



Existence and blow-up of global solutions for a class of fractional Lane–Emden heat flow system

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Abstract. In this paper, we consider a class of Lane–Emden heat flow system with the fractional Laplacian

$$\begin{cases} u_t + (-\Delta)^{\frac{\alpha}{2}} u = N_1(v) + f_1(x), & (x, t) \in Q, \\ v_t + (-\Delta)^{\frac{\alpha}{2}} v = N_2(u) + f_2(x), & (x, t) \in Q, \\ u(x, 0) = a(x), v(x, 0) = b(x), & x \in \mathbb{R}^N, \end{cases}$$

where $0 < \alpha \leq 2$, $N \geq 3$, $Q := \mathbb{R}^N \times (0, +\infty)$, $f_i(x) \in L^1_{loc}(\mathbb{R}^N)$ ($i = 1, 2$) are nonnegative functions. We study the relationship between the existence, blow-up of the global solutions for the above system and the indexes p, q in the nonlinear terms $N_1(v), N_2(u)$. Here, we first establish the existence and uniqueness of the global solutions in the supercritical case by using Duhamel’s integral equivalent system and the contraction mapping principle, and we further obtain some relevant properties of the global solutions. Next, in the critical case, we prove the blow-up of nonnegative solutions for the system by utilizing some heat kernel estimates and combining with proof by contradiction. Finally, by means of the test function method, we investigate the blow-up of negative solutions for the Cauchy problem of a more general higher-order nonlinear evolution system with the fractional Laplacian in the subcritical case.

Keywords: fractional Laplacian, Lane–Emden heat flow system, critical exponent, the contraction mapping principle, the test function method.


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1 Introduction

The classical Lane–Emden equation

$$-\Delta u = u^p, \quad x \in \mathbb{R}^N, \quad N > 2, \quad p > 1,$$

has been extensively studied, going back to the pioneering work of astronomers and astrophysicists Lane [32] and Emden [15]. It is one of the basic equations in the theory of stellar

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structure and originally used to compute the pressure, density and temperature on the surface of the Sun. It has been discussed by many scholars, see [13, 18, 35, 43, 47] and the references therein. The existence and nonexistence of the global solutions of the equation was once an significant research topic for scholars. For instance, Gidas and Spruck [22] proved that the equation has no positive classical solution in a bounded domain when $1 < p < \frac{N+2}{N-2}$, while the existence of the solution was solved by Caffarelli et al. in [6]. Thereafter, Chen and Li [8] found the form of the positive solution for $p = \frac{N+2}{N-2}$ ($N \geq 3$) in the whole space and obtained that only trivial solutions exist for $p < \frac{N+2}{N-2}$ by using the method of moving planes. In addition, as for $p > \frac{N+2}{N-2}$, Zou [49] proved that the equation has a unique positive radial symmetric solution with polynomial decay at infinity. Meanwhile, scholars also discussed the existence and nonexistence of solutions to nonlinear elliptic equation and system with a more general nonlinearity. In [4], Bernard studied the semilinear elliptic equation $-\Delta u = u^p + f(x)$ in the whole space. He obtained the blow-up of the global solutions for $1 < p \leq \frac{N}{N-2}$, while if $p > \frac{N}{N-2}$ and $f \in C^{0,\gamma}(\mathbb{R}^N)$ with $0 < \gamma \leq 1$, he showed that the equation has a bounded positive solution. Obviously, the Lane–Emden type system is the natural counterpart of the Lane–Emden equation

$$\begin{cases} -\Delta u = v|v|^{p-1}, & x \in \mathbb{R}^N, \\ -\Delta v = u|u|^{q-1} + f(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $p, q > 1$. When $f = 0$, Mitidieri [37] proved that there has no nontrivial radial positive solutions of class $C^2(\mathbb{R}^N)$ by contradiction if $1 < p \leq q$ and $\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}$, while if $1 < p \leq q$ and $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2}{N}$, the existence of positive (radial, bounded) classical solution for the system is fully solved by Serrin and Zou in [42]. As for more general cases, when $f \in L^{\frac{N(pq-1)}{2q(p+1)}}$, Ferreira et al. [19] showed the existence of the global solutions in the supercritical case $N > \max\{\frac{2q(p+1)}{pq-1}, \frac{2p(q+1)}{pq-1}\}$ by means of the fixed point theorem, here the range for (p, q) covers the critical and supercritical cases with respect to the hyperbola $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$. In case $N \leq \max\{\frac{2q(p+1)}{pq-1}, \frac{2p(q+1)}{pq-1}\}$, the nonexistence results has been pointed out by Mitidieri in [38]. For more researches on elliptic equations, please refer to [2, 11, 34, 40].

The parabolic equation corresponding to the classical Lane–Emden equation, namely the semilinear reaction-diffusion equation

$$u_t - \Delta u = u^q, \quad x \in \mathbb{R}^N, t > 0,$$

has been studied by many scholars, since the pioneering work [20] of Fujita in 1966, where it was shown that the Cauchy problem of the equation has two cases of the solution: if $q > q_c = 1 + \frac{2}{N}$, there exist both global and blow-up solutions, corresponding to small and large initial values, respectively; while if $q < q_c = 1 + \frac{2}{N}$, then the problem does not admit nonnegative global solution. The case of $q = q_c = 1 + \frac{2}{N}$ was decided by Hayakawa [28] for $N = 1, 2$ and Kobayashi et al. [31] for all $N \geq 1$ that the problem does not admit nontrivial nonnegative global solution. Thus, it can be seen that the range of index q plays an important role in the researches of existence and blow-up of the solutions. And q_c is called Fujita critical exponent. Since then, there have been a number of extensions to the research of critical exponent in several directions. For instance, Pascucci [39] considered a semilinear Cauchy problem on nilpotent Lie groups and obtained the sharp Fujita critical exponent, which generalized the results in [20, 28, 31].

As for the semilinear parabolic system, Escobedo and Herrero [16] discussed the Cauchy

problem of semilinear reaction-diffusion system in the whole space

$$\begin{cases} u_t - \Delta u = v^p, & (x, t) \in \mathbb{R}^N \times (0, \infty), \\ v_t - \Delta v = u^q, & (x, t) \in \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \mathbb{R}^N, N \geq 1. \end{cases}$$

They showed that the system has two significant curves, namely, the global existence curve $pq = 1$ and the Fujita curve $pq = 1 + \frac{2}{N} \max\{p+1, q+1\}$. If $0 < pq \leq 1$ or $pq > 1 + \frac{2}{N} \max\{p+1, q+1\}$ with suitably small initial values, then every solution is global by employing the integral equivalent system and Gronwall-type inequalities, respectively; while if $pq > 1 + \frac{2}{N} \max\{p+1, q+1\}$ with large initial values, the system possesses no nontrivial global solution. Meanwhile, the nonexistence of nontrivial global solution is proved based on some heat kernel estimates for $1 < pq \leq 1 + \frac{2}{N} \max\{p+1, q+1\}$. For some problems with boundary conditions or nonlinear terms different from the above, many scholars have also studied the existence and nonexistence of global solutions. For example, Deng and Fila [14] and Bai et al. [5] discussed the Fujita critical exponent of parabolic problems in the upper half space and bounded domain respectively. For more researches on parabolic system, see for example [17, 29, 30, 45].

In mathematical physics, nonlinear evolution equations with the fractional Laplacian are extensively used to describe anomalous diffusion, see [25, 26, 33] and the references therein. Therefore, it is of theoretical value and practical significance to study the existence of solutions of equations with the fractional Laplacian. Amor and Kenzizi [3] studied the Cauchy problem of the fractional perturbed heat equation on a bounded domain and obtained the necessary conditions for the existence of nonnegative global solution. In [23], Greco et al. concerned the Cauchy problem of the fractional heat equation $u_t + (-\Delta)^s u = 0$ in the whole space. It was showed that the problem has a global solution if the initial value subject to a certain growth condition. In addition, many scholars have also considered the fractional nonhomogeneous parabolic equation

$$u_t + (-\Delta)^\alpha u = f(t, u).$$

When $f(t, u) = h(t)u^p$, Guedda et al. [24] and Tan et al. [44] concerned the Cauchy problem of the equation by means of the integral equivalent equation and the contraction mapping principle, respectively. Their conclusions implied that the Fujita critical exponent is $1 + \frac{2\alpha(1+\sigma)}{N}$. Here, $p > 1$ and the function $h(t) \in C([0, \infty))$ satisfied $c_0 t^\sigma \leq h(t) \leq c_1 t^\sigma$ with $c_0, c_1 > 0$, $\sigma > -1$ for t large enough. Besides, the nonexistence of nontrivial nonnegative solutions and the asymptotic symmetry of the solution were obtained in [10] and [9] under suitable assumptions on $f(t, u)$ via narrow region principles and the method of moving planes, respectively. For more works about the fractional parabolic equation, see [1, 21, 36, 41] and the references therein.

Inspired by the above literature, we study the Cauchy problem of the Lane–Emden heat flow system with the fractional Laplacian

$$\begin{cases} u_t + (-\Delta)^{\frac{\alpha}{2}} u = N_1(v) + f_1(x), & (x, t) \in Q, \\ v_t + (-\Delta)^{\frac{\alpha}{2}} v = N_2(u) + f_2(x), & (x, t) \in Q, \\ u(x, 0) = a(x), v(x, 0) = b(x), & x \in \mathbb{R}^N. \end{cases} \quad (1.1)$$

This problem is used to describe the heat transfer of two mixed combustibles, where u and v represent the temperature of anomalous diffusion at which the two substances interact respectively. We are primarily concerned with the case $0 < \alpha \leq 2$, $Q := \mathbb{R}^N \times (0, +\infty)$, $N \geq 3$.

$f_i(x) \in L^1_{loc}(\mathbb{R}^N)$ for $i = 1, 2$ are nonnegative functions. The nonnegative coupling terms $N_1(v)$, $N_2(u)$ satisfying

$$C_1|v - \tilde{v}|(|v|^{p-1} + |\tilde{v}|^{p-1}) \leq |N_1(v) - N_1(\tilde{v})| \leq C_2|v - \tilde{v}|(|v|^{p-1} + |\tilde{v}|^{p-1}), \quad (1.2a)$$

$$\tilde{C}_1|u - \tilde{u}|(|u|^{q-1} + |\tilde{u}|^{q-1}) \leq |N_2(u) - N_2(\tilde{u})| \leq \tilde{C}_2|u - \tilde{u}|(|u|^{q-1} + |\tilde{u}|^{q-1}), \quad (1.2b)$$

where $p > 1$, $q > 1$, and $N_1(v) = N_2(u) = 0$ if $u = v = 0$. We shall assume henceforth that both $a(x) \in L^{\omega_1}(\mathbb{R}^N)$, $b(x) \in L^{\omega_2}(\mathbb{R}^N)$ are continuous, bounded, nonnegative and not-constant zero functions with $\omega_1, \omega_2 > 1$. Here u is a curve in $L^{\omega_3}(\mathbb{R}^N)$, $u: [0, \infty) \rightarrow L^{\omega_3}(\mathbb{R}^N)$ while v is a curve in $L^{\omega_4}(\mathbb{R}^N)$, $v: [0, \infty) \rightarrow L^{\omega_4}(\mathbb{R}^N)$ for $\omega_3, \omega_4 > 1$, which assumes only nonnegative values. We show the existence of a unique global solution for (1.1) in the supercritical case and the problem does not admit nonnegative global solutions in the critical case. As for the subcritical case, we consider the blow-up of the global solution for the following Cauchy problem of the higher-order nonlinear evolution system

$$\begin{cases} \frac{\partial^k u}{\partial t^k} + (-\Delta)^{\frac{\alpha}{2}} u = N_1(v) + f_1(x), & (x, t) \in Q, \\ \frac{\partial^k v}{\partial t^k} + (-\Delta)^{\frac{\beta}{2}} v = N_2(u) + f_2(x), & (x, t) \in Q, \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}(x, 0) \geq 0, \frac{\partial^{k-1} v}{\partial t^{k-1}}(x, 0) \geq 0, & x \in \mathbb{R}^N, \end{cases} \quad (1.3)$$

where $k > 1$, $0 < \alpha, \beta \leq 2$.

