Ground state solution for fractional problem with critical combined nonlinearities

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Abstract. This paper is concerned with the following nonlocal problem with combined critical nonlinearities

$$(-\Delta)^s u = -\alpha|u|^{q-2}u + \beta u + \gamma|u|^{2_s^*-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega,$$

where $s \in (0,1)$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ is a bounded $C^{1,1}$ domain with Lipschitz boundary, α is a positive parameter, $q \in (1,2)$, β and γ are positive constants, and $2_s^* = 2N/(N-2s)$ is the fractional critical exponent. For $\gamma > 0$, if $N \geqslant 4s$ and $0 < \beta < \lambda_{1,s}$, or $N > 2s$ and $\beta \geqslant \lambda_{1,s}$, we show that the problem possesses a ground state solution when α is sufficiently small.

Keywords: fractional problem, ground state solution, critical combined nonlinearities.

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1   Introduction

In this paper, we study ground state solution for the following fractional equation

$$\begin{cases} (-\Delta)^s u = -\alpha|u|^{q-2}u + \beta u + \gamma|u|^{2_s^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ is a bounded $C^{1,1}$ domain with Lipschitz boundary, $\alpha > 0$ is a parameter, $q \in (1,2)$, β and γ are positive constants, and $2_s^* = 2N/(N-2s)$ is the fractional critical exponent. The equation (1.1) is driven by the fractional Laplacian $(-\Delta)^s$ and exhibits combined nonlinearities and linear perturbation. $(-\Delta)^s$ is the nonlocal operator defined as follows

$$(-\Delta)^s u(x) := 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where $B_\varepsilon(x)$ denotes the open ball centered at x and of radius $\varepsilon > 0$. The operator $(-\Delta)^s$ arises in physics, biology, chemistry and finance and can be seen as the infinitesimal generators of

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Lévy stable diffusion process [3,4]. And, $(-\Delta + m^2)^{\frac{1}{2}}$ appears naturally in quantum mechanics, where m is the mass of the particle under consideration [35]. The study of nonlinear equations involving a fractional Laplacian has attracted much attention from many mathematicians working in different fields. We refer to [5,9,12,14–17,19,23–25,27–33,36,38–43] for more details on the fractional operator and applications.

From [42] we get that the spectrum of $(-\Delta)^s$ on $X_0^s(\Omega)$ consists of a sequence of eigenvalues $\{\lambda_{j,s}\}$ satisfying

$$0 < \lambda_{1,s} < \lambda_{2,s} \leqslant \lambda_{3,s} \leqslant \ldots \leqslant \lambda_{j,s} \leqslant \lambda_{j+1,s} \leqslant \ldots, \quad \lambda_{j,s} \to \infty \quad \text{as } j \to \infty,$$

where the space $X_0^s(\Omega)$ is given in [40].

For the problem (1.1), when $\alpha = 0$ and $\gamma = 1$, the equation is a fractional critical problem with linear perturbation term. For the critical problem, due to a lack of compactness occurs, there are serious difficulties when we try to find critical points by variational methods. Motivated by the pioneering work of Brezis and Nirenberg [8], the nonlocal fractional counterpart of the Laplacian equations involving critical nonlinearity were studied in [38–43], their model is the equation

$$\begin{cases} (-\Delta)^s u = \beta u + |u|^{2_s^* - 2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.2}$$

Servadei and Valdinoci have showed that problem (1.2) admits a nontrivial solution in the following case:

(i) $N > 4s$ and $\beta > 0$;

(ii) $N = 4s$ and $\beta \neq \lambda_{k,s}, k = 1, 2, \ldots$;

(iii) $2s < N < 4s$ and β is sufficiently large.

Moreover, the multiplicity result of (1.2) was proved by Fiscella et al. [24], where it was shown the number of solutions is at least twice the multiplicity of the $\lambda_{k,s}$, provided that β lies in a suitable neighborhood of $\lambda_{k,s}$, the authors also gave an estimate of the length of this neighborhood. Figueiredo et al. [23] proved the problem (1.2) has at least $cat_\Omega(\Omega)$ nontrivial solutions if $N \geqslant 4s$ and β is sufficiently small. For interesting results on the fractional Brezis–Nirenberg problem, we refer to [12,27] and the references therein.

For the problem (1.1), when $\alpha < 0$, $\beta = 0$ and $\gamma = 1$, the equation contains a sublinear term $|u|^{q-2} u$ and a critical superlinear term $|u|^{2_s^* - 2} u$, it belongs to the class of problems with competing nonlinearities, for instance sublinear-superlinear. An early example in this direction was given in [26] for the p-Laplacian operator. Other results for the classical Laplacian operator can be found in [1,6,13]. More generally, the problem with completely nonlinear operators has been studied in [10]. And we observed that Barrios et al. [5] have studied the critical fractional problem with concave-convex power nonlinearities, where they considered the following problem

$$\begin{cases} (-\Delta)^s u = -\alpha u^{q-1} + u^{2_s^* - 1}, \ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.3}$$

Main results show the existence and multiplicity of solutions to problem (1.3) for different values of α. To be more precise, assume that $N > 2s$, then there is $\alpha_3 < 0$, such that problem (1.3):

(i) has no solution for $\alpha < \alpha_3$;

(ii) if $\alpha = \alpha_3$ there exists at least one solution;

(iii) for $\alpha_3 < \alpha < 0$, there are at least two solutions, one of them is a minimal solution.

We refer to [15, 16, 21, 30] and references therein for more fractional problem with competing nonlinearities.

For the problem (1.1), if u in the critical term is the positive part of u, the problem becomes a nonlocal Dirichlet problem with asymmetric nonlinearities, that is

$$\begin{cases} (-\Delta)^s u = -\alpha |u|^{q-2} u + \beta u + \gamma (u^+)^{2_s^*-1} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.4}$$

Miyagaki et al. [36] studied the existence of at least three nontrivial solutions for problem (1.4). The corresponding local problem was studied by de Paiva and Presoto [37]. The study of equations with critical exponent and asymmetric nonlinearities was initiated by De Figueiredo and Yang [18] to investigate Ambrosetti–Prodi type problems involving critical growth. The Ambrosetti–Prodi type problems have a strong physical meaning because it appears in quantum mechanics models with asymmetric nonlinearities, see for instance [9, 11, 20, 28] and references therein. It can be seen from [36, Theorem 6], the two constant sign solutions of (1.4) are solutions for two corresponding auxiliary problems which are similar to problem (1.1). So solution of the problem (1.1) is valuable to study the Ambrosetti–Prodi type problem.

Motivated by the above works, in this paper, we consider the existence of ground state solutions of (1.1) which is affected by combined nonlinearities and linear perturbation. Our first main result can be stated as follows.

