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THE HYPER ORDER AND FIXED POINTS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we obtain a precise estimation of the hyper order of solutions for a class of higher order linear differential equation, and also investigate the exponents of convergence of the fixed points of solutions and their first derivatives for the second order case. These results generalize the results of Gundersen-Steinbart, Wittich and Chen-Shon.

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1. Introduction

In this paper, we will use standard notations from the value distribution theory of meromorphic functions (see [15] [20]). We suppose that $f(z)$ is a meromorphic function in whole complex plane $\mathbf{C}$. In addition, we denote the order of growth of $f(z)$ by $\sigma(f)$, and also use the notation $\sigma_2(f)$ to denote the hyper-order of $f(z)$, defined by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

To give the precise estimate of fixed points, we define the exponent of convergence of fixed points by $\tau(f)$

$$\tau(f) = \limsup_{r\to\infty} \frac{\log N(r, \frac{1}{(f-z)})}{\log r},$$

and also the hyper-exponent of convergence of (distinct) fixed points by $\tau_2(f)(\overline{\tau}_2(f))$

$$\tau_2(f) = \limsup_{r\to\infty} \frac{\log\log N(r, \frac{1}{(f-z)})}{\log r},$$

$$\overline{\tau}_2(f) = \limsup_{r\to\infty} \frac{\log\log \overline{N}(r, \frac{1}{(f-z)})}{\log r}.$$

Recently, many scholars devoted to investigating the growth of solutions of complex differential equations, see [1-3, 5-9, 11, 12, 16-19, 21].

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Consider the second order homogeneous linear periodic differential equation

$$f'' + P(e^z)f' + Q(e^z)f = 0, \tag{1.1}$$

where $P(z)$ and $Q(z)$ are polynomials in z and not both constants. It is well known that every solution of f is an entire.

Suppose $f \not\equiv 0$ is a solution of (1.1) and if f satisfies the condition

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0, \tag{1.2}$$

then, we say that f is a nontrivial subnormal solution of (1.1).

Wittich [17] investigated the subnormal solution of (1.1), and obtained the form of all subnormal solutions in the following theorem.

Theorem A. If $f(\not\equiv 0)$ is a subnormal solution of (1.1), then f must have the form

$$f(z) = e^{cz}(h_0 + h_1e^z + \cdots + h_m e^{mz}) \tag{1.3}$$

where $m \geq 0$ is an integer and $c, h_0, \cdots, h_m$ are constants with $h_0 \neq 0$ and $h_m \neq 0$.

Gundersen and Steinbart [12] refined Theorem A and got the following theorem.

Theorem B. Under the assumption of Theorem A, the following statements hold.
(i) if $\deg P > \deg Q$ and $Q \not\equiv 0$, then, any subnormal solution $f \not\equiv 0$ of (1.1) must have the form

$$f(z) = \sum_{k=0}^{m} h_k e^{-kz},$$

where $m \geq 1$ is an integer and $h_0, h_1, \cdots, h_m$ are constants with $h_0 \neq 0$ and $h_m \neq 0$.

(ii) if $\deg P \geq 1$ and $Q = 0$, then any subnormal solution of equation (1.1) must be a constant,
(iii) if $\deg P < \deg Q$, then the subnormal solution of equation (1.1) is $f = 0$.

Chen and Shon [6] investigate more general equation than (1.1), and get the following theorem.

Set

$$a_j(z) = a_{jd_j}z^{d_j} + a_{j(d_j-1)}z^{d_j-1} + \cdots + a_{j1}z + a_{j0} \tag{1.4}$$

$$b_k(z) = b_{kd_k}z^{d_k} + b_{k(d_k-1)}z^{d_k-1} + \cdots + b_{k1}z + b_{k0} \tag{1.5}$$

where $d_j \geq 0$, $m_k \geq 0\,(j = 1, \ldots, n,\ k = 1, \ldots, s)$ are integers. $a_{jd_j}, \ldots, a_{j0};\ b_{kd_k}, \ldots, b_{k0}$ are constants. $a_{jd_j} \neq 0,\ b_{kd_k} \neq 0$.

Theorem C. Let $a_n(z), \ldots, a_1(z), b_s(z), \ldots, b_1(z)$ be polynomials and satisfy (1.4) and (1.5), and $a_n(z)b_s(z) \neq 0$. Suppose that

$$P(e^z) = a_n(z)e^{nz} + \cdots + a_1(z)e^z, \quad Q(e^z) = b_s(z)e^{sz} + \cdots + b_1(z)e^z.$$

If $n \neq s$, then every solution $f\ (\not\equiv 0)$ of equation

$$f'' + P(e^z)f' + Q(e^z)f = 0 \tag{1.6}$$

satisfies $\sigma_2(f) = 1$.

For the higher-order linear homogeneous differential equation

$$f^{(k)} + P_{k-1}(e^z)f^{(k-1)} + \cdots + P_0(e^z)f = 0, \tag{1.7}$$

where $P_j(e^z)\,(j = 0, \ldots, k-1)$ are polynomials in z, many papers were devoted to investigate the solutions of (1.7) (see [3] [5] [7] [8] [16]).

In [7] Chen and Shon consider the existence of subnormal solution of (1.7) and obtain the following theorem.

Theorem D. Let $P_j(z)\,(j = 0, \ldots, k-1)$ be polynomials in z such that all constant terms of P_j are equal to zero and $\deg P_j = m_j$, that is,

$$P_j(e^z) = a_{jm_j}e^{m_jz} + a_{j(m_j-1)}e^{(m_j-1)z} + \cdots + a_{j1}e^z,$$

where $a_{jm_j},\ a_{j(m_j-1)},\ \ldots,\ a_{j1}$ are constants and $a_{jm_j} \neq 0$; $m_j \geq 1$ are integers. Suppose that there exists $m_s\ (s \in \{0, \ldots, k-1\})$ satisfying

$$m_s > \max\{m_j : j = 0, \ldots, s-1, s+1, \ldots, k-1\} = m.$$

Then one has the following properties.

