satisfying the strict condition of ess-uniform boundedness), then L and $(L, \beta(L, L^+_\sigma))$ (or $(L, \beta(L, L^b_\sigma))$) have the same bounded subsets.*

Proof. Bounded subsets of $(L, \beta(L, L^+_\sigma))$ are bounded in $(L, \sigma(L, L^+))$, and so are in L. If L is strictly s-barrelled, then bounded subsets of L are bounded in $(L, \beta(L, L^+_\sigma))$.

In Lemma 7, we proved that L, $(L, \sigma(L, L^+))$ and $(L, \sigma(L, L^b))$ have the same bounded subsets. A generalization is in the following.

Lemma 9. *If L is a LCS and M^* is a linear space with $L' \subseteqq M^* \subseteqq L^b$, then L, $(L, \sigma(L, M^*))$ and $(L, \beta(L, M^*_\beta))$ have the same bounded subsets, where $M^*_\beta = (M^*, \beta(M^*, L))$.*

We have the following permanence properties of the spaces constructed in this section (cf. Theorem 3, [2] and Theorem 8, [5]).

Theorem 7. *Let $\{L_\gamma\colon \gamma \in \Gamma\}$ be a non-empty collection of LCS and L be a Hausdorff LCS over $\mathbf{K}$ such that $\varphi_\gamma\colon L_\gamma \to L$ is a linear map for any $\gamma \in \Gamma$.*

(i) *If L_γ is strictly s-barrelled (or s-quasibarrelled) for any $\gamma \in \Gamma$, and L is the inductive limit of $\{(L_\gamma)_{\tau_{cs}}\colon \gamma \in \Gamma\}$: induced by $\{\varphi_\gamma\colon \gamma \in \Gamma\}$ (p. 157, [1]), then L is strictly s-barrelled (or s-quasibarrelled).*

(ii) *If L_γ satisfies the strict condition of ess-uniform boundedness for any $\gamma \in \Gamma$ and L is the inductive limit of $\{(L_\gamma)_{\tau_b}\colon \gamma \in \Gamma\}$, then L satisfies the strict condition of ess-uniform boundedness.*

Proof. If L_γ is strictly s-barrelled for any $\gamma \in \Gamma$, then $\varphi_\gamma\colon (L_\gamma)_{\tau_{cs}} \to L$ is continuous, and s-continuous. Hence $\varphi_\gamma\colon (L_\gamma)_{\tau_{cs}} = ((L_\gamma)_{\tau_{cs}})_{\tau_{cs}} \to L_{\tau_{cs}}$ is continuous by Lemma 4. If V is a strict s-barrel of L, then V is a barrel of $L_{\tau_{cs}}$, and $\varphi_\gamma^{-1}(V)$ is a barrel of $(L_\gamma)_{\tau_{cs}}$ which is a neighborhood of $0 \in (L_\gamma)_{\tau_{cs}}$ for any $\gamma \in \Gamma$. Thus V is a neighborhood of $0 \in L$ (p. 157, [1]), and a s-neighborhood of $0 \in L$. Thus L is strictly s-barrelled.

Corollary 11. *Let L_γ be a LCS over* $\mathbf{K}$ *for any* $\gamma \in \Gamma$.

(i) *If L_γ is strictly s-barrelled (or s-quasibarrelled) for any $\gamma \in \Gamma$, then so is* $\coprod_{\gamma \in \Gamma} (L_\gamma)_{\tau_{cs}}$, *the locally convex, direct sum of* $\{(L_\gamma)_{\tau_{cs}}: \gamma \in \Gamma\}$.

(ii) *If L_γ satisfies the strict condition of ess-uniform boundedness for any $\gamma \in \Gamma$, then so does* $\coprod_{\gamma \in \Gamma} (L_\gamma)_{\tau_b}$.

Corollary 12. *Let L be a LCS and M be a linear subspace of L with the quotient map* π: $L \to L/M$.

(i) *If L is strictly s-barrelled (or s-quasibarrelled), then so is $L_{\tau_{cs}}/M$ which is the inductive limit of $L_{\tau_{cs}}$ induced by* π.

(ii) *If L satisfies the strict condition of ess-uniform boundedness, then so does* L_{τ_b}/M.

The following theorem is on the mapping properties (cf. Theorem 4, [2]).

Theorem 8. *Let L_1, L_2 be LCS over* $\mathbf{K}$ *and* $\varphi \in \mathscr{B}(L_1, L_2)$.

(i) *If L_1 is strictly s-barrelled (or s-quasibarrelled) and φ: $(L_1)_{\tau_{cs}} \to (L_2)_{\tau_{cs}}$ is almost open (Definition* 3.17.1, [1]*), then L_2 is strictly s-barrelled (or s-quasibarrelled).*

(ii) *If L_1 satisfies the strict condition of ess-uniform boundedness and φ: $(L_1)_{\tau_b} \to (L_2)_{\tau_b}$ is almost open, then L_2 satisfies the strict condition of ess-uniform boundedness.*

We now consider the possibility of constructing the permanence properties of strictly s_ω-barrelled spaces.

Remarks. (iii) A LCS L is said to satisfy the strict condition of ω-uniform boundedness if bounded subsets of L^b_ω are ess-uniformly bounded on L (w.r.t. the original topology of L), where ω is a collection of bounded subsets of L covering L with $L^b_\omega \neq L^b_\beta$. We can check that Theorem 8 and Corollary 11 have no analogies for this class of spaces and strictly s_ω-barrelled spaces.

(iv) Let L_1, L_2 be LCS over $\mathbf{K}$ and φ: $L_1 \to L_2$ be a surjective, continuous, linear map such that φ: $(L_1)_{\tau_{cs}} \to (L_2)_{\tau_{cs}}$ is almost open. If ω_1 is a collection of bounded subsets of L_1 covering L_1, then $\omega_2 = \{\varphi(A): A \in \omega_1\}$ is a collection of bounded subsets of L_2 covering L_2. If L_1 is strictly s_{ω_1}-barrelled (or satisfies the strict condition of ω_1-uniform boundedness), then so is L_2 (or so does L_2).

(v) Let L be a LCS, M be a linear subspace of L and ω be a collection of bounded subsets of L covering L. If L is ω-barrelled (or ω-countably barrelled — Definition 2.2 (ii), [6]), then L/M is ω_q-barrelled (or ω_q-countably barrelled) since π is an open map, where $\omega_q = \{\pi(A): A \in \omega\}$.

The following result is actually used in Remark (iv): Let L_1, L_2 be LCS over **K**, ω_1 be a collection of bounded subsets of L_1 and φ: $L_1 \to L_2$ be a s-continuous (or ess-bounded), linear map. If $\{\psi_\gamma: \gamma \in \Gamma\}$ is a bounded subset of $(L_2)^+_{\omega_2}$ (or $(L_2)^b_{\omega_2}$), then $\{\psi_\gamma \circ \varphi: \gamma \in \Gamma\} = {}^t\varphi(\{\psi_\gamma: \gamma \in \Gamma\})$ is a bounded subset of $(L_1)^+_{\omega_1}$ (or $(L_1)^b_{\omega_1}$), where $\omega_2 = \{\varphi(A): A \in \omega_1\}$.

IV. Banach—Steinhaus theorems based on s-equicontinuity and ess-uniform boundedness

In this section, Banach—Steinhaus theorems of LCS which concern the continuity of the limiting function of a sequence of continuous, linear maps on a given space will be considered. These theorems of barrelled spaces have been proved (Proposition 3.6.5 with its corollary, [1], and Theorem 33.1 with its corollary, [10]). These theorems of other classes of barrelled spaces, e.g. countably barrelled, boundedly barrelled and convergently barrelled spaces, have also been obtained (Theorems 3 and 7, [5]; and Theorems 7 and 8 with Corollary 9, [2]). We first prove the filter and sequence versions of Banach—Steinhaus theorems for continuous (s-continuous or ess-bounded) linear maps and linear functionals on LCS. We then examplify how these theorems can be applied to the classes of spaces constructed in III. Auxiliary results will be led to Banach—Steinhaus theorems of generalizations of these classes of spaces, and also of c-sequential LCS, bornological spaces and special subclass of Montel spaces.

Theorem 9. *Let L be a LCS.*

(i) $\{\varphi_\gamma: \gamma \in \Gamma\}$ *is an equicontinuous (s-equicontinuous or ess-uniformly bounded) subset of* L' (L^+ *or* L^b) *iff there is a continuous (s-continuous or ess-bounded) seminorm p on L with* $|\varphi_\gamma(x)| \leqq p(x)$ *for* $x \in L$ *and* $\gamma \in \Gamma$.

(ii) *If* $\{\varphi_n: n=1, 2, \ldots\}$ *is an equicontinuous (s-equicontinuous or ess-uniformly bounded) sequence in* L' (L^+ *or* L^b) *such that* $\lim_{n \to +\infty} \varphi_n(x) = \varphi_0(x)$ *exists for any* $x \in L$, *then* $\varphi_0 \in L'$ ($\varphi_0 \in L^+$ *or* $\varphi_0 \in L^b$) *and* $\{\varphi_n: n=1, 2, \ldots\}$ *converges to* φ_0 *in* L'_λ ($L^+_{\lambda_s}$ *or* $L^b_{\lambda_b}$).

(iii) *If* $\{\varphi_\eta: 0 < \eta < \alpha\}$ *is an equicontinuous (s-equicontinuous or ess-uniformly bounded) subset of* L' (L^+ *or* L^b) *such that* $\lim_{\eta \to 0^+} \varphi_\eta(x) = \varphi_0(x)$ *exists for any* $x \in L$, *then* $\varphi_0 \in L'$ ($\varphi_0 \in L^+$ *or* $\varphi_0 \in L^b$) *and* $\{\varphi_\eta: 0 < \eta < \alpha\}$ *converges to* φ_0 *in* L'_λ ($L^+_{\lambda_s}$ *or* $L^b_{\lambda_b}$) *as* $\eta \to 0^+$.

Proof. The first case will be proved. The others can be derived from Lemmas 4 and 5.

(i) If $\{\varphi_\gamma: \gamma\in\Gamma\}\subseteqq L'$ is equicontinuous on L, then $\{\varphi_\gamma: \gamma\in\Gamma\}\subseteqq V^\circ$ for some open, convex, balanced neighborhood V of $0\in L$, where the polar is taken in L'. Let $p(x)$ be the Minkowski functional associated with V on L, then p is a continuous semi-norm on L and $V=p^{-1}([0, 1))$. The rest of the proof is similar to Theorem 7 (i), [2]. For the converse, there is a convex, balanced neighborhood V of $0\in L$ with $p(x)\leqq 1$ for $x\in V$. Hence $\{\varphi_\gamma: \gamma\in\Gamma\}\subseteqq V^\circ$ and $\{\varphi_\gamma: \gamma\in\Gamma\}$ is equicontinuous on L.

(ii) The proof is similar to Theorem 7 (ii), [2] with application of (i).

(iii) The proof is similar to Theorem 7 (iii), [2].

For an application of Theorem 9 (i), we need the following lemma.

Lemma 10. *Let L be a LCS and M be a linear subspace of L with the relative topology induced by L.*

(i) *If V is a convex balanced neighborhood of $0\in M$, then $V=W\cap M$ for some convex, balanced neighborhood W of $0\in L$.*

(ii) *If p is a continuous semi-norm on M, then $p=q|_M$ for some continuous semi-norm q on L.*

Proof. (i) Let W be the absolutely convex hull of $U\cup V$ in L, where U is a convex, balanced neighborhood U of $0\in L$ with $U\cap M\subseteqq V$, then W is a convex, balanced neighborhood of $0\in L$ and $V=W\cap M$ by the similar proof as Lemma 2.12.1 (i), [1].

(ii) $V=p^{-1}([0, 1))$ is a convex, balanced neighborhood of $0\in M$, and $V=W\cap M$ for some convex, balanced neighborhood W of $0\in L$. The Minkowski functional $q(x)$ associated with W on L satisfies the required properties.

Corollary 13. *Let L be a LCS and M be a linear subspace of L with the relative topology induced by L. If $\{\varphi_\gamma: \gamma\in\Gamma\}\subseteqq M'$ is equicontinuous on M, then there is an equicontinuous subset $\{\psi_\gamma: \gamma\in\Gamma\}$ of L' with $\psi_\gamma|_M=\varphi_\gamma$ for $\gamma\in\Gamma$.*

Proof. There exist continuous semi-norms p on M and q on L with $|\varphi_\gamma(x)|\leqq \leqq p(x)$ for $x\in M$ and $\gamma\in\Gamma$, and $q|_M=p$. Hence there is an $\psi_\gamma\in L^*$ with $\psi_\gamma|_M=\varphi_\gamma$ and $|\psi_\gamma(x)|\leqq q(x)$ for $x\in L$ (Theorem 3.1.1, [1]). Hence $\psi_\gamma\in L'$ for any $\gamma\in\Gamma$, and $\{\psi_\gamma: \gamma\in\Gamma\}$ is equicontinuous on L.

We can generalize the results of Theorem 9 to linear maps.

Theorem 10. *Let L_1, L_2 be LCS over $\mathbb{K}$ and L_2 be Hausdorff.*

(i) *If $\{\varphi_n: n=1, 2, \ldots\}$ is an equicontinuous (s-equicontinuous or ess-uniformly bounded) sequence in $\mathscr{B}(L_1, L_2)$, ($\mathscr{B}^+(L_1, L_2)$ or $\mathscr{B}^b(L_1, L_2)$) such that $\{\varphi_n: n=1, 2, \ldots\}$ converges to φ_0 pointwise on L_1, then φ_0 is in $\mathscr{B}(L_1, L_2)$ ($\mathscr{B}^+(L_1, L_2)$ or $\mathscr{B}^b(L_1, L_2)$) and $\{\varphi_n: n=1, 2, \ldots\}$ converges to φ_0 in $\mathscr{B}_\lambda(L_1, L_2)$ ($\mathscr{B}^+_{\lambda_s}(L_1, L_2)$ or $\mathscr{B}^b_{\lambda_b}(L_1, L_2)$).*

(ii) *If* $\{\varphi_\eta: 0<\eta<\alpha\}$ *is an equicontinuous (s-equicontinuous or ess-uniformly bounded) subset of* $\mathscr{B}(L_1, L_2)$ $(\mathscr{B}^+(L_1, L_2)$ *or* $\mathscr{B}^b(L_1, L_2))$ *such that* $\{\varphi_\eta(x): 0<\eta<\alpha\}$ *converges to* $\varphi_0(x)$ *as* $\eta \to 0^+$ *for any* $x \in L$ *then* φ_0 *is in* $\mathscr{B}(L_1, L_2)$ $(\mathscr{B}^+(L_1, L_2)$ *or* $\mathscr{B}^b(L_1, L_2))$ *and* $\{\varphi_\eta: 0<\eta<\alpha\}$ *converges to* φ_0 *in* $\mathscr{B}_\lambda(L_1, L_2)$ $(\mathscr{B}^+_{\lambda_s}(L_1, L_2)$ *or* $\mathscr{B}^b_{\lambda_b}(L_1, L_2))$ *as* $\eta \to 0^+$.

(iii) *If* $\mathscr{F}$ *is an equicontinuous (s-equicontinuous or ess-uniformly bounded) filter on* $\mathscr{B}(L_1, L_2)$ $(\mathscr{B}^+(L_1, L_2)$ *or* $\mathscr{B}^b(L_1, L_2))$ *such that* $\mathscr{F}(x)$ *converges to* $\varphi_0(x)$ *for any* $x \in L_1$, *then* φ_0 *is in* $\mathscr{B}(L_1, L_2)$ $(\mathscr{B}^+(L_1, L_2)$ *or* $\mathscr{B}^b(L_1, L_2))$ *and* $\mathscr{F}$ *converges to* φ_0 *in* $\mathscr{B}_\lambda(L_1, L_2)$ $(\mathscr{B}^+_{\lambda_s}(L_1, L_2)$ *or* $\mathscr{B}^b_{\lambda_b}(L_1, L_2))$.

Proof. We only prove the first case.

(i) $\varphi_0 \in \mathscr{L}(L_1, L_2)$ by the similar proof as Lemma 3.6.1, [1]. If W is a closed, convex, balanced neighborhood of $0 \in L_2$, then $V = \bigcap_{n=1}^{+\infty} \varphi_n^{-1}\left(\frac{1}{2}W\right)$ is a neighborhood of $0 \in L_1$, and $\varphi_0(V) \subseteqq W$. Thus $\varphi_0 \in \mathscr{B}(L_1, L_2)$ and $\{\varphi_n: n=1, 2, \ldots\}$ converges to φ_0 in $\mathscr{B}_\lambda(L_1, L_2)$ by the similar proof of Theorem 7 (ii), [2]).

(ii) The proof follows the same pattern in the proof of Theorem 7 (iii), [2].

(iii) $\varphi_0 \in \mathscr{L}(L_1, L_2)$ is clear. If $\mathscr{F}$ is equicontinuous on L_1 and W is a closed, convex, balanced neighborhood of $0 \in L_2$, then $V = \bigcap \left\{\varphi^{-1}\left(\frac{1}{2}W\right) : \varphi \in \mathscr{F}\right\}$ is a neighborhood of $0 \in L_1$ and $\varphi_0(V) \subseteqq W$. Hence $\varphi_0 \in \mathscr{B}(L_1, L_2)$ and $\mathscr{F}$ converges to φ_0 in $\mathscr{B}_\sigma(L_1, L_2)$, and so does in $\mathscr{B}_\lambda(L_1, L_2)$ (cf. Corollary 33.1, [10]).

The following remarks consider some generalizations of Theorems 9 and 10.

Remarks. (vi) Let L be a LCS and ω be a collection of bounded subsets of L covering L. If $\{\varphi_n: n=1, 2, \ldots\}$ is a sequence in Theorem 9 (ii) which converges to φ_0 uniformly on any $A \in \omega$, then φ_0 is in L' (L^+ or L^b) and $\{\varphi_n: n=1, 2, \ldots\}$ converges to φ_0 in L'_ω (L^+_ω or L^b_ω). Similar generalization of (iii) can be made.

(vii) We can make the similar generalizations (as those in (vi)) for Theorem 10 (i), (ii) and (iii).

We now prove Banach—Steinhaus theorems of spaces in III.