Theorem 1.1. *Let $\gamma > 0$, then there exists $\alpha_1 > 0$, such that for any $\alpha \in (0, \alpha_1)$, problem (1.1) has a ground state solution u_{m_α}, provided that*

- *$N \geqslant 4s$ and $0 < \beta < \lambda_{1,s}$ or*

- *$N > 2s$ and $\beta \geqslant \lambda_{1,s}$.*

It is well known that ground state solutions have important applications. For instance, to obtain the optimal constant in the Sobolev inequality and the interpolation estimates of the Gagliardo–Nirenberg inequality. To possess a global solution of nonlinear Schrödinger equation when L^2-norm of the initial value is sufficiently small. To overcome the loss of compactness when we consider some Schrödinger equation with potential and so on. There are several ways to get the ground state solution. The one in Theorem 1.1 is found by looking for the point at which infimum of the functional on Nehari manifold is attainable. Furthermore, under the same assumptions, we show that the functional possesses mountain pass geometry. By estimate of the minimax level, we have the following theorem.

Theorem 1.2. *Assume that the hypotheses of Theorem 1.1 are satisfied, problem (1.1) has a mountain pass ground state solution u_{c_α}.*

It is observed that there are some differences between the cases $\alpha = 0$ and $\alpha > 0$. Indeed, assume that $2s < N < 4s$. In case of $\alpha = 0$, the problem (1.1) translates into problem (1.2). Servadei et al. [39] have showed that problem (1.2) has a nontrivial solution when β is sufficiently large. If $\alpha > 0$ is small enough, owing to influence of sublinear term, Theorem 1.1

and Theorem 1.2 state that the problem (1.1) has solutions as long as $\beta \geqslant \lambda_{1,s}$ holds. Suppose that $N = 4s$, the problem (1.1) has solutions for any $\beta > 0$, which is also different from $\beta \neq \lambda_{k,s}, k = 1, 2, \ldots$ when $\alpha = 0$.

There are some similarities between the cases $\alpha < 0$ and $\alpha > 0$. Note that $\alpha < 0$ in problem (1.3), Barrios et al. [5] indicate that problem (1.3) has solutions when α is close to zero. For problem (1.1), if $\alpha < 0$ and $\beta \geqslant \lambda_{1,s}$, then it is easy to verify that it has no nontrivial solution since the corresponding Nehari manifold is empty, and it is unknown whether the Nehari manifold is nonempty in the case $0 < \beta < \lambda_{1,s}$. Thus in the present paper we study the case of $\alpha > 0$. Even though the sign of sublinearity in problem (1.1) is opposite to that of problem (1.3), Theorem 1.1 and 1.2 show that the problem (1.1) has ground state solution when α is small enough.

It can be seen from the comparison above, Theorem 1.1 and 1.2 are not only effective supplement to the main results of Barrios et al. [5], but also have some differences with Servadei et al. [39]. To the best of our knowledge, these results are novel and meaningful.

Since the problem (1.1) is affected by sublinearity, linearity and critical superlinearity at the same time, we have a different situation from (1.2) or (1.3). The minimax principle used by Servadei et al. in [38–43] and the method of obtaining the minimal solution in [5] cannot be applied directly to problem (1.1). Some other techniques and methods are used. In the proof of Theorem 1.1, an abstract result for existence of constrained extrema is used. So it is necessary to obtain that the infimum of the functional on the Nehari manifold is strictly less than admissible threshold for the (PS) condition. To confirm this result, a crucial point is to show a sufficiently small upper bound for the quotient

$$\frac{\|u_\varepsilon\|^2 - \beta \|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2_s^*}^2} \tag{1.5}$$

when $\varepsilon > 0$ is sufficiently small, and u_ε is given in [43]. The estimation of (1.5) in Lemma 3.1 is meticulous. In the proof of Theorem 1.2, due to influence of the sublinear term, it seems impossible to prove that the functional has mountain pass geometry directly according to the structures of the functional and the properties of $X_0^s(\Omega)$. We prove that 0 is the local minimum point of the functional in a special subspace of $X_0^s(\Omega)$.

The organization of this paper is as follows: In Section 2, we introduce some notations and preliminary lemmas which are needed later. Section 3 and Section 4 are devoted to the proof of Theorems 1.1–1.2, respectively.

2   Preliminaries

In this section, we recall a few notions and results that will be used later on. Throughout the paper, $|A|$ denotes the N-dimensional Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$, $L^r(\Omega)$ is usual Lebesgue space endowed with the norm $\| \cdot \|_r$ for $1 \leq r < \infty$. We recall that the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$[u]_s := \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}$$

provided the integral is finite. The fractional Sobolev space $H^s(\mathbb{R}^N)$ is introduced in [19] as

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : [u]_s < \infty \right\}$$

endowed with the norm $\|u\|_{H^s} = \left(\|u\|_2^2 + [u]_s^2\right)^{1/2}$ making it a Hilbert space. The relevant space to problem (1.1) is the closed subspace of $H^s(\mathbb{R}^N)$ given by

$$X_0^s(\Omega) := \left\{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\},$$

this Hilbert space was introduced in [40] with the scalar product

$$\langle u, v \rangle_{X_0^s(\Omega)} = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy$$

inducing the equivalent norm $\| \cdot \| = [\cdot]_s$.

It is known from [19], the following embedding results hold true:

$$\begin{aligned} X_0^s(\Omega) &\hookrightarrow L^v(\Omega) \quad \text{compactly for any } v \in [1, 2_s^*), \\ X_0^s(\Omega) &\hookrightarrow L^{2_s^*}(\Omega) \quad \text{continuously.} \end{aligned} \tag{2.1}$$

And the constant

$$S_s = \inf_{u \in H^s(\mathbb{R}^N) \backslash \{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy}{\left(\int_{\mathbb{R}^N} |u(x)|^{2_s^*} dx\right)^{2/2_s^*}} \tag{2.2}$$

is finite, by [14, Theorem 1.1] we know that S_s is attained by the function $(1/\|\tilde{u}\|_{2_s^*})\tilde{u}$ with $\tilde{u}(x) = (1 + |x|^2)^{-\frac{N-2s}{2}}, x \in \mathbb{R}^N$. For every $\varepsilon > 0$, we shall use the family of functions $\{U_\varepsilon\}$ introduced in [43] as