(i) If $P_0 \not\equiv 0$, then (1.7) has no nontrivial subnormal solution and every solution of (1.7) is of hyper order $\sigma_2(f) = 1$.

(ii) If $P_0 \equiv \cdots \equiv P_{d-1} \equiv 0$ and $P_d \not\equiv 0\,(d < s)$, then any polynomials with degree $\leq d-1$ are subnormal solutions of (1.7) and all other solutions f of (1.7) satisfy $\sigma_2(f) = 1$.

It is natural to ask the following question: whether the result of Theorem B can be generalized to the higher order case under the condition of Theorem C. In this paper, we first investigate the problem and obtain the following result.

Set

$$a_{jm_i}(z) = a_{jm_i d_{jm_i}} z^{d_{jm_i}} + a_{jm_i(d_{jm_i}-1)} z^{d_{jm_i}-1} + \cdots + a_{jm_i 1} z + a_{jm_i 0} \quad (1.8)$$

where $d_{jm_i} \geq 0 \, (j = 1, \ldots, n)$ are integers, $a_{jm_i d_{jm_i}}, \ldots, a_{jm_i 0}$ are constants, $a_{jm_i d_{jm_i}} \neq 0$.

Theorem 1. Let $a_{jm_i}(z)$ be polynomials and satisfy (1.8). Suppose that

$$P_j(e^z) = a_{jm_j}(z) e^{m_j z} + \cdots + a_{j1}(z) e^z \quad (1.9)$$

where $a_{jm_i}(z) \not\equiv 0$. If there exists an integer $s \, (s \in \{0, \ldots, k-1\})$ satisfying

$$m_s > \max\{m_j : j = 0, \ldots, s-1, s+1, \ldots, k-1\} = m, \quad (1.10)$$

then every nonconstant solution f of equation

$$f^{(k)} + P_{k-1}(e^z) f^{(k-1)} + \cdots + P_0(e^z) f = 0 \quad (1.11)$$

satisfies $\sigma_2(f) = 1$ if one of the following condition holds.

(1) $s = 0$ or 1.
(2) $s \geq 2$ and $\deg a_{0j}(z) > \deg a_{ij}(z) \, (i \neq 0)$.

For almost four decades, a lot of results have been obtained on the fixed points of general transcendental meromorphic function. However, there are few studies on fixed points of differential polynomials generated by solutions of differential equation. In 2000, Z.X.Chen [4] first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second-order linear differential equations with entire coefficients. In this paper, we continue to investigate the relation between the hyper-exponent of convergence of distinct fixed points and the rate of growth of solutions for a higher order case.

Theorem 2. Under the assumption of Theorem 1, if $zP_0(e^z) + P_1(e^z) \not\equiv 0$, then we have every nonconstant solution f of equation (1.11) satisfies

$$\tau_2(f) = \overline{\tau}_2(f) = \sigma_2(f) = 1.$$

In particular, we investigate the exponents of convergence of the fixed points of solutions and their first derivatives for a second order equation (1.6). we will prove the following theorems:

Theorem 3. Let $a_n(z), \ldots, a_1(z), b_s(z), \cdots, b_1(z)$ be polynomials and satisfy (1.4) and (1.5), and $a_n(z)b_s(z) \not\equiv 0$. Suppose that

$$P(e^z) = a_n(z)e^{nz} + \cdots + a_1(z)e^z, \quad Q(e^z) = b_s(z)e^{sz} + \cdots + b_1(z)e^z.$$

If $s \neq n$, then every solution $f\ (\not\equiv 0)$ of equation (1.6) satisfy $\lambda(f-z) = \lambda(f'-z) = \sigma(f) = \infty$ and $\lambda_2(f-z) = \lambda_2(f'-z) = \sigma_2(f)$.

2. Some Lemmas

Lemma 1. ([20]) Let $f_j(z)\,(j = 1, \ldots, n)\,(n \geq 2)$ be meromorphic functions, $g_j(z)\,(j = 1, \cdots, n)$ be entire functions, and satisfy
(1) $\sum_{j=1}^{n} e^{g_j(z)} \equiv 0$;
(2) when $1 \leq j < k \leq n$, then $g_i(z) - g_k(z)$ is not a constant;
(3) when $1 \leq j \leq n, 1 \leq h < k \leq n$, then

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or logarithmic measure. Then, $f_j(z) \equiv 0\,(j = 1, \cdots, n)$.

Lemma 2 Let $P_j(e^z),\ m_j,\ m_s,\ m$ and $a_{ij}(z)$satisfy the hypotheses of Theorem 1. Then equation (1.11) has no nonconstant polynomial solution.
Proof. Suppose that $f_0 = b_n z^n + \cdots + b_1 z + b_0$ ($n \geq 1$, $b_n, \ldots, b_0$ are constants, $b_n \neq 0$) is a nonconstant solution of (1.11).
If $n \geq s$, then $f_0^{(s)} \not\equiv 0$. Substituting f_0 into (1.11) and taking $z = r$, we conclude that

$$\begin{aligned} |a_{sm_s d_{sm_s}}|\ r^{d_{sm_s}}\ & e^{m_s r}|b_n n(n-1)\cdots(n-s+1)|r^{n-s}(1+o(1)) \\ \leq\ & |-P_s(e^z)f_0^{(s)}(z)| \\ \leq\ & |f_0^{(k)}(z)| + |P_{k-1}(e^z)f_0^{(k-1)}(z)| + \cdots + |P_{s+1}(e^z)f_0^{(s+1)}(z)| \\ & + |P_{s-1}(e^z)f_0^{(s-1)}(z)| + \cdots + |P_0(e^z)f_0(z)| \\ \leq\ & Mr^d e^{mr}(1+o(1)). \end{aligned} \tag{2.1}$$