For simplicity, throughout the paper, we denote by C a generic positive constant which may vary in value from line to line and even within the same line, but is independent of the terms which will take part in any limit process.

The following Duhamel's integral equivalent system [44] will be used to prove the existence of a global solution for (1.1) in the supercritical case and the blow-up result in the critical case for (1.1).

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}^N} \Gamma(x - y, t) a(y) dy + \int_0^t \int_{\mathbb{R}^N} \Gamma(x - y, t - s) N_1(v)(y, s) dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}^N} \Gamma(x - y, t - s) f_1(y) dy ds, \end{aligned} \quad (1.4)$$

$$\begin{aligned} v(x, t) &= \int_{\mathbb{R}^N} \Gamma(x - y, t) b(y) dy + \int_0^t \int_{\mathbb{R}^N} \Gamma(x - y, t - s) N_2(u)(y, s) dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}^N} \Gamma(x - y, t - s) f_2(y) dy ds, \end{aligned} \quad (1.5)$$

where $\Gamma(x, t)$ is the fundamental solution to $u_t + (-\Delta)^{\frac{\alpha}{2}} u = 0$. It is well known that $\Gamma(x, t)$ is given by

$$\int_{\mathbb{R}^N} \Gamma(x, t) e^{-iz \cdot x} dx = e^{-t|z|^\alpha}, \quad 0 < \alpha \leq 2,$$

From [48], we have

$$\Gamma(x, t) = \int_0^{+\infty} f_{t, \frac{\alpha}{2}}(s) T(x, s) ds, \quad 0 < \alpha \leq 2,$$

and $\Gamma(x, t) = T(x, t)$ if $\alpha = 2$, where

$$f_{t, \frac{\alpha}{2}}(s) = \frac{1}{2i\pi} \int_{\tau - i\infty}^{\tau + i\infty} e^{zs - tz^{\frac{\alpha}{2}}} dz \geq 0, \quad T(x, s) = \left(\frac{1}{4\pi s} \right)^{\frac{N}{2}} e^{-\frac{|x|^2}{4s}}, \quad \tau > 0, s > 0.$$

To facilitate writing, we set

$$u_0(x, t) = \int_{\mathbb{R}^N} \Gamma(x - y, t) a(y) dy, \quad (1.6)$$

and

$$v_0(x, t) = \int_{\mathbb{R}^N} \Gamma(x - y, t) b(y) dy. \quad (1.7)$$

Define

$$F(f_1) = \int_0^t \int_{\mathbb{R}^N} \Gamma(x - y, t - s) f_1(y) dy ds, \quad (1.8)$$

and

$$F(f_2) = \int_0^t \int_{\mathbb{R}^N} \Gamma(x - y, t - s) f_2(y) dy ds. \quad (1.9)$$

In this framework we can write the integral system (1.4)–(1.5) in the abstract form

$$(u, v) = (u_0, v_0) + B(u, v) + (F(f_1), F(f_2)), \quad (1.10)$$

where

$$B(u, v) = (B_1(v), B_2(u)), \quad (1.11)$$

and $B_1(v) = F(N_1(v))$, $B_2(u) = F(N_2(u))$.

If $N > \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$, we denote

$$P_{sc} := \frac{N(pq-1)}{\alpha(p+1)}, \quad Q_{sc} := \frac{N(pq-1)}{\alpha(q+1)},$$

and

$$P'_{sc} := \frac{N(pq-1)}{\alpha(p+1) + pq - 1}, \quad Q'_{sc} := \frac{N(pq-1)}{\alpha(q+1) + pq - 1}.$$

Below we assume the basic assumptions on the range of p_1 and q_1 :

$$\frac{p+1}{pq-1} - \frac{p+1}{pq+p} < \frac{N}{\alpha p_1} < \frac{p+1}{pq-1}, \quad p_1 \geq p, \quad (1.12)$$

$$\frac{q+1}{pq-1} - \frac{q+1}{pq+q} < \frac{N}{\alpha q_1} < \frac{q+1}{pq-1}, \quad q_1 \geq q, \quad (1.13)$$

and

$$\frac{N}{\alpha q_1} < \frac{Nq}{\alpha p_1} < 1 + \frac{N}{\alpha q_1}, \quad \frac{N}{\alpha p_1} < \frac{Np}{\alpha q_1} < 1 + \frac{N}{\alpha p_1},$$

that is

$$\frac{1}{q_1} < \frac{q}{p_1} < \frac{1}{q_1} + \frac{\alpha}{N}, \quad \frac{1}{p_1} < \frac{p}{q_1} < \frac{1}{p_1} + \frac{\alpha}{N}. \quad (1.14)$$

The range and some basic assumptions of the indexes p'_1 and q'_1 are

$$\frac{1}{\alpha} + \frac{p+1}{pq-1} - \frac{p+1}{pq+p} < \frac{N}{\alpha p'_1} < \frac{1}{\alpha} + \frac{p+1}{pq-1}, \quad p'_1 \geq p_1, \quad (1.15)$$

$$\frac{1}{\alpha} + \frac{q+1}{pq-1} - \frac{q+1}{pq+q} < \frac{N}{\alpha q'_1} < \frac{1}{\alpha} + \frac{q+1}{pq-1}, \quad q'_1 \geq q_1. \quad (1.16)$$

Therefore

$$\frac{1}{q'_1} < \frac{q}{p'_1}, \quad \frac{1}{p'_1} < \frac{p}{q_1}. \quad (1.17)$$

The above assumptions are used in the following statements. Our main results read

Theorem 1.1. Suppose that $N > \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$. Let $C_0(\mathbb{R}^N)$ denote the space of all continuous functions decaying to zero at infinity, and let $a(x) \in L^{P_{sc}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, $b(x) \in L^{Q_{sc}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, $f_1(x) \in L^{\frac{Q_{sc}}{p}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, $f_2(x) \in L^{\frac{P_{sc}}{q}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$.

(1) There exists $\delta > 0$ such that if $\|a(x)\|_{P_{sc}}, \|b(x)\|_{Q_{sc}}, \|f_1(x)\|_{\frac{Q_{sc}}{p}}, \|f_2(x)\|_{\frac{P_{sc}}{q}} \leq \frac{\delta}{3K_1}$, then the integral system (1.10) has a unique solution (u, v) satisfying $u \in L^{p_1}(\mathbb{R}^N)$, $v \in L^{q_1}(\mathbb{R}^N)$ and

$$\|u\|_{p_1}, \|v\|_{q_1} \leq 2\delta,$$

where the constant K_1 is as in Lemma 2.3.

(2) If $N > 1 + \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$, then (u, v) is a solution in the sense of distributions and satisfies

$$\nabla u \in L^{p'_1}(\mathbb{R}^N), \quad \nabla v \in L^{q'_1}(\mathbb{R}^N).$$

(3) Furthermore, if $a(x) \in L^{p_1}(\mathbb{R}^N)$, $b(x) \in L^{q_1}(\mathbb{R}^N)$, then $u, v \in C([0, \infty), C_0(\mathbb{R}^N))$.

Theorem 1.2. Suppose that $N = \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$. Then the problem (1.1) has no nonnegative global solution $u, v \in C^1(Q) \cap L^\infty(Q)$ such that $(-\Delta)^{\frac{\alpha}{2}}u(x, t), (-\Delta)^{\frac{\alpha}{2}}v(x, t) \in L^\infty(Q)$.

Theorem 1.3. Suppose that $N < \max\{\frac{q(\alpha+p\beta)}{pq-1} - \sigma, \frac{p(\alpha+q\beta)}{pq-1} - \sigma\}$ with $\sigma > \max\{\frac{\alpha}{k}, \frac{\beta}{k}\}$ and $f_i(x) \neq 0$ for $i = 1, 2$. Then (1.3) has no nonnegative global weak solution (see Definition 2.1 below).

Remark 1.4. It is worth noting that, compared with the semilinear reaction-diffusion system of the classical Laplacian in [16], the influence of the fractional operator and the nonlinear terms for (1.1) we consider on the estimates are more complicated. Hence, when we prove Theorem 1.2, we argue by contradiction, the integral related to the initial value is estimated skillfully, which reduces a large number of calculations generated by using the method in [16], and the method here is more convenient.

Remark 1.5. From Theorem 1.3, if $\alpha = \beta$ and $k = 1$, then hypothetical condition will correspondingly change to $N < \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$, which is consistent with the indexes in Theorems 1.1 and 1.2. So we can get that the critical curve for (1.1) is $N = \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$. Next, we give some comments about the critical curve(exponent) for (1.1).

(1) If $\alpha = 2$, then $N = \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$ becomes $pq = 1 + \frac{2}{N} \max\{p+1, q+1\}$, which is the critical curve for semilinear reaction-diffusion system in [16].

(2) If $u = v$ and $p = q$, then $N = \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$ becomes $p = 1 + \frac{\alpha}{N}$, which is the critical exponent for the corresponding single parabolic equation $u_t + (-\Delta)^{\frac{\alpha}{2}}u = u^p$ in [24, 44].

We conclude this introduction by describing the plan of the paper. Section 2 recalls some lemmas and some properties of the fundamental solution $\Gamma(x, t)$ which we shall use in the sequel. In Section 3, we use the contraction mapping principle to prove the existence of a unique global solution for (1.1) in the supercritical case, and further obtain some relevant properties of the global solution. The blow-up of the global solutions in the critical case is discussed via Duhamel's integral equivalent equations and combined with proof by contradiction, which is gathered in Section 4. As for the blow-up result for a more general higher-order system (1.3) in the subcritical case, we utilize the test function method to obtain and make up the content of Section 5. Section 6 is an appendix, in which we prove some lemmas given in Section 2 in detail.

2 Preliminaries

In this section, we mainly introduce some lemmas, as well as some properties and estimates related to the kernel function $\Gamma(x, t)$, which will be utilized in the following proofs. For general k , we first give the definition of weak solutions for (1.3).

Definition 2.1. Let $u, v \in L^1_{loc}(\mathbb{R}^N \times [0, \infty))$ with $N_1(v), N_2(u) \in L^1_{loc}(\mathbb{R}^N \times [0, \infty))$, and let the locally integrable traces $\frac{\partial^i u}{\partial t^i}(x, 0), \frac{\partial^i v}{\partial t^i}(x, 0), i = 1, 2, \dots, k-1$ on the hyperplane $t = 0$ are well defined. The function (u, v) is called a global weak solution for (1.3) in Q if for any nonnegative test function $\varphi(x, t) \in C_c^\infty(\mathbb{R}^N \times [0, \infty))$, the following integral equalities hold:

$$\begin{aligned} \iint_Q u \left[(-1)^k \frac{\partial^k \varphi}{\partial t^k} + (-\Delta)^{\frac{\alpha}{2}} \varphi \right] dx dt &= \iint_Q (N_1(v) + f_1(x)) \varphi(x, t) dx dt \\ &+ \sum_{i=1}^{k-1} (-1)^i \int_{\mathbb{R}^N} \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(x, 0) \frac{\partial^i \varphi}{\partial t^i}(x, 0) dx \\ &+ \int_{\mathbb{R}^N} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x, 0) \varphi(x, 0) dx, \end{aligned} \quad (2.1)$$

$$\begin{aligned} \iint_Q v \left[(-1)^k \frac{\partial^k \varphi}{\partial t^k} + (-\Delta)^{\frac{\beta}{2}} \varphi \right] dx dt &= \iint_Q (N_2(u) + f_2(x)) \varphi(x, t) dx dt \\ &+ \sum_{i=1}^{k-1} (-1)^i \int_{\mathbb{R}^N} \frac{\partial^{k-1-i} v}{\partial t^{k-1-i}}(x, 0) \frac{\partial^i \varphi}{\partial t^i}(x, 0) dx \\ &+ \int_{\mathbb{R}^N} \frac{\partial^{k-1} v}{\partial t^{k-1}}(x, 0) \varphi(x, 0) dx. \end{aligned} \quad (2.2)$$

According to [24, 27, 46], we collect the following propositions:

Proposition 2.2.