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}} \frac{1}{\|\tilde{u}\|_{2_s^*}} \tilde{u}\left(\frac{x}{\varepsilon S_s^{1/(2s)}}\right), \quad x \in \mathbb{R}^N,$$

which is a solution of problem $(-\Delta)^s u = |u|^{2_s^*-2}u$, in $\mathbb{R}^N$. Without loss of generality, we suppose that $0 \in \Omega$, let us fix $\delta > 0$ such that $B_{4\delta} \subset \Omega$, and let $\eta \in C^\infty(\mathbb{R}^N)$ be such that $0 \leqslant \eta \leqslant 1$ in $\mathbb{R}^N$, $\eta \equiv 1$ in B_δ and $\eta \equiv 0$ in $\mathbb{R}^N \backslash B_{2\delta}$, where $B_\delta = B(0, \delta)$. We denote by u_ε the following function

$$u_\varepsilon(x) = \eta(x) U_\varepsilon(x). \tag{2.3}$$

It is obvious that $u_\varepsilon \in X_0^s(\Omega)$, and the following estimates on the function u_ε were proved in [43, Proposition 21 and 22],

$$\|u_\varepsilon\|^2 \leqslant S_s^{\frac{N}{2s}} + O\left(\varepsilon^{N-2s}\right), \tag{2.4}$$

$$\|u_\varepsilon\|_{2_s^*}^{2_s^*} = S_s^{\frac{N}{2s}} + O\left(\varepsilon^N\right), \tag{2.5}$$

$$\|u_\varepsilon\|_2^2 \geqslant \begin{cases} C_s \varepsilon^{2s} + O\left(\varepsilon^{N-2s}\right) & \text{if } N > 4s, \\ C_s \varepsilon^{2s} |\log \varepsilon| + O\left(\varepsilon^{2s}\right) & \text{if } N = 4s, \end{cases} \tag{2.6}$$

as $\varepsilon \to 0$, for some positive constant C_s depending on s.

The Euler functional $\mathcal{I}_\alpha : X_0^s(\Omega) \to \mathbb{R}$ corresponding to problem (1.1) is given by

$$\mathcal{I}_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dx dy + \frac{\alpha}{q} \int_\Omega |u|^q dx - \frac{\beta}{2} \int_\Omega u^2 dx - \frac{\gamma}{2_s^*} \int_\Omega |u|^{2_s^*} dx. \tag{2.7}$$

It is easy to verify that $I_\alpha \in C^1(X_0^s(\Omega))$ with

$$\langle \mathcal{I}_\alpha'(u), v \rangle = \langle u, v \rangle_{X_0^s(\Omega)} + \alpha \int_\Omega |u|^{q-2} uv dx - \beta \int_\Omega uv dx - \gamma \int_\Omega |u|^{2_s^*-2} uv dx, \tag{2.8}$$

for $v \in X_0^s(\Omega)$. A direct computation shows that weak solution of (1.1) is critical point of I_α.

We say that the functional $\mathcal{I}_\alpha$ satisfies the Palais–Smale ((PS) for short) condition at level $c \in \mathbb{R}$ if any sequence $\{u_j\} \subset X_0^s(\Omega)$ such that

$$\mathcal{I}_\alpha(u_j) \to c \tag{2.9}$$

and

$$\sup\left\{\left|\langle \mathcal{I}_\alpha'(u_j), \varphi \rangle\right| : \varphi \in X_0^s(\Omega), \|\varphi\| = 1\right\} \to 0 \quad \text{as } j \to \infty \tag{2.10}$$

admits a subsequence which is convergent in $X_0^s(\Omega)$.

Now, we are ready to prove that the functional $\mathcal{I}_\alpha$ satisfies the (PS) condition in a suitable energy range involving the best fractional critical Sobolev constant S_s given in (2.2).

Lemma 2.1. *Assume that $1 < q < 2$, β and γ are positive constants, and $\alpha > 0$. Then the functional $\mathcal{I}_\alpha$ satisfies the (PS) condition at any level $c < \frac{s}{N}(S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}$.*

Proof. Let $\{u_j\} \subset X_0^s(\Omega)$ be a (PS) sequence for $\mathcal{I}_\alpha$, first of all, we show the $\{u_j\}$ is bounded in $X_0^s(\Omega)$. In fact, by (2.9) and (2.10), there is $\kappa > 0$ such that $\left|\mathcal{I}_\alpha(u_j)\right| \le \kappa, \left|\langle \mathcal{I}_\alpha'(u_j), u_j \rangle\right| \le \kappa \|u_j\|$. Taking into account that $1 < q < 2 < 2_s^*$, we have

$$
\begin{aligned}
\kappa\left(1 + \|u_j\|\right) &\ge \mathcal{I}_\alpha(u_j) - \frac{1}{2}\langle \mathcal{I}_\alpha'(u_j), u_j \rangle \\
&= \alpha\left(\frac{1}{q} - \frac{1}{2}\right)\int_\Omega |u_j(x)|^q \, dx + \gamma\left(\frac{1}{2} - \frac{1}{2_s^*}\right)\int_\Omega |u_j(x)|^{2_s^*} \, dx \\
&\ge \frac{\gamma s}{N}\|u_j\|_{2_s^*}^{2_s^*}.
\end{aligned}
$$

For $\bar{\kappa} := \frac{N\kappa}{\gamma s} > 0$, hence,

$$\|u_j\|_{2_s^*}^{2_s^*} \le \bar{\kappa}\left(1 + \|u_j\|\right) \quad \text{for } j \in \mathbb{N}. \tag{2.11}$$

Thus, by the Hölder inequality and (2.11), we get

$$\|u_j\|_2^2 \le |\Omega|^{\frac{2s}{N}} \|u_j\|_{2_s^*}^2 \le \bar{\kappa}^{\frac{2}{2_s^*}} |\Omega|^{\frac{2s}{N}} \left(1 + \|u_j\|\right)^{\frac{2}{2_s^*}} \le \hat{\kappa}\left(1 + \|u_j\|\right) \tag{2.12}$$

with $\hat{\kappa} := \bar{\kappa}^{\frac{2}{2_s^*}} |\Omega|^{\frac{2s}{N}}$. Thus, by (2.11) and (2.12) we conclude that

$$
\begin{aligned}
\kappa \ge \mathcal{I}_\alpha(u_j) &= \frac{1}{2}\|u_j\|^2 + \frac{\alpha}{q}\int_\Omega |u_j|^q dx - \frac{\beta}{2}\int_\Omega u_j^2 dx - \frac{\gamma}{2_s^*}\int_\Omega |u_j|^{2_s^*} dx \\
&\ge \frac{1}{2}\|u_j\|^2 - \frac{\beta}{2}\int_\Omega u_j^2 dx - \frac{\gamma}{2_s^*}\int_\Omega |u_j|^{2_s^*} dx \\
&\ge \frac{1}{2}\|u_j\|^2 - \left(\frac{\beta}{2}\hat{\kappa} + \frac{\gamma}{2_s^*}\bar{\kappa}\right)\left(1 + \|u_j\|\right).
\end{aligned}
$$

Hence, $\{u_j\}$ is bounded in $X_0^s(\Omega)$.