Since $m_s > m$ we see that (2.1) is a contradiction.
Obviously, when $s = 0$ or 1, we can get that the equation (1.11) has no nonconstant polynomial solution from the above process.
If $n < s$, then

$$P_n(e^z)f_0^{(n)}(z) + \cdots + P_0(e^z)f_0(z) = 0. \tag{2.2}$$

Set $\max\{m_i : i = 0, \cdots, n\} = h$. If $m_j < h$, then we can rewrite

$$\begin{aligned} P_j(e^z) = a_{jh}(z)e^{hz} + \cdots + a_{j(m_j+1)}(z)e^{(m_j+1)z} \\ + a_{jm_j}(z)e^{m_j z} + \cdots + a_{j1}(z)e^z \quad (j = 0, \ldots, n), \end{aligned} \tag{2.3}$$

where $a_{jh}(z) = \cdots = a_{j(m_j+1)}(z) = 0$.
Thus we conclude by (2.2) and (2.3) that

$$\begin{aligned}(a_{nh}(z)f_0^{(n)} &+ a_{(n-1)h}(z)f_0^{(n-1)} + \cdots + a_{0h}f_0)e^{hz} + \cdots \\ &+ (a_{nj}(z)f_0^{(n)} + a_{(n-1)j}(z)f_0^{(n-1)} + \cdots + a_{0j}f_0)e^{jz} + \cdots \\ &+ (a_{n1}(z)f_0^{(n)} + a_{(n-1)1}(z)f_0^{(n-1)} + \cdots + a_{01}f_0)e^{z} = 0 \quad (2.4)\end{aligned}$$

Set

$$Q_j(z) = a_{nj}(z)f_0^{(n)} + a_{(n-1)j}(z)f_0^{(n-1)} + \cdots + a_{0j}f_0 \quad (j = 1, \ldots, h). \quad (2.5)$$

Since f_0 and $a_{ij}(z)$ are polynomials, we see that

$$m(r, Q_j) = o\{m(r, e^{(\alpha-\beta)z})\} \quad (1 \leq \beta < \alpha \leq h). \quad (2.6)$$

By Lemma 1 and (2.4)-(2.6), we conclude that

$$Q_1(z) \equiv Q_2(z) \equiv \cdots \equiv Q_h(z) \equiv 0. \quad (2.7)$$

Since $\deg f_0 > \deg f_0' > \cdots > \deg f_0^{(n)}$ and $\deg a_{0j}(z) > \deg a_{ij}(z)$ $(i \neq 0)$, by (2.5) and (2.7), we get a contradiction.

Lemma 3. [11] Let $f(z)$ be an entire function and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then, there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \ldots)$, where $r_n \to \infty$, such that $f^{(k)}(z_n) \to \infty$ and

$$\frac{|f^{(j)}(z)|}{|f^{(k)}(z)|} \leq |z_n|^{(k-j)}(1 + o(1)) \quad (j = 0, \ldots, k-1). \quad (2.8)$$

Lemma 4. [10] Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$, Let $\Gamma = \{(k_1, j_1), \ldots, (k_m, j_m)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geq 0$ for $i = 1, 2, \ldots, m$. Also let $\varepsilon > 0$ be a given constant, then there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi) \setminus E$, then there is constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| \geq R_0$ and for all $(k, j) \in \Gamma$, we have

$$\frac{|f^{(k)}(z)|}{|f^{(j)}(z)|} \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \quad (2.9)$$

Remark 1 Obviously, in Lemma 4, if $\psi \in [0, 2\pi) \setminus E$ is replaced by $\psi \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \setminus E$, then (2.9) still holds.

Lemma 5. [5] Let $f(z)$ be an entire function with $\sigma(f) = \sigma < \infty$. Suppose that there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that for any ray $\arg z = \theta_0 \in [0, 2\pi) \setminus E$, $|f(re^{i\theta_0})| \leq Mr^k$ ($M = M(\theta_0) > 0$ is a constant and $k > 0$ is a constant independent of θ_0). Then $f(z)$ is a polynomial with $\deg f \leq k$.

Lemma 6. [8] Let $A_0, \ldots, A_{k-1}$ be entire functions of finite order. If $f(z)$ is a solution of equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0,$$

then $\sigma_2(f) \leq \max\{\sigma(A_j) : j = 0, \ldots, k-1\}$.

Lemma 7. [9] Let $g(z)$ be an entire function of infinite order with the hyper-order $\sigma_2 = \sigma$, and let $\nu(r)$ be the central index of g. Then,

$$\limsup_{r\to\infty} \frac{\log\log\nu(r)}{\log r} = \sigma_2(g) = \sigma.$$

Lemma 8. [6] Let $f(z)$ be an entire function of infinite order with $\sigma_2 = \alpha\,(0 \leq \alpha < \infty)$, and a set $E \subset [1, \infty)$ have a finite logarithmic measure. Then, there exists $\{z_k = r_k e^{i\theta_k}\}$, such that $|f(z_k)| = M(r_k, f)$, $\theta_k \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $\lim_{k\to\infty}\theta_k = \theta_0 \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $r_k \notin E$, and $r_k \to \infty$, such that (1) if $\sigma_2(f) = \alpha\,(0 < \alpha < \infty)$, then for any given $\varepsilon_1\,(0 < \varepsilon_1 < \alpha)$,

$$\exp\{r_k^{\alpha-\varepsilon_1}\} < \nu(r_k) < \exp\{r_k^{\alpha+\varepsilon_1}\}. \tag{2.10}$$

(2) if $\sigma(f) = \infty$ and $\sigma_2(f) = 0$, then for any given $\varepsilon_2\,(0 < \varepsilon_2 < \frac{1}{2})$, and any large $M\,(> 0)$, we have, as r_k sufficiently large,

$$r_k^M < \nu(r_k) < \exp\{r_k^{\varepsilon_2}\}. \tag{2.11}$$

Lemma 9. [10] Let f be a transcendental meromorphic function, and $\alpha > 1$ be a given constant. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and $i, j\,(i < j(i, j \in \mathbb{N}))$, such that for all z satisfying $|z| = r \notin E \cup [0, 1]$,

$$|\frac{f^{(j)}(z)}{f^{(i)}(z)}| \leq B(\frac{T(\alpha r, f)}{r}(\log^\alpha r)\log T(\alpha r, f))^{j-i}. \tag{2.12}$$

Remark 2 From the proof of Lemma 9, we can see that the exceptional set E satisfics that if a_n and b_m $(n, m = 1, 2, \ldots)$ denote all zeros and poles of f, respectively, $O(a_n)$ and $O(b_m)$ denote sufficiently small neighborhoods of a_n and b_m, respectively, then

$$E = \{|z| : z \in (\cup_{n=1}^{+\infty} O(a_n)) \cup (\cup_{m=1}^{+\infty} O(b_m))\}.$$

Hence, if $f(z)$ is a transcendental entire function, and z is a point that satisfies $|f(z)|$ to be sufficiently large, then (2.12) holds. For details see [7] Remark 2.10.