- (1) $\Gamma(x, ts) = t^{-\frac{N}{\alpha}} \Gamma(t^{-\frac{1}{\alpha}} x, s)$.
- (2) $\Gamma(x, t) \geq \left(\frac{s}{t}\right)^{-\frac{N}{\alpha}} \Gamma(x, s)$ for all $t \geq s$.
- (3) For all $x \in \mathbb{R}^N$ and $\alpha > 0$, $\Gamma(x, t)$ satisfies the following pointwise estimates

$$|\Gamma(x, 1)| \leq C(1 + |x|)^{-N-\alpha}, \quad \left| (-\Delta)^{\frac{\alpha}{2}} \Gamma(x, 1) \right| \leq C'(1 + |x|)^{-N-\alpha}.$$

- (4) $\|\Gamma(\cdot, t)\|_1 = 1$ for all $t > 0$, and $\Gamma(x, t)$ satisfies:

$$\Gamma(x, t) \in L^p(\mathbb{R}^N), \quad (-\Delta)^{\frac{\alpha}{2}} \Gamma(x, t) \in L^p(\mathbb{R}^N).$$

for all $t > 0$ and $1 \leq p \leq \infty$.

- (5) For all $x \in \mathbb{R}^N$ and $t, s > 0$, the following Chapman–Kolmogorov equation holds:

$$\int_{\mathbb{R}^N} \Gamma(x - z, s) \Gamma(z, t) dz = \Gamma(x, t + s).$$

- (6) If $\Gamma(0, t) \leq 1$ and $\tau \geq 2$, then $\Gamma\left(\frac{1}{\tau}(x - y), t\right) \geq \Gamma(x, t) \Gamma(y, t)$.

Lemma 2.3. *Let $1 \leq m \leq n \leq \infty$. Then for $t > 0$, $e^{-t(-\Delta)^{\frac{\alpha}{2}}}: L^m(\mathbb{R}^N) \rightarrow L^n(\mathbb{R}^N)$ is a bounded map. Furthermore, for any $T > 0$ and $h(x, t) \in L^m(\mathbb{R}^N)$, there are positive constants K_1 and K_2 depending only on m, n and l , such that*

$$\|\Gamma(x, t) * h(x, t)\|_n \leq K_1 t^{-\frac{N}{\alpha}(1-\frac{1}{r})} \|h(x, t)\|_m, \quad (2.3)$$

$$\left\| (-\Delta)^{\frac{l}{2}} \Gamma(x, t) * h(x, t) \right\|_n \leq K_2 t^{-\frac{l}{\alpha} - \frac{N}{\alpha}(1-\frac{1}{r})} \|h(x, t)\|_m, \quad (2.4)$$

for all $t \in (0, T]$ and any $l > 0$, where $1 + \frac{1}{n} = \frac{1}{m} + \frac{1}{r}$. In particular, if $l = 1$, then

$$\|\nabla \Gamma(x, t) * h(x, t)\|_n \leq K_2 t^{-\frac{1}{\alpha} - \frac{N}{\alpha}(1-\frac{1}{r})} \|h(x, t)\|_m, \quad \forall t \in (0, T].$$

Here and hereafter, “ $*$ ” stands for the convolution in the space variable.

See Appendix for detailed proof of Lemma 2.3.

Lemma 2.4 (See [7]). *Let $a \wedge b := \min\{a, b\}$ for $a, b \in \mathbb{R}$. Then there exist positive constants $C_{\alpha, N}$ and $C'_{\alpha, N}$, depending only on N and α , such that*

$$C'_{\alpha, N} \left(t^{-\frac{N}{\alpha}} \wedge \frac{t}{|x|^{N+\alpha}} \right) \leq \Gamma(x, t) \leq C_{\alpha, N} \left(t^{-\frac{N}{\alpha}} \wedge \frac{t}{|x|^{N+\alpha}} \right)$$

for all $(x, t) \in \mathbb{R}^N \times (0, +\infty)$ and $0 < \alpha < 2$.

Lemma 2.5. *Let (u, v) is a nonnegative solution to (1.1), then there exist positive constants t_0, C and τ such that*

$$u(x, t_0) \geq C\Gamma(x, \tau), \quad v(x, t_0) \geq C\Gamma^q(x, \tau), \quad (2.5)$$

for $q > 1$ and all $x \in \mathbb{R}^N$.

Similar estimates can be found in [24, Lemma 3.2]. To make the paper self-contained, we give the proof of Lemma 2.5 in Appendix.

3 Existence of the global solution for (1.1) in the supercritical case

In this section, we utilize (1.10) and the contraction mapping principle to prove the existence of a global solution for (1.1) in the supercritical case. To achieve this, we first derive a key lemma, which provides estimates for the integrals in (1.10).

Define

$$E := L^\infty \left((0, \infty), L^{p_1}(\mathbb{R}^N) \right) \times L^\infty \left((0, \infty), L^{q_1}(\mathbb{R}^N) \right).$$

For each $\delta > 0$ fixed we consider the space D defined by

$$D := \left\{ (u, v) \in E \mid \sup t^{b_1} \|u\|_{p_1} < 2\delta, \sup t^{b_2} \|v\|_{q_1} < 2\delta \right\},$$

where constants b_1, b_2 are given by formulas (3.9)–(3.10).

On the space D , we show the following lemma:

Lemma 3.1. *Let p_1, q_1 be as in (1.12)–(1.14) and p'_1, q'_1 be as in (1.15)–(1.16), $(u, v) \in D$. For all $v_1, v_2 \in L^{q_1}(\mathbb{R}^N)$ and $u_1, u_2 \in L^{p_1}(\mathbb{R}^N)$, there are positive constants $M_1, M_2, M'_1, M'_2 > 0$ such that*

$$(1) \quad \begin{aligned} & \|B_1(v_1) - B_1(v_2)\|_{p_1} \\ & \leq M_1 \int_0^t (t-s)^{-\frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_1} \right)} \|v_1 - v_2\|_{q_1} \left(\|v_1\|_{q_1}^{p-1} + \|v_2\|_{q_1}^{p-1} \right) ds, \end{aligned} \quad (3.1)$$

$$\begin{aligned} & \|B_2(u_1) - B_2(u_2)\|_{q_1} \\ & \leq M_2 \int_0^t (t-s)^{-\frac{N}{\alpha} \left(\frac{q}{p_1} - \frac{1}{q_1} \right)} \|u_1 - u_2\|_{p_1} \left(\|u_1\|_{p_1}^{q-1} + \|u_2\|_{p_1}^{q-1} \right) ds. \end{aligned} \quad (3.2)$$

$$(2) \quad \begin{aligned} & \|\nabla [B_1(v_1) - B_1(v_2)]\|_{p'_1} \\ & \leq M'_1 \int_0^t (t-s)^{-\frac{1}{\alpha} - \frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p'_1} \right)} \|v_1 - v_2\|_{q_1} \left(\|v_1\|_{q_1}^{p-1} + \|v_2\|_{q_1}^{p-1} \right) ds, \end{aligned} \quad (3.3)$$

$$\begin{aligned} & \|\nabla [B_2(u_1) - B_2(u_2)]\|_{q'_1} \\ & \leq M'_2 \int_0^t (t-s)^{-\frac{1}{\alpha} - \frac{N}{\alpha} \left(\frac{q}{p_1} - \frac{1}{q'_1} \right)} \|u_1 - u_2\|_{p_1} \left(\|u_1\|_{p_1}^{q-1} + \|u_2\|_{p_1}^{q-1} \right) ds. \end{aligned} \quad (3.4)$$

Proof. We will only prove the estimates in (3.1) and (3.3) because the ones in (3.2) and (3.4) can be obtained analogously.

(1) According to the definition of $B_1(v)$, together with (1.2a), one can calculate

$$\begin{aligned} & \|B_1(v_1) - B_1(v_2)\|_{p_1} \\ & = \left\| \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y, t-s) [N_1(v_1) - N_1(v_2)] dy ds \right\|_{p_1} \\ & \leq C_2 \int_0^t \left\| \int_{\mathbb{R}^N} \Gamma(x-y, t-s) |v_1 - v_2| \left(|v_1|^{p-1} + |v_2|^{p-1} \right) dy \right\|_{p_1} ds \\ & \leq K_1 C_2 \int_0^t (t-s)^{-\frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_1} \right)} \| |v_1 - v_2| \left(|v_1|^{p-1} + |v_2|^{p-1} \right) \|_{\frac{q_1}{p}} ds, \end{aligned} \quad (3.5)$$

here we have used Lemma 2.3 in the second inequality. By employing Hölder's inequality we have

$$\begin{aligned} & \left\| |v_1 - v_2| \left(|v_1|^{p-1} + |v_2|^{p-1} \right) \right\|_{\frac{q_1}{p}} \\ & \leq \left(\left(\int_{\mathbb{R}^N} |v_1 - v_2|^{q_1} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} \left(|v_1|^{p-1} + |v_2|^{p-1} \right)^{\frac{q_1}{p-1}} dx \right)^{\frac{p-1}{p}} \right)^{\frac{p}{q_1}} \\ & = \|v_1 - v_2\|_{q_1} \left(\int_{\mathbb{R}^N} \left(|v_1|^{p-1} + |v_2|^{p-1} \right)^{\frac{q_1}{p-1}} dx \right)^{\frac{p-1}{q_1}} \\ & \leq C \|v_1 - v_2\|_{q_1} \left(\|v_1\|_{q_1} + \|v_2\|_{q_1} \right)^{p-1} \\ & \leq C \|v_1 - v_2\|_{q_1} \left(\|v_1\|_{q_1}^{p-1} + \|v_2\|_{q_1}^{p-1} \right). \end{aligned} \quad (3.6)$$

Substitute (3.6) into (3.5), it yields (3.1), where $M_1 = K_1 C_2 C$.

(2) Using Lemma 2.3, similar to (3.5)–(3.6), we can get

$$\begin{aligned}
& \|\nabla [B_1(v_1) - B_1(v_2)]\|_{p'_1} \\
& \leq C_2 \int_0^t \left\| \int_{\mathbb{R}^N} \nabla \Gamma(x-y, t-s) |v_1 - v_2| \left(|v_1|^{p-1} + |v_2|^{p-1} \right) dy \right\|_{p'_1} ds \\
& \leq K_2 C_2 \int_0^t (t-s)^{-\frac{1}{\alpha} - \frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p'_1} \right)} \left\| |v_1 - v_2| \left(|v_1|^{p-1} + |v_2|^{p-1} \right) \right\|_{\frac{q_1}{p}} ds \\
& \leq M'_1 \int_0^t (t-s)^{-\frac{1}{\alpha} - \frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p'_1} \right)} \|v_1 - v_2\|_{q_1} \left(\|v_1\|_{q_1}^{p-1} + \|v_2\|_{q_1}^{p-1} \right) ds. \tag{3.7}
\end{aligned}$$

□

Using Lemma 3.1, we now give the proof of Theorem 1.1.