Consequently, passing to a subsequence if necessary, we may assume that

$$
\begin{aligned}
&u_j \rightharpoonup u_\infty \quad \text{in } X_0^s(\Omega), \qquad u_j \to u_\infty \quad \text{in } L^2(\Omega), \\
&u_j \to u_\infty \quad \text{in } L^q(\Omega) \quad \text{and} \quad u_j \to u_\infty \quad \text{for a.e. } x \in \Omega \text{ with some } u_\infty \in X_0^s(\Omega).
\end{aligned} \tag{2.13}
$$

Next, we show that u_∞ is a solution of (1.1) and $\mathcal{I}_\alpha(u_\infty) \ge 0$. Indeed, for any $\varphi \in X_0^s(\Omega)$, by (2.1) and (2.13), we have that

$$\int_\Omega |u_j(x)|^{2_s^*-2} u_j(x)\varphi(x) dx \to \int_\Omega |u_\infty(x)|^{2_s^*-2} u_\infty(x)\varphi(x) dx, \tag{2.14}$$

and

$$\int_\Omega |u_j(x)|^{q-2} u_j(x)\varphi(x)dx \to \int_\Omega |u_\infty(x)|^{q-2} u_\infty(x)\varphi(x)dx,$$

$$\int_\Omega u_j(x)\varphi(x)dx \to \int_\Omega u_\infty(x)\varphi(x)dx. \tag{2.15}$$

Thus, by (2.13), (2.14) and (2.15), we conclude that

$$\langle \mathcal{I}'_\alpha(u_j), \varphi \rangle \to \langle \mathcal{I}'_\alpha(u_\infty), \varphi \rangle.$$

In view of (2.10), we get

$$\langle \mathcal{I}'_\alpha(u_\infty), \varphi \rangle = 0, \tag{2.16}$$

namely, u_∞ is a solution of (1.1). Taking $\varphi = u_\infty$ as a test function in (2.16), we get

$$\|u_\infty\|^2 = -\alpha \int_\Omega |u_\infty|^q dx + \beta \int_\Omega u_\infty^2 dx + \gamma \int_\Omega |u_\infty|^{2_s^*} dx,$$

then $1 < q < 2 < 2_s^*$ implies that

$$\mathcal{I}_\alpha(u_\infty) = \alpha \left(\frac{1}{q} - \frac{1}{2} \right) \int_\Omega |u_\infty(x)|^q \, dx + \gamma \left(\frac{1}{2} - \frac{1}{2_s^*} \right) \int_\Omega |u_\infty(x)|^{2_s^*} \, dx \geqslant 0. \tag{2.17}$$

Finally, we show that $\{u_j\}$ converges to u_∞ in $X_0^s(\Omega)$. Note that $\{u_j\}$ is bounded in $X_0^s(\Omega)$, by (2.13), (2.1) and Brezis–Lieb lemma [7, Theorem 1], for $p \in (1, 2_s^*]$, we have

$$\int_\Omega |u_j|^p dx - \int_\Omega |u_j - u_\infty|^p dx = \int_\Omega |u_\infty|^p dx + o(1) \tag{2.18}$$

The boundedness of $\{u_j\}$ in $X_0^s(\Omega)$, (2.1), (2.10), (2.13), (2.16) and (2.18) imply that

$$\begin{aligned}
o(1) &= \langle \mathcal{I}'_\alpha(u_j) - \mathcal{I}'_\alpha(u_\infty), u_j - u_\infty \rangle \\
&= \|u_j - u_\infty\|^2 + \alpha \int_\Omega (|u_j|^{q-2}u_j - |u_\infty|^{q-2}u_\infty)(u_j - u_\infty)dx - \beta \int_\Omega |u_j - u_\infty|^2 dx \\
&\quad - \gamma \int_\Omega (|u_j|^{2_s^*-2}u_j - |u_\infty|^{2_s^*-2}u_\infty)(u_j - u_\infty)dx \\
&= \|u_j - u_\infty\|^2 + \alpha \int_\Omega |u_j - u_\infty|^q dx - \beta \int_\Omega |u_j - u_\infty|^2 dx - \gamma \int_\Omega |u_j - u_\infty|^{2_s^*} dx + o(1),
\end{aligned}$$

thus, by (2.13), we deduce that

$$\|u_j - u_\infty\|^2 - \gamma \int_\Omega |u_j - u_\infty|^{2_s^*} dx = o(1). \tag{2.19}$$

Since the sequence $\{\|u_j\|\}$ is bounded, we may assume that $\|u_j - u_\infty\|^2 \to L$ as $j \to +\infty$, in view of (2.19), $\int_\Omega |u_j(x) - u_\infty(x)|^{2_s^*} dx \to \frac{L}{\gamma}$ as $j \to +\infty$. So taking into account (2.2), we get $\left(\frac{L}{\gamma} \right)^{\frac{2}{2_s^*}} S_s \leqslant L$, then $L = 0$ or $L \geqslant (S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}$. Assume that $L \geqslant (S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}$. Since $u_j \rightharpoonup u_\infty$, we have

$$\|u_j - u_\infty\|^2 = \|u_j\|^2 - \|u_\infty\|^2 + o(1). \tag{2.20}$$

So (2.13), (2.20) and (2.18) yield

$$\begin{aligned}
\mathcal{I}_\alpha(u_j) &= \frac{1}{2}\|u_j\|^2 + \frac{\alpha}{q} \int_\Omega |u_j|^q dx - \frac{\beta}{2} \int_\Omega u_j^2 dx - \frac{\gamma}{2_s^*} \int_\Omega |u_j|^{2_s^*} dx \\
&= I_\alpha(u_\infty) + \frac{1}{2}\|u_j - u_\infty\|^2 - \frac{\gamma}{2_s^*} \int_\Omega |u_j - u_\infty|^{2_s^*} dx + o(1).
\end{aligned} \tag{2.21}$$

By (2.21), (2.19) and (2.17), we obtain

$$c = \lim_{j \to \infty} \mathcal{I}_\alpha(u_j) = I_\alpha(u_\infty) + \frac{1}{2}L - \frac{\gamma}{2_s^*}\frac{L}{\gamma} \geqslant \frac{s}{N}L \geqslant \frac{s}{N}(S_s)^{\frac{N}{2s}}\gamma^{\frac{2s-N}{2s}}$$

which contradicts the condition $c < \frac{s}{N}(S_s)^{\frac{N}{2s}}\gamma^{\frac{2s-N}{2s}}$. Thus $L = 0$, and so $\|u_j - u_\infty\| \to 0$ as $j \to +\infty$. $\qquad\square$

The manifold we are interested in this paper is the Nehari manifold associated with $I_\alpha(u)$, given by

$$\mathcal{N}_\alpha := \left\{ u \in X_0^s(\Omega)\backslash\{0\} : \langle \mathcal{I}_\alpha'(u), u \rangle = 0 \right\}.$$

First of all, we point out that $\mathcal{N}_\alpha$ is not empty.

Lemma 2.2. $\mathcal{N}_\alpha \neq \varnothing$. Precisely, for every $u \in X_0^s(\Omega)\backslash\{0\}$, then there exists a unique $t_u \in (0, +\infty)$, such that $t_u u \in \mathcal{N}_\alpha$.