Lemma 10.(See [14] and Satz 21.2 of [13]) Let g be a non-constant entire function, and let $0 < \delta < 1$. There exists a set $E \subset [1, \infty)$ of finite logarithmic measure with the following property. For $r \in [1, \infty) \setminus E$, the central index $\nu(r)$ of g satisfies

$$\nu(r) \leq (\log M(r, g))^{1+\delta}.$$

Lemma 11. [3] Let $A_0, \ldots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution the equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = F, \tag{2.13}$$

with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then f satisfies $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho$.

3. Proof of Theorem 1

Suppose that $f \not\equiv 0$ is a solution of (1.11), then, f is an entire function. By Lemma 2, we see that f is transcendental.
First step.we prove that $\sigma(f) = \infty$.
Assume that f is transcendental with $\sigma(f) < \infty$. By Lemma 4, we know that for any given $\varepsilon > 0$, there exists a set $E \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$ having linear measure zero, such that if $\psi \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \setminus E$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $argz = \psi$ and $|z| = r > R_0$, we have

$$|\frac{f^{(j)}(z)}{f^{(s)}(z)}| \leq r^{(\sigma-1+\varepsilon)(j-s)} \quad j = s+1, \ldots, k. \tag{3.1}$$

Case 1 Now we take a ray $\arg z = \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus E$. Then we have $\cos\theta > 0$. We assert that $|f^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 3, there exists a sequence $\{z_t = r_t e^{i\theta}\}$ such that as $r_t \to \infty$, $f^{(s)}(z_t) \to \infty$ and

$$|\frac{f^{(i)}(z_t)}{f^{(s)}(z_t)}| \leq r_t^{s-i}(1 + o(1)) \quad i = 0, \ldots, s-1. \tag{3.2}$$

By (1.11), we get that

$$-P_s(e^{z_t}) = \frac{f^{(k)}(z_t)}{f^{(s)}(z_t)} + \sum_{j=0, j\neq s}^{k-1} [P_j(e^{z_t})] \frac{f^{(j)}(z_t)}{f^{(s)}(z_t)}. \tag{3.3}$$

$$
\begin{aligned}
|P_s(e^{z_t})| &= |a_{sm_s}(z_t)e^{m_s z_t} + \cdots + a_1(z_t)e^{z_t}| \\
&\geq |a_{sm_s}(z_t)e^{m_s z_t}| - [|a_{s(m_s-1)}(z_t)e^{(m_s-1)z_t}| + \cdots + |a_{s1}(z_t)e^{z_t}|] \\
&= |a_{sm_s d_{sm_s}}| r^{d_{sms}} (1+o(1)) e^{m_s r_t \cos\theta} \\
&\quad - [|a_{s(m_s-1)d_{s(m_s-1)}}| r^{d_{s(m_s-1)}} e^{(m_s-1)r_t\cos\theta}(1+o(1)) + \cdots \\
&\quad + |a_{s1d_{s1}}| r^{d_{s1}} e^{r_t\cos\theta}(1+o(1))] \\
&\geq \frac{1}{2}|a_{sm_s d_{sm_s}}| r^{d_{sms}} e^{m_s r_t\cos\theta}(1+o(1)) \qquad (3.4)
\end{aligned}
$$

and

$$
|P_j(e^{z_t})| \leq 2|a_{jm_j d_{jm_j}}| r_t^{d_{jm_j}} e^{m r_t \cos\theta}(1+o(1)) \quad j \neq s. \qquad (3.5)
$$

By substituting (3.1) (3.2) (3.4) and (3.5) into (3.3), we obtain that

$$
\frac{1}{2}|a_{sm_s d_{sm_s}}| r^{d_{sms}} e^{m_s r_t\cos\theta}(1+o(1)) \leq 2|a_{jm_j d_{jm_j}}| r_t^{d_{jm_j}+k\sigma} e^{m r_t\cos\theta}(1+o(1)) \qquad (3.6)
$$

Since $m_s > m$ and $\cos\theta > 0$, we know that when $r_t \to \infty$, (3.6) is a contradiction.
Hence when $\arg z = \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus E$, we have $|f^{(s)(re^{i\theta})}| \leq M$, so, on the ray $\arg z = \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus E$,

$$
|f(re^{i\theta})| \leq Mr^s. \qquad (3.7)
$$

Case 2 Now we take a ray $\arg z = \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \setminus E$. Then we have $\cos\theta < 0$. We assert that $|f^{(k)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f^{(k)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 3, there exists a sequence $\{z_t = r_t e^{i\theta}\}$ such that as $r_t \to \infty$, $f^{(k)}(z_t) \to \infty$ and

$$
|\frac{f^{(i)}(z_t)}{f^{(k)}(z_t)}| \leq r_t^{k-i}(1+o(1)) \quad i = 0, \ldots, k-1. \qquad (3.8)
$$

By (1.11), we get that

$$
-1 = P_{k-1}(e^{z_t})\frac{f^{(k-1)}(z_t)}{f^{(k)}(z_t)} + \cdots + P_0(e^{z_t})\frac{f(z_t)}{f^{(k)}(z_t)}. \qquad (3.9)
$$