Proof of Theorem 1.1. (1) Due to $N > \max\left\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\right\}$, combining (1.12) and (1.13) we can obtain

$$p_1 > \frac{N(pq-1)}{\alpha(p+1)} = P_{sc} > 1, \quad q_1 > \frac{N(pq-1)}{\alpha(q+1)} = Q_{sc} > 1. \tag{3.8}$$

Let

$$b_1 = \frac{N}{\alpha} \left(\frac{1}{P_{sc}} - \frac{1}{p_1} \right) = \frac{N}{\alpha P_{sc}} - \frac{N}{\alpha p_1} = \frac{p+1}{pq-1} - \frac{N}{\alpha p_1}, \tag{3.9}$$

$$b_2 = \frac{N}{\alpha} \left(\frac{1}{Q_{sc}} - \frac{1}{q_1} \right) = \frac{N}{\alpha Q_{sc}} - \frac{N}{\alpha q_1} = \frac{q+1}{pq-1} - \frac{N}{\alpha q_1}. \tag{3.10}$$

Then using (1.12)–(1.14) and (3.8)–(3.10), we conclude that

$$\begin{aligned}
& b_1 > 0, \quad b_2 > 0, \\
& b_2 p - b_1 = 1 - \frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_1} \right), \quad b_1 q - b_2 = 1 - \frac{N}{\alpha} \left(\frac{q}{p_1} - \frac{1}{q_1} \right). \tag{3.11}
\end{aligned}$$

As $\|a(x)\|_{P_{sc}} \leq \frac{\delta}{3K_1}$, $\|b(x)\|_{Q_{sc}} \leq \frac{\delta}{3K_1}$, here K_1 is determined by Lemma 2.3. Applying Lemma 2.3, we obtain for any $t > 0$

$$\sup t^{b_1} \|u_0(x, t)\|_{p_1} = \sup t^{b_1} \|\Gamma(x, t) * a(x)\|_{p_1} \leq K_1 \|a(x)\|_{P_{sc}} \leq \frac{\delta}{3} < \infty, \tag{3.12}$$

$$\sup t^{b_2} \|v_0(x, t)\|_{q_1} = \sup t^{b_2} \|\Gamma(x, t) * b(x)\|_{q_1} \leq K_1 \|b(x)\|_{Q_{sc}} \leq \frac{\delta}{3} < \infty. \tag{3.13}$$

Since $\|f_1(x)\|_{\frac{Q_{sc}}{p}} \leq \frac{\delta}{3K_1}$, $\|f_2(x)\|_{\frac{P_{sc}}{q}} \leq \frac{\delta}{3K_1}$, combining (1.14) and (3.9), applying Lemma 2.3 with $n = p_1$ and $m = \frac{Q_{sc}}{p}$, we have

$$\begin{aligned}
\sup t^{b_1} \|F(f_1)\|_{p_1} & \leq \sup t^{b_1} \int_0^t \|\Gamma(x, t-s) * f_1(x)\|_{p_1} ds \\
& \leq K_1 \sup t^{b_1} \int_0^t (t-s)^{-\frac{N}{\alpha} \left(\frac{p}{Q_{sc}} - \frac{1}{p_1} \right)} \|f_1(x)\|_{\frac{Q_{sc}}{p}} ds \\
& \leq K_1 \|f_1(x)\|_{\frac{Q_{sc}}{p}},
\end{aligned}$$

namely

$$\sup t^{b_1} \|F(f_1)\|_{p_1} \leq K_1 \|f_1(x)\|_{\frac{Q_{sc}}{p}} \leq \frac{\delta}{3}. \tag{3.14}$$

One obtains in a similar way

$$\sup t^{b_2} \|F(f_2)\|_{q_1} \leq K_1 \|f_2(x)\|_{\frac{p_{sc}}{q}} \leq \frac{\delta}{3}. \quad (3.15)$$

Here u is a curve in $L^{p_1}(\mathbb{R}^N)$, $u: [0, \infty) \rightarrow L^{p_1}(\mathbb{R}^N)$, and v is also a curve in $L^{q_1}(\mathbb{R}^N)$, $v: [0, \infty) \rightarrow L^{q_1}(\mathbb{R}^N)$. For the above space E is endowed with the usual norm

$$\|(u, v)\|_E = \sup t^{b_1} \|u\|_{p_1} + \sup t^{b_2} \|v\|_{q_1},$$

we can define a map $\Phi: E \rightarrow E$ by

$$\Phi(u, v) = (u_0, v_0) + B(u, v) + (F(f_1), F(f_2)).$$

For each $\delta > 0$ fixed we consider the ball

$$B_\delta = \{u \in E \mid \|u\|_E < 2\delta\},$$

endowed with the metric

$$d_B(u, v) = \|u - v\|_E, \quad \forall u, v \in B_\delta.$$

Therefore, the metric space (B_δ, d_B) is complete. We will next prove that the operator $\Phi|_{B_\delta}$ is a strict contraction for some $\delta > 0$.

In fact, for any $(u_1, v_1), (u_2, v_2) \in B_\delta$, using (3.1) with $v_2 = 0$ we get

$$\begin{aligned} \sup t^{b_1} \|B_1(v_1)\|_{p_1} &\leq M_1 \sup t^{b_1} \int_0^t (t-s)^{-\frac{N}{\alpha}(\frac{p}{q_1} - \frac{1}{p_1})} \|v_1\|_{q_1}^p ds \\ &\leq M_1 (2\delta)^p \sup t^{b_1} \int_0^t (t-s)^{-\frac{N}{\alpha}(\frac{p}{q_1} - \frac{1}{p_1})} s^{-b_2 p} ds. \end{aligned} \quad (3.16)$$

Using (3.11), one obtains

$$\int_0^t (t-s)^{-\frac{N}{\alpha}(\frac{p}{q_1} - \frac{1}{p_1})} s^{-b_2 p} ds \leq Ct^{-b_1}. \quad (3.17)$$

Substituting (3.17) into (3.16) we get

$$\sup t^{b_1} \|B_1(v_1)\|_{p_1} \leq M_3 (2\delta)^p. \quad (3.18)$$

Similarly, we can arrive at

$$\sup t^{b_2} \|B_2(u_1)\|_{q_1} \leq M_4 (2\delta)^q. \quad (3.19)$$

Estimates (3.12)–(3.15), (3.18)–(3.19) and Minkowski's inequality yield

$$\begin{aligned} \|\Phi(u_1, v_1)\|_E &\leq \sup t^{b_1} \|u_0\|_{p_1} + \sup t^{b_1} \|B_1(v_1)\|_{p_1} + \sup t^{b_1} \|F(f_1)\|_{p_1} \\ &\quad + \sup t^{b_2} \|v_0\|_{q_1} + \sup t^{b_2} \|B_2(u_1)\|_{q_1} + \sup t^{b_2} \|F(f_2)\|_{q_1} \\ &\leq \left(\frac{4}{3} + M_3 2^p \delta^{p-1} + M_4 2^q \delta^{q-1} \right) \delta. \end{aligned} \quad (3.20)$$

Consequently, $\|\Phi(u_1, v_1)\|_E < 2\delta$ if $\frac{4}{3} + M_3 2^p \delta^{p-1} + M_4 2^q \delta^{q-1} < 2$. This shows that $\Phi(B_\delta) \subset B_\delta$.

For all $(u_1, v_1), (u_2, v_2) \in B_\delta$, we then have $\|(u_1, v_1) - (u_2, v_2)\|_E \leq 4\delta$. Combining (3.1) and (3.17) we get

$$\begin{aligned} \sup t^{b_1} \|B_1(v_1) - B_1(v_2)\|_{p_1} &\leq M_1 2^p \delta^{p-1} t^{b_1} \|(u_1, v_1) - (u_2, v_2)\|_E \\ &\quad \cdot \int_0^t (t-s)^{-\frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_1} \right)} s^{-b_2 p} ds \\ &\leq M_3 2^p \delta^{p-1} \|(u_1, v_1) - (u_2, v_2)\|_E. \end{aligned} \quad (3.21)$$

We can proceed this process similarly as in (3.21) to derive that

$$\sup t^{b_2} \|B_2(u_1) - B_2(u_2)\|_{q_1} \leq M_4 2^q \delta^{q-1} \|(u_1, v_1) - (u_2, v_2)\|_E. \quad (3.22)$$

By (3.21) and (3.22), it follows that

$$\begin{aligned} \|\Phi(u_1, v_1) - \Phi(u_2, v_2)\|_E &= \|B(u_1, v_1) - B(u_2, v_2)\|_E \\ &= \sup t^{b_1} \|B_1(v_1) - B_1(v_2)\|_{p_1} + \sup t^{b_2} \|B_2(u_1) - B_2(u_2)\|_{q_1} \\ &\leq \left(M_3 2^p \delta^{p-1} + M_4 2^q \delta^{q-1} \right) \|(u_1, v_1) - (u_2, v_2)\|_E. \end{aligned} \quad (3.23)$$

Combining (3.20) and (3.23) we obtain that the map $\Phi|_{B_\delta}$ is a strict contraction. So it has a fixed point in B_δ , which is the unique solution (u, v) for (1.10) satisfying $\|(u, v)\|_E < 2\delta$.

(2) If $N > 1 + \max\left\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\right\}$, using (1.15)–(1.16), we have

$$p'_1 > \frac{N(pq-1)}{\alpha(p+1)+pq-1} = P'_{sc} > 1, \quad q'_1 > \frac{N(pq-1)}{\alpha(q+1)+pq-1} = Q'_{sc} > 1. \quad (3.24)$$

Let

$$d_1 = \frac{N}{\alpha} \left(\frac{1}{P'_{sc}} - \frac{1}{p'_1} \right) = \frac{1}{\alpha} + \frac{p+1}{pq-1} - \frac{N}{\alpha p'_1}, \quad (3.25)$$

$$d_2 = \frac{N}{\alpha} \left(\frac{1}{Q'_{sc}} - \frac{1}{q'_1} \right) = \frac{1}{\alpha} + \frac{q+1}{pq-1} - \frac{N}{\alpha q'_1}. \quad (3.26)$$

Combining (1.15)–(1.17) and (3.24)–(3.26), we conclude that

$$d_1 > 0, \quad d_2 > 0,$$

and

$$b_2 p - d_1 = 1 - \frac{1}{\alpha} - \frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p'_1} \right), \quad b_1 q - d_2 = 1 - \frac{1}{\alpha} - \frac{N}{\alpha} \left(\frac{q}{p_1} - \frac{1}{q'_1} \right). \quad (3.27)$$

We consider the space $E_1 := L^\infty((0, \infty), L^{p'_1}(\mathbb{R}^N)) \times L^\infty((0, \infty), L^{q'_1}(\mathbb{R}^N))$ endowed with the usual norm

$$\|(u, v)\|_{E_1} = \sup t^{d_1} \|u\|_{p'_1} + \sup t^{d_2} \|v\|_{q'_1}.$$

It is easy to see that $E_1 \subset E$. Applying Lemma 2.3, similar to (1), we have

$$\sup t^{d_1} \|\nabla u_0(x, t)\|_{p'_1} = \sup t^{d_1} \|\nabla \Gamma(x, t) * a(x)\|_{p'_1} \leq K_2 \|a(x)\|_{P_{sc}} \leq \frac{\delta K_2}{3K_1} < \infty, \quad (3.28)$$

$$\sup t^{d_2} \|\nabla v_0(x, t)\|_{q'_1} = \sup t^{d_2} \|\nabla \Gamma(x, t) * b(x)\|_{q'_1} \leq K_2 \|b(x)\|_{Q_{sc}} \leq \frac{\delta K_2}{3K_1} < \infty, \quad (3.29)$$

for any $t > 0$. In view of the definitions of F and B , combining (1.17), (3.8), (3.3) with $v_2 = 0$, (3.25) and (3.27), we can calculate

$$\begin{aligned} \sup t^{d_1} \|\nabla F(f_1)\|_{p'_1} &\leq \sup t^{d_1} \int_0^t \|\nabla \Gamma(x, t-s) * f_1(x)\|_{p'_1} \, ds \\ &\leq K_2 \sup t^{d_1} \int_0^t (t-s)^{-\frac{1}{\alpha} - \frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p'_1}\right)} \|f_1(x)\|_{\frac{Q_{sc}}{p}} \, ds \\ &\leq \frac{\delta K_2}{3K_1}, \end{aligned} \quad (3.30)$$

and

$$\begin{aligned} \sup t^{d_1} \|\nabla B_1(v)\|_{p'_1} &\leq M'_1 \sup t^{d_1} \int_0^t (t-s)^{-\frac{1}{\alpha} - \frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p'_1}\right)} \|v\|_{q_1}^p \, ds \\ &\leq M'_1 (2\delta)^p \sup t^{d_1} \int_0^t (t-s)^{-\frac{1}{\alpha} - \frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p'_1}\right)} s^{-b_2 p} \, ds \\ &\leq M'_3 (2\delta)^p, \end{aligned} \quad (3.31)$$

analogously,

$$\sup t^{d_2} \|\nabla F(f_2)\|_{q'_1} \leq \frac{\delta K_2}{3K_1}, \quad (3.32)$$

and

$$\sup t^{d_2} \|\nabla B_2(u)\|_{q'_1} \leq M'_4 (2\delta)^q. \quad (3.33)$$

It follows that

$$\begin{aligned} \|\nabla \Phi(u, v)\|_{E_1} &\leq \sup t^{d_1} \|\nabla u_0\|_{p'_1} + \sup t^{d_1} \|\nabla B_1(v)\|_{p'_1} + \sup t^{d_1} \|\nabla F(f_1)\|_{p'_1} \\ &\quad + \sup t^{d_2} \|\nabla v_0\|_{q'_1} + \sup t^{d_2} \|\nabla B_2(u)\|_{q'_1} + \sup t^{d_2} \|\nabla F(f_2)\|_{q'_1} \\ &\leq \left(\frac{4K_2}{3K_1} + M'_3 2^p \delta^{p-1} + M'_4 2^q \delta^{q-1} \right) \delta, \end{aligned} \quad (3.34)$$

so $\nabla \Phi(u, v) \in E_1$. In view of the fact that (u, v) is the unique fixed point of Φ on E , thus $\nabla(u, v) \in E_1$, and likewise $\nabla u \in L^{p'_1}(\mathbb{R}^N)$, $\nabla v \in L^{q'_1}(\mathbb{R}^N)$.