Proof. Fix $u \in X_0^s(\Omega)\backslash\{0\}$, we consider the function $\varphi : [0, +\infty) \to \mathbb{R}$

$$\varphi(t) := \langle \mathcal{I}_\alpha'(tu), tu \rangle = t^2\|u\|^2 + \alpha t^q \int_\Omega |u|^q dx - \beta t^2 \int_\Omega u^2 dx - \gamma t^{2_s^*} \int_\Omega |u|^{2_s^*} dx = t^q \phi(t),$$

where

$$\phi(t) = \alpha \int_\Omega |u|^q dx + t^{2-q}(\|u\|^2 - \beta \int_\Omega u^2 dx) - \gamma t^{2_s^*-q} \int_\Omega |u|^{2_s^*} dx.$$

We have that $\phi \in C^1([0, +\infty))$ with $\phi(0) = \alpha \int_\Omega |u|^q dx > 0$ and $\lim_{t \to +\infty} \phi(t) = -\infty$. In the case of $0 < \beta < \lambda_{1,s}$, we have $\|u\|^2 - \beta \int_\Omega u^2 dx > 0$, ϕ has a unique maximum point

$$t_0 = \left(\frac{(2-q)(\|u\|^2 - \beta \int_\Omega u^2 dx)}{(2_s^* - q)\gamma \int_\Omega |u|^{2_s^*} dx} \right)^{\frac{1}{2_s^*-2}},$$

ϕ increases on $[0, t_0)$ and decreases on $(t_0, +\infty)$. In the case of $\beta \geqslant \lambda_{1,s}$, we can get that $\|u\|^2 - \beta \int_\Omega u^2 dx \leqslant 0$ and ϕ decreases on $[0, +\infty)$. Thus there is only one zero point in $(0, +\infty)$ to ϕ, namely, there exists a unique $t_u \in (0, +\infty)$, such that $t_u u \in \mathcal{N}_\alpha$. $\qquad\square$

The $\mathcal{N}_\alpha$ is a natural constraint for the functional $\mathcal{I}_\alpha$, since every constrained critical point of $\mathcal{I}_\alpha$ on $\mathcal{N}_\alpha$ is indeed a critical point of $\mathcal{I}_\alpha$. Precisely, the following result holds true.

Lemma 2.3. $\mathcal{I}_\alpha$ is bounded from below on $\mathcal{N}_\alpha$. And u is a critical point of $\mathcal{I}_\alpha$ constrained to $\mathcal{N}_\alpha$ if and only if u is a nontrivial critical point of $\mathcal{I}_\alpha$.

Proof. Notice that on $\mathcal{N}_\alpha$ the functional $\mathcal{I}_\alpha$ reads as follows

$$\mathcal{I}_\alpha(u) = \alpha\left(\frac{1}{q} - \frac{1}{2}\right)\int_\Omega |u|^q dx + \gamma\left(\frac{1}{2} - \frac{1}{2_s^*}\right)\int_\Omega |u|^{2_s^*} dx,$$

thank to $1 < q < 2 < 2_s^*$, so that $\inf_{u \in \mathcal{N}_\alpha} \mathcal{I}_\alpha(u) \geqslant 0$.

It is obvious that every nontrivial critical point of $\mathcal{I}_\alpha$ belongs to $\mathcal{N}_\alpha$. Let us show the converse. In the sequel we will denote by $G_\alpha : X_0^s(\Omega) \to \mathbb{R}$, the functional given by

$$G_\alpha(u) := \langle \mathcal{I}_\alpha'(u), u \rangle = \|u\|^2 + \alpha \int_\Omega |u|^q dx - \beta \int_\Omega u^2 dx - \gamma \int_\Omega |u|^{2_s^*} dx.$$

It is easy to verify that $G_\alpha \in C^1(X_0^s(\Omega))$ and

$$\langle G_\alpha'(u), u \rangle = 2\|u\|^2 + q\alpha \int_\Omega |u|^q dx - 2\beta \int_\Omega u^2 dx - \gamma 2_s^* \int_\Omega |u|^{2_s^*} dx,$$

so that, taking into account the definition of $\mathcal{N}_\alpha$, we have

$$\langle G_\alpha'(u), u \rangle = \alpha(q-2) \int_\Omega |u|^q dx + \gamma(2 - 2_s^*) \int_\Omega |u|^{2_s^*} dx < 0 \quad \text{for } u \in \mathcal{N}_\alpha. \tag{2.22}$$

Let u be a constrained critical point of $\mathcal{I}_\alpha$ on $\mathcal{N}_\alpha$, namely $u \in \mathcal{N}_\alpha$ and

$$\mathcal{I}_\alpha'(u) = \eta G_\alpha'(u) \tag{2.23}$$

for some $\eta \in \mathbb{R}$. Note that (2.23) yields

$$\langle \mathcal{I}_\alpha'(u), u \rangle = \eta \langle G_\alpha'(u), u \rangle. \tag{2.24}$$

Taking into account the fact that $u \in \mathcal{N}_\alpha$ and (2.22), by (2.24) we deduce that $\eta = 0$. Hence, again by (2.23), we get $\mathcal{I}_\alpha'(u) = 0$. $\qquad \square$

We say that the functional $\mathcal{I}_\alpha$ constrained on $\mathcal{N}_\alpha$ satisfies the (PS) condition at level $c \in \mathbb{R}$ if any sequence $\{u_j\} \subset \mathcal{N}_\alpha$ such that (2.9) holds and there exists $\{\eta_j\} \subset \mathbb{R}$ with

$$\sup \left\{ \left| \langle \mathcal{I}_\alpha'(u_j) - \eta_j G_\alpha'(u_j), \varphi \rangle \right| : \varphi \in X_0^s(\Omega), \|\varphi\| = 1 \right\} \to 0 \tag{2.25}$$

as $j \to +\infty$ admits a subsequence which is convergent in $X_0^s(\Omega)$.

By Lemma 2.1 we know that the functional $\mathcal{I}_\alpha$ satisfies the (PS) condition at level $c < \frac{s}{N}(S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}$. Now, we are ready to show that the functional $\mathcal{I}_\alpha$ constrained on $\mathcal{N}_\alpha$ satisfies the (PS) condition at the same level.