Theorem 11. *Let L be a strictly s-barrelled space (or LCS satisfying the strict condition of ess-uniform boundedness).*

(i) *If* $\{\varphi_\gamma: \gamma \in \Gamma\}$ *is a subset of* L^+ *(or* L^b*) such that* $\{\varphi_\gamma(x): \gamma \in \Gamma\}$ *is bounded in* $\mathbf{K}$ *for any* $x \in L$, *then* $\{\varphi_\gamma: \gamma \in \Gamma\}$ *is a s-equicontinuous (or ess-uniformly bounded) subset of* L^+ *(or* L^b*), and there is a s-continuous (or ess-bounded) semi-norm* p *on* L *with* $|\varphi_\gamma(x)| \leqq p(x)$ *for* $x \in L$ *and* $\gamma \in \Gamma$.

(ii) *If* $\{\varphi_n: n=1, 2, \ldots\}$ *is a sequence in* L^+ *(or* L^b*) which converges to* φ_0 *point-*

wise on L, then $\varphi_0 \in L^+$ (or $\varphi_0 \in L^b$) and $\{\varphi_n\colon n=1, 2, \ldots\}$ is a s-equicontinuous (or ess-uniformly bounded) sequence in L^+ (or L^b) which converges to φ_0 in $L^+_{\lambda_s}$ (or $L^b_{\lambda_b}$).

(iii) *If $\{\varphi_\eta\colon 0<\eta<\alpha\}$ is a subset of L^+ (or L^b) such that $\lim_{\eta\to 0^+} \varphi_\eta(x)=\varphi_0(x)$ exists for any $x\in L$, then $\varphi_0\in L^+$ (or $\varphi_0\in L^b$) and $\{\varphi_\eta\colon 0<\eta<\alpha\}$ converges to φ_0 in $L^+_{\lambda_s}$ (or $L^b_{\lambda_b}$) as $\eta\to 0^+$.*

(iv) *If $\mathscr{F}$ is a bounded filter on L^+_σ (or L^b_σ), i.e. $\mathscr{F}$ contains a bounded subset of L^+_σ (or L^b_σ), such that $\mathscr{F}(x)$ converges to $\varphi_0(x)$ for any $x\in L$, then $\varphi_0\in L^+$ (or $\varphi_0\in L^b$) and $\mathscr{F}$ is a s-equicontinuous (or ess-uniformly bounded) filter on L^+ (or L^b) which converges to φ_0 in $L^+_{\lambda_s}$ (or $L^b_{\lambda_b}$).*

(v) *If $\mathscr{F}$ is a filter on L^+ (or L^b) with a countable base $\{A^+_n$ (or A^b_n): $n=1, 2, \ldots\}$ such that $\mathscr{F}(x)$ converges to $\varphi_0(x)$ for any $x\subseteq L$, then $\varphi_0\in L^+$ (or $\varphi_0\in L^b$) and $\mathscr{F}$ converges to φ_0 in $L^+_{\lambda_s}$ (or $L^b_{\lambda_b}$).*

Proof. Since bounded subsets of L^+_σ (or L^b_σ) are s-equicontinuous (or ess-uniformly bounded) on L, (i) and (ii) follow from Theorem 9 (i) and (ii). The proofs of (iii) and (v) are similar to Theorem. 7 (iii) and Corollary 9, [2].

Remark. (viii) All the statements of Theorem 11 with proper modifications (e.g. the set $\{\varphi_\gamma\colon \gamma\in\Gamma\}$ in (i) should satisfy $\sup\{|\varphi_\gamma(x)|\colon x\in A$ and $\gamma\in\Gamma\}<+\infty$ for any $A\in\omega$; and the convergence is uniform on $A\in\omega$) are Banach—Steinhaus theorems of strictly s_ω-barrelled spaces and LCS satisfying strict condition of ω-uniform boundedness, and in particular, are these theorems of strictly s-quasibarrelled spaces when $\omega=\{A\colon A$ is bounded in $L\}$.

Corollary 14. *Let L be a c-sequential LCS (or bornological space) over* $\mathbf{K}$.

(i) *If $\{\varphi_\gamma\colon \gamma\in\Gamma\}$ is a s-equicontinuous (or ess-uniformly bounded) subset of L' such that $\{\varphi_\gamma(x)\colon \gamma\in\Gamma\}$ is bounded in* $\mathbf{K}$ *for any $x\in L$, then $\{\varphi_\gamma\colon \gamma\in\Gamma\}$ is equicontinuous on L and there is a continuous semi-norm p on L with $|\varphi_\gamma(x)|\leqq p(x)$ for any $x\in L$ and $\gamma\in\Gamma$.*

(ii) *If $\{\varphi_n\colon n=1, 2, \ldots\}$ is a s-equicontinuous (or ess-uniformly bounded) sequence in L' converging to φ_0 pointwise on L, then $\varphi_0\in L'$ and $\{\varphi_n\colon n=1, 2, \ldots\}$ is an equicontinuous sequence in L' converging to φ_0 in L'_λ.*

(iii) *If $\{\varphi_\eta\colon 0<\eta<\alpha\}$ is a s-equicontinuous (or ess-uniformly bounded) subset of L' such that $\lim_{\eta\to 0^+} \varphi_\eta(x)=\varphi_0(x)$ exists for any $x\in L$, then $\varphi_0\in L'$ and $\{\varphi_\eta\colon 0<\eta<\alpha\}$ converges to φ_0 in L'_λ as $\eta\to 0^+$.*

(iv) *If $\mathscr{F}$ is a s-equicontinuous (or ess-uniformly bounded) filter on L' such that $\mathscr{F}(x)$ converges to $\varphi_0(x)$ for any $x\in L$, then $\varphi_0\in L'$ and $\mathscr{F}$ is an equicontinuous filter on L' converging to φ_0 in L'_λ.*

Proof. These are clear since $L=L_{\tau_{cs}}$, $L'=L^+$, $L'_\lambda=L^+_{\lambda_s}$ (or $L=L_{\tau_b}$, $L'=L^b$, $L'_\lambda=L^b_{\lambda_b}$) and Lemma 4 (or 5).

Corollary 15. *Let L be a c-sequential, Montel space and* $L_w=(L, \sigma(L, L'))$.

(i) *All the statements of Theorem* 11 *(or Corollary* 14*) are the sequence and filter versions of Banach—Steinhaus theorems for s-continuous (or continuous), linear functionals on L.*

(ii) *All the statements of Theorem* 11 *with all the L^+ replaced by $(L_w)^+$ are the sequence and filter versions of Banach—Steinhaus theorems for s-continuous, linear functionals on L_w.*

Proof. These are clear since L and L_w are strictly s-barrelled by Example 3 (iii).

We now generalize the results from Theorem 11 to Corollary 14 to linear maps.

Theorem 12. *Let L_1 be a strictly s-barrelled space (or LCS satisfying the strict condition of ess-uniform boundedness) and L_2 be a Hausdorff LCS over* $\mathbf{K}$, *then the conclusions of the statements* (ii)$\sim$(v) *in Theorem* 11 *are true if the given assumptions in the indicated statements can be modified verbatim and properly. For example, if* $\{\varphi_n\colon n=1, 2, \ldots\}$ *is a sequence in* $\mathscr{B}^+(L_1, L_2)$ *(or* $\mathscr{B}^b(L_1, L_2)$*) converging to φ_0 pointwise on L_1, then φ_0 is in* $\mathscr{B}^+(L_1, L_2)$ *(or* $\mathscr{B}^b(L_1, L_2)$*),* $\{\varphi_n\colon n=1, 2, \ldots\}$ *is s-equicontinuous (or ess-uniformly bounded) on L_1 and converges to φ_0 in* $\mathscr{B}^+_{\lambda_s}(L_1, L_2)$ *(or* $\mathscr{B}^b_{\lambda_b}(L_1, L_2)$*).*

Proof. For (ii), we can prove that $\{\varphi_n\colon n=1, 2, \ldots\}$ is bounded in $\mathscr{B}^+_\sigma(L_1, L_2)$ (or $\mathscr{B}^b_\sigma(L_1, L_2)$), and hence s-equicontinuous (or ess-uniformly bounded) on L_1 by Theorem 5 (v) (or 6 (v)). The conclusions thus follow from Theorem 10 (iii). The proofs of (iii), (iv) and (v) are clear.

The similar remark as Remark (viii) can be made for Theorem 12.

Corollary 16. *Let L_1 be a c-sequential LCS (or bornological space) and L_2 be a Hausdorff LCS over* $\mathbf{K}$, *then the conclusions of the statements* (ii)$\sim$(iv) *in Corollary* 14 *are true if the given assumptions in the indicated statements can be modified verbatim and properly. For example, if* $\{\varphi_n\colon n=1, 2, \ldots\}$ *is a s-equicontinuous (or ess-uniformly bounded) sequence in* $\mathscr{B}(L_1, L_2)$ *converging to φ_0 pointwise on L_1, then* $\varphi_0\in\mathscr{B}(L_1, L_2)$, $\{\varphi_n\colon n=1, 2, \ldots\}$ *is equicontinuous on L_1 and converges to φ_0 in* $\mathscr{B}_\lambda(L_1, L_2)$.

Proof. These are clear since $L_1=(L_1)_{\tau_{cs}}$, $\mathscr{B}(L_1, L_2)=\mathscr{B}^+(L_1, L_2)$, $\mathscr{B}_\lambda(L_1, L_2)=\mathscr{B}^+_{\lambda_s}(L_1, L_2)$ (or $L_1=(L_1)_{\tau_b}$, $\mathscr{B}(L_1, L_2)=\mathscr{B}^b(L_1, L_2)$, $\mathscr{B}_\lambda(L_1, L_2)=\mathscr{B}^b_{\lambda_b}(L_1, L_2)$) and Lemma 4 (or 6).

Corollary 17. *Let L_1 be a c-sequential, Montel space and L_2 be a Hausdorff LCS over* $\mathbf{K}$. *Let* $(L_1)_w=(L_1, \sigma(L_1, L_1'))$.

(i) *All the statements of Theorem* 12 *(or Corollary* 16*) are the sequence and filter versions of Banach—Steinhaus theorems for s-continuous (or continuous), linear maps from L_1 into L_2.*

(ii) *All the statements of Theorem* 12 *with* $\mathscr{B}^+(L_1, L_2)$ *replaced by* $\mathscr{B}^+((L_1)_w, L_2)$ *are the sequence and filter versions of Banach—Steinhaus theorems for s-continuous, linear maps from* $(L_1)_w$ *into* L_2.

This completes the main purpose of this paper mentioned in the beginning of § I and the auxiliary purpose, namely, the construction of Banach—Steinhaus theorems for continuous, linear functionals and maps on c-sequential LCS, bornological spaces, and c-sequential, Montel spaces.

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Compact and Fredholm composite multiplication operators

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1. Introduction. Let X be a nonempty set and $V(X)$ be a vector space of complex valued functions on X under the pointwise operations of addition and scalar multiplication. Let T be a mapping of X into X such that $f\circ T$ is in $V(X)$ whenever f is in $V(X)$. Define the composition transformation C_T on $V(X)$ as $C_T f = f\circ T$ for every f in $V(X)$. If $V(X)$ has a Banach space structure and C_T is bounded, then C_T is called the composition operator on $V(X)$ induced by T. Let $\theta\colon X\to\mathbf{C}$ be a function such that M_θ, defined as $M_\theta f=\theta\cdot f$ for every f in $V(X)$ is a bounded linear operator on $V(X)$. Then the product $M_\theta C_T$ which becomes a bounded operator on $V(X)$ is called a composite multiplication operator.

The study of composite multiplication operators becomes significant and interesting due to the fact that the class of composite multiplication operators includes composition operators, multiplication operators, weighted composition operators. LAMBERT and QUINN [4] initiated the study of weighted composition process on L^1-space, having resemblence with composite multiplication operators. HADWIN, NORDGREN, RADJAVI and ROSENTHAL [2] proved that there exists on operator belonging to the class of composite multiplication operators, which does not satisfy Lomonosov's hypothesis [5] pertaining to the wellknown invariant subspace problem in operator theory.

In this paper the necessary and sufficient conditions for $M_\theta C_T\in B(L^2(\lambda))$ to be a compact operator and a Fredholm operator are obtained in case $V(X)$ is an L^2-space of a sigma-finite measure space.

By $\mathscr{B}(\mathfrak{H})$, we mean the Banach algebra of all bounded operators on a Hilbert space $\mathfrak{H}$. If $(X, \mathscr{S}, \lambda)$ is a measure space and $T\colon X\to X$ is a measurable transformation such that $C_T\in\mathscr{B}(L^2(\lambda))$, then the measure λT^{-1}, defined as $\lambda T^{-1}(E)=\lambda(T^{-1}(E))$ for every E in $\mathscr{S}$, is absolutely continuous with respect to the measure λ [7]. Let f_0 denote the Radon—Nikodym derivative of λT^{-1} with respect to λ. If $C_T\in\mathscr{B}(L^2(\lambda))$, then $C_T^*C_T=M_{f_0}$ [7]. The symbols Ker A and Ran A denote the

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kernel and the range of the operator $A\in\mathscr{B}(\mathfrak{H})$ and Z_δ^θ denotes the closed subspace of $L^2(\lambda)$ consisting of all those functions which vanish outside $X_\delta^\theta=\{x\in X\,|\,|\theta(x)|>\delta\}$. By Z_θ, we mean the set $\{x\in X\,|\,\theta(x)=0\}$ and Z'_θ is the complement of Z_θ. In this paper we consider $(X, \mathscr{S}, \lambda)$ to be a σ-finite measure space.

2. Some basic results. In this section we present some essential results which are often used in the presentation of this paper.

Theorem 2.1. *Let $C_T\in\mathscr{B}(L^2(\lambda))$. Then C_T has dense range if and only if $C_TC_T^*=M_{f_0\circ T}$.*

Proof. Suppose that C_T has dense range. Then for every f in $L^2(\lambda)$ we have a sequence $\{f_n\}$ with $f=\lim_n C_Tf_n$ and we get

$$C_TC_T^*f = \lim_n C_TC_T^*C_Tf_n = \lim_n C_TM_{f_0}f_n = \lim_n C_T(f_0\cdot f_n) =$$

$$= \lim_n (f_0\circ T)(f_n\circ T) = \lim_n M_{f_0\circ T}C_Tf_n = M_{f_0\circ T}C_Tf.$$

Hence $C_TC_T^*=M_{f_0\circ T}$.

Conversely, let $C_TC_T^*=M_{f_0\circ T}$. Then since $f_0\circ T\neq 0$ [11], we can conclude from Lemma 1.2 of [9] that $M_{f_0\circ T}$ is an injection. Hence C_T^* is an injection. So the fact that $\{0\}=\operatorname{Ker} C_T^*=(\overline{\operatorname{Ran} C_T})^\perp$ proves that C_T has dense range. Hence the proof is complete.

Theorem 2.2. *Let $M_\theta C_T\in\mathscr{B}(L^2(\lambda))$. Then $M_\theta C_T=0$ if and only if θ vanishes on $T^{-1}(E)$ almost everywhere whenever $\lambda(E)<\infty$.*

Proof. In case θ vanishes on $T^{-1}(E)$ a.e. whenever $\lambda(E)<\infty$, we get $M_\theta=0$. Hence $M_\theta C_T=0$. For the converse suppose $M_\theta C_T=0$. Since X is σ-finite measure space, we can write $X=\bigcup_{i=1}^{\infty} E_i$, where $\{E_i\}$ is the sequence of disjoint sets such that $\lambda(E_i)<\infty$ for each i, $1\leqq i<\infty$. Now $M_\theta C_T\chi_{E_i}=0$, i.e. $M_\theta\chi_{T^{-1}(E_i)}=0$. Hence

$$\theta=0 \quad \text{on} \quad T^{-1}(E_i) \quad \text{for each} \quad i,\ 1\leqq i<\infty.$$

3. Compact composite multiplication operators. Let us recall that an operator $A\in\mathscr{B}(\mathfrak{H})$ is compact if $\{Af\colon f\in\mathfrak{H}$ and $\|f\|<1\}$ is a precompact subset of $\mathfrak{H}$. A measure λ is called atomic if every element E of $\mathscr{S}$ with $\lambda(E)\neq 0$ contains an atom. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathfrak{H})$ is transitive if $\mathscr{A}$ is weakly closed, contains the identity operator and $\operatorname{Lat}\mathscr{A}=\{0, \mathfrak{H}\}$ where $\operatorname{Lat}\mathscr{A}=\bigcap_{A\in\mathscr{A}}\operatorname{Lat} A$.

Theorem 3.1. *Suppose $C_T\in\mathscr{B}(L^2(\lambda))$ has dense range. Then $M_\theta C_T\in\mathscr{B}(L^2(\lambda))$ is compact if and only if $Z_\delta^{|\theta|^2 f_0\circ T}$ is finite dimensional for every $\delta>0$.*

Proof. The operator $M_\theta C_T$ is compact if and only if $(M_\theta C_T)(M_\theta C_T)^*$ is compact. So by using the Theorem 2.1, the operator $M_\theta C_T$ becomes compact if and only if $M_{|\theta|^2 f_0\circ T}$ is compact. Hence by the Lemma 1.1 of [10], $M_\theta C_T$ is compact if and only if $Z_\delta^{|\theta|^2 f_0\circ T}$ is finite dimensional for every $\delta>0$.

Corollary 3.2. *Let $T: N\to N$ be an injection. Then $M_\theta C_T \in \mathscr{B}(l^2(N))$ is compact if and only if $Z_\delta^{|\theta|^2}$ is finite dimensional for every $\delta>0$.*

Proof. Since T is an injection, C_T has dense range [8] and $f_0\circ T=1$. Hence the proof is immediate.

The main theorem on compact composite multiplication operator on $l^2(N)$ is given below.

Theorem 3.3. *Let $M_\theta C_T\in\mathscr{B}(l^2(N))$. Then $M_\theta C_T$ is compact if and only if $\{\theta(n)\}\to 0$ as $n\to\infty$.*

Proof. Suppose $M_\theta C_T$ is compact. Let $\{e^{(n)}\}$ be the sequence defined by $e^{(n)}(m)=\delta_{nm}$, the Kronecker delta. Since $e^{(n)}\to 0$ weakly and $(M_\theta C_T)^*$ is compact we have

$$\|(M_\theta C_T)^* e^{(n)}\| = |\theta(n)|\,\|C_T^* e^{(n)}\| \to 0.$$

Since $\|C_T^* e^{(n)}\| = \|e^{T(n)}\| = 1$, we get $\{\theta(n)\}\to 0$ as $n\to\infty$.

The converse is trivial.