Let $\phi(x, t) \in C_c^\infty(\mathbb{R}^N)$ be a nonnegative test function. Multiplying the integral equation (1.4) by $(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}})\phi(x, t)$, and then integrating on Q , we obtain

$$\begin{aligned} &\iint_Q u \left(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \right) \phi(x, t) \, dx dt \\ &= \iint_Q [u_0(x, t) + F(N_1(v) + f_1)] \left(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \right) \phi(x, t) \, dx dt \\ &= \iint_Q u_0(x, t) \left(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \right) \phi(x, t) \, dx dt \\ &\quad + \iint_Q F(N_1(v) + f_1) \left(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \right) \phi(x, t) \, dx dt \\ &=: A_1 + A_2. \end{aligned} \quad (3.35)$$

One can invert the order of integration and utilize the self-adjointness of $(-\Delta)^{\frac{\alpha}{2}}$ to obtain that

$$\begin{aligned} A_1 &= - \int_{\mathbb{R}^N} \int_0^\infty u_0(x, t) \varphi_t(x, t) dt dx + \iint_Q (-\Delta)^{\frac{\alpha}{2}} u_0(x, t) \varphi(x, t) dx dt \\ &= \int_{\mathbb{R}^N} u_0(x, 0) \varphi(x, 0) dx + \iint_Q \left(\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \right) u_0(x, t) \cdot \varphi(x, t) dx dt. \end{aligned} \quad (3.36)$$

Furthermore, A_2 is estimated as follows

$$\begin{aligned} A_2 &= - \int_{\mathbb{R}^N} \int_0^\infty F(N_1(v) + f_1) \varphi_t(x, t) dt dx + \iint_Q (-\Delta)^{\frac{\alpha}{2}} F(N_1(v) + f_1) \varphi(x, t) dx dt \\ &= \iint_Q \left(\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \right) F(N_1(v) + f_1) \cdot \varphi(x, t) dx dt, \end{aligned} \quad (3.37)$$

where we have used the fact that $\varphi \in C_c^\infty(Q)$ and $F(N_1(v) + f_1)(x, 0) = 0$ in the last equality. Plug (3.36) and (3.37) into (3.35), one obtains

$$\begin{aligned} &\iint_Q u \left(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \right) \varphi(x, t) dx dt \\ &= \iint_Q \left(\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \right) (u_0(x, t) + F(N_1(v) + f_1)) \cdot \varphi(x, t) dx dt \\ &\quad + \int_{\mathbb{R}^N} u_0(x, 0) \varphi(x, 0) dx \\ &= \iint_Q \left(\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \right) u(x, t) \cdot \varphi(x, t) dx dt + \int_{\mathbb{R}^N} u_0(x, 0) \varphi(x, 0) dx \\ &= \iint_Q (N_1(v) + f_1(x)) \varphi(x, t) dx dt + \int_{\mathbb{R}^N} u(x, 0) \varphi(x, 0) dx, \end{aligned} \quad (3.38)$$

here we have used (1.4) with $t = 0$ in the last equality. Obviously, (3.38) is (2.1) when $k = 1$.

In the same vein, (2.2) with $k = 1$ can be deduced from the integral equation (1.5), which can be derived through a similar process as in the proof for (3.38). As a result, (u, v) satisfies (1.1) in the sense of distributions.

(3) For $0 < T < \infty$, if $a(x) \in L^{p_1}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, $b(x) \in L^{q_1}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, then repeating the fixed point argument, it is easily conclude that

$$u \in C\left([0, T], L^{p_1}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)\right), \quad v \in C\left([0, T], L^{q_1}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)\right).$$

Next, we show that $u, v \in C([T, \infty), C_0(\mathbb{R}^N))$ by a bootstrap argument.

Indeed, for $t \geq T$, we write

$$\begin{aligned} u(x, t) - u_0(x, t) &= \int_0^T \int_{\mathbb{R}^N} \Gamma(x - y, t - s) [N_1(v) + f_1(y)] dy ds \\ &\quad + \int_T^t \int_{\mathbb{R}^N} \Gamma(x - y, t - s) [N_1(v) + f_1(y)] dy ds \\ &:= I_1(x, t) + I_2(x, t), \end{aligned} \quad (3.39)$$

$$\begin{aligned} v(x, t) - v_0(x, t) &= \int_0^T \int_{\mathbb{R}^N} \Gamma(x - y, t - s) [N_2(u) + f_2(y)] dy ds \\ &\quad + \int_T^t \int_{\mathbb{R}^N} \Gamma(x - y, t - s) [N_2(u) + f_2(y)] dy ds \\ &:= J_1(x, t) + J_2(x, t). \end{aligned} \quad (3.40)$$

Since $u, v \in C([0, T], C_0(\mathbb{R}^N))$, $f_1(x), f_2(x) \in C_0(\mathbb{R}^N)$, it follows that

$$I_1(x, t), J_1(x, t) \in C([T, \infty), C_0(\mathbb{R}^N)).$$

Also, if $t \geq T$, then $t^{-b_1} \leq T^{-b_1} < \infty$, $t^{-b_2} \leq T^{-b_2} < \infty$. On the metric space (B_δ, d_B) , using (3.14) and (3.18), we can get

$$\begin{aligned} \sup t^{b_1} \|I_1(x, t)\|_{p_1} &\leq \sup t^{b_1} \|B_1(v)\|_{p_1} + \sup t^{b_1} \|F(f_1)\|_{p_1} \\ &\leq M_3 (2\delta)^p + \frac{\delta}{3}, \end{aligned} \quad (3.41)$$

it implies that

$$\|I_1(x, t)\|_{p_1} \leq t^{-b_1} \left(M_3 (2\delta)^p + \frac{\delta}{3} \right) \leq T^{-b_1} \left(M_3 (2\delta)^p + \frac{\delta}{3} \right).$$

Similarly, we obtain

$$\|J_1(x, t)\|_{q_1} \leq T^{-b_2} \left(M_4 (2\delta)^q + \frac{\delta}{3} \right). \quad (3.42)$$

As a result,

$$I_1(x, t) \in C([T, \infty), L^{p_1}(\mathbb{R}^N)), \quad J_1(x, t) \in C([T, \infty), L^{q_1}(\mathbb{R}^N)).$$

Hence,

$$\begin{aligned} I_1(x, t) &\in C([T, \infty), L^{p_1}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)), \\ J_1(x, t) &\in C([T, \infty), L^{q_1}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)). \end{aligned}$$

Next, from (1.14) we can get

$$0 < \frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_1} \right) < 1, \quad 0 < \frac{N}{\alpha} \left(\frac{q}{p_1} - \frac{1}{q_1} \right) < 1.$$

Thus, there exists $p_2 > p_1, q_2 > q_1$ such that

$$0 < \frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_2} \right) < 1, \quad 0 < \frac{N}{\alpha} \left(\frac{q}{p_1} - \frac{1}{q_2} \right) < 1.$$

Due to $u \in L^\infty((0, \infty), L^{p_1}(\mathbb{R}^N))$, $v \in L^\infty((0, \infty), L^{q_1}(\mathbb{R}^N))$, then

$$u^q \in L^\infty((0, \infty), L^{\frac{p_1}{q}}(\mathbb{R}^N)), \quad v^p \in L^\infty((0, \infty), L^{\frac{q_1}{p}}(\mathbb{R}^N)).$$

For $t \geq T$, using (3.8)–(3.10), we can get

$$b_1 + 1 - \frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_2} \right) < b_2 p, \quad b_1 + 1 - \frac{N}{\alpha} \left(\frac{p}{Q_{sc}} - \frac{1}{p_2} \right) < 0. \quad (3.43)$$

Taking into account that the fixed point is in B_δ , employing Lemma 2.3, Hölder's inequality and (3.43), we have

$$\begin{aligned}
\sup t^{b_1} \|I_2(x, t)\|_{p_2} &\leq C \sup t^{b_1} \int_T^t \|\Gamma(x, t-s) * v^p\|_{p_2} ds \\
&\quad + \sup t^{b_1} \int_T^t \|\Gamma(x, t-s) * f_1(x)\|_{p_2} ds \\
&\leq C \sup t^{b_1} \int_0^t (t-s)^{-\frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_2}\right)} \|v\|_{q_1}^p ds \\
&\quad + K_1 \sup t^{b_1} \int_T^t (t-s)^{-\frac{N}{\alpha} \left(\frac{p}{Q_{sc}} - \frac{1}{p_2}\right)} \|f_1(x)\|_{\frac{Q_{sc}}{p}} ds \\
&\leq C \sup t^{b_1} (2\delta)^p \int_0^t (t-s)^{-\frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_2}\right)} s^{-b_2 p} ds \\
&\quad + K_1 \sup t^{b_1+1-\frac{N}{\alpha} \left(\frac{p}{Q_{sc}} - \frac{1}{p_2}\right)} \|f_1(x)\|_{\frac{Q_{sc}}{p}} \\
&\leq C (2\delta)^p + \frac{\delta}{3}, \tag{3.44}
\end{aligned}$$

therefore,

$$\|I_2(x, t)\|_{p_2} \leq T^{-b_1} \left((2\delta)^p + \frac{\delta}{3} \right),$$

that is $I_2(x, t) \in C([T, \infty), L^{p_2}(\mathbb{R}^N))$. Similarly, we can get $J_2(x, t) \in C([T, \infty), L^{q_2}(\mathbb{R}^N))$.

Combining (3.9) and $p_2 > p_1$, it yields $b_1 - \frac{N}{\alpha} \left(\frac{1}{p_{sc}} - \frac{1}{p_2}\right) < 0$. Similar to (3.12), (3.14) and (3.16), we have

$$\sup t^{b_1} \|u_0(x, t)\|_{p_2} \leq K_1 \sup t^{b_1 - \frac{N}{\alpha} \left(\frac{1}{p_{sc}} - \frac{1}{p_2}\right)} \|a(x)\|_{p_{sc}} \leq \frac{\delta}{3} T^{b_1 - \frac{N}{\alpha} \left(\frac{1}{p_{sc}} - \frac{1}{p_2}\right)}, \tag{3.45}$$

$$\sup t^{b_1} \|F(f_1)\|_{p_2} \leq K_1 \sup t^{b_1+1-\frac{N}{\alpha} \left(\frac{p}{Q_{sc}} - \frac{1}{p_2}\right)} \|f_1(x)\|_{\frac{Q_{sc}}{p}} \leq \frac{\delta}{3} T^{b_1+1-\frac{N}{\alpha} \left(\frac{p}{Q_{sc}} - \frac{1}{p_2}\right)}, \tag{3.46}$$

and

$$\sup t^{b_1} \|B_1(v)\|_{p_2} \leq M_3 (2\delta)^p \sup t^{1+b_1-\frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_2}\right) - b_2 p} \leq M_3 (2\delta)^p T^{1+b_1-\frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_2}\right) - b_2 p}. \tag{3.47}$$

Using (3.44)–(3.47), we can calculate

$$\begin{aligned}
\sup t^{b_1} \|u(x, t)\|_{p_2} &\leq \sup t^{b_1} \|u_0(x, t)\|_{p_2} + \sup t^{b_1} \|I_1(x, t)\|_{p_2} + \sup t^{b_1} \|I_2(x, t)\|_{p_2} \\
&\leq \frac{\delta}{3} T^{b_1 - \frac{N}{\alpha} \left(\frac{1}{p_{sc}} - \frac{1}{p_2}\right)} + M_3 (2\delta)^p T^{1+b_1-\frac{N}{\alpha} \left(\frac{p}{q_1} - \frac{1}{p_2}\right) - b_2 p} \\
&\quad + \frac{\delta}{3} T^{b_1+1-\frac{N}{\alpha} \left(\frac{p}{Q_{sc}} - \frac{1}{p_2}\right)} + C (2\delta)^p + \frac{\delta}{3} \\
&\leq C,
\end{aligned}$$

consequently,

$$\|u(x, t)\|_{p_2} \leq C t^{-b_1} \leq C T^{-b_1}.$$

Obviously, $u(x, t) \in C([T, \infty), L^{p_2}(\mathbb{R}^N))$. Analogously, we can obtain that $v(x, t) \in C([T, \infty), L^{q_2}(\mathbb{R}^N))$.