Lemma 2.4. *Assume that $1 < q < 2$, β and γ are positive constants, and $\alpha > 0$. Then the functional $\mathcal{I}_\alpha$ constrained on $\mathcal{N}_\alpha$ satisfies the (PS) condition at any level $c < \frac{s}{N}(S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}$.*

Proof. Let $\{u_j\} \subset \mathcal{N}_\alpha$ be a sequence such that (2.9) holds and there exists $\{\eta_j\} \subset \mathbb{R}$ for which (2.25) is satisfied. First of all, we claim that $\{u_j\}$ is bounded in $L^q(\Omega)$ and $L^{2_s^*}(\Omega)$. Indeed, by (2.9) there exists a positive constant M such that

$$\left| \mathcal{I}_\alpha(u_j) \right| \leqslant M, \tag{2.26}$$

for any $j \in \mathbb{N}$. By (2.26) and the fact that $u_j \in \mathcal{N}_\alpha$, we obtain that

$$\begin{aligned}
M &\geqslant \mathcal{I}_\alpha(u_j) \\
&= \mathcal{I}_\alpha(u_j) - \frac{1}{2} \langle \mathcal{I}_\alpha'(u_j), u_j \rangle \\
&= \alpha \left(\frac{1}{q} - \frac{1}{2} \right) \int_\Omega |u_j|^q dx + \gamma \left(\frac{1}{2} - \frac{1}{2_s^*} \right) \int_\Omega |u_j|^{2_s^*} dx,
\end{aligned}$$

thus $\{u_j\}$ is bounded in $L^q(\Omega)$ and $L^{2_s^*}(\Omega)$. Hence, taking into account (2.22), we conclude that $\{\langle G_\alpha'(u_j), u_j \rangle\}$ is bounded in $\mathbb{R}$ and there exists $\theta \in (-\infty, 0]$ such that, up to a subsequence

$$\langle G_\alpha'(u_j), u_j \rangle \to \theta, \quad \text{as } j \to \infty. \tag{2.27}$$

Now, suppose that $\theta < 0$. Then, by (2.25), the fact that $u_j \in \mathcal{N}_\alpha$ and (2.27) we deduce that $\eta_j \to 0$ as $j \to \infty$. Hence, again by (2.25) we obtain that (2.10) holds. So, $\{u_j\} \subset \mathcal{N}_\alpha$ is a PS sequence for the functional $\mathcal{I}_\alpha$, the assertion of Lemma 2.4 follows from Lemma 2.1.

Finally, suppose that $\theta = 0$. By (2.22) and (2.27) we get that

$$\int_\Omega |u_j|^q dx \to 0 \quad \text{and} \quad \int_\Omega |u_j|^{2_s^*} dx \to 0, \quad \text{as } j \to \infty,$$

since $u_j \in \mathcal{N}_\alpha$, we get that $\|u_j\| \to 0$ as $j \to \infty$. Thus, $u_j \to 0$ in $X_0^s(\Omega)$ as $j \to \infty$. □

In order to obtain a ground state solution of (1.1), here we will use a theory which is introduced by Ambrosetti and Malchiodi in [2, Theorem 7.12].

Lemma 2.5. *Let E be a Banach space and $J \in C^{1,1}(E, \mathbb{R})$. If there exist $G \in C^{1,1}(E, \mathbb{R})$ such that $M = G^{-1}(0)$ with $G'(u) \neq 0$ for any $u \in M$. Moreover, suppose that J is bounded from below on M and satisfies $(PS)_m$ condition, where*

$$m := \inf_{u \in M} J(u) > -\infty.$$

Then the infimum m is achieved. Precisely, there is $z \in M$ such that $J(z) = m$ and $\nabla_M J(z) = 0$.

3 Proof of Theorem 1.1

In order to show that the equation (1.1) has a ground state solution, it suffices to verify that the infimum of $\mathcal{I}_\alpha$ on $\mathcal{N}_\alpha$ is attainable, in which the estimation of the energy of $\mathcal{I}_\alpha$ on $\mathcal{N}_\alpha$ is essential. Now, we have the following result.

Lemma 3.1. *Suppose that $\gamma > 0$, Then there exists $\alpha_1 > 0$, such that for any $\alpha \in (0, \alpha_1)$, there holds the estimate*

$$\inf_{u \in \mathcal{N}_\alpha} \mathcal{I}_\alpha(u) < \frac{s}{N} (S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}, \tag{3.1}$$

provided that

- $N \geqslant 4s$ *and* $0 < \beta < \lambda_{1,s}$, *or*

- $N > 2s$ *and* $\beta \geqslant \lambda_{1,s}$.

Proof. In order to prove (3.1) it is enough to show that there exists $u_0 \in \mathcal{N}_\alpha$ such that

$$\mathcal{I}_\alpha(u_0) < \frac{s}{N} (S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}. \tag{3.2}$$

Firstly, let us consider the case $0 < \beta < \lambda_{1,s}$. Let $\varepsilon > 0$ and u_ε be as in (2.3). By Lemma 2.2 there exists $t_\varepsilon > 0$ such that $t_\varepsilon u_\varepsilon \in \mathcal{N}_\alpha$, namely, that is

$$\langle \mathcal{I}'_\alpha(t_\varepsilon u_\varepsilon), t_\varepsilon u_\varepsilon \rangle = \alpha t_\varepsilon^q \int_\Omega |u_\varepsilon|^q dx + t_\varepsilon^2 \left(\|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx \right) - \gamma t_\varepsilon^{2_s^*} \int_\Omega |u_\varepsilon|^{2_s^*} dx = 0. \tag{3.3}$$

Then, in view of $0 < \beta < \lambda_{1,s}$ and (2.4), we obtain that

$$0 < \|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx \leqslant \|u_\varepsilon\|^2 \leqslant S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s}). \tag{3.4}$$

It follows from Hölder's inequality and (2.5) that

$$0 < \int_\Omega |u_\varepsilon|^q dx \leqslant |\Omega|^{\frac{2_s^*-q}{2_s^*}} \|u_\varepsilon\|_{2_s^*}^q \leqslant |\Omega|^{\frac{2_s^*-q}{2_s^*}} \left(S_s^{\frac{N}{2s}} + O\left(\varepsilon^N\right) \right)^{\frac{q}{2_s^*}}. \tag{3.5}$$

So (3.4) and (3.5) imply that there exists $K > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \max \left\{ \int_\Omega |u_\varepsilon|^q dx, \|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx, \int_\Omega |u_\varepsilon|^{2_s^*} dx \right\} \leqslant K. \tag{3.6}$$

Thank to $1 < q < 2 < 2_s^*$, by (3.3) and (3.6) we conclude that there exists $t_0 > 0$ such that

$$t_\varepsilon \in (0, t_0) \quad \text{for } \varepsilon \in (0, \varepsilon_0). \tag{3.7}$$

Let the function $f : [0, +\infty) \to \mathbb{R}$ given by

$$f(t) := \frac{1}{2}t^2 \left(\|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx \right) - \frac{\gamma}{2_s^*} t^{2_s^*} \int_\Omega |u_\varepsilon|^{2_s^*} dx,$$

then f admits the maximum point

$$t_{max} = \left(\frac{\|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx}{\gamma \int_\Omega |u_\varepsilon|^{2_s^*} dx} \right)^{\frac{1}{2_s^*-2}}$$

with the maximum value

$$f(t_{max}) = \frac{s}{N} \gamma^{\frac{2s-N}{2s}} \left(\frac{\|u_\varepsilon\|^2 - \beta\|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2_s^*}^2} \right)^{\frac{N}{2s}}. \tag{3.8}$$