Corollary 3.4. *If $\mathscr{A}$ is a transitive algebra of $\mathscr{B}(l^2)$ containing $M_\theta C_T$ such that $\{\theta(n)\}\to 0$ as $n\to\infty$, then $\mathscr{A}=\mathscr{B}(l^2)$.*

Proof. Since $\mathscr{A}$ is a transitive algebra of $\mathscr{B}(l^2)$ and contains the compact operator $M_\theta C_T$, $\mathscr{A}=\mathscr{B}(l^2)$, [6].

Example 3.5. Let $X=N$ and λ be the counting measure. Define $T: N\to N$ as $T(n)=\begin{cases} n, & \text{if } n=1 \\ n-1, & \text{if } n\geqq 2\end{cases}$ and define $\theta: N\to \mathbf{C}$ as $\theta(n)=1/n^2$. Then $M_\theta C_T\in\mathscr{B}(l^2)$ is compact by an application of the Theorem 3.3.

Theorem 3.6. *Suppose $(X, \mathscr{S}, \lambda)$ is a nonatomic measure space and $C_T\in\mathscr{B}(L^2(\lambda))$ has dense range. Then $M_\theta C_T\in\mathscr{B}(L^2(\lambda))$ is compact if and only if $\theta=0$ on $Z'_{f_0\circ T}$.*

Proof. Let $M_\theta C_T$ be compact. Then in view of the Theorem 2.1 $(M_\theta C_T)C_T^*(=M_{\theta\cdot f_0\circ T})$ is compact. Thus $\theta\cdot f_0\circ T=0$ a.e. by a theorem of [10]. If $\theta\neq 0$ on $Z'_{f_0\circ T}$, then $f_0\circ T=0$ on $Z'_{f_0\circ T}$. Hence $f_0\circ T=0$ a.e. This is a contradiction to the fact that $f_0\circ T\neq 0$ a.e. for $C_T\in\mathscr{B}(L^2(\lambda))$ [11]. Hence $\theta=0$ on $Z'_{f_0\circ T}$.

Conversely, if $\theta=0$ on $Z'_{f_0\circ T}$, then $|\theta|^2 f_0\circ T=0$ a.e. Hence the operator

$$M_{|\theta|^2 f_0\circ T}(=(M_\theta C_T)(M_\theta C_T)^*)$$

is compact. This proves that $M_\theta C_T$ is compact.

Theorem 3.7. *Let* $\theta\in L^\infty(\lambda)$ *be such that* $|\theta|=1$ *a.e. and* $M_\theta C_T\in\mathscr{B}(L^2(\lambda))$, *Then* $M_\theta C_T$ *is an injective compact operator only if* X *is an atomic measure space.*

Proof. Since $C_T^*C_T=M_{f_0}$, [7], we get $\operatorname{Ker} M_\theta C_T=\operatorname{Ker}(M_\theta C_T)^*(M_\theta C_T)=$ $=\operatorname{Ker} M_{f_0}$. Also the operator $M_\theta C_T$ is compact if and only if $(M_\theta C_T)^*(M_\theta C_T)$ $(=M_{f_0})$ is compact. Since $M_\theta C_T$ is an injective compact operator, we get M_{f_0} to be an injective compact multiplication operator. Then by a result of [10], we conclude that X is an atomic measure space.

Theorem 3.8. *Let* $\theta\in L^\infty(\lambda)$ *be such that* $|\theta|=1$ *a.e. and suppose* $M_\theta C_T\in\mathscr{B}(L^2(\lambda))$. *Then the following are equivalent:*

(i) $M_\theta C_T$ *is compact,*

(ii) C_T *is compact,*

(iii) $Z_\delta^{f_0}$ *is finite dimensional for every* $\delta>0$.

Proof. Obvious.

4. Fredholm composite multiplication operator. Let $\mathscr{C}(\mathfrak{H})$ be the ideal of compact operators in $\mathscr{B}(\mathfrak{H})$ and π be the natural homomorphism from $\mathscr{B}(\mathfrak{H})$ into $\mathscr{B}(\mathfrak{H})/\mathscr{C}(\mathfrak{H})$ which is known as the Calkin algebra. Then an operator $A\in\mathscr{B}(\mathfrak{H})$ is said to be a Fredholm operator if $\pi(A)$ is invertible in $\mathscr{B}(\mathfrak{H})/\mathscr{C}(\mathfrak{H})$.

Atkinson Theorem. [1] *If* $\mathfrak{H}$ *is a Hilbert space, then* $T\in\mathscr{B}(\mathfrak{H})$ *is a Fredholm operator if and only if the range of* T *is closed,* $\dim\ker T$ *is finite and* $\dim\ker T^*$ *is finite.*

Theorem 4.1. *Let* $\theta\in L^\infty(\lambda)$ *be bounded away from zero and* C_T^*, *the adjoint of* $C_T\in\mathscr{B}(L^2(\lambda))$ *be a composition operator. Then* $M_\theta C_T\in\mathscr{B}(L^2(\lambda))$ *is a Fredholm operator if and only if* C_T *is a Fredholm operator.*

Proof. Since $\operatorname{Ker} M_\theta C_T=\operatorname{Ker} C_T$ and $\operatorname{Ker}(M_\theta D_T)^*=\operatorname{Ker} C_T^*$, in the light of Atkinson's theorem it is enough to prove that $M_\theta C_T$ has closed range if and only if C_T has closed range. For this, suppose $M_\theta C_T$ has closed range. Let $f\in\overline{\operatorname{Ran} C_T}$. Then there exists a sequence $\{f_n\}$ in $L^2(\lambda)$ such that $C_Tf_n\to f$. Hence $M_\theta C_Tf_n\to M_\theta f$. Since $M_\theta C_T$ has closed range, $M_\theta C_Tf_n\to M_\theta C_Tg$ for some g in $L^2(\lambda)$. Hence $M_\theta f=$ $=M_\theta C_Tg$. Since M_θ is invertible, $f=C_Tg$. This proves that C_T has closed range. The converse can be proved similarly.

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A note on local spectra and multicyclic hyponormal operators

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0. Introduction. For a compact subset E of the complex plane, $R(E)$ denotes the set of rational functions with poles off E. An operator A acting on a Hilbert space $\mathfrak{H}$ is said to be n-multicyclic if there are n vectors $g_1, \ldots, g_n \in \mathfrak{H}$, called generating vectors, such that $\mathfrak{H} = \bigvee \{r(A)g_i:\ r \in R(\sigma(A)),\ 1 \leqq i \leqq n\}$. The following theorem of BERGER and SHAW [1] is very well known.

Theorem A. *Let* $A \in \mathcal{B}(\mathfrak{H})$ *be hyponormal, with n-multicyclic generating vectors* $g_1, \ldots, g_n$. *Then*

$$\operatorname{tr}[A^*, A] \leqq (n/\pi)\omega(\sigma(A)),$$

where $[A^*, A] = A^*A - AA^*$, *and* ω *denotes the planar Lebesgue measure.*

The purpose of this paper is to sharpen this theorem as follows:

Main Theorem. *Let* $A \in \mathcal{B}(\mathfrak{H})$ *be hyponormal, with n-multicyclic generating vectors* $g_1, \ldots, g_n$. *Then*

$$\operatorname{tr}[A^*, A] \leqq (1/\pi)\left[\omega(\sigma_A(g_1)) + \ldots + \omega(\sigma_A(g_n))\right],$$

where $\sigma_A(g_i)$, $i = 1, 2, \ldots, n$, *are local spectra of* A.

This formulation is due to the consideration of the operator $A = T_z \oplus T_{z/2}$ defined on $H^2(\chi_{\mathbf{D}}\omega) \oplus H^2(\chi_{\mathbf{D}}\omega)$ by multiplication by z and $z/2$ respectively, where $\mathbf{D}$ is the unit disk. It is clear that A is a 2-multicyclic hyponormal operator, with generating vectors $g_1 = 1 \oplus 0$ and $g_2 = 0 \oplus 1$, and

$$\operatorname{tr}[A^*, A] = (1/\pi)[\omega(\mathbf{D}) + \omega(\mathbf{D})/4] = (1/\pi)\left[\omega(\sigma_A(g_1)) + \omega(\sigma_A(g_2))\right].$$

This shows that our Main Theorem is sharper then Theorem A. As for the proof, it is carried out by "localizing" that given in [1].

Remark. In [5], D. VOICULESCU has extended Theorem A to cover also operators whose self-commutators possess trace-class negative parts. Since these oper-

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ators may not satisfy property (C) (defined below) even when they are cyclic (sample: the backward shift), it seems to be difficult to sharpen this generalized version according to our scheme.

Throughout this paper, all operators are bounded, acting on complex separable Hilbert space of infinite dimension.

1. Preliminaries. The following notions and lemmas come from Dunford and Schwartz [2], p. 2171.

Definition. Let $A\in\mathscr{B}(\mathfrak{K})$. For each $x\in\mathfrak{K}$ the symbol $[x]$ will be used for the closed linear manifold spanned by all vectors $(\lambda I-A)^{-1}x$ with $\lambda\in\varrho(A)$; $\mathfrak{M}(\sigma)$ denotes the set of all x whose spectrum is contained in the set σ: $\sigma_A(x)\subset\sigma$.

Note here that if $A\in\mathscr{B}(\mathfrak{K})$ is an n-multicyclic operator, with generating vectors $g_1, \dots, g_n$, then $\mathfrak{K}=[g_1]\vee\dots\vee[g_n]$.

Lemma A. *$x\in[x]$ and $f(A)[x]\subset[x]$ for $f\in F(\sigma(A))$, where $F(\sigma(A))$ denotes the set of all complex functions which are single valued and analytic on an open set containing $\sigma(A)$.*

Lemma B. *If A has property* (C) *(i.e., $\mathfrak{M}(\sigma)$ is closed when σ is closed), then for $x\in\mathfrak{K}$ we have $\sigma(A|_{[x]})=\sigma_A(x)$, the local spectrum of A at x.*

The next theorem is due to Stampfli [4] for $\sigma(A)=\sigma_C(A)$, the continuous spectrum of A; Radjabalipour [3] put the finishing touch by showing that it remains valid for $\sigma(A)\neq\sigma_C(A)$.

Theorem B. *If A is a hyponormal operator then A satisfies property* (C).

Combining Lemma B and Theorem B, one sees immediately that if A is hyponormal then $\sigma(A|_{[x]})=\sigma_A(x)$. This observation makes possible the "localization" of the Subspace Dominance Lemma of Berger and Shaw [1]. Indeed, due to the observation, it makes sense to introduce the following notation for hyponormal operators:

$$[x; A', E] = \vee\{(\lambda I-A')^{-1}x|\lambda\notin E\},$$

where $A'=A|_{[x]}$ and $E\supset\sigma_A(x)$ $(=\sigma(A'))$. At the same time, it is crucial to notice that $[x]=[x; A', \sigma(A')]$. (Proof: $[x]\supset[x; A', \sigma(A')]$ is obvious since A' is an operator from $[x]$ to $[x]$. The reverse inclusion can be established by observing that $x\in[x]$ and $(\lambda I-A)^{-1}x=(\lambda I-A')^{-1}x$ for all $\lambda\in\varrho(A)$.)

To end this section, we list lemmas from [1], which are needed in the proof of the Main Theorem.

Structure Lemma. *Let T and A be hyponormal operators on $\mathfrak{H}$ and $\mathfrak{K}$ respectively, and let W: $\mathfrak{H}\to\mathfrak{K}$ be a trace class operator with dense range, such that $WT=AW$. Then* $\operatorname{tr}[A^*, A]\leqq\operatorname{tr}[T^*, T]$.

Intertwining Lemma. *Let (U, k_z, x) be an analytic evaluation for $T\in\mathscr{B}(\mathfrak{H})$ and suppose that x is a 1-multicyclic vector for T. If $u\in\mathfrak{H}$, let $\hat{u}(z)=(u, k_z)$, for $z\in U$. Let $A\in\mathscr{B}(\mathfrak{K})$ such that $\sigma(A)\subset U$ and let $y\in\mathfrak{K}$. Define $W\colon \mathfrak{H}\to\mathfrak{K}$, $Wu=\hat{u}(A)y$. The $WT=AW$ and W lies in trace class.*

For convenience, we copy the definition of analytic evaluation here from [1]. Let $T\in\mathscr{B}(\mathfrak{H})$. Suppose there is a map $z\mapsto k_z$, from the open set U to $\mathfrak{H}$, which is conjugate analytic as a map into $\mathfrak{H}$ in the strong topology, and such that there is a vector $x\in\mathfrak{H}$ satisfying $\langle r(T)x,\ k_z\rangle=r(z)$ for all rational functions with poles off $\sigma(T)$, and all $z\in U$. Then the triple (U, k_z, x) will be called an analytic evaluation for T, if $T^*k_z=\bar{z}k_z$ for all $z\in U$.

Second Computational Lemma. *Let $U_1, \dots, U_n$ be open sets with disjoint closures, each bounded by finitely many disjoint smooth Jordan curves. Let $U=\bigcup_{i=1}^{n} U_i$ and $\mathfrak{H}=R^2(\chi_U\text{-}\omega)$ (the closure of $R(\chi_U\text{-})$ in $L^2(\chi_U\text{-}\omega)$). Then T_z on $\mathfrak{H}$ satisfies* $\operatorname{tr}[T_z^*, T_z]\leq\pi^{-1}\omega(U)$.

2. Proof of the main theorem. To start with, it is necessary to "localize" the Subspace Dominant Lemma in [1].

Lemma. *Let $A\in\mathscr{B}(\mathfrak{H})$ be an n-multicyclic hyponormal operator, with generating vectors $g_1, \dots, g_n$. Thus $\mathfrak{H}=[g_1]\vee\dots\vee[g_n]=[g_1;\ A_1, \sigma(A_1)]\vee\dots\vee[g_n;\ A_n, \sigma(A_n)]$, where $A_i=A|_{[g_i]}$. Let E_i be a compact set containing $\sigma_A(g_i)$ $(=\sigma(A_i))$ for $i=1, 2, \dots, n$, and let $\mathfrak{V}=[g_1\colon A_1, E_1]\vee\dots\vee[g_n;\ A_n, E_n]$. Then $\mathfrak{V}$ is an invariant subspace for A, $A|_{\mathfrak{V}}$ is hyponormal, $\sigma(A_i|_{[g_i;\ A_i, E_i]})\subset E_i$ for $i=1, 2, \dots, n$, $A|_{\mathfrak{V}}$ is n-multicyclic with generating vectors $g_1, g_2, \dots, g_n$ and* $\operatorname{tr}[A^*, A]\leq\operatorname{tr}[(A|_{\mathfrak{V}})^*, A|_{\mathfrak{V}}]$.

Proof. Assume $\operatorname{tr}[(A|_{\mathfrak{V}})^*, A|_{\mathfrak{V}}]<\infty$. Let $\{a_{ij}\}_{j=1}^{\infty}$ be a sequence of points in $E_i-\sigma_A(g_i)$ which land densely in each component of $\sigma_A(g_i)^C$ which lies entirely in E_i. Let

$$r_{im}(z) = \prod_{j=1}^{m}(z-a_{ij})^{-1}.$$

Let $\mathfrak{V}_{im}=r_{im}(A_i)[g_i;\ A_i, E_i]$, $\mathfrak{V}_{i0}=[g_i;\ A_i, E_i]$ and let $\mathfrak{V}_m=\bigvee_{i=1}^{n}\mathfrak{V}_{im}$. Clearly $\mathfrak{V}_{m+1}\supset\mathfrak{V}_m$, $\operatorname{rank}(\mathfrak{V}_{m+1}-\mathfrak{V}_m)\leq n$ and $\mathfrak{V}_m\uparrow\mathfrak{H}$ strongly. The rest of the proof is identical to that of Berger and Shaw's and thus omitted.

Proof of the Main Theorem. Let U_i, for $i=1, \dots, n$, be open sets bounded by a finite number of disjoint smooth Jordan curves such that $\sigma_A(g_i)\subset U_i$ and $\omega(U_i)-\omega(\sigma_A(g_i))$, $i=1, \dots, n$, are small. Let $\mathfrak{K}'$ be the subspace spanned by $[g_1;\ A_1, U_1^-], \dots, [g_n;\ A_n, U_n^-]$. Let $A'=A|_{\mathfrak{K}'}$. A' is hyponormal, $\sigma(A_i|_{[g_i, A_i, U_i^-]})\subset U_i$ for $i=1, \dots, n$, and $\{g_i\}$ is a set of n-multicyclic vectors for A'. Now let

$T=\sum_{i=1}^{n}\oplus T_z$ acting on $\mathfrak{H}=\sum_{i=1}^{n}\oplus R^2(\chi_{U_i^-}\omega)$. It is enough to establish:

$$\operatorname{tr}[A^*, A] \leqq \operatorname{tr}[A'^*, A'] \leqq \operatorname{tr}(T^*, T] \leqq (1/\pi)[\omega(U_1)+\ldots+\omega(U_n)].$$

The first and the third inequalities are due to the "local" subspace dominance lemma and the Second Computational Lemma, respectively. The second inequality can be claimed by producing an intertwining map between T and A' satisfying the conditions of the Structure Lemma.

$R^2(\chi_{U_i^-}\omega)$ has reproducing kernel k_z at each $z\in U_i$, for $i=1, \ldots, n$. The maps $z\mapsto k_z$ are strongly conjugate analytic, and the triples $(U_i, k_z, 1)$, $i=1, \ldots, n$, are analytic evaluations. Thus the map W_i: $R^2(\chi_{U_i^-}\omega)\to[g_i; A_i, U_i^-]$ defined by $W_i f=\hat{f}(A_i')g_i$ lies in trace class and $W_i T_z=A_i' W_i$ where $A_i'=A_i|_{[g_i; A_i, U_i^-]}$. Define W: $\sum\oplus R^2(\chi_{U_i^-}\omega)\to K'$ by $W=\sum W_i$. W lies in trace class and $WT=A'W$. Indeed,

$$WT(f_1\oplus\ldots\oplus f_n)=W(T_z f_1\oplus\ldots\oplus T_z f_n)=A_1'\hat{f}_1(A_1')g_1+\ldots+A_n'\hat{f}_n(A_n')g_n,$$

$$A'W(f_1\oplus\ldots\oplus f_n)=A'[\hat{f}_1(A_1')g_1+\ldots+\hat{f}_n(A_n')g_n]=A_1'\hat{f}_1(A_1')g_1+\ldots+A_n'\hat{f}_n(A_n')g_n.$$

The last equality holds because $\hat{f}_i(A_i')g_i\in[g_i; A_i, U_i^-]$ for $i=1, \ldots, n$. Clearly the range of W is dense in $\mathfrak{K}'$. The proof is complete.