Iterating this procedure a finite number of times, we deduce that

$$u(x, t), v(x, t) \in C([T, \infty), C_0(\mathbb{R}^N)).$$

This completes the proof. \square

4 Blow-up of nonnegative solutions for (1.1) in the critical case

Throughout this section, we shall assume $1 < p \leq q$ for definiteness. The following estimate of the solution for (1.1) is the key step in proving the blow-up theorem for (1.1) in the critical case.

Lemma 4.1. *Assume $u, v \in C^1(Q) \cap L^\infty(Q)$, and let (u, v) is a nonnegative global solution for (1.1) and satisfies $(-\Delta)^{\frac{\alpha}{2}}u(x, t), (-\Delta)^{\frac{\alpha}{2}}v(x, t) \in L^\infty(Q)$, $u_0(x, t), v_0(x, t)$ be as in (1.6)–(1.7), then there exists a constant C , depending on only p and q , such that*

$$t^{\frac{p+1}{pq-1}}u_0(x, t) \leq C, \quad t^{\frac{q+1}{pq-1}}v_0(x, t) \leq C. \quad (4.1)$$

Proof. We will only show the first estimate in (4.1) because the proof of the second one is similar. Arguing as in Lemma 2.5 one has

$$v(x, t) \geq Ct \left| \int_{\mathbb{R}^N} \Gamma(x-y, t) a(y) dy \right|^q. \quad (4.2)$$

We now substitute (4.2) into (1.4), drop the first and third terms on the right there, and use (1.2a), Jensen's inequality and Tonelli's theorem to obtain

$$\begin{aligned} u(x, t) &\geq C_1 \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y, t-s) v^p(y, s) dy ds \\ &\geq C \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y, t-s) \left(s \left| \int_{\mathbb{R}^N} \Gamma(y-z, s) a(z) dz \right|^q \right)^p dy ds \\ &\geq C \int_0^t s^p \left(\int_{\mathbb{R}^N} \Gamma(x-y, t-s) \int_{\mathbb{R}^N} \Gamma(y-z, s) a(z) dz dy \right)^{pq} ds \\ &= \frac{C}{p+1} t^{p+1} (u_0(x, t))^{pq}. \end{aligned} \quad (4.3)$$

We next substitute (4.3) into (1.5). Ignoring again the first and third terms, we have

$$\begin{aligned} v(x, t) &\geq \frac{C}{(p+1)^q} \int_0^t s^{(p+1)q} \int_{\mathbb{R}^N} \Gamma(x-y, t-s) \left(\int_{\mathbb{R}^N} \Gamma(y-z, s) a(z) dz \right)^{pq^2} dy ds \\ &\geq \frac{C}{(p+1)^q} \cdot \frac{t^{(p+1)q+1}}{(p+1)q+1} (u_0(x, t))^{pq^2}. \end{aligned} \quad (4.4)$$

Plugging (4.4) into (1.4) we obtain in turn

$$u(x, t) \geq \frac{C}{(p+1)^{pq}} \cdot \frac{1}{((p+1)q+1)^p} \cdot \frac{t^{(p+1)(pq+1)}}{(p+1)(pq+1)} (u_0(x, t))^{p^2q^2}. \quad (4.5)$$

Iterating the previous procedure, it follows that for any integer k

$$u(x, t) \geq A_k B_k t^{(p+1)(1+pq+(pq)^2+\dots+(pq)^{k-1})} (u_0(x, t))^{(pq)^k}, \quad (4.6)$$

where

$$A_k = \frac{C}{(p+1)(pq)^{k-1}} \left(\frac{1}{(p+1)(1+pq)} \right)^{(pq)^{k-2}} \left(\frac{1}{(p+1)(1+pq+p^2q^2)} \right)^{(pq)^{k-3}} \cdots \left(\frac{1}{(p+1)(1+pq+p^2q^2+\cdots+(pq)^{k-1})} \right), \quad (4.7)$$

$$B_k = \left(\frac{1}{(p+1)q+1} \right)^{\frac{(pq)^{k-1}}{q}} \left(\frac{1}{(p+1)(1+pq)q+1} \right)^{\frac{(pq)^{k-2}}{q}} \cdot \left(\frac{1}{(p+1)(1+pq+p^2q^2)q+1} \right)^{\frac{(pq)^{k-3}}{q}} \cdots \left(\frac{1}{(p+1)(1+pq+p^2q^2+\cdots+(pq)^{k-2})q+1} \right)^p, \quad (4.8)$$

here constant C in A_k is changed one by one according to the different k .

We next note that for any positive integers k and l , the following equalities hold

$$\begin{aligned} \sum_{i=0}^{k-1} (pq)^i &= \frac{(pq)^k - 1}{pq - 1}, \\ \sum_{i=0}^{l-1} (l-i)(pq)^i &= pq \frac{(pq)^l - 1}{(pq - 1)^2} - \frac{l}{pq - 1}, \\ \sum_{i=1}^{l-1} (l-i)(pq)^i &= pq \frac{(pq)^l - 1}{(pq - 1)^2} - \frac{pql}{pq - 1}. \end{aligned}$$

Now set $\lambda = pq$, then (4.7) can be written as

$$A_k = C \left(\frac{1}{p+1} \right)^{\frac{\lambda^k - 1}{\lambda - 1}} \prod_{i=1}^{k-1} \left(\frac{\lambda - 1}{\lambda^{i+1} - 1} \right)^{\lambda^{k-i-1}}. \quad (4.9)$$

We note that $(p+1)(1+\lambda+\lambda^2+\cdots+\lambda^i) > 1$ for any integer $i > 1$, then

$$\begin{aligned} B_k &\geq \left(\frac{1}{(p+1)(q+1)} \right)^{\frac{\lambda^{k-1}}{q}} \left(\frac{1}{(p+1)(q+1)(1+\lambda)} \right)^{\frac{\lambda^{k-2}}{q}} \\ &\quad \cdot \left(\frac{1}{(p+1)(q+1)(1+\lambda+\lambda^2)} \right)^{\frac{\lambda^{k-3}}{q}} \\ &\quad \cdots \left(\frac{1}{(p+1)(q+1)(1+\lambda+\lambda^2+\cdots+\lambda^{k-2})} \right)^{\frac{\lambda}{q}}, \end{aligned}$$

and therefore

$$B_k \geq \left(\frac{1}{(p+1)(q+1)} \right)^{\frac{\lambda^{k-1}}{q(\lambda-1)}} \cdot \prod_{i=1}^{k-1} \left(\frac{\lambda - 1}{\lambda^{i+1} - 1} \right)^{\frac{\lambda^{k-i-1}}{q}}. \quad (4.10)$$

Substitution of (4.9) and (4.10) into (4.6) yields

$$\begin{aligned} u(x, t) &\geq Ct^{\frac{(p+1)(\lambda^k - 1)}{\lambda - 1}} (u_0(x, t))^{\lambda^k} \left(\frac{1}{p+1} \right)^{\frac{\lambda^k - 1}{\lambda - 1} \left(1 + \frac{1}{q}\right)} \\ &\quad \times \left(\frac{1}{q+1} \right)^{\frac{\lambda^k - 1}{q(\lambda - 1)}} \left(\prod_{i=1}^{k-1} \left(\frac{\lambda - 1}{\lambda^{i+1} - 1} \right)^{\lambda^{k-i-1}} \right)^{1 + \frac{1}{q}}, \end{aligned}$$

hence

$$t^{\frac{(p+1)(\lambda^k-1)}{(\lambda-1)\lambda^k}} u_0(x, t) \leq C(p+1)^{\frac{\lambda^k-1}{(\lambda-1)\lambda^k} \left(1+\frac{1}{q}\right)} (q+1)^{\frac{\lambda^k-1}{q(\lambda-1)\lambda^k}} \|u(x, t)\|_{\infty}^{\lambda^{-k}} \times \left(\prod_{i=1}^{k-1} \left(\frac{\lambda-1}{\lambda^{i+1}-1} \right)^{\lambda^{k-i-1}} \right)^{-(1+\frac{1}{q})\lambda^{-k}}. \quad (4.11)$$

Since $\|u(x, t)\|_{\infty} < +\infty$ for any $t \in [0, \infty)$, letting $k \rightarrow \infty$ in (4.11) and recalling that $\lambda = pq$, we finally arrive at

$$t^{\frac{p+1}{pq-1}} u_0(x, t) \leq C < +\infty$$

for some constant C that only depends on p and q . \square

In the critical case, applying Lemma 4.1, the blow-up theorem (Theorem 1.2) of the non-negative solutions for (1.1) is proved as follows.

Proof of Theorem 1.2. Let $1 < p \leq q$, $N = \max\left\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\right\} = \frac{\alpha(q+1)}{pq-1}$. We suppose by contradiction that there exists a nonnegative global solution $u, v \in C^1(Q) \cap L^\infty(Q)$ for (1.1) such that $(-\Delta)^{\frac{\alpha}{2}} u(x, t), (-\Delta)^{\frac{\alpha}{2}} v(x, t) \in L^\infty(Q)$. Using Lemma 4.1, there exists a constant C which depends only on p and q such that

$$t^{\frac{q+1}{pq-1}} v_0(x, t) \leq C,$$

that is

$$t^{\frac{N}{\alpha}} v_0(0, t) \leq C. \quad (4.12)$$

By employing Fatou’s lemma and Lemma 2.4, we can derive

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{\frac{N}{\alpha}} v_0(0, t) &\geq \int_{\mathbb{R}^N} \lim_{t \rightarrow \infty} t^{\frac{N}{\alpha}} \Gamma(-y, t) b(y) dy \\ &\geq C'_{\alpha, N} \int_{\mathbb{R}^N} b(y) dy. \end{aligned} \quad (4.13)$$

Estimates (4.12) and (4.13) yield

$$\|b(x)\|_1 \leq C, \quad (4.14)$$

where $C > 0$ depends only on α and N . Regarding $v(\cdot, t)$ as initial value, by (4.14) we get

$$\|v(\cdot, t)\|_1 \leq C, \quad \forall t \geq 0. \quad (4.15)$$

Let $t_0 > 0$ be as in Lemma 2.5. For $t > 1$, we set $\tilde{u}(\cdot, t) = u(\cdot, t + t_0)$, $\tilde{v}(\cdot, t) = v(\cdot, t + t_0)$. Obviously, (\tilde{u}, \tilde{v}) is also a solution for (1.1). Applying (1.4) and Lemma 2.5, it follows that

$$\begin{aligned} \tilde{u}(x, t) &\geq \int_{\mathbb{R}^N} \Gamma(x - y, t) \tilde{u}(y, 0) dy \\ &\geq C \int_{\mathbb{R}^N} \Gamma(x - y, t) \Gamma(y, \tau) dy \\ &= C \Gamma(x, t + \tau), \end{aligned} \quad (4.16)$$

here we have used Proposition 2.2 (5) in the last equality. By means of (1.5), (1.2b), Tonelli's theorem and Proposition 2.2 (4), we have

$$\begin{aligned} \|\tilde{v}(x, t)\|_1 &\geq \tilde{C}_1 \int_{\mathbb{R}^N} \left(\int_0^t \int_{\mathbb{R}^N} \Gamma(x - y, t - s) \tilde{u}^q(y, s) dy ds \right) dx \\ &= \tilde{C}_1 \int_0^t \int_{\mathbb{R}^N} \tilde{u}^q(y, s) dy ds. \end{aligned} \quad (4.17)$$

Substituting (4.16) into (4.17), combining with Proposition 2.2 (2), and observing that $N = \frac{\alpha(q+1)}{pq-1}$, we can estimate

$$\begin{aligned} \|\tilde{v}(x, t)\|_1 &\geq C \int_1^t \int_{\mathbb{R}^N} \Gamma^q(y, s + \tau) dy ds \\ &\geq C \int_1^t (s + \tau)^{\frac{N}{\alpha}q} ds \int_{\mathbb{R}^N} \Gamma^q(y, 1) dy \\ &= C \left[(t + \tau)^{\frac{q(q+1)}{pq-1} + 1} - (1 + \tau)^{\frac{q(q+1)}{pq-1} + 1} \right]. \end{aligned} \quad (4.18)$$

In the proceeding estimate, we consider the integral $\int_{\mathbb{R}^N} \Gamma^q(y, 1) dy$ as a constant.