We note that

$$\mathcal{I}_\alpha(t_\varepsilon u_\varepsilon) = \frac{\alpha}{q} t_\varepsilon^q \int_\Omega |u_\varepsilon|^q dx + \frac{1}{2} t_\varepsilon^2 \left(\|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx \right) - \frac{\gamma}{2_s^*} t_\varepsilon^{2_s^*} \int_\Omega |u_\varepsilon|^{2_s^*} dx. \tag{3.9}$$

From (3.9) and (3.8) it turns out

$$\mathcal{I}_\alpha(t_\varepsilon u_\varepsilon) \leqslant \frac{\alpha}{q} t_\varepsilon^q \int_\Omega |u_\varepsilon|^q dx + \frac{s}{N} \gamma^{\frac{2s-N}{2s}} \left(\frac{\|u_\varepsilon\|^2 - \beta\|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2_s^*}^2} \right)^{\frac{N}{2s}}. \tag{3.10}$$

Suppose that $N > 4s$, in view of (2.4)–(2.6), and by using the mean value theorem for the

function $(1+t)^{\frac{N-2s}{N}}$, we find that

$$
\begin{aligned}
\frac{\|u_\varepsilon\|^2 - \beta \|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2_s^*}^2} &\leqslant \frac{\left(S_s^{\frac{N}{2s}} + O\left(\varepsilon^{N-2s}\right)\right) - \beta\left(C_s\varepsilon^{2s} + O\left(\varepsilon^{N-2s}\right)\right)}{\left(S_s^{\frac{N}{2s}} + O\left(\varepsilon^N\right)\right)^{\frac{N-2s}{N}}} \\
&= S_s + \frac{S_s^{\frac{N}{2s}}\left(1 - \left(1 + S_s^{-\frac{N}{2s}}O\left(\varepsilon^N\right)\right)^{\frac{N-2s}{N}}\right) + O\left(\varepsilon^{N-2s}\right)}{\left(S_s^{\frac{N}{2s}} + O\left(\varepsilon^N\right)\right)^{\frac{N-2s}{N}}} \\
&\quad - \varepsilon^{2s}\frac{\beta\left(C_s + O\left(\varepsilon^{N-4s}\right)\right)}{\left(S_s^{\frac{N}{2s}} + O\left(\varepsilon^N\right)\right)^{\frac{N-2s}{N}}} \\
&= S_s + \frac{O\left(\varepsilon^N\right) + O\left(\varepsilon^{N-2s}\right)}{\left(S_s^{\frac{N}{2s}} + O\left(\varepsilon^N\right)\right)^{\frac{N-2s}{N}}} - \varepsilon^{2s}\frac{\beta\left(C_s + O\left(\varepsilon^{N-4s}\right)\right)}{\left(S_s^{\frac{N}{2s}} + O\left(\varepsilon^N\right)\right)^{\frac{N-2s}{N}}} \\
&< S_s
\end{aligned}
\tag{3.11}
$$

with $\varepsilon > 0$ sufficiently small. Now assume that $N = 4s$, in this case, by (2.4)–(2.6), we get

$$
\begin{aligned}
\frac{\|u_\varepsilon\|^2 - \beta \|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2_s^*}^2} &\leqslant \frac{\left(S_s^2 + O\left(\varepsilon^{2s}\right)\right) - \beta\left(C_s\varepsilon^{2s}|\log\varepsilon| + O\left(\varepsilon^{2s}\right)\right)}{\left(S_s^2 + O\left(\varepsilon^{4s}\right)\right)^{\frac{1}{2}}} \\
&= S_s + \varepsilon^{2s}\frac{\left(O\left(\varepsilon^{2s}\right) + O(1)\right) - \beta\left(C_s|\log\varepsilon| + O(1)\right)}{\left(S_s^2 + O\left(\varepsilon^{4s}\right)\right)^{\frac{1}{2}}} \\
&< S_s
\end{aligned}
\tag{3.12}
$$

when $\varepsilon > 0$ is small enough, since $|\log\varepsilon| \to +\infty$ as $\varepsilon \to 0$.

So we can choose $\varepsilon > 0$ sufficiently small such that (3.11), (3.12) and $\varepsilon < \varepsilon_0$ hold. For this ε, let $N \geqslant 4s$, $u_0 = t_\varepsilon u_\varepsilon$. By (3.6), (3.7) and (3.10), then there is $\alpha_4 > 0$, if $0 < \alpha < \alpha_4$, such that (3.2) holds.

Secondly, in the case of $\beta \geqslant \lambda_{1,s}$. Fix $u \in X_0^s(\Omega)\backslash\{0\}$, by Lemma 2.2, there exists a unique $t_u \in (0, +\infty)$, such that

$$
\left\langle \mathcal{I}_\alpha'\left(t_u u\right), t_u u \right\rangle = 0,
\tag{3.13}
$$

Hölder inequality and (3.13) imply that

$$
\gamma t_u^{2_s^*} \int_\Omega |u|^{2_s^*} dx \leqslant \alpha t_u^q \int_\Omega |u|^q dx \leqslant \alpha|\Omega|^{\frac{2_s^*-q}{2_s^*}} t_u^q \|u\|_{2_s^*}^q,
$$

thus

$$
t_u \leqslant \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2_s^*-q}} \frac{|\Omega|^{\frac{1}{2_s^*}}}{\|u\|_{2_s^*}}.
\tag{3.14}
$$

Hence, by the fact that $\beta \geqslant \lambda_{1,s}$, (3.14) and Hölder's inequality we conclude that

$$
\mathcal{I}_\alpha(t_u u) \leqslant \frac{\alpha}{q} t_u^q \int_\Omega |u|^q dx \leqslant \frac{\alpha}{q}\left(\frac{\alpha}{\gamma}\right)^{\frac{q}{2_s^*-q}} \frac{|\Omega|^{\frac{q}{2_s^*}}}{\|u\|_{2_s^*}^q}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\|u\|_{2_s^*}^q = \frac{\alpha^{\frac{2_s^*}{2_s^*-q}}}{q\gamma^{\frac{q}{2_s^*-q}}}|\Omega|
$$

So we choose $u_0 = t_u u$, there exists $\alpha_5 > 0$ such that (3.2) holds provided that $0 < \alpha < \alpha_5$.

Let $\alpha_1 = \min\{\alpha_4, \alpha_5\}$, Assume $N \geqslant 4s$ and $0 < \beta < \lambda_{1,s}$, or $N > 2s$ and $\beta \geqslant \lambda_{1,s}$. Then there exists $u_0 \in \mathcal{N}_\alpha$, if $\alpha \in (0, \alpha_1)$, such that (3.2) holds. $\qquad\square$

Finally we are ready to apply the above lemmas to prove the first main result.