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The asymptotic log likelihood function for a class of stationary processes

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The study of the weak consistency of maximum likelihood (ML) estimators for stationary processes in the scalar case was initiated by WHITTLE [7]. The strong consistency of the ML estimates for parameters of ARMA processes has intensively been dealt with by some authors. HANNAN [5] and DUNSMUIR and HANNAN [2] have given the strong laws of large numbers and the central limit theorem for ML estimates of ARMA processes. RISSANEN and CAINES [6] constructed the likelihood function via the innovation process. They proved the uniform P a.s. convergence of the log likelihood function over a compact set of parameters with fixed McMillan degree or Kronecker indices. A similar problem has been investigated by ARATÓ [1] in the continuous time case. These results show that one of the possible methods for proving the strong consistency of ML estimators is to show the P a.s. uniform convergence of the log likelihood function to the asymptotic one.

Our main aim in this paper is to extend the earlier results on the P a.s. uniform convergence of the log likelihood function. In a natural parameter domain the corresponding set of spectral densities would contain a sequence of spectral densities of stationary processes approaching to nonstationary processes, which is not allowed here. However we have that (i) the convergence holds not only over a special compact set of parameters, but on an arbitrary compact set of spectral densities, (ii) the convergence is shown for a wider class of processes then the ARMA processes.

1. Introduction. We shall consider discrete time r-dimensional stationary processes having exponentially bounded covariances. For $0<K, 0<\alpha<1$, let $S(K, \alpha)$ denote the set of spectral densities $\Phi(\omega)$, $\omega\in[-\pi, \pi]$ such that the sequences of covariance matrices

$$C_t = \int_{-\pi}^{\pi} \Phi(\omega) e^{it\omega}\, d\omega, \quad C_t^+ = \int_{-\pi}^{\pi} \Phi(\omega)^{-1} e^{it\omega}\, d\omega, \quad t\in\mathbf{Z}$$

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are uniformly bounded by the powers of α:

$$\|C_t\| \leqq K\alpha^{|t|} \quad \|C_t^+\| \leqq K\alpha^{|t|} \quad t\in \mathbf{Z} \tag{1.1}$$

where $\|v\|^2=v_1^2+\ldots+v_n^2$ is the norm of a vector $v=(v_1, \ldots, v_n)'$ and $\|A\|^2= \sup_{\|u\|=1} \|Au\|^2$ for a matrix A. The transpose of A will be denoted by A'. Let $\mathcal{S}=\bigcup_{K>0,\alpha<1} S(K, \alpha)$.

The log likelihood function is defined as usual by

$$L_n(y_n, \Phi) = \log\det \Gamma_n(\Phi)+\frac{1}{2}\, y_n'\Gamma_n(\Phi)^{-1}y_n \tag{1.2}$$

where

$$\Gamma_n = \begin{bmatrix} C_0 & C_1 \ldots C_{n-1} \\ C_{-1} & C_0 \ldots C_{n-2} \\ \vdots & \ddots \vdots \\ C_{-n+1} & C_0 \end{bmatrix}$$

is the Toeplitz matrix (see Grenander and Szegő [3]) composed from the autocovariance matrix sequence C_t, $t\in\mathbf{Z}$. In the following $\xrightarrow{P}$ denotes convergence in probability. Convergence with probability 1 will be written as P a.s.

2. Convergence of the log likelihood function. Introduce on $\mathcal{S}$ the metric ϱ

$$\varrho(\Phi, \Psi) = \operatorname*{ess\,sup}_{\omega\in[-\pi,\pi]} \sup_{1\leqq j,k\leqq r} |\Phi_{jk}(\omega)-\Psi_{jk}(\omega)| \quad \Phi, \Psi\in\mathcal{S}.$$

Theorem 1. *Let $S\subseteqq S(K,\alpha)$ be compact. If y_t, $t\in\mathbf{Z}$ is a Gaussian stationary process with spectral density $\Phi_0\in\mathcal{S}$ then with probability* 1

$$L_n(y_n, \Phi) \to \frac{1}{2\pi}\int_{-\pi}^{\pi} [\log\det \Phi(\omega)+\operatorname{tr} \Phi^{-1}(\omega)\Phi_0(\omega)]\,d\omega \tag{2.1}$$

as $n\to\infty$, uniformly for $\Phi\in S$.

Proof. The proof is based on some lemmas and properties of Toeplitz matrices.

Lemma 1. *With the above notations*

$$\frac{1}{n}\log\det\Gamma_n(\Phi) \to \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det\Phi(\omega)\,d\omega \tag{2.2}$$

as $n\to\infty$ uniformly for $\Phi\in S$.

Proof. This statement is an extension of Szegő's classical theorem and we refer to Gyires [4] for its proof.

Lemma 2. *For each* $\Phi \in S$ *holds the convergence*

$$\bar{y}_n = \frac{1}{n} \underline{y}_n' \Gamma_n(\Phi)^{-1} \underline{y}_n \xrightarrow{P} \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{tr} \Phi^{-1}(\omega) \Phi_0\{\omega) \, d\omega, \tag{2.3}$$

where Φ_0 *is the spectral density of* y_t, $t \in \mathbf{Z}$.

Proof. The proof of this lemma is a straightforward modification for the vectorial case of a result of GRENANDER and SZEGŐ [3] in Section 11.5.

It will be proved by Lemmas 3.1—3.5 that $\bar{y}_n$ converges P a.s. uniformly to some function, then by Lemma 2 the limit function is the right hand side of (2.3) P a.s.

The proof of Theorem 1 will be based on an approximation of the matrix Γ_n with another matrix L_n defined in the following way. Let U_n be an orthogonal matrix of order nr composed from the r-order matrices

$$[U_n]_{\mu\nu} = n^{-1/2} e^{2\pi i \nu\mu} I_r, \quad \mu, \nu = 1, 2, \ldots, r,$$

where I_r is the r-order identity matrix. We define Φ_p by

$$\Phi_p(ix) = \sum_{-p}^{p} \left(1 - \frac{|\nu|}{p}\right) C_\nu e^{i\nu x}, \quad x \in [-\pi, \pi], \quad p = 1, 2, \ldots.$$

Let D_n be an nr-order matrix with the r-order matrices $[D_n]_{\nu\nu} = \Phi_p(2\pi i\nu/n)$ in the diagonal and 0 everywhere else, i.e. $[D_n]_{\mu\nu} = 0$, if $\mu \neq \nu$.

Now we define L_n by $L_n = U_n^* D_n U_n$ and $\underline{C}_\nu$ is given by $\underline{C}_\nu = (1 - |\nu|/p) C_\nu$ if $|\nu| < p$ and $\underline{C}_\nu = 0$ if $|\nu| \geqq p$.

Lemma 3.1. *Let* p *be in the above definition of* L_n $p = p(n) = [n^{1/2+\varepsilon}]$, *where* $1/4 < \varepsilon < 1/2$ *is a once for all fixed, but arbitrary number. Then for all natural numbers* k

$$\frac{1}{n} \underline{y}_n' (\Gamma_n^k - L_n^k) \underline{y}_n \to 0$$

as $n \to 0$ *uniformly over* S P *a.s.*

Proof. For the proof we shall consider the matrices K_n defined by

$$[K_n]_{\mu\nu} = \underline{C}_{\nu-\mu} \quad \text{for} \quad \nu, \mu = 1, 2, \ldots, n,$$

and the convergence of the sequences

$$\frac{1}{n} \underline{y}_n' (L_n^k - K_n^k) \underline{y}_n \tag{2.4}$$

and

$$\frac{1}{n} \underline{y}_n' (K_n^k - \Gamma_n^k) \underline{y}_n \tag{2.5}$$

will be examined.

First we show the convergence of (2.4). Using the notations

$$W_n = L_n^{k-1} + L_n^{k-2} K_n + \dots + K_n^{k-1}$$

and

$$M = \sup_{\Phi \in S} \operatorname{ess\,sup}_{x \in [-\pi, \pi]} \|\Phi(x)\|$$

we have

(2.6)
$$\|W_n\| \leqq k \cdot M^{k-1}$$

because of the inequalities

$$\|L_n\| \leqq \sup_{x \in [-\pi, \pi]} \|\Phi_p(ix)\| \leqq \operatorname{ess\,sup}_{x \in [-\pi, \pi]} \|\Phi(x)\| \leqq M$$

and

$$\|K_n\| \leqq \sup_{x \in [-\pi, \pi]} \|\Phi_p(ix)\| \leqq \operatorname{ess\,sup}_{x \in [-\pi, \pi]} \|\Phi(x)\|.$$

Introduce the notation $\underline{z}_n = W_n \underline{y}_n$, then the expression in (2.4) takes the form

(2.7)
$$\frac{1}{n} \underline{y}_n'(L_n - K_n) \underline{z}_n.$$

Since the process $\{y_n, n \in \mathbf{Z}\}$ is stationary and ergodic so is the scalar process $\{\|y_n\|^2 : n \in \mathbf{Z}\}$. This implies that the averages

$$n^{-1}(\|y_1\|^2 + \|y_2\|^2 + \dots + \|y_n\|^2) = n^{-1} \|\underline{y}_n\|^2$$

converge P a.s., and therefore the sequence $\{n^{-1} \|\underline{y}_n\|^2 : n \in \mathbf{N}\}$ is bounded P a.s. by a number $K(\omega)$, which depends on the elementary event ω. This implies that the sequence $n^{-1} \|\underline{z}_n\|^2$, $n \in \mathbf{N}$ is bounded P a.s., indeed

$$n^{-1} \|\underline{z}_n\|^2 \leqq n^{-1} \|W_n\|^2 \|y_n\|^2 < k \cdot M^{k-1} n^{-1} \|\underline{y}_n\|^2 \leqq k \cdot M^{k-1} K\{\omega).$$

The r-order matrix block of L_n at place (ν, μ) can easily be computed as

$$[L_n]_{\nu, \mu} = \frac{1}{n} \sum_{j=1}^{n} e^{-2\pi i \nu j} I_r \Phi_p(2\pi i j / n) e^{2\pi i j \mu} I_r = \sum_{m=-\infty}^{\infty} \underline{C}_{\nu - \mu + mn}.$$

Now we deduce the following sequence of inequalities

(2.8)
$$\left| \frac{1}{n} \underline{y}_n'(L_n - K_n) \underline{z}_n \right| =$$

$$= \left| \frac{1}{n} (y_1' \underline{C}_{p-1} z_{n-p+1} + y_2' \underline{C}_{p-1} z_{n-p+2} + \dots + y_p' \underline{C}_{p-1} z_n) + \right.$$

$$+ \frac{1}{n} (y_1' \underline{C}_{p-2} z_{n-p+2} + \dots + y_{p-1}' \underline{C}_{p-2} z_n) + \dots + \frac{1}{n} y_1' \underline{C}_0 z_n +$$

$$+ \frac{1}{n} (y_{n-p+1}' \underline{C}_{1-p} z_p + y_{n-p+2}' \underline{C}_{1-p} z_{p-1} + \dots + y_n' \underline{C}_{n-p} z_1) +$$

$$+\frac{1}{n}(y'_{n-p+2}\underline{C}_{2-p}z_{p-1}+\ldots+y'_n\underline{C}_{2-p}z_1)+\ldots+\frac{1}{n}y'_n\underline{C}_0z_1\Big| \leqq$$

$$\leqq n^{-1}\|\underline{C}_{p-1}\|(\|y_1\|\,\|z_{n-p+1}\|+\|y_2\|\,\|z_{n-p+2}\|+\ldots+\|y_p\|\,\|z_n\|)+$$

$$+n^{-1}\|\underline{C}_{p-2}\|(\|y_1\|\,\|z_{n-p+2}\|+\ldots+\|y_{p-1}\|\,\|z_n\|)+\ldots+n^{-1}\|\underline{C}_0\|\,\|y_1\|\,\|z_n\|+$$

$$+n^{-1}\|\underline{C}_{1-p}\|(\|y_{n-p+1}\|\,\|z_p\|+\|y_{n-p+2}\|\,\|z_{p-1}\|+\ldots+\|y_n\|\,\|z_1\|+$$

$$+n^{-1}\|\underline{C}_{2-p}\|(\|y_{n-p+2}\|\,\|z_{p-1}\|+\ldots+\|y_n\|\,\|z_1\|)+\ldots+n^{-1}\|\underline{C}_0\|\,\|y_n\|\,\|z_1\| \leqq$$

$$\leqq (p/n)^{1/2}\|\underline{C}_{p-1}\|\,p^{-1/2}\|y_p\|\,n^{-1/2}(\|z_{n-p+1}\|^2+\ldots+\|z_n\|^2)^{1/2}+$$

$$+(p/n)^{1/2}\|C_{p-2}\|\,p^{-1/2}\|y_{p-1}\|\,n^{-1/2}(\|z_{n-p+2}\|^2+\ldots+\|z_n\|^2)^{1/2}+\ldots$$

$$\ldots+(p/n)^{1/2}\|C_0\|\,p^{-1/2}\|y_0\|\,n^{-1/2}\|z_n\|+$$

$$+(p/n)^{1/2}\|C_{1-p}\|\,n^{-1/2}(\|y_{n-p+1}\|^2+\ldots+\|y_n\|^2)^{1/2}p^{-1/2}\|z_p\|+\ldots$$

$$\ldots+(p/n)^{1/2}\|C_0\|\,n^{-1/2}\|y_n\|\,p^{-1/2}\|z_1\| \leqq$$

$$\leqq (p/n)^{1/2}\Big(\sum_{\nu=0}^{p-1}\|C_\nu\|\Big)K(\omega)^{1/2}K_0(\omega)^{1/2}+$$

$$+(p/n)^{1/2}\Big(\sum_{\nu=1-p}^{0}\|C_\nu\|\Big)K_0(\omega)^{1/2}K(\omega)^{1/2}$$

P a.s., where the notation $K_0(\omega)=k\cdot M^{k-1}$ was used. The last inequality follows from the simple relations

$$n^{-1}(\|z_{n-p+i}\|^2+\ldots+\|z_n\|^2) \leqq K_0(\omega), \quad i=1,2,\ldots,p$$

and

$$n^{-1}(\|y_{n-p+i}\|^2+\ldots+\|y_n\|^2) \leqq K(\omega), \quad i=1,2,\ldots,p.$$

But $\sum_{\nu=-\infty}^{\infty}\|C_\nu\|<\infty$ and therefore both summands in (2.8) converge to 0 as $n\to\infty$ P a.s. which implies that the expression in (2.7) converges to 0 P a.s. which was to be proved.

For proving (2.5) the following lemma can be applied, which gives an approximation of the powers of Toeplitz matrices.

Lemma 3.2. *For each* $k\geqq 1$

$$\|K_n^k-\Gamma_n^k\|\to 0 \quad as \quad n\to\infty. \tag{2.9}$$

Proof. Define the norm $|\cdot|$ for n-order symmetric matrices Γ by $|\Gamma|^2=\frac{1}{n}\sum_i \Gamma_{i,j}^2$. Using the inequality $|\Gamma|\leqq\|\Gamma\|$ it follows that

$$|K_n-\Gamma_n|^2 = 2n^{-1}\sum_{\nu=1}^{p}(\nu^2/p^2)(n-\nu)|C_\nu|^2+2n^{-1}\sum_{\nu=p}^{n}(n-\nu)|C_\nu|^2 \leqq$$

$$\leqq 2\cdot p^{-2}\sum_{\nu=1}^{p}\nu^2\|C_\nu\|^2+2n^{-1}(n-p)\sum_{\nu=p}^{\infty}\|C_\nu\|^2.$$

In the last sum

$$\sum_{\nu=1}^{\infty} \nu^2 \|C_\nu\|^2 < \infty \quad \text{and} \quad \sum_{\nu=p}^{\infty} \|C_\nu\|^2 \leqq O(\alpha^p).$$

Thus

$$|K_n - \Gamma_n|^2 \leqq O(n^{-2\varepsilon-1}) + O(\alpha^{[n^{1/2+\varepsilon}]}),$$

from which we can conclude that

$$|K_n - \Gamma_n| \leqq o(n^{-1/2}).$$

Denote the eigenvalues of the matrix $V_n = n^{-1}(K_n - \Gamma_n)$ by $\lambda_1^{(n)}, \ldots, \lambda_{nr}^{(n)}$. Then

$$|V_n|^2 = \frac{1}{nr} \sum_{i=1}^{nr} |\lambda_i^{(n)}|^2 \geqq \frac{1}{nr} \|V_n\|^2 \tag{2.10}$$

and by the preceding inequality

$$|V_n| \leqq o(n^{-3/2}). \tag{2.11}$$

By (2.10) and (2.11)

$$\|V_n\| \leqq o(n^{-1}). \tag{2.12}$$

Since $\|K_n\| \leqq M$ and $\|\Gamma_n\| \leqq M$

$$n^{-1}\|K_n^k - \Gamma_n^k\| \leqq n^{-1}\|K_n - \Gamma_n\| \cdot k \cdot M^{k-1} = \|V_n\| k \cdot M^{k-1}.$$

Finally it follows from (2.11) and (2.12) that

$$\|n^{-1}(K_n^k - \Gamma_n^k)\| \leqq o(n^{-1})$$

and therefore Lemma 3.2 can be concluded. Applying (2.9) we have

$$|n^{-1} y_n'(K_n^k - \Gamma_n^k) y_n| \leqq \|y_n\|^2 o(n^{-1}) = n^{-1}\|y_n\|^2 o(1). \tag{2.13}$$

But $n^{-1}\|y_n\| < K(\omega)$ P a.s., and by the inequality (2.13) it yields

$$\frac{1}{n} y_n'(K_n^k - \Gamma_n^k) y_n \to 0,$$

as $n \to \infty$ P a.s., and this was to be proved.

This means that both expressions in (2.4) and (2.5) tend to 0 as $n \to \infty$ P a.s., and this completes the proof of Lemma 3.

For later proofs we introduce the term

$$C(n, j) = \frac{1}{n} \sum_{\nu=1}^{n} e^{-2\pi i j\nu/n} \Phi^k_{p(n)}(2\pi i\nu/n), \quad n = 1, 2, \ldots, \quad j = 0, \pm 1, \ldots,$$

which plays an important role in the theory of Toeplitz matrices and the following two statements are valid.