In addition, estimate (4.15) also holds for the function \tilde{v} , which conflicts with (4.18) as t large enough. \square

5 Blow-up of nonnegative solutions for (1.3) in the subcritical case

Next, we prove the nonexistence of nonnegative solutions for (1.3) in the subcritical case.

Proof of Theorem 1.3. Assume (1.3) admits a nonnegative global solution (u, v) , we argue by contradiction. Take $\varphi(x, t) = \phi^{s_1}(\frac{|x|}{R})\phi^{ks_2}(\frac{t}{R^\sigma})$ as the test function in (2.1) and (2.2), where $s_1, s_2 \geq \max\{\frac{p}{p-1}, \frac{q}{q-1}\}$, $\sigma \geq \max\{\frac{\alpha}{k}, \frac{\beta}{k}\}$, $\phi(\rho) \in C_c^\infty(\mathbb{R})$ is the "standard cut-off function" with the following properties:

$$0 \leq \phi(\rho) \leq 1, \quad \text{and} \quad \phi(\rho) = \begin{cases} 1, & \text{if } \rho \leq 1, \\ 0, & \text{if } \rho \geq 2. \end{cases}$$

Substituting $\varphi(x, t)$ into (2.1), thanks to $\frac{\partial^i \varphi}{\partial t^i}(x, 0) \equiv 0, i = 1, 2, \dots, k-1$, it follows that

$$\begin{aligned} &\iint_Q (N_1(v) + f_1(x)) \varphi(x, t) dx dt + \int_{\mathbb{R}^N} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x, 0) \varphi(x, 0) dx \\ &= \iint_Q u \left[(-1)^k \frac{\partial^k \varphi}{\partial t^k} + (-\Delta)^{\frac{\alpha}{2}} \varphi \right] dx dt \\ &\leq \iint_Q u \left[(-1)^k \phi^{s_1} \left(\frac{|x|}{R} \right) \frac{d^k}{dt^k} \left(\phi^{ks_2} \left(\frac{t}{R^\sigma} \right) \right) \right. \\ &\quad \left. + s_1 \phi^{ks_2} \left(\frac{t}{R^\sigma} \right) \phi^{s_1-1} \left(\frac{|x|}{R} \right) (-\Delta)^{\frac{\alpha}{2}} \phi \left(\frac{|x|}{R} \right) \right] dx dt, \end{aligned} \quad (5.1)$$

here we have used Ju's inequality in the last inequality. Since $u(x, t) \geq 0$, $N_1(v) \geq 0$ and (1.2a), we obtain

$$\iint_Q N_1(v) \varphi(x, t) dx dt \geq \iint_Q (C_1 |v|^p + N_1(0)) \varphi(x, t) dx dt \geq C_1 \iint_Q |v|^p \varphi(x, t) dx dt.$$

By $\frac{\partial^{k-1}u}{\partial t^{k-1}}(x, 0) \geq 0$, $f_1(x) \geq 0$ and $f_1(x) \not\equiv 0$, applying Hölder's inequality, we get that

$$\begin{aligned} & \iint_Q v^p \varphi(x, t) dx dt \\ & < C \left(\iint_Q u^q \varphi(x, t) dx dt \right)^{\frac{1}{q}} \cdot \left(\iint_Q \left| (-1)^k \phi^{s_1} \left(\frac{|x|}{R} \right) \frac{d^k}{dt^k} \left(\phi^{ks_2} \left(\frac{t}{R^\sigma} \right) \right) \phi^{\frac{1-q'}{q'}} \right|^{q'} \right. \\ & \quad \left. + \left| s_1 \phi^{ks_2} \left(\frac{t}{R^\sigma} \right) \phi^{s_1-1} \left(\frac{|x|}{R} \right) (-\Delta)^{\frac{\alpha}{2}} \phi \left(\frac{|x|}{R} \right) \phi^{\frac{1-q'}{q'}} \right|^{q'} dx dt \right)^{\frac{1}{q'}}, \end{aligned} \quad (5.2)$$

here we have used C_p inequality in the last inequality. Set

$$\begin{aligned} B_1 &:= C \iint_Q \left| \phi^{s_1} \left(\frac{|x|}{R} \right) \frac{d^k}{dt^k} \left(\phi^{ks_2} \left(\frac{t}{R^\sigma} \right) \right) \phi^{\frac{1-q'}{q'}} \right|^{q'} dx dt, \\ B_2 &:= C \iint_Q \left| s_1 \phi^{ks_2} \left(\frac{t}{R^\sigma} \right) \phi^{s_1-1} \left(\frac{|x|}{R} \right) (-\Delta)^{\frac{\alpha}{2}} \phi \left(\frac{|x|}{R} \right) \phi^{\frac{1-q'}{q'}} \right|^{q'} dx dt. \end{aligned}$$

Then we have

$$\iint_Q v^p \varphi(x, t) dx dt < \left(\iint_Q u^q \varphi(x, t) dx dt \right)^{\frac{1}{q}} \cdot (B_1 + B_2)^{\frac{1}{q'}}. \quad (5.3)$$

Now according to the expression of $\varphi(x, t)$, we get

$$\begin{aligned} B_1 &= C \iint_Q \phi^{s_1} \left(\frac{|x|}{R} \right) \left| \frac{1}{R^{k\sigma}} \left(\phi^{ks_2} \right)^{(k)} \left(\frac{t}{R^\sigma} \right) \right|^{q'} \phi^{ks_2(1-q')} \left(\frac{t}{R^\sigma} \right) dx dt \\ &= C \int_{R^\sigma}^{2R^\sigma} \left| \frac{1}{R^{k\sigma}} \left(\phi^{ks_2} \right)^{(k)} \left(\frac{t}{R^\sigma} \right) \right|^{q'} \phi^{ks_2(1-q')} \left(\frac{t}{R^\sigma} \right) dt \int_{\{x \in \mathbb{R}^N: |x| < 2R\}} \phi^{s_1} \left(\frac{|x|}{R} \right) dx. \end{aligned} \quad (5.4)$$

Since $\phi(\rho) \in [0, 1]$, and if $s_2 \geq \max\{\frac{p}{p-1}, \frac{q}{q-1}\}$, namely $s_2 \geq q'$, for any $1 \leq j \leq k$, we obtain

$$\left| \left(\phi^{ks_2} \right)^{(j)}(\rho) \right|^{q'} \leq \left| \sum_{i=1}^j C \frac{(ks_2)!}{(ks_2-i)!} \phi^{ks_2-i}(\rho) \right|^{q'} \leq C \phi^{ks_2(q'-1)}(\rho). \quad (5.5)$$

Substituting (5.5) into (5.4), it yields

$$B_1 \leq C \int_{R^\sigma}^{2R^\sigma} \frac{1}{R^{k\sigma q'}} dt \int_{\{x \in \mathbb{R}^N: |x| < 2R\}} \phi^{s_1} \left(\frac{|x|}{R} \right) dx \leq CR^{N+\sigma-k\sigma q'}. \quad (5.6)$$

Furthermore, B_2 is estimated as follows

$$\begin{aligned} B_2 &= Cs_1^{q'} \int_0^{2R^\sigma} \phi^{ks_2} \left(\frac{t}{R^\sigma} \right) dt \int_{\{x \in \mathbb{R}^N: |x| < 2R\}} \phi^{s_1-q'} \left(\frac{|x|}{R} \right) \left| (-\Delta)^{\frac{\alpha}{2}} \phi \left(\frac{|x|}{R} \right) \right|^{q'} dx \\ &\leq 2Cs_1^{q'} R^\sigma \int_{\{x \in \mathbb{R}^N: |x| < 2R\}} \phi^{s_1-q'} \left(\frac{|x|}{R} \right) \left| (-\Delta)^{\frac{\alpha}{2}} \phi \left(\frac{|x|}{R} \right) \right|^{q'} dx. \end{aligned} \quad (5.7)$$

Let $y = \frac{x}{R}$. Using the definition of fractional Laplacian [12], we obtain

$$(-\Delta)^{\frac{\alpha}{2}} \phi \left(\frac{x}{R} \right) = R^{-\alpha} (-\Delta)^{\frac{\alpha}{2}} \phi(|y|).$$

Plugging the above equality into (5.7), it follows that

$$\begin{aligned}
B_2 &\leq CR^\sigma \int_{\{x \in \mathbb{R}^N: |x| < 2R\}} \left| (-\Delta)^{\frac{\alpha}{2}} \phi \left(\frac{|x|}{R} \right) \right|^{q'} dx \\
&\leq CR^\sigma \int_{\{y \in \mathbb{R}^N: |y| < 2\}} R^{N-\alpha q'} \left| (-\Delta)^{\frac{\alpha}{2}} \phi (|y|) \right|^{q'} dy \\
&\leq CR^{N+\sigma-\alpha q'}.
\end{aligned} \tag{5.8}$$

Here we consider the integral $\int_{\{y \in \mathbb{R}^N: |y| < 2\}} \left| (-\Delta)^{\frac{\alpha}{2}} \phi (|y|) \right|^{q'} dy$ as a constant.

When $\sigma \geq \max \left\{ \frac{\alpha}{k}, \frac{\beta}{k} \right\}$, the powers in (5.6) and (5.8) satisfy the following inequality:

$$N + \sigma - k\sigma q' \leq N + \sigma - \alpha q'.$$

Thus, we eventually arrive at

$$B_1 + B_2 \leq CR^{N+\sigma-\alpha q'} \tag{5.9}$$

for sufficiently large R .

Analogously, we next substitute $\varphi(x, t)$ into (2.2), use Hölder's inequality and the definition of the global weak solution to obtain

$$\iint_Q u^q \varphi(x, t) dx dt < \left(\iint_Q v^p \varphi(x, t) dx dt \right)^{\frac{1}{p}} \cdot (B_3 + B_4)^{\frac{1}{p'}}, \tag{5.10}$$

where

$$B_3 = C \iint_Q \left| \phi^{s_1} \left(\frac{|x|}{R} \right) \frac{d^k}{dt^k} \left(\phi^{ks_2} \left(\frac{t}{R^\sigma} \right) \right) \phi^{\frac{1-p'}{p'}} \right|^{p'} dx dt,$$

and

$$B_4 = C \iint_Q \left| s_1 \phi^{ks_2} \left(\frac{t}{R^\sigma} \right) \phi^{s_1-1} \left(\frac{|x|}{R} \right) (-\Delta)^{\frac{\beta}{2}} \phi \left(\frac{|x|}{R} \right) \phi^{\frac{1-p'}{p'}} \right|^{p'} dx dt.$$

Similar to (5.6) and (5.8), we can derive that

$$B_3 \leq CR^{N+\sigma-k\sigma p'}, \quad B_4 \leq CR^{N+\sigma-\beta p'},$$

since $\sigma \geq \max \left\{ \frac{\alpha}{k}, \frac{\beta}{k} \right\}$, we derive

$$B_3 + B_4 \leq CR^{N+\sigma-\beta p'} \tag{5.11}$$

for sufficiently large R .