Since when $r_t \to \infty$,

$$
\begin{aligned}
|P_j(e^{z_t})| &= |a_{jm_j}(z_t)e^{m_j z_t} + a_{jm_j-1}(z_t)e^{(m_j-1)z_t} + \cdots + a_{j1}(z_t)e^{z}| \\
&\leq |a_{jm_j d_{jm_j}}| r_t^{d_{jm_j}} e^{m_j r_t\cos\theta}(1+o(1)) + \cdots \\
&\quad + |a_{j1d_{j1}}| r^{d_{j1}} e^{r_t\cos\theta}(1+o(1)). \qquad (3.10)
\end{aligned}
$$

By substituting (3.8) and (3.10) into (3.9), we obtain that

$$\begin{aligned} 1 \le r_t[&|a_{k-1m_{k-1}d_{k-1m_{k-1}}}|r_t^{d_{k-1m_{k-1}}}e^{m_{k-1}r_t\cos\theta_t}(1+o(1)) + \cdots \\ &+ |a_{k-11d_{k-11}}|r^{d_{k-11}}e^{r_t\cos\theta_t}(1+o(1))] + \cdots \\ &+ r_t^k[|a_{0m_0d_{0m_0}}|r_t^{d_{0m_0}}e^{m_0r_t\cos\theta_t}(1+o(1)) \\ &+ \cdots + |a_{01d_{01}}|r_t^{d_{01}}e^{r_t\cos\theta_t}(1+o(1))]. \end{aligned} \tag{3.11}$$

Since $\cos\theta_t < 0$, when $r_t \to \infty$, by (3.11), we get $1 \le 0$. This is a contradiction. Hence $|f^{(k)}(re^{i\theta})| \le M$ on the ray $\arg z = \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \backslash E$. So, on the ray $\arg z = \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \setminus E$, we have

$$|f(re^{i\theta})| \le Mr^k. \tag{3.12}$$

Since the linear measure of $E \cup \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ is zero, by Lemma 5, (3.7) and (3.12), we know that $f(z)$ is a polynomial. This contradicts our assumption that $f(z)$ is transcendental. Therefore $\sigma(f) = \infty$.

Second step. We prove that $\sigma_2(f) = 1$.

By Lemma 6 and $\sigma(P_i(e^z)) = 1\,(j = 0, \dots, k-1)$, we see that

$$\sigma_2(f) \le \max\{\sigma(P_i(e^z))\} = 1. \tag{3.13}$$

Now we suppose that there exists a solution f_0 satisfies $\sigma_2(f_0) = \alpha < 1$. Then we have

$$\limsup_{r\to\infty} \frac{\log T(r, f_0)}{r} = 0. \tag{3.14}$$

By Lemma 9, we see that there exists a subset $E_1 \subset (1, \infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$,

$$|\frac{f_0^{(j)}(z)}{f_0(z)}| \le M_0[T(2r, f_0)]^{k+1},\ j = 1, \dots, k, \tag{3.15}$$

where $M(>0)$ is some constant.

From the Wiman-Valiron theory, there is a set $E_2 \subset (1, \infty)$ having logarithmic measure $lmE_2 < \infty$, such that we can choose a z satisfying $|z| = r \notin [0, 1] \cup E_2$ and $|f_0(z)| = M(r, f_0)$, then we get

$$\frac{f_0^{(j)}(z)}{f_0(z)} = (\frac{\nu(r)}{z})^j(1+o(1)), \quad j = 0, \dots, k-1. \tag{3.16}$$

where $\nu(r)$ is the central index of $f_0(z)$.
By Lemma 8, we see that there exists a sequence $\{z_t = r_te^{i\theta_t}\}$ such that $|f_0(z_t)| = M(r_t, f_0)$, $\theta_t \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $\lim \theta_t = \theta_0 \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $r_t \notin$

$[0,1] \cup E_1 \cup E_2$, $r_t \to \infty$, and if $\alpha > 0$, then for any given $\varepsilon_1\,(0 < \varepsilon_1 < \min\{\alpha,\, 1-\alpha\})$ and for sufficiently large r_k, we get by (2.10) that

$$\exp\{r_t^{\alpha-\varepsilon_1}\} < \nu(r_t) < \exp\{r_t^{\alpha+\varepsilon_1}\}; \tag{3.17}$$

if $\alpha = 0$, then by $\sigma(f_0) = \infty$ and (2.11), we see that for any $\varepsilon_2\,(0 < \varepsilon_2 < \frac{1}{2})$, and any large $M\,(> 0)$, we have, as r_t sufficiently large,

$$r_t^M < \nu(r_t) < \exp\{r_t^{\varepsilon_2}\}. \tag{3.18}$$

Since θ_0 may belong to $(-\frac{\pi}{2}, \frac{\pi}{2})$, or $(\frac{\pi}{2}, \frac{3\pi}{2})$, or $\{-\frac{\pi}{2}, \frac{\pi}{2}\}$, we devide this proof into three cases.

Case 1. Suppose $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $\cos\theta_0 > 0$. We take $\delta = \frac{1}{4}(\frac{\pi}{2} - |\theta_0|)$. Thus $[\theta_0 - \delta, \theta_0 + \delta] \subset (-\frac{\pi}{2}, \frac{\pi}{2})$. By $\theta_t \to \theta_0$, we see that there is a constant $N(> 0)$, such that as $t > N$, $\theta_t \in [\theta_0 - \delta, \theta_0 + \delta]$, and $0 < \cos(|\theta_0| + \delta) \le \cos\theta_t$. By (3.14), we see that for any given $\varepsilon_3\,(0 < \varepsilon_3 < \frac{1}{4(k+1)}\cos(|\theta_0| + \delta))$,

$$[T(2r_t, f_0)]^{k+1} \le e^{\varepsilon_3(k+1)2r_t} \le e^{\frac{1}{2}\cos(|\theta_0|+\delta)r_t} \le e^{\frac{1}{2}\cos\theta_t r_t} \tag{3.19}$$

holds for $n > N$.
By (3.15) (3.16) and (3.19), we see that

$$(\frac{\nu(r_t)}{r_t})^{k-s}(1 + o(1)) = |\frac{f_0^{(k-s)}(z_t)}{f_0(z_t)}| \le M_0[T(2r_t, f_0)]^{k+1} \le M_0 e^{\frac{1}{2}\cos\theta_t r_t}. \tag{3.20}$$