Proof of Theorem 1.1. Taking into account the definitions of $\mathcal{I}_\alpha$ and $\mathcal{N}_\alpha$, it is easy to verify that $\mathcal{I}_\alpha, G_\alpha \in C^{1,1}(X_0^s(\Omega))$, whose proof is similar to that of [22, 8.5.2 Theorem 3]. Lemma 2.3 imply that

$$\mathcal{N}_\alpha = G_\alpha^{-1}(0), \quad \langle G_\alpha'(u), u \rangle < 0 \text{ for } u \in \mathcal{N}_\alpha \text{ and } \inf_{u \in \mathcal{N}_\alpha} \mathcal{I}_\alpha(u) \geqslant 0.$$

By Lemma 3.1, we know that there exists $\alpha_1 > 0$ such that

$$m_\alpha := \inf_{u \in \mathcal{N}_\alpha} \mathcal{I}_\alpha(u) < \frac{s}{N}(S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}$$

for $\alpha \in (0, \alpha_1)$ provided that $N \geqslant 4s$ and $0 < \beta < \lambda_{1,s}$, or $N > 2s$ and $\beta \geqslant \lambda_{1,s}$. In view of Lemma 2.4, we deduce that the functional $\mathcal{I}_\alpha$ constrained on $\mathcal{N}_\alpha$ satisfies the $(PS)_{m_\alpha}$ condition. According to Lemma 2.5, let E and M be $X_0^s(\Omega)$ and $\mathcal{N}_\alpha$ respectively, then there exists $u_{m_\alpha} \in \mathcal{N}_\alpha$ such that

$$\mathcal{I}_\alpha(u_{m_\alpha}) = m_\alpha \quad \text{and} \quad \mathcal{I}_\alpha'|_{\mathcal{N}_\alpha}(u_{m_\alpha}) = 0.$$

Moreover, Lemma 2.3 implies that $\mathcal{I}_\alpha'(u_{m_\alpha}) = 0$, thus u_{m_α} is a ground state solution of problem (1.1). $\qquad\square$

4   Proof of Theorem 1.2

Proof. We first prove that the functional $\mathcal{I}_\alpha$ has mountain pass geometry when the conditions of Theorem 1.1 are satisfied. Let $\alpha_1 > 0$ be given in Theorem 1.1, it suffices to show that the following assertions hold provided that $0 < \alpha < \alpha_1$.

(i) there are $\rho, r > 0$ such that for $u \in X_0^s(\Omega)$ with $\|u\| = \rho$, we have $\mathcal{I}_\alpha(u) \geqslant r$.

(ii) there exists $e \in X_0^s(\Omega)$ such that $\|e\| > \rho$ and $\mathcal{I}_\alpha(e) < 0$.

We claim that $u \equiv 0$ is a strict local minimizer of the functional $\mathcal{I}_\alpha$. In virtue of [31, Theorem 1.1], it suffices to prove this claim in the space $C_s^0(\bar{\Omega}) \cap X_0^s(\Omega)$, where

$$C_s^0(\bar{\Omega}) = \left\{ w \in C^0(\bar{\Omega}) : \|w\|_{C_s^0} := \left\| \frac{w}{\delta^s} \right\|_{L^\infty} < \infty \right\}$$

with $\delta(x) := \text{dist}(x, \partial\Omega)$. Notice that $\sup_{x \in \Omega} \delta(x) \leqslant \text{diam}(\Omega)$, then for any $u \in C_s^0(\bar{\Omega}) \cap X_0^s(\Omega)$ we have that

$$\int_\Omega u^2 dx = \int_\Omega \left(\frac{|u|}{\delta^s} \right)^{2-q} (\delta^s)^{2-q} |u|^q dx \leq C_1 \|u\|_{C_s^0}^{2-q} \int_\Omega |u|^q dx \tag{4.1}$$

and

$$\int_\Omega |u|^{2_s^*} dx = \int_\Omega \left(\frac{|u|}{\delta^s} \right)^{2_s^* - q} (\delta^s)^{2_s^* - q} |u|^q dx \leq C_2 \|u\|_{C_s^0}^{2_s^* - q} \int_\Omega |u|^q dx \tag{4.2}$$

with positive constants C_1 and C_2. From (4.1) and (4.2) we obtain

$$\mathcal{I}_\alpha(u) \geq \frac{1}{2}\|u\|^2 + \left(\frac{\alpha}{q} - \frac{\beta C_1}{2}\|u\|_{C_s^0}^{2-q} - \frac{\gamma C_2}{2_s^*}\|u\|_{C_s^0}^{2_s^* - q} \right) \int_\Omega |u|^q dx. \tag{4.3}$$

Since β and γ are positive constants and $1 < q < 2 < 2_s^*$, by (4.3) we deduce that $u \equiv 0$ is a strict local minimizer of $\mathcal{I}_\alpha$ in $C_s^0(\bar{\Omega}) \cap X_0^s(\Omega)$ for any $\alpha > 0$. Thus the assertion (i) holds.

Next, we show that the assertion (ii) is true. Let u_{m_α} be the ground state solution obtained in Theorem 1.1. For $t > 0$, we have

$$\mathcal{I}_\alpha(tu_{m_\alpha}) = \frac{t^2}{2}(\|u_{m_\alpha}\|^2 - \beta \int_\Omega u_{m_\alpha}^2 dx) + \frac{\alpha}{q}t^q \int_\Omega |u_{m_\alpha}|^q dx - \frac{\gamma}{2_s^*}t^{2_s^*} \int_\Omega |u_{m_\alpha}|^{2_s^*} dx. \qquad (4.4)$$

For any $\alpha \in (0, \alpha_1)$, thanks to $1 < q < 2 < 2_s^*$ and (4.4), there is $t_0 \in (0, +\infty)$ sufficiently large such that $\|t_0 u_{m_\alpha}\| > \rho$ and $\mathcal{I}_\alpha(t_0 u_{m_\alpha}) < 0$. So we complete the proof of (ii) by choosing $e = t_0 u_{m_\alpha}$.

Set the minimax value

$$c_\alpha := \inf_{h\in\Gamma} \max_{t\in[0,1]} I_\alpha(h(t)),$$

where

$$\Gamma = \{h \in C([0,1], X_0^s(\Omega)) : h(0) = 0 \text{ and } h(1) = e\}$$

where $e = t_0 u_{m_\alpha}$ is given in (ii). By Lemma 2.2 and Lemma 3.1, we have that

$$c_\alpha \leqslant \max_{t\in[0,t_0]} \mathcal{I}_\alpha(tu_{m_\alpha}) = \mathcal{I}_\alpha(u_{m_\alpha}) < \frac{s}{N}(S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}.$$

So, the functional $\mathcal{I}_\alpha$ possesses mountain path geometry, by Lemma 2.1, the functional $\mathcal{I}_\alpha$ satisfies the (PS) condition at the level c_α. Therefore, in view of the Mountain Pass theorem, we conclude that c_α is a critical value of $\mathcal{I}_\alpha$. According to (i), we have $c_\alpha \geqslant r > 0$, even it is obvious that $c_\alpha = \mathcal{I}_\alpha(u_{m_\alpha})$. Hence problem (1.1) has a ground state solution u_{c_α} with $I_\alpha(u_{c_\alpha}) = c_\alpha$. $\qquad \square$

Acknowledgements

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