Plugging (5.3) into (5.10) we obtain

$$\left(\iint_Q u^q \varphi(x, t) dx dt \right)^{1-\frac{1}{pq}} < (B_1 + B_2)^{\frac{1}{pq'}} \cdot (B_3 + B_4)^{\frac{1}{p'}} \leq CR^{\frac{N+\sigma}{pq'} - \frac{\alpha}{p} + \frac{N+\sigma}{p'} - \beta} \tag{5.12}$$

for sufficiently large R . Similarly, substituting (5.10) into (5.3) gives

$$\left(\iint_Q v^p \varphi(x, t) dx dt \right)^{1-\frac{1}{pq}} < CR^{\frac{N+\sigma}{qp'} - \frac{\alpha}{q} + \frac{N+\sigma}{q'} - \beta} \tag{5.13}$$

for sufficiently large R . Letting $R \rightarrow \infty$ in (5.12) and (5.13). Using the hypothesis of Theorem 1.3, we can see that $\frac{N+\sigma}{pq'} - \frac{\alpha}{p} + \frac{N+\sigma}{p'} - \beta < 0$ or $\frac{N+\sigma}{qp'} - \frac{\alpha}{q} + \frac{N+\sigma}{q'} - \beta < 0$, we eventually have

$$\iint_Q u^q dx dt = \lim_{R \rightarrow +\infty} \iint_Q u^q \varphi(x, t) dx dt \leq 0,$$

or

$$\iint_Q v^p dx dt = \lim_{R \rightarrow +\infty} \iint_Q v^p \varphi(x, t) dx dt \leq 0.$$

Therefore $u(x, t) \equiv 0$ or $v(x, t) \equiv 0$ in Q . This is a contradiction with the assumption that $f_i(x) \not\equiv 0$ for $i = 1, 2$, which ends the proof. \square

A Appendix

Below we give the complete proof of Lemma 2.3.

Proof of Lemma 2.3. (1) We first prove (2.3). When $1 + \frac{1}{n} = \frac{1}{m} + \frac{1}{r}$, using Proposition 2.2 (4), we obtain $\Gamma(x, t) \in L^r(\mathbb{R}^N)$. Since $h(x, t) \in L^m(\mathbb{R}^N)$, by applying generalized Young's inequality, we get

$$\|\Gamma(x, t) * h(x, t)\|_n \leq \|\Gamma(x, t)\|_r \|h(x, t)\|_m = \left(\int_{\mathbb{R}^N} \Gamma^r(x, t) dx \right)^{\frac{1}{r}} \cdot \|h(x, t)\|_m.$$

By Proposition 2.2 (1) with $s = 1$, namely $\Gamma(x, t) = t^{-\frac{N}{\alpha}} \Gamma(t^{-\frac{1}{\alpha}} x, 1)$, we have

$$\|\Gamma(x, t) * h(x, t)\|_n \leq t^{-\frac{N}{\alpha}} \left(\int_{\mathbb{R}^N} \Gamma^r(t^{-\frac{1}{\alpha}} x, 1) dx \right)^{\frac{1}{r}} \cdot \|h(x, t)\|_m. \quad (\text{A.1})$$

Utilizing Proposition 2.2 (3), we estimate

$$\begin{aligned} \int_{\mathbb{R}^N} \Gamma^r(t^{-\frac{1}{\alpha}} x, 1) dx &\leq C \int_{\mathbb{R}^N} \frac{1}{\left(1 + t^{-\frac{1}{\alpha}} |x|\right)^{(N+\alpha)r}} dx \\ &\leq C \int_{\mathbb{R}^N} \frac{t^{\frac{N}{\alpha}}}{\left(1 + t^{-\frac{1}{\alpha}} |x|\right)^{(N+\alpha)r}} d\left(t^{-\frac{1}{\alpha}} x\right). \end{aligned} \quad (\text{A.2})$$

Denote $y = t^{-\frac{1}{\alpha}} x$, then (A.2) can be reduced to

$$\int_{\mathbb{R}^N} \Gamma^r(t^{-\frac{1}{\alpha}} x, 1) dx \leq C t^{\frac{N}{\alpha}} \int_{\mathbb{R}^N} \frac{1}{(1 + |y|)^{(N+\alpha)r}} dy. \quad (\text{A.3})$$

Due to $1 + \frac{1}{n} = \frac{1}{m} + \frac{1}{r}$ and $1 \leq m \leq n \leq \infty$, we have $r \geq 1$. Otherwise, it conflicts with $m \leq n$. Therefore, $(N + \alpha)r > N$. Consequently, the integral on the right hand side of inequality (A.3) is integrable. We now substitute (A.3) into (A.1) to obtain

$$\|\Gamma(x, t) * h(x, t)\|_n \leq K_1 t^{-\frac{N}{\alpha} \left(1 - \frac{1}{r}\right)} \|h(x, t)\|_m, \quad (\text{A.4})$$

where $K_1 = \left(C \int_{\mathbb{R}^N} \frac{1}{(1 + |y|)^{(N+\alpha)r}} dy\right)^{\frac{1}{r}}$.

(2) The proof of (2.4) is as follows. Similarly, by using generalized Young's inequality and Proposition 2.2 (1), we conclude

$$\begin{aligned} \left\| (-\Delta)^{\frac{l}{2}} \Gamma(x, t) * h(x, t) \right\|_n &\leq \left\| (-\Delta)^{\frac{l}{2}} \Gamma(x, t) \right\|_r \|h(x, t)\|_m \\ &= \left(\int_{\mathbb{R}^N} \left((-\Delta)^{\frac{l}{2}} \Gamma(x, t) \right)^r dx \right)^{\frac{1}{r}} \cdot \|h(x, t)\|_m \\ &= t^{-\frac{N}{\alpha}} \left(\int_{\mathbb{R}^N} \left((-\Delta)^{\frac{l}{2}} \Gamma(t^{-\frac{1}{\alpha}} x, 1) \right)^r dx \right)^{\frac{1}{r}} \cdot \|h(x, t)\|_m. \end{aligned}$$

Let $y = t^{-\frac{1}{\alpha}} x$. Using the definition of fractional Laplacian, we get that

$$\begin{aligned} \int_{\mathbb{R}^N} \left((-\Delta)^{\frac{l}{2}} \Gamma(t^{-\frac{1}{\alpha}} x, 1) \right)^r dx &= t^{-\frac{lr}{\alpha} + \frac{N}{\alpha}} \int_{\mathbb{R}^N} \left((-\Delta)^{\frac{l}{2}} \Gamma(y, 1) \right)^r dy \\ &\leq C' t^{-\frac{lr}{\alpha} + \frac{N}{\alpha}} \int_{\mathbb{R}^N} \frac{1}{(1 + |y|)^{(N+l)r}} dy, \end{aligned} \quad (\text{A.5})$$

here we have used Proposition 2.2 (3) in the last inequality. Since $r \geq 1$, then $(N + l)r > N$. Therefore, the integral on the right hand side of inequality (A.5) is integrable. It follows that

$$\left\| (-\Delta)^{\frac{l}{2}} \Gamma(x, t) * h(x, t) \right\|_n \leq K_2 t^{-\frac{l}{\alpha} - \frac{N}{\alpha} (1 - \frac{1}{r})} \|h(x, t)\|_m, \quad (\text{A.6})$$

where $K_2 = (C' \int_{\mathbb{R}^N} \frac{1}{(1 + |y|)^{(N+l)r}} dy)^{\frac{1}{r}}$.

In particular, substituting $l = 1$ into (A.6), we have

$$\|\nabla \Gamma(x, t) * h(x, t)\|_n \leq K_2 t^{-\frac{1}{\alpha} - \frac{N}{\alpha} (1 - \frac{1}{r})} \|h(x, t)\|_m, \quad \forall t \in (0, T].$$

This completes the proof of Lemma 2.3. \square

Based on the method in the proof of Lemma 3.2 in [24], we now show the complete proof of Lemma 2.5 as follows.

Proof of Lemma 2.5. Let $t_0 > 0$ be such that $\Gamma(0, t_0) \leq 1$. We obtain

$$\Gamma(x - y, t_0) = \Gamma\left(\frac{1}{2}(2x - 2y), t_0\right).$$

Proposition 2.2 (1) and (6) yield

$$\Gamma\left(\frac{1}{2}(2x - 2y), t_0\right) \geq \Gamma(2x, t_0) \Gamma(2y, t_0) = 2^{-N} \Gamma\left(x, \frac{t_0}{2^\alpha}\right) \Gamma(2y, t_0),$$

that is

$$\Gamma(x - y, t_0) \geq 2^{-N} \Gamma\left(x, \frac{t_0}{2^\alpha}\right) \Gamma(2y, t_0). \quad (\text{A.7})$$

Plugging (A.7) into (1.4), and dropping the second and third terms on the right side, we obtain

$$\begin{aligned} u(x, t_0) &\geq \int_{\mathbb{R}^N} \Gamma(x - y, t_0) a(y) dy \\ &\geq \int_{\mathbb{R}^N} 2^{-N} \Gamma\left(x, \frac{t_0}{2^\alpha}\right) \Gamma(2y, t_0) a(y) dy \\ &= C \Gamma(x, \tau), \end{aligned}$$

where $C = \int_{\mathbb{R}^N} 2^{-N} \Gamma(2y, t_0) a(y) dy$, $\tau = \frac{t_0}{2^\alpha} > 0$.

In order to get the corresponding result of $v(x, t)$, for $q > 1$, by employing Jensen’s inequality and Tonelli’s theorem we get

$$\begin{aligned} v(x, t) &\geq \int_0^t \int_{\mathbb{R}^N} \Gamma(x - y, t - s) N_2(u)(y, s) dy ds \\ &\geq \tilde{C}_1 \int_0^t \int_{\mathbb{R}^N} \Gamma(x - y, t - s) \left| \int_{\mathbb{R}^N} \Gamma(y - z, s) a(z) dz \right|^q dy ds \\ &\geq C \int_0^t \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Gamma(x - y, t - s) \Gamma(y - z, s) a(z) dz dy \right|^q ds \\ &= C \int_0^t \left| \int_{\mathbb{R}^N} \Gamma(x - z, t) a(z) dz \right|^q ds, \end{aligned}$$

hence

$$v(x, t) \geq Ct \left| \int_{\mathbb{R}^N} \Gamma(x - y, t) a(y) dy \right|^q. \tag{A.8}$$

For the above t_0 , we then have

$$\begin{aligned} v(x, t_0) &\geq Ct_0 \left| \int_{\mathbb{R}^N} \Gamma(x - y, t_0) a(y) dy \right|^q \\ &\geq Ct_0 \left| \int_{\mathbb{R}^N} 2^{-N} \Gamma\left(x, \frac{t_0}{2^\alpha}\right) \Gamma(2y, t_0) a(y) dy \right|^q \\ &= C\Gamma^q(x, \tau), \end{aligned}$$

where $\tau = \frac{t_0}{2^\alpha} > 0$. We consider the integral $\left| \int_{\mathbb{R}^N} 2^{-N} \Gamma(2y, t_0) a(y) dy \right|^q$ as a constant. □

Remark A.1. In Section 4, we assumed that $1 < p \leq q$. If $p \geq q > 1$, then the conclusion of Lemma 2.5 needs to be changed as follows:

$$v(x, t_0) \geq C\Gamma(x, \tau), \quad u(x, t_0) \geq C\Gamma^p(x, \tau).$$

Its proof is similar to the proof of Lemma 2.5. Therefore, in the case of $p \geq q > 1$, through the homologous proof of Theorem 1.2, we can still get the conclusion of Theorem 1.2.

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