Lemma 3.3. *Using the previous notation for $p(n)$ there is an $L_0 > 0$ such that*

$$\|C(n, j)\| \leqq L_0(2p(n))^{k-1}\alpha^j$$

holds.

Lemma 3.4. *There are numbers A and B, which do not depend on n or j such that*

$$\|C(j) - C(n, j)\| \leqq A \cdot p(n)^{-1} + B \cdot jn^{-1}$$

holds, where

$$C(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi^k(x) \cdot e^{-ij\omega}\, d\omega, \quad j = 0, \pm 1, \pm 2, \ldots.$$

The proofs of Lemma 3.3 and Lemma 3.4 can easily be given using the definitions of $C(n, j)$ and $C(j)$.

Lemma 3.5. *The sequence*

$$\frac{1}{n} \underline{y}'_n L^k_n \underline{y}_n, \quad n = 1, 2, \ldots$$

converges as $n \to \infty$ uniformly over S P *a.s..*

Proof. Taking into consideration the previous definitions

$$(2.14) \qquad \frac{1}{n} \underline{y}'_n L^k_n \underline{y}_n = \frac{1}{n} \sum_{m, l=1}^{n, n} \sum_{\nu=1}^{n} \frac{1}{n} e^{2\pi i (m-l)\nu/n} y'_l \Phi^k_p(2\pi i\nu/n) y_m =$$

$$= \frac{1}{n} \sum_{m, l=1}^{n, n} y'_l C(n, l-m) y_m = \frac{1}{n} \sum_{j=-n+1}^{n-1} \sum_{\nu=a}^{f} y'_\nu C(n, j) y_{\nu+j},$$

where the notations $a = \max(1, -j+1)$ and $f = \min(n-j, n)$ were used.

Using the stationary and ergodic property of $\{y_n : n \in \mathbf{Z}\}$ and Lemma 7 for fixed j, with the notation $\gamma_j = \operatorname{tr} \Gamma_j C(j)$

$$(2.15) \qquad \frac{1}{n} \sum_{\nu=a}^{f} y'_\nu C(n, j) y_{\nu+j} \to E(y'_1 C(j) y_{1+j}) = \gamma_j$$

holds uniformly over S P a.s..

To prove this convergence we show that taking in (2.14) $C(j)$ in place of $C(n,j)$ we have uniformly the same limit P a.s.. This comes from

(2.16)
$$\left|\frac{1}{n}\sum_{j=-n+1}^{n-1}\sum_{v=a}^{f} y'C(n,j)y_{v+j}-\frac{1}{n}\sum_{j=-n+1}^{n-1}\sum_{v=a}^{f} y'_vC(j)y_{v+j}\right| \leqq$$
$$\leqq \frac{1}{n}\sum_{j=-n+1}^{n-1}\sum_{v=a}^{f} \|y_v\|\,\|C(n,j)-C(j)\|\,\|y_{v+j}\| \leqq$$
$$\leqq \frac{1}{n}\sum_{l=-j(n)}^{j(n)}\sum_{v=a}^{f} \|y_v\|\,\|C(n,l)-C(l)\|\,\|y_{v+j}\| +$$
$$+\frac{1}{n}\sum_{n\geqq|l|>j(n)}\sum_{v=a}^{f} \|y_v\|(\|C(n,l)\|+\|C(l)\|)\|y_{v+j}\| \leqq$$
$$\leqq \sum_{l=-j(n)}^{j(n)} p(n,j)n^{-1}\sum_{v=1}^{n}\|y_v\|^2+\sum_{n\geqq l>j(n)} 2L_0(2p(n))^{k-1}\alpha^{|l|}\frac{1}{n}\sum_{v=1}^{n}\|y_v\|^2 \leqq$$
$$\leqq (2j(n)+1)p(n,j)\cdot K(\omega)+2L_0(2p(n))^{k-1}K(\omega)\frac{\alpha^{j(n)}}{1-\alpha}$$

P a.s. for all elementary events, where the notation $p(n,j)=Ap(n)^{-1}+Bjn^{-1}$ was used. Now we choose $j(n)$ so that $j(n)/p(n)\to 0$ as $n\to\infty$. Let e.g. $j(n)=[n^{1/2-\varepsilon/2}]$. The on the right hand side of (2.16) both expressions tend to 0. Indeed, the convergence of the first term is obvious, and the convergence of the second easily follows from

$$p(n)^{k-1}\alpha^{[n^{1/2-\varepsilon/2}]}\to 0, \quad \text{as} \quad n\to\infty.$$

Now choose an arbitrary fixed $j_0>1$ and take the limit of the first $2j_0+1$ terms in

(2.17)
$$\frac{1}{n}\sum_{l=-n+1}^{n-1}\sum_{v=a}^{f} y'_vC(l)y_{v+l} =$$
$$=\sum_{l=-j_0}^{j_0}\frac{1}{n}\sum_{v=a}^{f} y'_vC(l)y_{v+l}+\sum_{n\geqq|l|>j_0}\frac{1}{n}\sum_{v=a}^{f} y'_vC(l)y_{v+l}.$$

The second term can be majorized with the aid of (1.3)

(2.18)
$$\left|\sum_{n\geqq|l|>j_0}\frac{1}{n}\sum_{v=a}^{f} y'_vC(l)y_{v+l}\right| \leqq 2V_0\alpha^{j_0}(1-\alpha)^{-1}K(\omega).$$

Then it follows by (2.15), (2.17) and (2.18) that

$$\sum_{l=-j_0}^{j_0}\gamma_l-V(j_0)\leqq \underline{\lim}_{n\to\infty},\ \overline{\lim}\,\frac{1}{n}\sum_{l=-n+1}^{n-1}\sum_{v=a}^{f} y'_vC(l)y_{v+l}\leqq \sum_{l=-j_0}^{j_0}\gamma_l+V(j_0)$$

uniformly over S P a.s. for all $j_0>1$, where the notation $V(j_0)=2V_0\alpha^{j_0}(1-\alpha)^{-1}K(\omega)$ was used. This implies by (2.16) that

$$\lim_{n\to\infty}\frac{1}{n}y_n' L_n^k y_n=\lim_{n\to\infty}\frac{1}{n}\sum_{l=-n+1}^{n-1}\sum_{v=a}^{f}y_v' C(l)y_{v+l}=\sum_{l=-\infty}^{\infty}\gamma_l$$

uniformly over S P a.s., which completes the proof of the lemma.

Lemma 3. $\frac{1}{n}y_n'\Gamma_n^{-1}y_n$ *converges uniformly over* S P *a.s. as* $n\to\infty$.

Proof. By Lemma 3.1 $n^{-1}y_n'\Gamma_n^k y_n$ has the same limit as $n^{-1}y_n' L_n^k y_n$ uniformly over S P a.s.. Therefore

$$\frac{1}{n}y_n(I_n-c\Gamma_n)^k y_n, \tag{2.19}$$

converges uniformly P a.s. as $n\to\infty$, too. Here I_n denotes the unit matrix of order nr. We shall choose $c>0$ so that

$$\|I_n-c\Gamma_n(\Phi)\|<\chi,\quad n=1,2,\ldots \tag{2.20}$$

be valid with a fixed $0<\chi<1$ over S. The existence of such $c>0$ will be assured by $0<\min_{\Phi\in S}\operatorname{ess\,sup}_{x\in[-\pi,\pi]}\|\Phi(x)\|\leqq\max_{\Phi\in S}\operatorname{ess\,sup}_{x\in[-\pi,\pi]}\|\Phi(x)\|$. Indeed,

$$\|I_n-c\Gamma_n\|=\max_{\|u\|=1}|u'(I_n-c\Gamma_n)u|=$$

$$=\max\left\{1-c\cdot\min_{\|u\|=1}u'\Gamma_n u,\ c\cdot\max_{\|u\|=1}u'\Gamma_n u-1\right\}=\chi_n^c(\Phi)$$

and therefore it is enough to choose $c=c_0$ so that

$$c_0\cdot\max_{\Phi\in S,\,x\in[-\pi,\pi]}\|\Phi(x)\|-1<1$$

be valid.

It follows that there is a fixed $\chi<1$ such that

$$\|I_n-c_0\cdot\Gamma_n(\Phi)\|=\chi_n^c(\Phi)<\chi$$

uniformly over S for all $n=1,2,\ldots$.

Now we can apply a natural expansion of Γ_n^{-1}:

$$\Gamma_n^{-1}=c_0\big(I_n+(I_n-c_0\Gamma_n)+(I_n-c_0\Gamma_n)^2+\ldots\big).$$

Thus we get the series

$$\frac{1}{n}y_n'\Gamma_n^{-1}y_n=c_0\sum_{k=0}^{\infty}\frac{1}{n}y_n'(I_n-c_0\Gamma_n)^k y_n. \tag{2.21}$$

Using Lemmas 3—6 we conclude that the terms in the series (2.21) converge uniformly P a.s., an can be evaluated by

$$\left|\frac{1}{n}y_n'(I_n-c_0\Gamma_n)^k y_n\right| \leqq \frac{1}{n}\sum_{v=1}^{n}\|y_v\|^2\chi^k \leqq K(\omega)\chi^k. \tag{2.22}$$

By the previous convergence results we may use the notation

$$r(k) = \lim_{n\to\infty}\frac{1}{n}y_n'(I_n-c_0\Gamma_n)^k y_n \quad \text{P a.s.} \quad k = 1, 2, \ldots. \tag{2.23}$$

Then by (2.21), (2.22) and (2.23) for all fixed $k_0 \in \mathbf{N}$

$$\sum_{k=0}^{k_0} r(k) - K(\omega)\chi^{k_0}(1-\chi)^{-1} \leqq \underline{\lim}_{n\to\infty}, \overline{\lim}_{n\to\infty}\frac{1}{n}y_n'\Gamma_n^{-1}y_n \leqq$$

$$\leqq \sum_{k=0}^{k_0} r(k) + K(\omega)\chi^{k_0}(1-\chi)^{-1}$$

holds uniformly over S P a.s.. This implies

$$\lim_{n\to\infty}\frac{1}{n}y_n'\Gamma_n^{-1}(\Phi)y_n = \sum_{k=0}^{\infty} r(k) \tag{2.24}$$

where the convergence is uniform over S P a.s., completing the proof of Lemma 3 and thus the proof of Theorem 1 too.

3. Strong consistency. As a consequence of Theorem 1 the following consistency theorem can be concluded for processes with exponentially stable covariances.

Theorem 2. *Let $S \subseteq S(K, \alpha)$ be a compact set of spectral densities and let y_t, $t \in \mathbf{Z}$ be a Gaussian stationary process with spectral density $\Phi_0 \in S$. Then for the estimates $\hat{\Phi_n}$ obtained by minimizing $L_n(y_n, \Phi)$ over S*

$$\hat{\Phi_n} \to \Phi_0 \tag{3.1}$$

P *a.s. as $n \to \infty$, where the convergence is considered in metric ϱ of the uniform convergence on S.*

Proof. The proof follows by a standard argumentation from Theorem 1 and the following lemma.

Lemma 4. *$L(\Phi, \Phi_0)$ is continuous with Φ as a variable on S and attains its minimum value over S only at $\Phi = \Phi_0$.*

Proof. For all $x \in [-\pi, \pi]$ the matrices $\Phi(x)$ and $\Phi_0(x)$ are positive definite. Therefore the matrix $\Phi^{-1}\Phi_0(x)$ must have positive eigenvalues $\lambda_1(x), \lambda_2(x), \ldots$ $\ldots, \lambda_r(x)$, although $\Phi^{-1}\Phi_0(x)$ is not necessarily symmetric.

By the inequality $\log\lambda \leqq \lambda - 1$, $\lambda > 0$ we have

$$\sum_{i=1}^{r} \log \lambda_i(x) - \sum_{i=1}^{r} \lambda_i(x) + r \leqq 0 \tag{3.2}$$

and thus

$$\log\det \Phi^{-1}\Phi_0(x) - \mathrm{tr}\,[\Phi^{-1}\Phi_0(x)] + r \leqq 0$$

that can be written in the form

$$r + \log\det \Phi_0(x) - \big(\log\det \Phi(x) + \mathrm{tr}\,[\Phi^{-1}\Phi_0(x)]\big) \leqq 0.$$

Taking the integral of both sides over $[-\pi, \pi]$ we have

$$L(\Phi_0, \Phi_0) \leqq L(\Phi, \Phi_0)$$

and equality is here only if equality holds in (3.2) for all $x \in [-\pi, \pi]$, which implies $\lambda_1(x) = \lambda_2(x) = \ldots = \lambda_r(x) = 1$, $x \in [-\pi, \pi]$ and this is equivalent to $\Phi(X) = \Phi_0(X)$, $x \in [-\pi, \pi]$.

Remark. Assume that the topological space $\underline{\Theta}$ is a parametrization for stationary processes with exponentially bounded covariances, i.e. there is an injective continuous map $\tau\colon \underline{\Theta} \to S(K, \alpha)$ such that the process with parameter $\Theta \in \underline{\Theta}$ has spectral density $\tau(\Theta)$. Let $\Theta^c \subseteqq \underline{\Theta}$ be a compact subset of parameters. If the observed process y_t, $t \in \mathbf{Z}$ has parameter $\Theta_0 \in \Theta^c$ then by Theorem 2 the estimates computed by minimizing $L_n\big(y_n, \tau(\Phi)\big)$ over Θ^c are strongly consistent.

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One-dimensional perturbations of singular unitary operators

N. G. MAKAROV

Introduction and results. Let $\mathbf{T}$ denote the unit circle and m be the normalized Lebesgue measure on $\mathbf{T}$. Recall that a closed subset e of $\mathbf{T}$ is said to be a *Carleson set* if

$$\int \log [\operatorname{dist}(\zeta, e)] \, \mathrm{dm}(\zeta) > -\infty.$$

These sets arise as sets of nonuniqueness for functions analytic in the unit disc and smooth up to the boundary, see [1]. Also we introduce the class (C_σ) consisting of all countable unions of Carleson sets.

This class plays a crucial role in the description of point spectrum of almost unitary operators acting on a *separable* Hilbert space. It was proved in [3] that if U is a unitary and K is a trace class operators, then

$$\sigma_p(U+K) \cap \mathbf{T} \in (C_\sigma).$$

In the opposite direction, given $e \in (C_\sigma)$, there is a one-dimensional perturbation of the shift operator $f(z) \mapsto zf(z)$ on $L^2 \equiv L^2(m)$ with point spectrum equal to e.

It is not immediatly clear from the proof whether the appearing of an uncountable point spectrum relies on the absolutely continuous properties of the unitary operator. The question seems also natural from the viewpoint of spectral analysis of general noncontractive operators (cf. [4]), and it was stated in [2] p. 120 as a reseach problem. In the present paper we give an answer to this question.

A unitary operator is said to be *singular* if its spectral measure is singular with respect to the Lebesgue measure.

Theorem 1. *Let $e \in (C_\sigma)$. There exist a singular unitary operator U and an operator K of rank one such that $e \subset \sigma_p(U+K)$ and, moreover, each point ζ in e is an eigenvalue of $U+K$ having infinite multiplicity (i.e. for any positive integer n,*

$$\ker (U+K-\zeta I)^{n+1} \neq \ker (U+K-\zeta I)^n).$$

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As an application, we consider a question concerning inner functions. By z we denote the identity mapping of the unit disc and by H^2 the standard Hardy space. Let φ_1 and φ_2 be two nonequal inner functions. On which subsets e of $\mathbf{T}$ can such functions "coincide" in the sense that $(z-\zeta)^{-1}(\varphi_1-\varphi_2)\in H^2$ for all $\zeta\in e$? As it follows from Theorem 2 in [3], e has to be of class (C_σ). On the other hand, e is at most countable if, for instance, $\varphi_1=1$. One possible way to see this is as follows.

Assume, for simplicity, that $\varphi(0)=0$, $\varphi\equiv\varphi_2$. Let P denote the orthogonal projection in H^2 onto $H^2\ominus\varphi H^2$. The point $\zeta\in\mathbf{T}$ is an eigenvalue of the unitary operator

$$f\mapsto Pzf+\langle f,\bar z\varphi\rangle 1$$

acting on $H^2\ominus\varphi H^2$ if and only if $(z-\zeta)^{-1}(\varphi-1)\in H^2$. Hence, the set of all such points is at most countable.

By similar reasoning, we shall obtain from Theorem 1 the following result.

Theorem 2. *Let $e\in(C_\sigma)$. There exist two nonequal inner functions φ_1 and φ_2 such that for any $\zeta\in e$ and any integer n, the function $(z-\zeta)^{-n}(\varphi_1-\varphi_2)$ belongs to H^2.*

At the same time, the author does not dispose of any explicit construction of such functions.

The proof of both theorems appeals to some properties of almost unitary operators, and thus this work could be considered as an illustration to the theory presented in [4].

Proof of Theorem 1. Fix $e\in(C_\sigma)$. There exists a bounded analytic function h, $h(0)=-1$, satisfying $(z-\zeta)^{-n}h\in H^2$ for all integer n and $\zeta\in e$. In case e is a Carleson set, for h, one can take an infinitely smooth up to the boundary analytic function which vanishes on e together with all its derivatives. For an arbitrary $e\in(C_\sigma)$, one can consider an appropriate product of smooth functions, see [3] for a detailed proof.

Let $w=h+\bar h$ and the operator L_0 be defined on L^2 by the equality

$$L_0f=zf+\langle f,\bar z\rangle w.$$

If $\zeta\in e$ and $n\in\mathbf{N}$, then $w_{\zeta,n}\equiv(z-\zeta)^{-n}w\in L^2$ and

$$\langle w_{\zeta,n},\bar z\rangle=\left\langle\frac{zh}{(z-\zeta)^n},1\right\rangle+\left\langle\frac{\bar z^{n-1}\bar h}{(1-\bar z\zeta)^n},1\right\rangle=\begin{cases}-1, & n=1\\ 0, & n\geqq 2.\end{cases}$$

Therefore,

$$(L_0-\zeta I)w_{\zeta,n}=\begin{cases}0, & n=1\\ w_{\zeta,n-1}, & n\geqq 2\end{cases}$$

and ζ is an eigenvalue of infinite multiplicity. Remark that the operator L_0 is invertible, since otherwise the origin would be an eigenvalue of L_0 and hence $\langle w, 1\rangle=-1$; on the other hand, by construction, $\langle w, 1\rangle=-2$.