By (1.11), we get

$$-\frac{f_0^{(s)}(z_t)}{f_0(z_t)} P_s(e^{z_t}) = \frac{f_0^{(k)}(z_t)}{f_0(z_t)} + \sum_{j=0, j\neq s}^{k-1} P_j(e^{z_t}) \frac{f_0^{(j)}(z_t)}{f_0(z_t)}. \tag{3.21}$$

Because $\cos\theta_t > 0$ and (1.9), we get that

$$|P_s(e^{z_t})| = |a_{sm_s d_{sm_s}}| r^{d_{sm_s}} e^{m_s r_t \cos\theta_t}(1 + o(1)) \tag{3.22}$$

and

$$|P_j(e^{z_t})| \le M_1 r_t^{d_{jm_j}} e^{m r_t \cos\theta_t}(1 + o(1)) \quad (j = 0, \ldots, s-1, s+1, \ldots, k-1). \tag{3.23}$$

Substituting (3.16) (3.22) and (3.23) into (3.21), we get for sufficiently large r_t,

$$(\frac{\nu(r_t)}{r_t})^s |a_{sm_s d_{sm_s}}| r^{d_{sm_s}} e^{m_s r_t \cos\theta_t}(1 + o(1)) \le (\frac{\nu(r_t)}{r_t})^k (1 + o(1))$$
$$+ M_1 e^{m r_t \cos\theta_t} \sum_{j=0, j\neq s}^{k-1} r_t^{d_{jm_j}} (\frac{\nu(r_t)}{r_t})^j (1 + o(1)). \tag{3.24}$$

By (3.17) or (3.18),

$$\nu(r_t) > r_t^M > r_t. \tag{3.25}$$

By (3.20) (3.24) and (3.25), we get

$$|a_{sm_sd_{sm_s}}| r^{d_{sm_s}} e^{(m_s-m)r_t\cos\theta_t}(1+o(1)) \le kM_1\left(\frac{\nu(r_t)}{r_t}\right)^{k-s} r_t^d(1+o(1)) \\ \le M_2 r_t^d e^{\frac{1}{2}r_t\cos\theta_t}. \tag{3.26}$$

Since $m_s - m \ge 1 > \frac{1}{2}$ and $\cos\theta_t > 0$, we see that (3.26) is a contradiction.

Case 2.Suppose $\theta_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$. By $\cos\theta_0 < 0$ and $\theta_t \to \theta_0$, we see that for sufficiently large t, we have $\cos\theta_t < 0$. By (1.11) (3.16) and $\cos\theta_t < 0$, we get for sufficiently large r_t,

$$e^{-m_sz_t}\frac{f^{(k)}(z_t)}{f(z_t)} = e^{-m_sz_t}P_{k-1}(e^{z_t})\frac{f^{(k-1)}(z_t)}{f(z_t)} + \cdots + e^{-m_sz_t}P_0(e^z). \tag{3.27}$$

From (1.9) and $\cos\theta_t < 0$, we get

$$|e^{-m_sz_t}P_j(e^{z_t})| = |a_{jm_j}(z_t)e^{-(m_s-m_j)z_t} + \cdots + a_{j1}(z_t)e^{-(m_s-1)z_t}| \\ \le Br_t^{d_{j1}}e^{-(m_s-1)r_t\cos\theta_t}(1+o(1)). \tag{3.28}$$

Substituting (3.16) (3.28) into (3.27), from (3.25) we have

$$e^{-m_sr_t\cos\theta_t}\nu(r_t) \le M_3r_t^d e^{-(m_s-1)r_t\cos\theta_t}(1+o(1)). \tag{3.29}$$

If $\alpha > 0$, from (3.17) we have

$$\exp\{r_t^{\alpha-\varepsilon_3}\}e^{-m_sr_t\cos\theta_t} \le M_3r_t^d e^{-(m_s-1)r_t\cos\theta_t}(1+o(1)). \tag{3.30}$$

Since $\cos\theta_t < 0$ and $\alpha < 1$, we see (3.30) is a contradiction.
If $\alpha = 0$, from (3.18) we have

$$r_t^M e^{-m_sr_t\cos\theta_t} \le M_3r_t^d e^{-(m_s-1)r_t\cos\theta_t}(1+o(1)). \tag{3.31}$$

Since $\cos\theta_t < 0$, we see (3.31) is also a contradiction.

Case 3. Suppose that $\theta_0 = \frac{\pi}{2}$ or $\theta_0 = -\frac{\pi}{2}$. Since the proof for $\theta_0 = -\frac{\pi}{2}$ is the same as the proof for $\theta_0 = \frac{\pi}{2}$, we only prove the case that $\theta_0 = \frac{\pi}{2}$. Since $\theta_t \to \theta_0$, for any given ε_4 $(0 < \varepsilon_4 < \frac{1}{10})$, we see that there is an integer $K\,(>0)$, as $t > K$, $\theta_t \in [\frac{\pi}{2} - \varepsilon_4, \frac{\pi}{2} + \varepsilon_4]$, and

$$z_t = r_te^{i\theta_t} \in \overline{\Omega} = \{z : \frac{\pi}{2} - \varepsilon_4 \le \arg \le \frac{\pi}{2} + \varepsilon_4\}. \tag{3.32}$$

By Lemma 9, we see that there exist a subset $E_3 \subset (1, \infty)$ having logarithmic measure $lmE_3 < \infty$, and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0,1] \cup E_3$, we have

$$|\frac{f_0^{(i)}(z)}{f_0^{(s)}(z)}| \leq B[T(2r, f_0^{(s)})]^{k-s+1} \quad (i = s+1, \ldots, k). \tag{3.33}$$

Now we consider the property of $f_0(re^{i\theta})$ on a ray $\arg z = \theta \in \overline{\Omega} \setminus \{\frac{\pi}{2}\}$. If $\theta \in [\frac{\pi}{2} - \varepsilon_3, \frac{\pi}{2})$, then $\cos\theta > 0$.

