Congruences on Finite Automata

Zarko Popović

For a class $\boldsymbol{C}$ of algebras and an algebra A, a congruence relation ϱ on A is called a *$\boldsymbol{C}$-congruence* on A if the factor algebra A/ϱ belongs to $\boldsymbol{C}$. A class $\boldsymbol{P}$ of finite algebras is called a *pseudovariety* if it is closed under formation of subalgebras, homomorphisms and finite direct products. It is known (see [1] and [4]) that if $\boldsymbol{P}$ is a pseudovariety and A is any algebra, then the set $\mathrm{Con}_{\boldsymbol{P}}(A)$ of $\boldsymbol{P}$-congruences on A is a filter of the congruence lattice $\mathrm{Con}(A)$ of A, so any finite algebra has the *least $\boldsymbol{P}$-congruence*. The main aim of this paper is to find the least congruences which correspond to certain particular psudovarieties of finite automata.

An automaton A is called *directable* if there exists an input word which takes all states of A into one single state, it is *trapped* if there exists an input word which takes any state of A into some trap of A, it is *trap-directable* if it is directable and has a trap, it is *locally trap-directable* if any monogenic subautomaton of A is directable, and it is *generalized directable* if there exists an input word u such that $au = 3Dauvu$, for every state $a \in A$ and every input word v. Imreh and Steinby in [6], 1995, determined the least $\boldsymbol{Dir}$-congruence on a finite automaton, where $\boldsymbol{Dir}$ denotes the pseudovariety of *directable automata*. Here we shall determine the least congruences corresponding to the classes $\boldsymbol{Trap}$, of trapped automata, $\boldsymbol{TDir}$, of trap-directable automata, $\boldsymbol{LDir}$, locally directable automata, $\boldsymbol{GDir}$, generalized directable automata, etc. These classes were introduced and studied by Petković, Ćirić and Bogdanović in [9], 1998, and more information about them can be also found in the survey article by Bogdanović, Imreh, Ćirić and Petković [2].

One of the starting point of our investigation is the result due to Kovacević, Ćirić, Petković and Bogdanović [7], which says that every finite automaton can be uniquely represented as an extension of a reversible automaton by a trap-directable automaton, where a *reversible* automaton is the one which can be represented as a direct sum of sum of strongly connected automata. Using this result, we prove the following theorems.

Theorem 7 *Let a finite automaton A be represented as an extension of a reversible automaton B by a trap-directable automaton, where B is represented as a direct sum of strongly connected automata $B_\alpha, \alpha \in Y$. Then the relation τ on A defined by*

$$(a,b) \in \tau \;\Leftrightarrow\; a = 3Db \;\textit{ or }\; (a,b) \in B_\alpha,\ \textit{for some } \alpha \in Y,$$

is the least $\boldsymbol{Trap}$-congruence on A.

Theorem 8 *Let a finite automaton A be represented as an extension of a reversible automaton B by a trap-directable automaton C. Then the Rees congruence ρ_B on A determined by B is the least $\boldsymbol{TDir}$-congruence on A.*

The least $\boldsymbol{GDir}$-congruence on a finite automaton is characterized in terms of the least $\boldsymbol{Dir}$-congruence, described by Imreh and Steinby in [6].

Theorem 9 *Let a finite automaton A be represented as an extension of an automaton B by a trap-directable automaton, let B be represented as a direct sum of strongly connected automata $B_\alpha, \alpha \in Y$, and for each $\alpha \in Y$ let θ_α be the least $\boldsymbol{Dir}$-congruence on B_α. Then the relation θ on A defined by*

$$(a,b) \in \theta \;\Leftrightarrow\; a = 3Db \;\textit{ or }\; (a,b) \in \theta_\alpha,\ \textit{for some } \alpha \in Y,$$

is the least $\boldsymbol{GDir}$-congruence on A.

If $\boldsymbol{P} \subseteq \boldsymbol{Dir}$ is a pseudovariety, then the class $L(\boldsymbol{P})$ of all finite automata whose any monogenic subautomaton belongs to $\boldsymbol{P}$ is also a pseudovariety, and it consists of automata which are direct sums of automata from $\boldsymbol{P}$ (see [3] and [8]). Using this fact and a theorem proved by Ćirić and Bogdanović in [5], 1999, which says that every automaton can be uniquely represented as a direct sum of direct sum indecomposable automata, we prove the following theorem.

Theorem 10 *Let $\boldsymbol{P} \subseteq \boldsymbol{Dir}$ be a pseudovariety, let a finite automaton A be represented as a direct sum of direct sum indecomposable automata $A_\alpha, \alpha \in Y$, and for each $\alpha \in Y$ let θ_α be the least $\boldsymbol{P}$-congruence on A_α. Then the relation θ on A defined by*

$$(a, b) \in \theta \;\Leftrightarrow\; (a, b) \in \theta_\alpha, \text{ for some } \alpha \in Y,$$

is the least $L(\boldsymbol{P})$-congruence on A.

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