Let $E=\text{span}\,\{\ker(L_0-\zeta I)^n\colon \zeta\in e,\ n\in\mathbf{N}\}$. It is a hyperinvariant subspace of L_0. Consider the imbedding $j\colon E\to L^2$ and define the operator L on E by $L=j^*L_0 j$. Obviously, L is invertible and any $\zeta\in e$ is its eigenvalue of infinite multiplicity. Also consider the one-dimensional operator $K=\langle\cdot, a\rangle b$ with

$$a=\frac{j^*\bar z}{\|j^*\bar z\|},\quad b=La-\frac{L^{*-1}a}{\|L^{*-1}a\|}.$$

(Note that $j^*\bar z\neq 0$ because $\langle w_{\zeta,1}, \bar z\rangle=-1$ for $\zeta\in e$ and so $\bar z$ is not orthogonal to E.) Let $U=L-K$. We shall prove that U is a unitary operator and that it is singular.

If $f\in E$, then

$$Uf=L(f-\langle f, a\rangle a)+\langle f, a\rangle\|L^{*-1}a\|^{-1}L^{*-1}a.$$

Observe that the terms on the right are orthogonal. Hence

$$\|Uf\|^2=\|L(f-\langle f, a\rangle a)\|^2+|\langle f, a\rangle|^2=$$
$$=\|f-\langle f, a\rangle a\|^2+|\langle f, a\rangle|^2=\|f\|^2.$$

Since U is a Fredholm operator of index zero, it is unitary.

To prove the singularity of U, it suffices to verify that for all f and g in E,

(1) $\quad\langle (U-r\eta I)^{-1}f-(U-r^{-1}\eta I)^{-1}f, g\rangle\to 0\quad\text{as}\quad r\to 1\quad\text{for a.e.}\quad \eta\in\mathbf{T},$

cf. Proposition 6.7 and Remark 6.10 in [4]. Let $\lambda\notin\sigma(L)\cup\mathbf{T}$ and R_λ denote $(L-\lambda I)^{-1}$. Direct calculation gives $\langle R_\lambda b, a\rangle\neq 1$ and

$$(U-\lambda I)^{-1}=R_\lambda+\langle\cdot, R_\lambda^* a\rangle\big(1-\langle R_\lambda b, a\rangle\big)^{-1}R_\lambda b.$$

Consequently, (1) follows from the corresponding fact concerning L. But the latter is obvious since linear combinations of root vectors of L are dense in E and for $f\in\ker(L-\zeta I)^n$, $\langle R_\lambda f, g\rangle$ is a polynomial in $(\lambda-\zeta)^{-1}$.

Proof of Theorem 2. By Theorem 1, given $e\in(C_\sigma)$ there exists an operator L, one-dimensional perturbation of a singular unitary operator, such that any point in e is its eigenvalue of infinite multiplicity. Without loss of generality, we can assume that L is completely nonunitary, i.e. it has no reducing subspaces on which it is unitary. (Otherwise, L is the direct sum of a unitary and a completely nonunitary operators, and we can take the latter instead of L. Obviously, all the required properties would persist.) Such an operator admits a representation

$$L=T+\Omega_* A\Omega^*$$

where T is a completely nonunitary partial isometry with two-dimensional defect subspaces $\mathfrak{D}=\text{im}\,(I-T^*T)$ and $\mathfrak{D}_*=\text{im}\,(I-TT^*)$, $\Omega\colon \mathbf{C}^2\to\mathfrak{D}$ and $\Omega_*\colon \mathbf{C}^2\to\mathfrak{D}_*$ are some unitary operators and A is a (2×2)-matrix, cf. [4] §3.

Let Θ denote the characteristic function of T. Since T is a finite-dimensional perturbation of a singular unitary operator, Θ is an inner function, see [4] §§ 5 and 6. Since T is partially isometric, $\Theta(0)=0$. We shall replace T by its functional model [5]. Thus we shall assume that T acts on $K_\Theta \equiv H^2(\mathbf{C}^2)\ominus \Theta H^2(\mathbf{C}^2)$ by the formula $Tf= =Pzf$ where P is the orthogonal projection in $H^2(\mathbf{C}^2)$ onto K_Θ. In this model representation, L is given by

$$Lf = zf-(\Theta-A)x_f, \quad x_f = \langle z\Theta^* f, 1\rangle \in \mathbf{C}^2. \tag{2}$$

Lemma. *Let the operator L be defined on K_Θ by* (2). *If* $\zeta\in\mathbf{T}$, $n\in\mathbf{N}$ *and* $\ker (L-\zeta I)^n \neq \ker (L-\zeta I)^{n-1}$, *then*

$$(z-\zeta)^{-n}\det(\Theta-A)\in H^2. \tag{3}$$

Proof. If $f\neq 0$ is in $\ker(L-\zeta I)$, then $(z-\zeta)f=(\Theta-A)x_1$ for some $x_1\neq 0$ in $\mathbf{C}^2$. Hence, $(z-\zeta)^{-1}(\Theta-A)x_1=f\in H^2(\mathbf{C}^2)$. If $f\in\ker(L-\zeta I)^2\setminus\ker(L-\zeta I)$, then, for some $x_1, x_2\in\mathbf{C}^2$,

$$(z-\zeta)f-(\Theta-A)x_2 = (L-\zeta I)f = (z-\zeta)^{-1}(\Theta-A)x_1, \quad x_1\neq 0,$$

and

$$(\Theta-A)[(z-\zeta)^{-2}x_1+(z-\zeta)^{-1}x_2]\in H^2(\mathbf{C}^2).$$

Proceeding by induction, we obtain

$$(\Theta-A)[(z-\zeta)^{-n}x_1+\ldots+(z-\zeta)^{-1}x_n]\in H^2(\mathbf{C}^2), \quad x_1\neq 0.$$

Let V be an analytic matrix-function such that $V(\Theta-A)=[\det(\Theta-A)]I$. Then

$$(z-\zeta)^{-n}\det(\Theta-A)[x_1+\ldots+(z-\zeta)^{n-1}x_n]\in H^2(\mathbf{C}^2).$$

Because of $x_1\neq 0$, we have (3).

Now we are able to complete the proof of Theorem 2. Let δ_1 denote $\det(\Theta-A)$. By the established lemma, the function $(z-\zeta)^{-n}\delta_1$ belongs to H^2 for any $\zeta\in e_0$ and $n\in\mathbf{N}$. Fix a positive number C greater than $\sup|\delta_1|$. Then $h_1\equiv\delta_1+C$ is an outer function. Let δ denote the inner function $\det\Theta$ and h the function $\det(I-A^*\Theta)+C\delta$. We have $h_1=\delta\bar h$. Consider the inner-outer factorization $h=h_i h_0$ of the function h. Since $|h_1|=|h_0|$, we can assume that $h_1=h_0$. Hence $\delta h_i^{-1}=h_1\bar h_1^{-1}$ and

$$\delta-h_i = h_i\bar h_1^{-1}(\delta_1-\bar\delta_1).$$

Therefore, $(z-\zeta)^{-n}(\delta-h_i)$ is in H^2 for all $\zeta\in e$ and $n\in\mathbf{N}$. It remains to observe that $\delta\not\equiv h_i$. Indeed, if it were not so then the last equality would imply that $\delta_1\equiv\text{const}$ $(\neq 0)$ and $(z-\zeta)^{-1}\delta_1\notin H^2$ for any $\zeta\in\mathbf{T}$. The assertion now follows with $\varphi_1=\delta$ and $\varphi_2=h_i$.

Acknowledgement. The author expresses his gratitude to the referee for many useful remarks.

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Addendum to "The lattice variety $\mathbf{D} \circ \mathbf{D}$"

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In our paper, this Journal, vol. **51** (1987), pp. 73—80, the Corollary to Theorem 4 in Section 3 (referred to in the Introduction) was inadvertently left out.

Corollary. *Let P be a set of odd prime numbers. Let $\mathbf{M}_P$ denote the set of all modular lattices not containing any finite projective geometry over $GF(p)$ as a sublattice where $p \in P$. Then $\mathbf{M}_P$ is a lattice variety closed under gluing. There are continuumly many distinct varieties of the form $\mathbf{M}_P$. Thus, there are continuumly many lattice varieties $\mathbf{V}$ such that $\mathbf{V} \circ \mathbf{D}$ is a variety.*

Proof. R. Freese (see reference [1] in our paper) proved that, in the class of modular lattices, any finite projective geometry over $GF(p)$ is projective. It follows immediately, that $\mathbf{M}_P$ is a variety, and $\mathbf{M}_P$ obviously determines P.

$\mathbf{M}_P$ is closed under gluing. Indeed, if L is formed by gluing $A \in \mathbf{M}_P$ and $B \in \mathbf{M}_P$ over S (S is a dual ideal of A, and an ideal of B) and L contains the finite projective geometry G, then we can assume that the zero, 0, of G is in $A-B$ while the unit, 1, of G is in $B-A$. If two of the atoms of G are in B, then so is their meet, 0, a contradiction. So all but one of the atoms of G must be in A, and then so is their join, 1 a contradiction. Thus $\mathbf{M}_P$ is a lattice variety closed under gluing, and by Theorem 4 of our paper, $\mathbf{M}_P \circ \mathbf{D}$ is a variety. This completes the proof of the Corollary.

We would like to point out a misprint: in Section 4 (p. 80), "Theorem 4" should read "Theorem 5".

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Bibliographie

Amorphous Polymers and Non-Newtonian Fluids, Edited by Constantine Dafermos, Jerry L. Ericksen and David Kinderlehrer (The IMA Volumes in Mathematics and its Applications, Volume 6), XII+195 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

This, and the preceding IMA Volumes 2 and 4 are in part proceedings of a series of IMA workshops held during 1984—85 on Continuum Physics and Partial Differential Equations. The book includes 10 separate papers, clustered mainly around concepts, models and mathematical problems in the theory of viscoelastic flow of polymers. There is a brief introduction to the kinetic theory of polymeric liquids in order to show the kinds of differential equations that arise for the configuration-space distribution functions. The aim of the second paper is to study Lagrangian concepts which can be of use in the finite element simulation of viscoelastic flows. The main result of the paper on Solutions with Shocks for Conservation Laws is contained in a proposition, which states that when the "memory" response is appropriately dissipative then the total variation of the solution is bounded independently of the variation of the initial data. The initial value problem of the motion of linear and nonlinear viscoelastic materials are discussed with special emphasis on the development and smoothing of singularities.

This monograph level book is of interest to mathematicians and physicists interested in the continuum physics and the applications of partial differential equations.

I. K. Gyémánt (Szeged)

Automata, Languages and Programming (Proceedings, Karlsruhe, 1987). Edited by T. Ottmann (Lecture Notes in Computer Science, 267), X+565 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1987.

This book contains the presentations of the 14th International Colloquium on Automata, Languages and Programming (ICALP 87) held at the University of Karlsruhe, from July 13 to July 17, 1987.

ICALP 87 is a broadly based conference covering all aspects of Theoretical Computer Science including topics like Algorithms and Data Structures, Automata and Formal Languages, Computability and Complexity Theory, Semantics of Programming Languages, Program Specification, Transformation and Verification, Theory of Data Bases, Logic Programming, Theory of Logical Design and Layout, Parallel and Distributed Computation, Theory of Concurrency, Symbolic and Algebraic Computation, Term Rewriting Systems, Cryptography and Theory of Robotics.

These proceedings consist of three invited papers and 46 contributed ones. The list of invited addresses is: J. Karhumäki: On Recent Trends in Formal Language Theory; J. T. Schwartz and

M. Sharir: On the Bivariate Function Minimization Problem and its Applications to Motion Planning; L. G. Valiant: Recent Developments in the Theory of Learning.

This well edited volume presents the state of art in Theoretical Computer Science. It is recommended for everybody interested in the latest results of the field.

S. Vágvölgyi (Szeged)

E. Behrends, Maß und Integrations theory (Hochschultext), VII+260 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

The text is divided into five chapters. Chapter 1 is concerned with the basic concepts of measure and integral theory. The theorem on measure extension is proved and at the end of the chapter the integral is defined. Chapter 2 deals with the fundamental theorems of measure and integral theory. The convergence theorems, the Radon—Nikodym theorem are proved. Also the product of measures, the Fubini theorem, and the Hahn and Jordan decompositions are given. Chapter 3 introduces the Lebesgue—Stieltjes measures in R^n and characterizes the functions which are integrable in Riemannian sense. Chapter 4 is devoted to the description of the L^p spaces and their dual spaces. The final Chapter 5 deals with measures in topological spaces, contains the Riesz representation theorem and characterizes the dual space of the space of continuous functions defined on a compact space. Two short Appendices are concerned with the analytic sets and with the projections of Borel sets.

László Gehér (Szeged)

B. Benninghofen—S. Kemmerich—M. M. Richter, Systems of Reductions (Lecture Notes in Computer Science, 277), VII+265 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1987.

Recently there has been considerable interest in rewriting systems because of their applications to theorem proving, specifications of abstract datatypes, algebraic simplification, etc.

Most of the results in these notes were obtained in the years after 1978 at the Technical University of Aachen. The last part of this book was written by F. Otto, the material is a part of his Habilitationsschrift at the University of Kaiserslautern.

"There are two main lines of research here. On the one hand one studies the completion algorithm and searches for criteria which ensure its termination. As the completion algorithm in many (one is tempted to say 'most') cases fails to terminate this leads to the investigation of infinite systems. In many cases these can be finitely described and are as useful as finite systems.

The other type of investigations is concerned with the use of complete systems. A complete system certainly provides an answer to the word problem but unravels much more of the structure of the algebra under investigation. This turns out to be most apparent in the case of groups."

Titles of the chapters describe well the topics involved: I. General Concepts from Universal Algebra; II. Finite Sets of Reductions; III. Infinite Sets of Reductions; IV. Automata and Reductions; V. Deciding Algebraic Properties of Finitely Presented Monoids.

This nice book may be recommended to everybody interested in rewriting systems.

S. Vágvölgyi (Szeged)

János Bolyai, Appendix, The Theory of Space, 239 pages, Akadémiai Kiadó, 1987.

The bimillennial hope to deduce Euclid's Fifth Postulate from the remaining part of his foundations vanished ultimately when, in the twenties of the last century, J. Bolyai, Lobachevsky, and

Gauss simultaneously and independently "have created another world, a new world of nothing": the world of non-Euclidean geometries, the Fifth Postulate is not valid in.

The words between quotation marks are taken from a letter written by János Bolyai, a 21 year old Hungarian military engineer. The youngest of the great trinity, who started to elaborate his new geometry in 1823, and, although he lived further 37 years, his fate is commensurable with that of Évariste Galois. Really, during his life, his discovery received no appreciation, and he died with the dreadful sense of complete indifference and incomprehension from the side of his native country and of scientific community.

This book is a facsimile edition of J. Bolyai's pioneering work, which appeared as an appendix to his father's mathematical textbook in 1832. It contains also the English translation of the Latin original, and, in a compact and well-readable form, the most important information on the history of Euclid's Fifth Postulate including summaries on the related results of Gauss and Lobachevsky, as well as concise comments on each paragraph of the Appendix. Furthermore, the book comprises a part on how J. Bolyai's work is reflected by subsequent research and how large influence it had on the evolution of mathematics in our century. These additional chapters are written by Prof. F. Kárteszi.

Finally, the reader can also appreciate a supplement due to Prof. B. Szénássy, painting a colorful historical and biographical background to this wonderful scientific breakthrough.

The book is recommended to everybody interested in geometry or history of mathematics. It can also serve as a base for a university course on the foundations of geometry.

Rozália Juhász (Szeged)

I. Borg—J. Lingoes, Multidimensional Similarity Structure Analysis, XIV+390 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

Multidimensional similarity structure analysis (SSA) comprises a class of models that represent the similarity among entities (for example, variables, items, objects, persons, etc.) in multidimensional space to permit one to more easily grasp the interrelations and patterns present in one's data.

The book is divided into the following chapters: Construction of SSA Representations; Ordinal SSA by Iterative Optimization; Monotone Regression; SSA Models, Measures of Fit, and Their Optimization; Three Applications of SSA; SSA and Facet Theory; Degenerate Solutions in Ordinal SSA; Computer Simulation Studies on SSA Multidimensional Unfolding; Generalized and Metric Unfolding; Generalized SSA Procedures; Confirmatory SSA (1); Confirmatory SSA (2); Psysical and Psychological Spaces; SSA as Multidimensional Scaling; Scalar Products; Matrix Algebra for SSA; Mappings of Data in Distances; Procrustes Procedures; Individual Differences Models.

"The book is oriented to both researchers who have little or no previous exposure to data scaling and have no more than a high school background in mathematics and to investigators who would like to extend their analyses in the direction of hypothesis and theory testing or to more intimately understand these analytic procedures. The book is replete with examples and illustrations of the various techniques drawn largely, but not restrictively, from the social sciences, with a heavy emphases on the concrete, geometric, or spatial aspect of the data representations."

J. Csirik (Szeged)

N. Bourbaki, Topological Vector Spaces (Chapters 1—5), VII+364 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

This is the English translation of the original French edition. In the first chapter the notion of topological vector spaces over a valued division ring is introduced, linear varieties and subspaces are defined and properties of metrisable topological vector spaces are given. In Chapter 2 locally convex spaces are considered over the field of real numbers. Here the Hahn—Banach theorem in algebraic and geometric forms can be found, the dual space and weak topologies are introduced and the Krein—Milman theorem is proved. The last paragraph of this chapter deals with topological vector spaces over the field of complex numbers. Chapter 3 introduces the notion of bornology in topological vector-spaces, investigates the spaces of continuous linear mappings. The Banach—Steinhaus theorem and Borel graph theorem are also proved. Chapter 4 is devoted to the study of the duality in topological vector spaces, to the topologies compatible with duality, and the bidual and reflexive spaces and gives compactness criteria. In an appendix fixed points of groups of affine transformations are considered. Chapter 5 contains the elementary theory of Hilbert spaces and some classes of operators in Hilbert spaces. At the end of all chapters a rich collection of exercises can be found.

László Gehér (Szeged)

Nigel P. Chapman, LR Parsing, Theory and Practice, VIII+228 pages, Cambridge University Press, Cambridge—New York—New Rochelle—Melbourne—Sydney, 1987.

Linear time deterministic parsing methods have been widely used in syntax analysis. *LR* parsing, initiated by D. E. Knuth in the mid 60's, seems to be appropriate for most practical problems. This volume successfully brings together the theory and practice of *LR* parsing with emphasis on parser construction and implementation.