Since $\sigma_2(f_0) < 1$, we get that f_0 satisfy (3.14). From $T(r, f_0^{(s)}) \leq (s+1)T(r, f_0)$, we get that $f_0^{(s)}$ also satisfies (3.14). So for any given ε_5 $(0 < \varepsilon_5 < \frac{1}{4(k-s+1)} \cos\theta)$, we have

$$[T(2r, f_0^{(s)})]^{k-s+1} \leq e^{\varepsilon_5 (k-s+1) 2r} \leq e^{\frac{1}{2} r \cos\theta}. \tag{3.34}$$

We assert that $|f_0^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta \in [\frac{\pi}{2} - \varepsilon_3, \frac{\pi}{2})$. If $|f_0^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 3, there exists a sequence $\{y_j = R_j e^{i\theta}\}$ such that as $R_j \to \infty$, $f_0^{(s)}(y_j) \to \infty$ and

$$|\frac{f_0^{(i)}(y_j)}{f_0^{(s)}(y_j)}| \leq R_j^{s-i}(1 + o(1)) \quad i = 0, \ldots, s-1. \tag{3.35}$$

By Remark 2 and $f_0^{(s)}(y_j) \to \infty$, we know that $|y_j| = R_j \notin E_3$. By (3.33) and (3.34), we have for sufficiently large j,

$$|\frac{f_0^{(j)}(y_j)}{f_0^{(s)}(y_j)}| \leq [T(2R_j, f_0^{(s)})]^{k-s+1} \leq Be^{\frac{1}{2} R_j \cos\theta} \quad j = s+1, \ldots, k. \tag{3.36}$$

Substituting (3.35) and (3.36) into (1.11), we get

$$\begin{aligned} |a_{sm_s d_{sm_s}}| R_j^{d_{sm_s}} e^{m_s R_j \cos\theta}(1 + o(1)) &= |-P_s(e^{y_j})| \\ &\leq s M_4 R_j^{d_2} e^{m R_j \cos\theta}(1 + o(1)) \\ +(k-s) M_5 e^{\frac{1}{2} R_j \cos\theta} R_j^{d_1} e^{m R_j \cos\theta}(1 + o(1)). \end{aligned} \tag{3.37}$$

Since $m_s > m + \frac{1}{2}$ and $\cos\theta > 0$, we get (3.37) is a contradiction.

Hence $|f_0^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta \in [\frac{\pi}{2} - \varepsilon_4, \frac{\pi}{2})$. Set $|f_0^{(s)}(re^{i\theta})| \leq M_6$, then on the ray $\arg z = \theta \in [\frac{\pi}{2} - \varepsilon_4, \frac{\pi}{2})$,

$$|f_0(re^{i\theta})| \leq M_7 r^s. \tag{3.38}$$

On the other hand, since $r_t \notin [0,1] \cup E_1 \cup E_2 \cup E_3$, by Lemma 10, and (3.17) or (3.18), we see that for sufficiently large r

$$\log M(r_t, f_0) \geq (\nu(r_t))^{\frac{1}{2}} \geq r_t^{\frac{M}{2}},$$

where $M(> 1)$ is some constant. Since $\{z_t\}$ satisfies $|f_0(z_t)| = M(r_t, f_0)$,

$$|f_0(z_t)| \geq \exp\{r_t^{\frac{M}{2}}\}. \tag{3.39}$$

By (3.38) and (3.39), we see that for sufficiently large t, $\theta_t \notin [\frac{\pi}{2}-\varepsilon_3, \frac{\pi}{2})$, i.e.,

$$\theta_t \in [\frac{\pi}{2}, \frac{\pi}{2}+\varepsilon_3]. \tag{3.40}$$

Thus there are two subcases: Subcase (i) there are infinitely many θ_t in $(\frac{\pi}{2}, \frac{\pi}{2}+\varepsilon_3]$; Subcase (ii) there are only finitely many θ_t in $(\frac{\pi}{2}, \frac{\pi}{2}+\varepsilon_3]$. Now consider Subcase (i), all $\theta_t \in (\frac{\pi}{2}, \frac{\pi}{2}+\varepsilon_3]$ form a subsequence θ_{t_j} of θ_t and a corresponding subsequence $z_{t_j} = r_{t_j} e^{i\theta_{t_j}}$ of z_t. For a subsequence $\{z_{t_j}\} \subset \{z : \frac{\pi}{2} < \arg z = \theta \le \frac{\pi}{2}+\varepsilon_3\}$, using a similar method to that in the proof of Case 2, we can get a contradiction.
Consider Subcase (ii), we see that for sufficiently large t,

$$\theta_t = \frac{\pi}{2}.$$

Thus, for sufficiently large t, $\cos\theta_t = 0$ and

$$\begin{aligned}|P_j(e^{z_t})| &= |a_{jm_j}(z_t)e^{m_j z_t} + \cdots + a_{j1}(z_t)e^{z_t}| \\ &\le |a_{jm_j}(z_t)| + \cdots + |a_{j1}(z_t)| \le M_8 r^d,\end{aligned} \tag{3.41}$$

where $j = 0, \ldots, k-1$ and M_8 is a constant.
By (1.11) (3.16) (3.17) (or (3.18)) and (3.41), we get that

$$|-(\frac{\nu(r_t)}{z_t})^k(1+o(1))| = |-\frac{f_0^{(k)}(z_t)}{f_0(z_t)}| \le kM_9 r_t^{d_3}(\frac{\nu(r_t)}{r_t})^{k-1}(1+o(1)),$$

i.e.,

$$\nu(r_t)(1+o(1)) \le kM_9 r_t^{d_3+1}(1+o(1)).$$

By (3.17) (or (3.18)), this is also a contradiction.

So we have $\sigma_2(f) = \alpha = 1$.