The book consists of ten chapters, the first one is providing an introduction with historical notes. Chapter 2 contains the necessary elements of formal languages and automata, including right linear grammars and finite state machines, as well as context free languages and pushdown automata. Chapter 3 is a good introduction to $LR(0)$ and $SLR(1)$ parsing. Chapter 4 starts with a parser oriented definition of $LR(k)$ grammars and provides necessary and sufficient conditions on a grammar to be $LR(k)$ for a given integer k. After discussing the canonical $LR(k)$ parser construction, it culminates in a brief discussion on the relation of $LR(k)$ languages to deterministic context free languages, the complexity of $LR(k)$ parsing, as well as the inefficiency of the canonical $LR(k)$ parser construction. This motivates the need for defining $LALR(k)$ grammars in Chapter 5, an intermediate class between $SLR(k)$ grammars and $LR(k)$ grammars. After a brief account of some aspects of the definition, the second part of Chapter 5 deals with practical *LALR* parser constructions and a general method for *LR* parser construction.

Chapters 6 to 10 are concerned with more or less practical matters, such as data structures, optimization of parser tables, the relation of *LR* parsers to other system components, semantic actions during *LR* parsing, error handling, some extensions of the *LR* technique, and automatic generation of *LR* parsers. Algorithms for computing the reflexive transitive closure of a relation are exploited in the Appendix.

The Bibliography contains more than 100 items relevant to *LR* parsing and gives a good source for further reading. A carefully compiled Index helps guide the reader in looking up notions and notations.

The book is written in a nice style. Numerous examples are worked out. It can be recommended to graduate students and computer scientists with interest in formal languages and/or compiler techniques.

Z. Ésik (Szeged)

Aleksei A. Dezin, Partial Differential Equations (An Introduction to a General Theory of Linear Boundary Value Problems), XII+163 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

One often hears that generalization by abstraction in analysis "does nothing really new and finds no new results". Although there is a valid basis behind this opinion, we can find such generalizations which contain originally new things. As an example we recommend this book.

It is well known that mathematical physics, the study of boundary value problems of partial differential equations is the source of some new notions of analysis. Usually the authors in this branch of mathematics investigate restricted classes of equations. In the wide range of applications new and new problems arise which do not belong to the known types. These suggest the necessity of the more general way of putting the question. Briefly summarizing, this book studies the dependence of the solvability of given linear partial differential equations from the choice of the boundary conditions by using the methods of functional analysis especially the theory of linear operators in Hilbert space. The first two chapters give a concise, clear summary of the main notions and theorems of functional analysis which are necessary in the further study. This was a hard, master's work. The most important part of the book is Chapter 4—6 titled Model Operators; First-Order Operator Equations; Operator Equations in Higher Order. The investigated problems are of fundamental importance and the results are remarkable. The discussion is carried out with elegance and it is a striking example of the interplay between partial differential equations and functional analysis. In order to put the case more clearly several remarks — introductory and concluded ones at the beginning and at the end of some chapters, respectively — make the difficulties, the importance of the theorems clear and constitute a very good reference source for further study.

Nothing can prove better the success of the method applied in this book than the Appendix 2, in which the translater R. P. Boas sums up some results having been achieved in this theme since the first publication of this book in Russian. For experienced reader R. P. Boas' name can be a guarantee as well that this is a good book, otherwise it is not likely that he would have undertaken the translation.

Lajos Pintér (Szeged)

Differential Geometry, Proceedings, Lingby, 1985. Edited by V. L. Hansen (Lecture Notes in Mathematics, 1263), X+288 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

This volume contains the lectures held at the Nordic Summer School that took place at the Technical University of Denmark in Lingby: P. Braam, Quantum field theory: the bridge between the mathematics and the physical world; J. P. Bourguignon, Yang—Mills theory: the differential geometric side; F. Burstall, Twistor methods for harmonic maps; J. Rawnsley, Twistor methods; J. L. Kazdan, Partial differential equations in differential geometry; K. Grove, Metric differential geometry; R. Greene, Complex differential geometry. "The main reason for choosing differential geometry as the subject for the 1985 Nordic Summer School in mathematics was that the last two decades have witnessed a new strong interaction between mathematics and field theories in physics" — the editor writes in the Preface. The lectures have introductory character and present important mathematical tools and results necessary for making research into the applications of differential geometry in physics.

Péter T. Nagy (Szeged)

Differential Geometry and Differential Equations, Proceedings, Shanghai, China, 1985. Edited by Gu Chaohao, M. Berger and R. L. Bryant (Lecture Notes in Mathematics, 1255), XII+243 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

The Sixth Symposium on Differential Geometry and Differential Equations was held from June 21 to July 6, 1985 in Fudan University, Shanghai, China. This volume contains the proceedings of this conference. The topics cover a wide range of differential geometry: global submanifold theory of Riemannian manifolds, extremal surfaces in Minkowski spaces, the imbedding problems of symmetric spaces, the geometric theory of harmonic maps, Lie transformation groups, gauge theory, spectral geometry, etc.

The book gives a good overview of some important fields of differential geometry and makes us acquainted with the scientific activity of high level in this traditional subject in China.

Péter T. Nagy (Szeged)

Z. Ditzian—V. Totik, Moduli of Smoothness (Springer Series in Computational Mathematics, 9), IX+225 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

The subject of this book is the introduction and application of a very useful new type of moduli of smoothness of functions. As the theorems included in the book prove this new measure of smoothness gives a better tool to deal with the rate of best approximation, inverse theorems and imbedding theorems. The fundamental feature of this new modulus is the replacement of h in $\omega^r(f; t) = \sup_{0<h\leqq t} \|\Delta_h^r f\|$ by $h\cdot\varphi(x)$ to obtain $\omega_\varphi^r(f; t)_p = \sup_{0<h<t} \|\Delta_{\varphi h} f\|_{L^p}$ where the choice of $\varphi(x)$ is depending on the problem that has to be solved.

Here we pick up just three advantages of this new modulus. The first one is that it can easily be used to characterize the particular class of functions for which more smoothness is required inside the interval than near its endpoints (see especially the cases of weighted polynomial approximation in L_p). The new modulus furthermore is suitable to solve some basic problems in approximation theory related to the characterization of the class of functions defined by the rate of approximation by known operators (for example by the Kantorovich operators). The third fact that should be noted is that this new modulus plays very important role in the theory of interpolation spaces (for example in the problem of characterization of K-functionals introduced by J. Peetre for investigation of interpolation spaces between two Banach spaces). The book is divided into two parts and thirteen chapters. In Part I the following investigations are included: equivalence relation of the new modulus with the K-functional; the introduction of the main-part modulus and its relation to ω_φ^r; the extension of all important properties of the classical modulus to the new one; weighted moduli of smoothness. Part II contains the applications for the best polynomial approximation on $[-1, 1]$; for the rate of convergence of various operators; for the best weighted polynomial approximation on R; for the best polynomial approximation on simple polytopes.

The book is well organized, its style is clear. The results are new and complete proofs are given. Certainly this book will be very useful for researchers interested in approximation theory.

József Németh (Szeged)

Functional Analysis II (with contribution by J. Hoffmann—Jørgensen et al.), Edited by S. Kurepa, H. Kraljević and D. Butković (Lecture Notes in Mathematics, 1242), VII+432 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Tokyo, 1987.

This volume contains seven papers. Four of them, essentially lecture notes, are as follows: A. Dijksma, H. Langer and H. de Snoo, Unitary colligations in Krein spaces and their role in the extension theory of isometries and symmetric linear relations in Hilbert spaces; S. Kurepa, Quadratic and sesquilinear forms. Contributions to characterizations of inner product spaces; J. Hoffmann—Jørgensen, The general marginal problem; Z. R. Pop—Stojanović, Energy in Markov processes.

The corresponding four series of lectures were given at postgraduate school and conference on Functional Analysis held from November 3 to November 17, 1985 at the Inter-University Center of Postgraduate Studies, Dubrovnik, Yugoslavia.

The remaining three papers, namely: S. Suljagić, Invariant subspaces of shifts in quaternionic Hilbert space; D. Butković, H. Kraljević and N. Sarapa, On the almost convergence; N. Elezović, p-nuclear operators and cylindrical measures on tensor products of Banach spaces; are connected with one-hour lectures presented at the same school and conference.

As the titles of the papers already show, this collection deals with several branches of functional analysis, operator theory and their applications. Beside its expository content it contains also some new results with proofs.

The volume can be useful for postgraduate students, and first of all for researchers interested in one or more topics discussed in it.

E. Durszt (Szeged)

Johan Grasman, Asymptotic Methods for Relaxation Oscillations and Applications (Applied Mathematical Sciences, 63), XIV+221 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

Relaxation oscillations are present in various fields of chemistry and biology. In a typical relaxation oscillation some of the variables may vary rapidly during a short time interval and the others fluctuate regularly. The differential equation models contain a "small parameter". The solution of the reduced system (the system with $\varepsilon=0$) gives the regular approximation, which gives a good impression of the qualitative behaviour of the solution apart from the rapid variation during the short time interval. For the purpose of making a quantitative approximation, expansions with respect to the small parameter are necessary. In this book the author shows that the method of matched asymptotic expansions makes it possible to describe quantitatively phenomena such as chaotic dynamics of physical and biological systems.

In the Introduction examples for phenomena of relaxation oscillation are presented. In Section 2 the definition of a relaxation oscillation and a review of the proofs of existence of periodic solutions of singularly perturbed systems are given, and an asymptotic analysis of the Van der Pol oscillator and of the Volterra—Lotka equations are made. A chaotic relaxation oscillator is constructed, as well. In Section 3 a rigorous theory for the existence of entrained solutions for systems of coupled relaxation oscillators, and interpretation of entrainment phenomena in biological systems are given. In Section 4 asymptotic approximations are constructed for the Van der Pol oscillator with sinusoidal forcing term, and equivalence between solutions and iterates of an interval mapping is established.

Appendices and appropriate references to most recent results complete this book, which is warmly recommended to mathematicians, physicists and biologists interested in applications of the theory of dynamical systems.

I. K. Gyémánt (Szeged)

Hydrodynamic Behavior and Interacting Particle Systems, Edited by George Papanicolaou (The IMA Volumes in Mathematics and Its Applications, Volume 9), VI+215 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

This is the third IMA volume (out of four) with papers presented at a workshop on Stochastic Equations and Their Applications. The workshop was held in 1986 at the Institute for Mathematics and Its Applications at the University of Minnesota. Research of several different directions are contained in these papers. The table of contents: 1. R. E. Caflisch: Stochastic Modelling of a Dilute Fluid-Particle Suspension. 2. P. M. Chaikin, W. D. Dozier and H. M. Lindsay: Experiments on Suspensions of Interacting Particles in Fluids. 3. D. A. Dawson: Stochastic Models of Parallel Systems for Global Optimization. 4. R. Figari, G. Papanicolaou and J. Rubinstein: Remarks on the Point Interaction Approximation. 5. K. F. Freed, S. Wang and J. F. Douglas: Renormalization Group Treatment of the Hydrodynamics of Polymer Chains in the Rigid Body Approximation. 6. J. Fritz: On the Hydrodynamic Limit of a Scalar Ginzburg—Landau Lattice Model: The Resolvent Approach. 7. J. Goodman: Convergence of the Random Vortex Method. 8. L. G. Gorostiza: Supercritical Branching Random Fields. Asymptotics of a Process Involving the Past. 9. D. E. Loper and P. H. Roberts: A Simple Mathematical Model of a Slurry. 10. H. Osada: Limit Points of Empirical Distributions of Vorticies with Small Viscosity. 11. S. Ozawa: Mathematical Study of Spectra in Random Media. 12. J. Rubinstein: Hydrodynamic Screening in Random Media. 13. H. Spohn: Interacting Brownian Particles: A Study of Dyson's Model. 14. A. S. Sznitman: A Propagation of Chaos Result for Burgers' Equation. 15. H. Tanaka: Limit Distributions for One-Dimensional Diffusion Processes in Self-Similar Random Environments.

Introduction to modern mathematical methods is contained in papers 6 and 13. Analytical methods currenty used in the physics and chemistry literature are presented in paper 5. The continuum limit of boundary value problems in regions with small inclusions is analyzed in 4, 11 and 12. In papers 3, 8 and 15 the probabilistic aspects of particle systems on random media are discussed. The vortex method is treated in 7 and 10.

This monograph level book is of interest to researchers in applied mathematics, engineering, and physics.

I. K. Gyémánt (Szeged)

I. M. James, Topological and Uniform spaces (Undergraduate Texts in Mathematics), IX+163 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

The book is divided into 13 chapters. The text starts with a preliminary chapter dealing with certain aspects of the theory of sets. The first two chapters are concerned with some basic axioms, with continuity and with topological product. Also the topological groups are introduced. Subspaces and quotient spaces are considered in Chapter 3. Chapter 4 deals with functions which are structure preserving in the direct image sense. Specifically open and closed functions are considered. In Chapter 5 the notion of compactness is introduced and the characterization of compact spaces in terms of filters is given. Chapter 6 is concerned with the separation axioms and the basic properties of Hausdorff regular and normal spaces are established. Chapter 7 and 8 contain the definition of the uniform spaces with illustrations taken from topological groups and metric spaces, and introduce the uniform continuity of functions and discuss the Cauchy condition both for sequences and for filters. Chapter 10 deals with the two countability axioms. Also σ-compactness, sequential compactness, Lindelöf property and separability are considered. Chapter 11 returns to the separation axioms, furthermore introduces the complete regularity and shows that this property is necessary and sufficient for a topological space to be uniformisable. At the end of this chapter the Urysohn theorem is proved. The last chapter is concerned with completeness and completion of metric and uniform spaces.

László Gehér (Szeged)

J. Lindenstrauss—V. D. Millman, Geometrical Aspects of Functional Analysis (Lecture Notes in Mathematics, 1267), 212 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

The book contains 16 papers the material of which are based on lectures held in the Israel Seminar on Geometrical Aspects of Functional Analysis between October 1985 and June 1986. Most of the papers are based on original research which have not been published elsewhere, the others are of expository nature. The basic topics are: imbedding problems, extension of Lipschitz maps and the study of convex sets in R^n and Banach spaces, which play a central role in the subject.

The book is highly recommended to researchers interested in the Banach space theory.

László Gehér (Szeged)

Moshe S. Livšic—Leonid L. Waksman, Commuting Nonselfadjoint Operators in Hilbert Space (Lecture Notes in Mathematics, 1272), 114 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

The text consists of two independent parts. The first one is written by Livšic, and the second one by Waksman. The first part investigates operator colligations and collective motions of open systems. It turns out that a deep connection between the theory of commuting nonselfadjoint operators and the problem of wave equations can be found. The second part deals with harmonic analysis of multi-parameter semigroups of contractions. Firstly the strongly continuous isometric representations of multi-parameter semigroups $K \subset R^n$ in Hilbert space are considered, and then multi-parameter semigroups of contractions admitting dilations are investigated. In the Appendix triangular models of pairs of commuting operators are given.

László Gehér (Szeged)

Mathematical Ecology. An Introduction, Edited by Thomas G. Hallam and Simon A. Levin (Biomathematics, 17), XII+457 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1986.

"The study of ecology has its roots in the basic investigations of naturalists, who seek to understand the ecological and evolutionary relationships among species and their relationships to their environment. These studies usually have been retrospective, aimed at understanding how the universe we observe came to be. To explain why we see what we see, we must imbed our studies in a broader context, encompassing both what is and what is not. We must abstract and imagine, and construct a feasible world much bigger than reality; only then can we explain why evolution has taken the course it has ... Studies of this sort have been the mainstays of theoretical ecology, and occupy a major portion of this book."

These lecture notes reflect a nucleus of the material from the lectures presented during the first weeks of the Autumn Course on Mathematical Ecology, held at the International Centre for Theoretical Physics, Miramare—Trieste, Italy, November—December 1982.

First of all the ecological and mathematical foundations of the areas of physiological, population, community and ecosystem ecology are introduced in detail in this book. Moreover, some past and current problems are presented in the important fundamental topics. Speculations on possible directions for future research are also contained. Not only the theoretical aspects are explained but also some applied fields are developed.

The book is divided into five parts. Part I contains an overview on ecology by L. J. Gross. Two introductory papers by L. J. Gross on physiological and behavioral ecology are given in Part II. Papers in Part III (by T. G. Hallam, R. M. Nisbet, W. S. C. Gurney, J. C. Frauenthal, S. S. Levin and L. M. Ricciardi) are concerned with the dynamical and stochastic approach to population ecology. Part IV is devoted to the theory of communities and ecosystems. Here the authors are: T. G. Hallam, A. Hastings, S. A. Levin, M. Turelly and R. R. Lassiter. Two topics (resource management and infectious diseases, epidemiology) from applied mathematical ecology are developed and discussed by J. M. Conrad and R. M. May in Part V.

These very well written lecture notes will certainly be interesting and useful for both researchers in these areas and those interested readers wanting to understand the foundations and the basic problems of mathematical ecology.

T. Krisztin (Szeged)

Vladimir A. Marchenko, Sturm—Liouville Operators and Applications (Operator Theory: Advances and Applications Vol. 22), XI+367 pages, Birkhauser Verlag, Basel—Boston—Stuttgart, 1986.

In the various branches of mathematics there exist ever-living problems, inexhaustible sources (see for example the various problems of prime numbers, the solution of equations etc.). In the theory of differential equations such an eternal question is the now so called Sturm—Liouville equation: $y''+q(x)y=zy$ and the allied Sturm—Liouville operator $L=-d^2/dx^2+q(x)$. The first results concerning this equation go back to D. Bernoulli and L. Euler. Since then this equation has been constantly presented in the literature. In the middle of this century the transformation operators appeared in the theory of the Sturm—Liouville equation. As the results of e.g. A. Ya. Povzner, I. M. Gelfand, B. M. Levitan, B. Ya. Levin and V. A. Marchenko show, this tool became more and more important. In the Preface the author says: "The main goal of this monograph is to show what can be achieved with the aid of transformation operators in spectral theory, as well as in its recently revealed untraditional applications." The chapter headings are: The Sturm—Liouville equation and transformation operators; The Sturm—Liouville boundary value problem on the half line; The boundary value problem of scattering theory; Nonlinear equations.

Unfortunately sometimes it happens that mathematicians in the West or mathematicians in the East don't know the results of each other (e.g. language difficulties occur etc.). This book contains such questions in the theme in which the major part of the theorems belong to Soviet mathematicians. Often the original publications are not easily accessible. Perhaps this book helps to solve a part of these problems.

At last I'd like to mention the interesting examples and their hints which in some sense remind the reviewer of the examples being in the world famous book of Pólya and Szegő.

Lajos Pintér (Szeged)

J. M. Montesinos, Classical Tessalations and Three-Manifolds (Universitext), XIII+230 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

This book is devoted to a self-contained study of the interaction between the classical geometry of tessalations in euclidean and non-euclidean spaces and the topology of 3-manifolds. The origin of this relationship is the construction of a non-classical homological 3-sphere given by H. Poincaré in addition to the formulation of his famous conjecture in 1904 about the identity of homological and geometric spheres in higher dimensions. The homological 3-sphere can be interpreted as the manifold of positions of a dodecahedron inscribed in a 2-sphere. Similarly, the configuration spaces of platonic solids give interesting examples of other 3-manifolds. As the author says: "This is the type of topic we deal with in this book, only that instead of restricting our attention to the dodecahedron, we also consider the remaining platonic solids, and the euclidean and hyperbolic tessalations for which analogous constructions of three-manifolds can be developed in a similar way. At this stage one might also ask what can be considered new here. In fact, there is nothing new except the point of view. What I had in mind in writing this book was to use these constructions as a "pretext" for talking about three-manifolds and teaching geometrical intuition, which is crucial in forming our students to be able to make new discoveries in mathematics."

Really, the original and new view-point and entertaining style of this very nice book with numerous exercises, problems and illustrations yield a very good introduction to the intuitive geometry and topology. It can be highly recommended to graduate students and researchers interested in these fields.

Péter T. Nagy (Szeged)

V. V. Nikulin—I. R. Shafarevich, Geometries and Groups (Universitext), VI+251 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

The expression "geometrical" is used everywhere in both the theoretical and applied sciences without a well-defined sense. It means something visuable thing having analogue with the structure of the physical space. But different models of physical space are formulated using various mathematical notions: classical axiom systems of elementary geometry, discrete and continuous transformation group theory, classical differential geometry, manifold theory, surface topology etc. This excellent book, which is a translation of the Russian edition (Nauka, Moscow, 1983), gives an elementary introduction into intuitive geometry, based on a unification of the above approaches from the view-point of modern mathematics.

In Chapter I the authors formulate the main problems and illustrate them by the Euclidean description of the geometry on the sphere, cylinder and torus. Chapter II contains the classification of 2-dimensional locally Euclidean geometries: the plane, cylinder, torus, twisted cylinder and Klein bottle. The proof uses the description of uniformly discontinuous motion groups on the plane and an elementary introduction into the covering space construction. Chapter III is devoted to the space geometry and the crystallographic group theory. In the final Chapter IV there is given a treatment of lattice geometry and an introduction into Bolyai—Lobachevsky geometry using complex numbers and some modular group theory.

The book contains many exercises, hystorical remarks, very good illustrative figures and references for further study. Only familiarity with the knowledge of school mathematics is supposed. Certainly this book is very interesting and useful for mathematicians (both students and teachers) and non-mathematicians interested in the development of the sciences.

Péter T. Nagy (Szeged)

D. H. Pitt—A. Poigné—D. E. Rydeheard, Category Theory and Computer Science (LNCS, 283), V+300 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1987.

This volume is the proceedings of the Conference on Category Theory and Computer Science held in Edingburgh, September 7—9, 1987. Most papers reflect the fact that logical aspects of category theory have become the main issue in category theory applied to computer science.

Contributions are: G. Rosolini: Categories and Effective Computations; A. M. Pitts: Polymorphism is Set Theoretic, Constructively; Th. Coquand, Th. Ehrhard: An Equational Presentation of Higher Order Logic; S. Kasangian, A. Labella, A. Pettorossi: Enriched Categories for Local and Interaction Calculi; D. B. Benson: The Category of Milner Processes is Exact; G. Winskel: Relating Two Models of Hardware; D. E. Rydeheard, J. G. Stell: Foundations of Equational Deduction: A Categorical Treatment of Equational Proofs and Unification Algorithms; T. Hagino: A Typed Lambda Calculus with Categorical Type Constructors; L. S. Moss, J. Meseguer, J. A. Goguen: Final Algebras, Cosemicomputable Algebras, and Degrees of Unsolvability; G. Bernot: Good Functors ... are Those Preserving Philosophy; C. Beierle, A. Voss: Viewing Implementations as an Institution; S. Martini: An Interval Model for Second-Order Lambda Calculus; E. Robinson: Logical Aspects of Denotational Semantics; M. Proietti: Connections Between Partial Maps Categories and Tripos Theory; S. Vickers: A Fixpoint Construction of the p-adic Domain; J. M. McDill, A. C. Melton, G. E. Strecker: A Category of Galois Connections.

The volume will be useful to specialists interested in category theory or categorical aspects of computer science.

Z. Ésik (Szeged)

W. Purkert—H. J. Ilgauds, Georg Cantor (Vita Mathematica 1), 262 pages, Birkhäuser Verlag, Basel—Boston—Stuttgart, 1987.

This new series "Vita Mathematica" of the Birkhäuser Verlag is in some sense a continuation of the 16 "supplements" to the journal Elemente the Mathematik published by the Birkhäuser Verlag between 1947 and 1980. The difference from the supplement is not only formal (considerably greater length in book form). The aim of the new series is to present technical biographies of great mathematicians from antiquity to modern times, taking into account relevant research carried out in recent decades. In the forthcoming volumes we will read on Pascal, Dirichlet, Felix Klein and Euler among others.

The last (the sixth) part of Cantor's fundamental work "Über unendliche lineare Punktmannichfaltigkeiten" appeared in the Mathematischen Annalen about 100 years ago. This was the birth of Set Theory with an essentially new approach to the infinity in mathematics which was embodied in the theory of transfinite numbers. D. Hilbert described it as "the most marvellous flower of the mathematical spirit and really one of the highest achievements, pure reasonable human activity".

In the two first (and short) chapters of the book (written in German) we can read on Cantor's childhood and his studies in Zürich, Göttingen and Berlin. The third and main chapter is the "Genesis der Mengenlehre" (The Genesis of Set Theory). In the subsequent chapters we can read on Cantor's illness, on his personality and philosophy, on the antinomies and his final years.

A subsequent chapter deals with researches due to Zermelo, Hilbert and others which were striven to avoid the antinomies.

The book ends with numerous documents (letters to and from Cantor), a chronology and a detailed bibliography.

Finally, we cite the last paragraph of the Editorial of the book: "May the series Vita Mathematica help to promote interest in the history of science in our time when consciousness of history is deficient and decline in the use of language is evident. Thereby we may contribute in a small way to our culture."

We hope that the forthcoming volumes of this series will serve this aim as good as the first excellent one.

Lajos Klukovits (Szeged)

H. Riesel, Prime Numbers and Computer Methods for Factorization (Progress in Mathematics, 57) XVI+464 pages, Birkhäuser, Boston—Basel—Stuttgart, 1987 (revised and corrected second printing).

Applications of number theory have growing interest nowdays. It can be used in several areas of science and engineering, e.g., in communications, coding theory and cryptology.

In number theory there are several easily formulated problems, solutions of which are rather advanced. The author's aim is to write a book on this topic suitable for mathematically inclined layman, as well as for a more advanced student. For this reason not all results are proved, but there are detailed bibliographical references to serve the readers interested in the proofs. There are references to recent original works as well.

The main text has six essentially independent chapters. The Number of Primes Below a Given Limit; The Primes Viewed at Large; Subleties in the Distribution of Primes; The Recognition of Primes; Factorization, Prime Numbers and Cryptography.

While number theory is a small part of the basic mathematical courses only, the book has six additional chapters (appendices) which contain all the algebra and number theory (basic concepts in higher algebra and arithmetic, quadratic residues, arithmetic of quadratic fields, continued fractions, algebraic fractions) required for the main text. There are another three appendices, two on computational questions (multiple-precision arithmetic, fast multiplication of large integers) and one on the Stieltjes integral.

For those readers who have accesses to computers, the author has provided computer programs written in the high-level programming language PASCAL for many of the methods (and algorithms) described in the text.

At the end of the book a large amount of results are collected in 34 tables, e.g., primes below 12 553 and between 10^n and 10^n+1000 $(n=5, 6, \ldots, 15)$, factors of Fermat numbers and of Mersenne numbers, factors of integers of types a^n+b^n for some small a and b, quadratic residues.

This carefully written and excellently printed book will be enjoyed by both mathematicians and non-mathematicians, everybody who are interested in number theory and its applications.

Lajos Klukovits (Szeged)

K. P. Rybakowski, The Homotopy Index and Partial Differential Equations (Universitext), IX+208 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

This book grew out of lectures held by the author at various universities. Recently the homotopy index theory has become a useful tool in perturbation problems involving ordinary differential equations. The homotopy index generalizes the Morse index, it was developed by Ch. Conley for twosided flows on compact spaces. It was a natural thing to try the application of the theory for partial differential equations as well. But this problem requires further extension of the homotopy index theory. This was done by the author who published it previously in several papers. This book is a clear presentation of the results written not only for experienced researchers but for readers having only modest knowledge of algebraic topology.

The book consists of three chapters. In Chapter 1 the author presents the main concepts of the categorial Morse index and the homotopy index. This chapter is especially useful for beginners in this field. Several examples make the introduced notions more understandable. In Chapter 2 applications are given on parabolic partial differential equations and on functional differential equations. The third, relatively brief, chapter contains selected topics.

This is an interesting book on the application of a modern notion promising further new results.

Lajos Pintér (Szeged)

Masahiro Shiota, Nash Manifolds (Lecture Notes in Mathematics, 1269), VI+223 pages, Springer-Verlag, Berlin—Heidelberg—New York—London, 1987.

The purpose of this book is to construct a theory of real manifolds equipped with "algebraic" structures. The fundamental ideas are the following:

A semialgebraic subset of R^n is by definition a finite union of sets of the form

$$\{x \in ER^n\colon f_1(x) = \ldots = f_k(x) = 0,\ g_1(x) > 0, \ldots, g_k(x) > 0\},$$

where $f_1, \ldots, f_k, g_k$ are polynomials. (For example a compact polyhedron in R^n.) A C^r map between two semialgebraic subset of R^n and R^m is C^r Nash-map if its graph is semialgebraic in $R^n \times R^m$. A C^r manifold with a finite system of coordinate neighbourhoods $\{\psi_i\colon U_i \to R^m\}$ is a C^r Nash-manifold of dimension m if for each i and j, $\psi_i(U_i \cap U_j)$ is an open semialgebraic subset of R^m and the map $\psi_j \circ \psi_i^{-1}$ is a C^r Nash-diffeomorphism.

The main result of this subject has been proved by Nash. Namely he showed that a compact C^1 manifold M can be imbedded in a Euclidean space R^n and such a C^ω Nash-manifold structure on M is unique up to C^ω Nash-diffeomorphism. Hence we can endow a compact C^1 manifold with "algebraic" properties, which appears to contribute to differential topology. Really, there are several applications of this result.

This book is clearly and accurately written. Certainly it will be interesting for researchers working in differential topology, *PL* topology or Nash-manifold.

Árpád Kurusa (Szeged)

Trends, Techniques, and Problems in Theoretical Computer Science (Selected Contributions, Smolenice, Czechoslovakia, 1986), Edited by A. Kelemenová and J. Kelemen (Lecture Notes in Computer Science, 281), VI+213 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1987.

This volume contains a selected collection of papers presented at the scientific programme of the Fourth International Meeting of Young Computer Scientists (IMYCS 86) held at Smolenice Castle Czechoslovakia, October 13—17, 1986.

"Organized biennially since 1980, the meetings are intended to stimulate the scientific activity of beginners in computer science, mainly that of both university students in the final years of their studies and of graduates. Therefore, the scientific programme of the meetings include tutorials and more invited lectures than it is usual at conferences."

In this book the texts of the tutorial of IMYCS 86 as well as the texts of all invited talks are included together with some selected short communications presented during the meeting's regular and informal evening sessions. Thematically, the volume is divided into four chapters:

Chapter 1. VLSI and Formal Languages: J. Hromkovič: Lower bound techniques for VLSI algorithms; J. Karhumäki: The equivalence of mappings on languages; J. Sakarovitch: Kleene's theorem revisited; Z. Tuza: Some combinatorial problems concerning finite languages.

Chapter 2. Theory of Formal Grammars: E. Csuhaj—Varjú: A connection between descriptional complexity of context-free grammars and grammar form theory; H. C. M. Kleijn: Basic ideas of selective substitution grammars; G. Păun: Some recent restrictions in the derivation of context-free grammars.

Chapter 3. Biologically Motivated Structures: V. Aladyev: Recent results on the theory of homogeneous structures; M. Král'ová: A note on the ratio function in DOL systems; A. Lindenmayer: Models for multicellular development: Characterization, inference and complexity of *L*-systems.

Chapter 4. Artificial Intelligence: J. Kalaš: A formal model of knowledge-based systems; F. N. Springsteel: Basic complexity analysis of hypothesis formation; P. Szeredi: Perspectives of logic programming.

We warmly recommend this interesting volume to everybody who works in Theoretical Computer Science.

S. Vágvölgyi (Szeged)

W. Van Assche, Asymptotics for Orthogonal Polynomials (Lecture Notes in Mathematics, 1265), VI+201 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

Recently there has been a great deal of interest in the theory of orthogonal polynomials. The number of books dealing with the subject, however, is limited. This monograph contains some results on the asymptotic behaviour of orthogonal polynomials when the degree tends to infinity. Only a basic knowledge of real and complex analysis is assumed. In Chapter 1 the asymptotic behaviour of orthogonal polynomials on a compact set is discussed. Results are given for orthogonal polynomials on the interval $[-1, 1]$ especially those belonging to the Szegő class. In Chapter 2 among others recurrence relations are given for the orthogonal polynomials in the case when the recurrence coefficients are asymptotically periodic. In Chapter 3 a new method based on well-known theorems of probability theory is given to obtain asymptotic formulas for sequences of polynomials. Chapter 4 is devoted to study the orthogonal polynomials on infinite intervals. The results involve the zero distribution for orthogonal polynomials with exponential weights (asymptotic results for the largest zeros, for the leading coefficient are given). Chapter 5 deals with some consequences of the existence of the asymptotic zero distribution. In the final Chapter 6 some applications of the theory given in the previous chapters can be found.

The book is warmly recommended to both researchers and graduate students interested in approximation theory, orthogonal polynomials and mathematical physics.

József Németh (Szeged)

Joachim Weidmann, Spectral Theory of Ordinary Differential Operators (Lecture Notes in Mathematics, 1258), VI+303 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

This volume presents a general and rather complete spectral theory for selfadjoint ordinary differential operators with motivations and some applications in physics. The generating differential expressions are of order n, operate on $\mathbf{C}^m$-valued functions ($n, m \in \mathbf{N}$), and are sufficiently general in order to cover the "classical" cases.

The selfadjoint realizations in certain L^2 spaces of the considered differential expressions are based essentially on the notion of quasi derivatives and a quite general existence and uniqueness theorem for first order systems. The discussion of the induced selfadjoint operators starts with the determination of the maximal and (closed) minimal ones (denoted by T and T_0, respectively). Then the deficiency indices and the selfadjoint extensions of T_0 are studied, mainly by means of the boundary conditions of the solutions of $(\tau-\lambda)u=0$ (τ is the generating differential expression). For these extensions a spectral theory is developed: the general forms of the resolvent, the spectral representation and the spectral resolution are studied, the spectral multiplicity and the absolute continuous spectrum is discussed. Attention is paid to differential operators with periodic coefficients. An oscillation theory is developed for Sturm—Liouville operators and Dirac systems, and this is applied in studying their spectral properties. Finally explicite solutions are given for some problems concerned with special cases of Sturm—Liouville operators and Dirac systems.

Mathematicians or physicists, postgraduate students and researchers will certainly find the generality of the treatise as well as the many-sided discussion to be of interest. The book contains also new results; its method is functional analytic whenever possible. The reader has to be familiar with basic facts of analysis and needs some knowledge of the abstract theory of selfadjoint operators.

E. Durszt (Szeged)

Marisa Venturi Zilli, Mathematical Models for the Semantics of Parallelism (Lecture Notes in Computer Science, 280), IV+231 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1987.

The volume contains eight papers from the material presented at the Advanced School on Mathematical Models for the Semantics of Parallelism, Rome, September 24—October 1, 1986. The papers discuss diverse approaches to concurrent systems.

Table of contents:

L. Aceto, R. De Nicola and A. Fantechi: Testing equivalences for event structures, p. 1—20. Three extensional models of concurrency are defined in the common framework of event structures. These models correspond to different kinds of observations: sequences of actions, sequences of multisets of actions, and partial orderings of actions. Some basic relationships are established.

P. America and J. de Bakker: Designing equivalent semantic models for process creation, p. 21—80. This long paper provides a detailed analysis of certain models for concurrent languages with process creation. The languages fall into four categories according to their uniform/nonuniform and static/dynamic nature. The models are defined in metric structures involving either linear or branching time semantics.

E. Astesiano and G. Reggio: An outline of the SMoLCS approach, p. 81—113. The paper elaborates a methodology for the specification of concurrent systems and languages. The methodology has both algebraic and denotational flavour.

M. Broy and T. Streicher: Views of distributed systems, p. 114—143. This is a rather informal paper on the various issues on distributed systems, focusing around the notion of a process, sequentiality, functionality, and some aspects of semantics.

P. Degano, R. De Nicola and U. Montanari, CCS is an (augmented) contact free C/E system, p. 144—165. It is shown how Milner's CCS can be modeled by a class of Petri nets in a way which corresponds to the original interleaving semantics.

J.-Y. Girard: Linear logic and parallelism, p. 166—182. An informal paper on the relevance of a kind of intuitionistic logic to concurrent computations.

A. Labella and A. Pettorossi: Universal models in categories for process synchronization, p. 183—198. Processes are defined as objects of a category with morphisms labelled by the elements of a free monoid. The notion of synchronization is then captured by that of a functor. The category of (synchronization) trees is related to behaviours of processes.

G. Mirkowska and A. Salwicki: On axiomatic definition of Max-model of concurrency, p. 199—230. The admissible parallel executions of a concurrent program are shown to provide an optimal Kripke model of a set of model formulas determined by the program itself.

The volume can be recommended to graduate students and researchers with interest in concurrency.

Z. Ésik (Szeged)

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