4. Proof of Theorem 2

From Theorem 1, we get $\sigma_2(f) = 1$.
Let $g = f - z$, then $f = g + z$. Substituting it into (1.11), we have

$$g^{(k)} + P_{k-1}(e^z)g^{(k-1)} + \cdots + P_0(e^z)g = -zP_0(e^z) - P_1(e^z). \tag{4.1}$$

Since $zP_0(e^z) - P_1(e^z) \not\equiv 0$, from Lemma 11 and $\sigma_2(g) = 1$ we conclude $\overline{\lambda}_2(g) = \lambda_2(g) = \sigma_2(g) = 1$. So we have $\overline{\tau}_2(f) = \tau_2(f) = \sigma_2(f) = 1$.

5. Proof of Theorem 3

From Theorem 1, we get $\sigma(f) = \infty$.
(i)Let $g = f - z$, then $f = g + z$. Substituting it into (1.6), we have

$$g'' + P(e^z)g' + Q(e^z)g = -P(e^z) - Q(e^z)z. \tag{5.1}$$

Since $n \ne s$, we get that $-P(e^z) - Q(e^z)z \not\equiv 0$. From Lemma 11, we get $\lambda(g) = \sigma(g) = \sigma(f) = \infty$ and $\lambda_2(g) = \sigma_2(g) = \sigma_2(f) = \sigma$. i.e.,

$\lambda(f-z)=\infty$ and $\lambda_2(f-z)=\sigma$.

(ii) Differentiating both sides of (1.6), we get that

$$f''' + P(e^z)f'' + (P'(e^z) + Q(e^z))f' + Q'(e^z)f = 0. \quad (5.2)$$

By (1.6), we get that

$$f = -\frac{f'' + P(e^z)f'}{Q(e^z)}. \quad (5.3)$$

Substituting (5.3) into (5.2), we get

$$f''' + [P(e^z) - \frac{Q'(e^z)}{Q(e^z)}]f'' + [P'(e^z) + Q(e^z) - \frac{Q'(e^z)}{Q(e^z)}P(e^z)]f' = 0. \quad (5.4)$$

Let $g = f' - z$, then $f' = g + z$, $f'' = g' + 1$, $f''' = g''$. Substituting these into (5.4), we get that

$$g'' + [P(e^z) - \frac{Q'(e^z)}{Q(e^z)}]g' + [P'(e^z) + Q(e^z) - \frac{Q'(e^z)}{Q(e^z)}P(e^z)]g$$
$$= \frac{Q'(e^z)}{Q(e^z)} - P(e^z) - [P'(e^z) + Q(e^z) - \frac{Q'(e^z)}{Q(e^z)}P(e^z)]z = h(z). \quad (5.5)$$

Next we prove that $h(z) \not\equiv 0$.
If $h(z) \equiv 0$, then $\frac{Q'(e^z)}{Q(e^z)} - P(e^z) \equiv [P'(e^z) + Q(e^z) - \frac{Q'(e^z)}{Q(e^z)}P(e^z)]z$.
Since $Q(z) \not\equiv 0$, we have

$$Q'(e^z) - P(e^z)Q(e^z) \equiv [P'(e^z)Q(e^z) + Q^2(e^z) - Q'(e^z)P(e^z)]z. \quad (5.6)$$

If $n < s$, taking $z = r$, we have

$$e^{2sr}(1 + o(1)) \leq e^{(n+s)r}(1 + o(1)).$$

This is a contradiction.
So we have $h(z) \not\equiv 0$.
From Lemma 11, we get $\lambda(g) = \sigma(g) = \sigma(f' - z) = \sigma(f) = \infty$ and $\lambda_2(g) = \sigma_2(g) = \sigma_2(f' - z) = \sigma_2(f) = \sigma$. i.e., $\lambda(f' - z) = \infty$ and $\lambda_2(f' - z) = \sigma$.

If $n > s$, taking $z = r$, we have

$$P(e^r) = a_n(r)e^{nr} + \cdots + a_1(r)e^r, \quad Q(e^r) = b_s(r)e^{sr} + \cdots + b_1(r)e^r.$$

We get

$$P'(e^r) = (a_n'(r) + na_n(r))e^{nr} + (a_{n-1}'(r) + (n-1)a_{n-1}(r))e^{(n-1)r} + \cdots$$

and

$$Q'(e^r) = (b_s'(r) + sb_s(r))e^{sr} + (b_{s-1}'(r) + (s-1)b_{s-1}(r))e^{(s-1)r} + \cdots.$$

So we have

$$|P(e^r)Q(e^r) + P'(e^r)Q(e^r)r - P(e^r)Q'(e^r)r| = |(n-s)ra_n(r)b_s(r) + [a_n(r)b_s(r) + (a_n'(r)b_s(r) - b_s'(r)a_n(r))r]|e^{(n+s)r}(1+o(1)). \quad (5.7)$$

Since $a_n(r)$, $b_s(r)$ are polynomials and $n > s$, we get

$\deg((n-s)ra_n(r)b_s(r)) > \deg[a_n(r)b_s(r) + (a_n'(r)b_s(r) - b_s'(r)a_n(r))r]$.

So we have

$$|(n-s)ra_n(r)b_s(r) + [a_n(r)b_s(r) + (a_n'(r)b_s(r) - b_s'(r)a_n(r))r]| = Mr^{d_1}(1+o(1)) \not\equiv 0.$$

From (5.6), we have

$$Mr^{d_1}e^{(n+s)r}(1+o(1)) = |P(e^r)Q(e^r) + P'(e^r)Q(e^r)r - P(e^r)Q'(e^r)r| = |Q'(e^r) - Q^2(e^r)r| \leq Br^{d_2}e^{2sr}(1+o(1)). \quad (5.8)$$

Since $n > s$, we get a contradiction.
So we also have $h(z) \not\equiv 0$.
From Lemma 11, we get $\lambda(g) = \sigma(g) = \sigma(f' - z) = \sigma(f) = \infty$ and $\lambda_2(g) = \sigma_2(g) = \sigma_2(f' - z) = \sigma_2(f) = \sigma$. i.e., $\lambda(f' - z) = \infty$ and $\lambda_2(f' - z) = \sigma$.

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